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PAPER

Relative completeness of arithmetical numberings

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Abstract

This paper studies two approaches to relativizing the notions of complete and precomplete numbering. The first one was introduced by Selivanov in the late 1980s, and it strengthens the standard definitions of complete and precomplete numbering. The second one was introduced by Badaev, Goncharov, and Sorbi in the early 2000s, and it is the full relativization of these two concepts. In the first part of the paper, we study how these two approaches differ from each other. In the second part, we study Mal'cev's object uniquely for the relativized complete numberings.

Keywords: Arithmetical numbering; complete numbering; precomplete numbering; principal numbering

1. Introduction

This paper is concerned with the notion of completeness of numberings introduced by Mal'cev (1963) and its relativizations. A numbering ν of a set S (i.e., an arbitrary surjective mapping of the set of all non-negative integers $\mathbb{N}$ onto S) is said to be *complete with respect to* $X \in S$ if for every partial recursive function ψ there exists a recursive h such that

$$\nu(h(x)) = \begin{cases} \nu(\psi(x)), & \text{if } \psi(x)\downarrow, \\ X, & \text{if } \psi(x)\uparrow, \end{cases} \tag{1}$$

for each x. If for every partial recursive function ψ there exists a recursive h satisfying (1) for each x, then X is called a *special object* of the numbering ν. We say that a numbering is *complete* if it has a special object.

The proofs of Kleene's recursion theorem, Rice's theorem, the infinite padding property, and many other well-known results in recursion theory rely on the completeness with respect to $\varnothing$ of the standard Gödel numberings $x \mapsto W_x$ and $x \mapsto \varphi_x$ of the families of all r.e. sets and partial recursive functions, respectively. Therefore, any complete numbering satisfies each of these theorems. In addition, complete numberings find applications in control structures theory (Jain and Nessel 2001), computable analysis (Selivanov 2021a), and other areas of computability (cf., e.g., Ershov 1969a; Faizrahmanov 2023a; Terwijn 2024), and they make it possible to combine important results in recursion theory due to Muchnik (1963), Myhill (1955), Rogers (1958), and Smullyan (1961), which previously seemed unrelated to each other.

A weaker property than the completeness of numberings is the property of their *precompleteness* introduced by Ershov (1970, 1973). He called a numbering ν *precomplete* if for every partial recursive function ψ there exists a recursive h such that

$$\psi(x)\downarrow \Rightarrow \nu(h(x)) = \nu(\psi(x)) \tag{2}$$

for each x. The precomplete numberings can be characterized by the following theorem.

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Theorem 1 (Ershov's recursion theorem (Ershov 1977)). *A numbering ν is precomplete if and only if for every binary partial recursive function ψ there exists a recursive function h such that $\nu(\psi(x, h(x))) = \nu(h(x))$ for each x with $\psi(x, h(x)) \downarrow$.*

Even though not every precomplete numbering is complete, precomplete numberings still have many properties of complete numberings. Ershov's recursion theorem implies that they satisfy Kleene's recursion theorem. In addition, they have infinite padding (see Ershov (1977)) and other properties common in recursion theory (cf., e.g., Selivanov (2021b)).

In this paper, we study the relativized properties of completeness and precompleteness of numberings from two perspectives. The first one is the full relativization of these two concepts.

Definition 1 (Badaev et al. 2003). *Given an oracle A, a numbering ν is said to be* A-precomplete *(A-complete with respect to $X \in \nu(\mathbb{N})$) if for any partial A-recursive function ψ there exists an A-recursive h satisfying condition (2) (condition (1), respectively) for each x.*

The precompleteness property, relativized in this way, was also studied by Golov and Terwijn in their recent paper (Golov and Terwijn 2022). There, the following question was raised (see Golov and Terwijn 2022, Question 3.8).

Question 1. *Does there exist a set A and an A-precomplete numbering that is not precomplete?*

A positive answer to this question is given by Theorem 2 of this paper. It proves that there are even A-complete numberings that are not precomplete.

Another relativization of (pre)completeness was previously studied by Selivanov (1988, 2021b). In his definition, Selivanov requires the totalization h of ψ to be absolutely recursive, rather than merely A-recursive.

Definition 2. *Given an oracle A, a numbering ν is said to be* Selivanov A-precomplete *(*Selivanov A-complete with respect to $X \in \nu(\mathbb{N})$*) if for any partial A-recursive function ψ there exists a recursive h satisfying condition (2) (condition (1), respectively) for each x.*

If a numbering ν is A-complete or Selivanov A-complete with respect to $X \in \nu(\mathbb{N})$, then the element X will also be called a *special object* of this numbering, and from the context it will always be clear what version of the completeness we are talking about.

Theorem 5 and Corollary 4 of this paper study Mal'cev's problem (see Denisov and Lavrov 1970) of whether an arbitrary complete numbering determines its special object uniquely for A-complete and Selivanov A-complete numberings. From Corollary 4, it follows that given an oracle A, for every nonempty at most countable set S and for any of its nonempty subsets $S_0 \subseteq S$, there exists a numbering ν of S that is both A-complete and Selivanov A-complete and whose set of special objects is S_0.

In the notation and terminology of recursion theory, we mainly follow Soare's monograph (Soare 2016). By $\Phi_e^A(x)$, we denote the partial Turing functional with the Gödel number e. For a partial function ψ, we write $\psi(x) \downarrow$ if the value of ψ on the argument x is defined, and $\psi(x) \uparrow$ otherwise. We denote its domain and range by dom ψ and ran ψ, respectively. Let $c(x, y)$ denotes the recursive pairing function $2^x(2y + 1) - 1$. Instead of $c(x, c(y, z))$ we will simply write $c(x, y, z)$.

2. Preliminaries on numberings

For the main concepts and notions of the theory of numberings, we refer to the book by Ershov (1977) and his paper Ershov (1999). Given numberings μ and ν with $\mu(\mathbb{N}) \subseteq \nu(\mathbb{N})$, we say that μ

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is *reducible* to ν, abbreviated $\mu \leqslant \nu$, if there exists a recursive function f such that

$$\mu = \nu \circ f.$$

Definition 3 (Goncharov and Sorbi (1997)). *A numbering ν of a family $\mathscr{A}$ of Σ^0_{n+1}-sets, $n \in \mathbb{N}$, is said to be* Σ^0_{n+1}-computable *if the set of pairs*

$$G_\nu = \{\langle x, y\rangle \in \mathbb{N} \times \mathbb{N} : y \in \nu(x)\}$$

is a Σ^0_{n+1}-set.

We say that $\mathscr{A}$ is Σ^0_{n+1}-*computable* if it has a Σ^0_{n+1}-computable numbering. For $n = 0$, Σ^0_{n+1}-computable numberings and families are called *computable*.

In the theory of numberings, the completeness property is often studied in Σ^0_{n+2}-computable numberings, $n \in \mathbb{N}$, (see Badaev et al. 2003, 2008; Khisamiev 2006). This is because every Σ^0_{n+2}-computable family has a complete Σ^0_{n+2}-computable numbering, while a computable family has a complete computable numbering only if it contains a least element under inclusion (cf., e.g., Jain and Nessel 2001; Badaev et al. 2003). Moreover, every Σ^0_{n+2}-computable principal numbering is complete with respect to each of its elements (see Badaev et al. 2003). Additionally, for all n, any Σ^0_{n+2}-computable family has a Selivanov $\varnothing^{(n)}$-complete Σ^0_{n+2}-computable numbering. Indeed, the *completion* (with respect to $X \in \alpha(\mathbb{N})$) of any Σ^0_{n+2}-computable numbering α (see Ershov 1970), relativized with respect to $\varnothing^{(n)}$ (see Badaev et al. 2003), that is, the numbering

$$\alpha^{\mathbf{0}^{(n)}}_X(c(e,x)) = \begin{cases} \alpha(\Phi_e^{\varnothing^{(n)}}(x)), & \text{if } \Phi_e^{\varnothing^{(n)}}(x)\downarrow, \\ X, & \text{if } \Phi_e^{\varnothing^{(n)}}(x)\uparrow, \end{cases}$$

is obviously Selivanov $\varnothing^{(n)}$-complete with respect to $X \in \alpha(\mathbb{N})$ and Σ^0_{n+2}-computable. For these reasons, in this paper, we also limit ourselves to considering Σ^0_{n+2}-computable numberings, $n \in \mathbb{N}$.

Definition 4. *We say that a Σ^0_{n+1}-numbering ν is* principal *if $\mu \leqslant \nu$ for every Σ^0_{n+1}-numbering μ such that $\mu(\mathbb{N}) \subseteq \nu(\mathbb{N})$.*

If the family $\mathscr{A}$ contains at least two sets, then we will call it *non-trivial*. We say that a numbering ν is *decidable* if its *numeration equivalence*

$$\eta_\nu = \{\langle x, y\rangle \in \mathbb{N} \times \mathbb{N} : \nu(x) = \nu(y)\}$$

is recursive.

If in the definition of the A-complete numbering ν, we take as ψ the partial function

$$c(e,x) \mapsto \Phi_e^A(x),$$

then we obtain that there exists a binary A-recursive function f satisfying the condition

$$\nu(f(e,x)) = \begin{cases} \nu(\Phi_e^A(x)), & \text{if } \Phi_e^A(x)\downarrow, \\ X, & \text{if } \Phi_e^A(x)\uparrow, \end{cases} \tag{3}$$

for all e and x.

3. Relativized complete and precomplete numberings

The following theorem implies that there exists an oracle A and an A-complete numbering that is not precomplete. Thus, it solves Question 1.

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Theorem 2. *Every non-trivial Σ^0_{n+2}-computable family $\mathscr{A}$ with a least element under inclusion has an $\varnothing^{(n+1)}$-complete Σ^0_{n+2}-computable numbering that is not $\varnothing^{(m)}$-precomplete for each $m \leqslant n$.*

Proof. Let α be a Σ^0_{n+2}-computable numbering of $\mathscr{A}$ and let X be the least element of $\mathscr{A}$ under inclusion. Fix $Z \in \mathscr{A}$ such that $Z \neq X$. It was proved by Ershov (1977) that every precomplete numbering β is *cylindrical*, that is, there exists a binary injective recursive function g such that $\beta(x) = \beta(g(x,t))$ for all x and t. This result can be relativized to an arbitrary oracle. To prove the statement of the theorem, we are going to define an $\varnothing^{(n+1)}$-complete Σ^0_{n+2}-computable numbering ν of $\mathscr{A}$ that is not $\varnothing^{(n)}$-cylindrical.

Let $\{g_u\}_{u\in\mathbb{N}}$ be an $\varnothing^{(n+1)}$-recursive sequence of all binary injective $\varnothing^{(n)}$-recursive functions. Such a sequence can be obtained by an obvious modification of an $\varnothing^{(n+1)}$-recursive sequence of all $\varnothing^{(n)}$-recursive functions. Now, we choose $\varnothing^{(n+1)}$-recursive sequences of integers $\{s_u\}_{u\in\mathbb{N}}$, $\{t_u\}_{u\in\mathbb{N}}$, and $\{y_u\}_{u\in\mathbb{N}}$ such that

$$g_0(0,s_0) < g_0(0,t_0) < y_0 < y_0 + 1 < g_1(1,s_1) < g_1(1,t_1) < y_1 < \ldots,$$

and for every u, define

$$\nu(g_u(u,s_u)) = X,\ \ \nu(g_u(u,t_u)) = Z,\ \ \nu(y_u) = \alpha(u).$$

Thus, ν is not $\varnothing^{(n)}$-cylindrical and $\nu(\mathbb{N}) = \mathscr{A}$. Further, we define the binary $\varnothing^{(n+1)}$-recursive f by letting

$$f(e,x) = y_{c(e,x)} + 1$$

for all e and x. For every

$$z \notin \bigcup_u \{g_u(u,s_u), g_u(u,t_u), y_u\} \cup \operatorname{ran} f,$$

we let $\nu(z) = X$.

Let us determine the values of ν on $f(e,x)$, $e,x \in \mathbb{N}$, in such a way that condition (3), where $A = \varnothing^{(n+1)}$, is met for all e and x. To do this, we define the relation η as the equivalence relation on $\mathbb{N}$ generated by the binary relation

$$\left\{\left\langle f(e,x), \Phi_e^{\varnothing^{(n+1)}}(x)\right\rangle : e,x \in \mathbb{N}\ \&\ \Phi_e^{\varnothing^{(n+1)}}(x)\downarrow\right\}.$$

Thus, η is $\varnothing^{(n+1)}$-r.e. It is easy to check by direct verification that for any e and x there exists at most one z such that

$$\left\langle f(e,x), z\right\rangle \in \eta\ \&\ z \notin \operatorname{ran} f. \tag{4}$$

Hence, for any e and x, we can define

$$\nu(f(e,x)) = \begin{cases} \nu(z), & \text{if } \left\langle f(e,x), z\right\rangle \in \eta\ \&\ z \notin \operatorname{ran} f, \\ X, & \text{otherwise.} \end{cases} \tag{5}$$

Since the function $c(e,x) \mapsto f(e,x)$ is strictly increasing, the predicate "$z \notin \operatorname{ran} f$" is $\varnothing^{(n+1)}$-recursive. Since, besides this, X is the least element of $\mathscr{A}$ under inclusion, ν is Σ^0_{n+2}-computable.

Let us show that condition (3), where $A = \varnothing^{(n+1)}$, is met for all e and x. If $\Phi_e^{\varnothing^{(n+1)}}(x)\uparrow$, then, by the definition of η, there is no z satisfying (4). Therefore,

$$\nu(f(e,x)) = X.$$

Suppose $\Phi_e^{\varnothing^{(n+1)}}(x)\downarrow$. If there is z satisfying (4), then either $\Phi_e^{\varnothing^{(n+1)}}(x) = z$ or $\Phi_e^{\varnothing^{(n+1)}}(x) = f(j,w)$ for some j and w with $\left\langle f(j,w), z\right\rangle \in \eta$. Hence,

$$\nu(z) = \nu\left(\Phi_e^{\varnothing^{(n+1)}}(x)\right) = \nu(f(e,x)).$$

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If there is no z satisfying (4), then for every y such that $\langle f(e,x), y\rangle \in \eta$, we have $\nu(y) = X$. It follows that $\nu(\Phi_e^{\varnothing^{(n+1)}}(x)) = X = \nu(f(e,x))$. □

The following corollary follows from the proof of Theorem 2.

Corollary 1. *Let $\mathscr{A}$ be a non-trivial Σ^0_{n+3}-computable family. Then for all $k \leqslant n+1$ with $k > 0$ and $X \in \mathscr{A}$, the family $\mathscr{A}$ has a Σ^0_{n+3}-computable numbering $\varnothing^{(k)}$-complete with respect to X that is not $\varnothing^{(m)}$-precomplete for each $m < k$.*

Proof. Let ν be the non-$\varnothing^{(k-1)}$-cylindrical numbering $\varnothing^{(k)}$-complete with respect to X defined by (5) (where f and η are defined exactly as in the proof of Theorem 2, and $\{g_u\}_{u\in\mathbb{N}}$ is an $\varnothing^{(k)}$-recursive sequence of all binary injective $\varnothing^{(k-1)}$-recursive functions). Since η is $\varnothing^{(n+1)}$-r.e., it is $\varnothing^{(n+2)}$-recursive. Therefore, ν is Σ^0_{n+3}-computable. This completes the proof of the corollary. □

Corollary 2. *For every n there exists an $\varnothing^{(n+1)}$-complete Σ^0_{n+2}-computable numbering that is not Selivanov $\varnothing^{(n+1)}$-precomplete.*

Proof. Since any Selivanov $\varnothing^{(n+1)}$-precomplete numbering is $\varnothing^{(m)}$-precomplete for each $m \leqslant n$, the statement of the corollary follows immediately from Theorem 2. □

Now we will show that Σ^0_{n+2}-computable families may not even have $\varnothing^{(n+1)}$-precomplete Σ^0_{n+2}-computable numberings.

Proposition 3. *Let $\mathscr{A}$ be a non-trivial Σ^0_{n+2}-computable family of (graphs of) total functions. Then $\mathscr{A}$ has no $\varnothing^{(n+1)}$-precomplete Σ^0_{n+2}-computable numberings.*

Proof. Since $\mathscr{A}$ is a non-trivial family of total functions, the numeration equivalence of any of its Σ^0_{n+2}-computable numberings is co-$\varnothing^{(n+1)}$-r.e. Hence, for every Σ^0_{n+2}-computable numbering α of $\mathscr{A}$ there exists a function $h \leqslant_T \varnothing^{(n+1)}$ such that $\alpha(x) \neq \alpha(h(x))$ for each x. But it was proved by Golov and Terwijn (see Golov and Terwijn (2022, Theorem 3.2) that for every set A, any A-recursive function has a fixed point in A-precomplete numberings. Therefore, α is not $\varnothing^{(n+1)}$-precomplete. □

In Golov and Terwijn (2022), it was proved that there exists a set A and a numbering ν such that ν is precomplete, but not A-precomplete. It follows from the following theorem that even complete numberings may not be A-precomplete.

Theorem 3. *Let $\mathscr{A}$ be a non-trivial Σ^0_{n+2}-computable family that has an element $X \in \mathscr{A}$ such that the family of all $Y \in \mathscr{A}$ with $Y \neq X$ is Σ^0_{n+2}-computable. Then for all integers $k < m \leqslant n+1$, the family $\mathscr{A}$ has a Selivanov $\varnothing^{(k)}$-complete Σ^0_{n+2}-computable numbering that is not $\varnothing^{(m)}$-precomplete.*

Proof. Let X be an element of $\mathscr{A}$ such that the family $\mathscr{A}\setminus\{X\}$ is Σ^0_{n+2}-computable and let γ be a Σ^0_{n+2}-computable numbering of $\mathscr{A}\setminus\{X\}$. Fix arbitrary integers k and m with $k < m \leqslant n+1$.

In order to define an $\varnothing^{(k)}$-complete Σ^0_{n+2}-computable numbering ν of $\mathscr{A}$ that is not $\varnothing^{(m)}$-precomplete, we need the $\varnothing^{(k)}$-r.e. equivalence relation η generated by the binary relation

$$\left\{\left\langle c(e,x), \Phi_e^{\varnothing^{(k)}}(x)\right\rangle : e, x \in \mathbb{N} \;\&\; \Phi_e^{\varnothing^{(k)}}(x)\downarrow\right\}.$$

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Using the recursion theorem and its uniformity, we choose an injective recursive sequence of integers $\{p_t\}_{t\in\mathbb{N}}$ such that

$$\Phi_{p_t}^{\varnothing^{(k)}}(0) = c(p_t, 0),\ t \in \mathbb{N}.$$

It is not hard to verify (for example, using the arguments from the proof of (Faizrahmanov 2023b, Theorem 1)) that

$$\langle c(p_t, 0), c(p_u, 0)\rangle \notin \eta \tag{6}$$

for all distinct integers t and u and

$$\langle c(p_t, 0), c(e, x)\rangle \notin \eta \tag{7}$$

for each t and all e, x with $\Phi_e^{\varnothing^{(k)}}(x)\uparrow$. Now, we define

$$\nu(y) = \begin{cases} \gamma(t), & \text{if } \langle c(p_t, 0), y\rangle \in \eta, \\ X, & \text{if } \forall t\left[\langle c(p_t, 0), y\rangle \notin \eta\right], \end{cases}$$

for each y. Since $k < n + 1$, this numbering is Σ^0_{n+2}-computable. To show that it is Selivanov $\varnothing^{(k)}$-complete, we are going to prove that

$$\nu(c(e, x)) = \begin{cases} \nu(\Phi_e^{\varnothing^{(k)}}(x)), & \text{if } \Phi_e^{\varnothing^{(k)}}(x)\downarrow, \\ X, & \text{if } \Phi_e^{\varnothing^{(k)}}(x)\uparrow, \end{cases} \tag{8}$$

for all e and x. If $\Phi_e^{\varnothing^{(k)}}(x)\uparrow$, then, by (7),

$$\nu(c(e, x)) = X.$$

Suppose $\Phi_e^{\varnothing^{(k)}}(x)\downarrow$. If

$$\langle c(p_t, 0), \Phi_e^{\varnothing^{(k)}}(x)\rangle \notin \eta$$

for each t, then since $\langle c(e, x), \Phi_e^{\varnothing^{(k)}}(x)\rangle \in \eta$, we have that

$$\langle c(p_t, 0), c(e, x)\rangle \notin \eta$$

for each t as well. Therefore,

$$X = \nu(\Phi_e^{\varnothing^{(k)}}(x)) = \nu(c(e, x)).$$

Assume that there exists t such that

$$\langle c(p_t, 0), \Phi_e^{\varnothing^{(k)}}(x)\rangle \in \eta.$$

Then

$$\langle c(p_t, 0), c(e, x)\rangle \in \eta$$

as well. It follows that

$$\gamma(t) = \nu\left(\Phi_e^{\varnothing^{(k)}}(x)\right) = \nu(c(e, x)).$$

Thus, (8) is met for all e, x and, hence, ν is Selivanov $\varnothing^{(k)}$-complete.

To prove that ν is not $\varnothing^{(m)}$-precomplete, we define a function $h \leqslant_T \varnothing^{(m)}$ such that

$$\nu(h(y)) \neq \nu(y) \tag{9}$$

for each y. Fix z with $\nu(z) = X$ and for every y, define

$$h(y) = \begin{cases} z, & \text{if } \exists t\left[\langle c(p_t, 0), y\rangle \in \eta\right], \\ c(p_0, 0), & \text{otherwise.} \end{cases}$$

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By the definition of η, for every y, we have

$$\nu(y) = X \Leftrightarrow \forall t\, [\langle c(p_t, 0), y\rangle \notin \eta].$$

Since $\nu(c(p_t, 0)) = \gamma(t) \neq X$ for each t, (9) is met for each y. □

Note that any non-trivial family $\mathscr{A}$ with a decidable Σ^0_{n+2}-computable numbering satisfies the statement of Theorem 3. In particular, this statement is satisfied by the family of all Σ^0_{n+2}-sets and by any non-trivial finite family of Σ^0_{n+2}-sets.

It was proved in Badaev and Goncharov (2014) by Badaev and Goncharov that for every n there exists a family without a least element under inclusion which has a Σ^0_{n+2}-computable principal numbering. Since this family has no least element under inclusion, it has no $\varnothing^{(n+1)}$-complete numberings (cf., e.g., Badaev et al. 2003). The following theorem shows that this family nevertheless has a Selivanov $\varnothing^{(n+1)}$-precomplete Σ^0_{n+2}-computable numbering. This theorem also proves that a Selivanov $\varnothing^{(n+1)}$-complete Σ^0_{n+2}-computable numbering may not be principal. Previously, Badaev, Goncharov, and Sorbi showed (Badaev et al. 2008) that there exists a family with the least element under inclusion that has a complete Σ^0_{n+2}-computable non-principal numbering.

Theorem 4.
1) *Every Σ^0_{n+2}-computable principal numbering is Selivanov $\varnothing^{(n+1)}$-precomplete.*
2) *For every n there exists a family $\mathscr{A}$ with the least element under inclusion that has a Selivanov $\varnothing^{(n+1)}$-complete Σ^0_{n+2}-computable numbering, but has no Σ^0_{n+2}-computable principal numbering.*

Proof.
(1) Let ν be a Σ^0_{n+2}-computable principal numbering ant let ψ be an arbitrary partial $\varnothing^{(n+1)}$-recursive function. To construct a recursive h that for each x satisfies (2), we fix an $\varnothing'$-recursive sequence $\{h_m\}_{m\in\mathbb{N}}$ of all recursive functions and define the Σ^0_{n+2}-computable numbering μ of a subfamily of $\nu(\mathbb{N})$ by letting

$$\mu(c(m, x, s)) = \begin{cases} \nu(\psi_s(x)), \text{ if } \psi_s(x)\downarrow, \\ \nu(h_m(c(m, x, s+1))), \text{ if } \psi_s(x)\uparrow, \end{cases}$$

for all m, x, and s. Since ν is principal, we can fix an index e such that $\mu = \nu \circ h_e$. Further, we choose an arbitrary x. If $\psi(x)\downarrow$, then we fix the least t such that $\psi_t(x)\downarrow$. Then

$$\nu(h_e(c(e, x, s))) = \mu(c(e, x, s)) = \nu(h_e(c(e, x, s+1)))$$

for each $s < t$ and

$$\nu(\psi_t(x)) = \mu(c(e, x, t)) = \nu(h_e(c(e, x, t))).$$

Thus, if $\psi(x)\downarrow$, then $\nu(\psi(x)) = \nu(h_e(c(e, x, 0)))$.

Therefore, the recursive function

$$h : x \mapsto h_e(c(e, x, 0))$$

satisfies (2) for each x.

(2) Let $\{X_m\}_{m\in\mathbb{N}}$ be an $\varnothing^{(n+1)}$-r.e. sequence of sets such that

$$X_m \subseteq X_{m+1} \,\&\, X_m \neq \bigcup_k X_k$$

for each m. Relativizing Lachlan's result (see Lachlan 1964; Badaev and Goncharov 2014), we obtain that if a family has a Σ^0_{n+2}-computable principal numbering, then it is closed under unions of $\varnothing^{(n+1)}$-r.e. sequences of elements non-decreasing under inclusion. Hence, the

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Σ^0_{n+2}-computable family

$$\mathscr{A} = \{X_m : m \in \mathbb{N}\}$$

has no Σ^0_{n+2}-computable principal numberings.

To show that $\mathscr{A}$ has a Selivanov $\varnothing^{(n+1)}$-complete Σ^0_{n+2}-computable numbering, we define an $\varnothing^{(n+1)}$-r.e. equivalence relation η as the equivalence generated by the binary relation

$$\left\{\left\langle c(e,x), \Phi_e^{\varnothing^{(n+1)}}(x)\right\rangle : e, x \in \mathbb{N} \,\&\, \Phi_e^{\varnothing^{(n+1)}}(x)\downarrow\right\}.$$

By the recursion theorem, there exists a recursive sequence of integers $\{p_t\}_{t\in\mathbb{N}}$ such that

$$\Phi_{p_t}^{\varnothing^{(n+1)}}(0) = c(p_t, 0)$$

for each t. In the same way as in the proof of Theorem 3, we have that (6) is met for all distinct integers t and u and (7) is met for each t and all e, x with $\Phi_e^{\varnothing^{(n+1)}}(x)\uparrow$, and also that the Σ^0_{n+2}-computable numbering ν of $\mathscr{A}$ defined by

$$\nu(y) = \begin{cases} X_t, & \text{if } \langle c(p_t,0), y\rangle \in \eta, \\ X_0, & \text{if } \forall t\, [\langle c(p_t,0), y\rangle \notin \eta],\ y \in \mathbb{N}, \end{cases}$$

is Selivanov $\varnothing^{(n+1)}$-complete with respect to X_0. □

4. Families of special objects

As mentioned earlier, a computable numbering of a family $\mathscr{A}$ is complete with respect to $X \in \mathscr{A}$ if and only if X is the least element of $\mathscr{A}$ under inclusion. It was proved by Ershov (1968) that this is no longer true for non-computable numberings. Namely, he showed that for every n there exists a numbering having exactly n special objects. Later, Denisov and Lavrov proved (Denisov and Lavrov 1970) that complete numberings can have infinitely many special objects (the question of the existence of such numberings was raised by Ershov (1969b). A joint generalization of the results by Ershov, Denisov, and Lavrov was obtained by (2006), who proved that any subset of a given at most countable nonempty set S is the set of special objects of some complete numbering of S. Moreover, he showed that for all Σ^0_{n+2}-computable families $\mathcal{C}$ and $\mathcal{D}$ with $(\mathcal{C} \cup \{\varnothing\}) \setminus \mathcal{D} \neq \varnothing$ there exists a Σ^0_{n+2}-computable numbering of the family $\mathcal{C} \cup \mathcal{D} \cup \{\varnothing\}$ whose family of special objects is $\mathcal{C} \cup \{\varnothing\}$. The following theorem generalizes Khisamiev's result and implies that given an oracle A, every subset of a nonempty at most countable set S is the set of special objects of some (Selivanov) A-complete numbering of S.

Theorem 5. *Let $\mathscr{A}$ and $\mathcal{B} \subseteq \mathscr{A}$ be nonempty Σ^0_{n+2}-computable families. Then for every $m \leqslant n$, the family $\mathscr{A}$ has a Σ^0_{n+2}-computable numbering ν that for any $X \in \mathcal{B}$ is Selivanov $\varnothing^{(n)}$-complete with respect to X and for no $Y \in \mathscr{A} \setminus \mathcal{B}$ is $\varnothing^{(m)}$-complete with respect to Y.*

Proof. Let α and β be Σ^0_{n+2}-computable numberings of the families $\mathscr{A}$ and $\mathcal{B}$, respectively. Since the statement of the theorem is obvious for $|\mathscr{A}| = 1$, we will assume that $|\mathscr{A}| > 1$, and also, without loss of generality, that

$$\alpha(0) \neq \alpha(1).$$

Fix an arbitrary integer $m \leqslant n$. To define the numbering ν satisfying the statement of the theorem, we construct a sequence $\{\nu_s\}_{s\in\mathbb{N}}$ of partial mappings of $\mathbb{N}$ to $\mathscr{A}$ such that $\nu_s \subseteq \nu_{s+1}$ for each s and $\bigcup_s \nu_s$ is a Σ^0_{n+2}-computable numbering of $\mathscr{A}$, satisfying the statement of the theorem. After that, we let $\nu = \bigcup_s \nu_s$.

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In the construction of ν, for all z, we will satisfy the following requirements:

$$N_z: \text{ (There is a binary recursive } f \text{ satisfying (3) for all } x \text{ and } e,$$
$$\text{where } A = \varnothing^{(n)} \text{ and } X = \beta(z)),$$
$$P_z: \exists x\,[\nu(x) = \alpha(z)],$$

$$R_z: (\Phi_z^{\varnothing^{(m)}} \text{ is total} \Rightarrow \text{there exist } e \text{ and } x \text{ such that for every } X \in \mathscr{A} \setminus \mathcal{B},$$
$$\text{condition (3) does not hold, where } f = \Phi_z^{\varnothing^{(m)}} \text{ and } A = \varnothing^{(m)}).$$

Obviously, having met all these requirements, we obtain that ν satisfies the statement of the theorem.

Construction

Stage 0. We set ν_0 equal to the partial mapping of $\mathbb{N}$ to $\mathscr{A}$ with empty domain. At each stage $s+1$ of the construction, we will also define some $\varnothing^{(n)}$-r.e. equivalence η_{s+1} and a strictly increasing recursive sequence of integers $\{a_i^{s+1}\}_{i\in\mathbb{N}}$. Let η_0 be the equality relation on $\mathbb{N}$ and let $a_i^0 = i$ for each i.

Stage $s+1$. Assume by induction that for $u = s$ the following conditions are met:

(i) $\nu_v \subseteq \nu_u$ for each $v < u$,

(ii) the sequence $\{a_i^u\}_{i\in\mathbb{N}}$ is strictly increasing and recursive,

(iii) $\langle a_i^u, a_j^u\rangle \notin \eta_u$ for all distinct integers i and j,

(iv) $\operatorname{dom}\nu_u = \mathbb{N} \setminus (\bigcup_i [a_i^u]_{\eta_u})$, where $[a_i^u]_{\eta_u}$ is the η_u-equivalence class of a_i^u,

(v) for all $x, y \in \operatorname{dom}\nu_u$, if $\langle x, y\rangle \in \eta_u$, then $\nu_u(x) = \nu_u(y)$.

It is not hard to see that when $u = 0$ conditions (i)–(v) are met. Now, let us consider three cases.

1. $s = 3z$ for some z. In this case, we satisfy the requirement N_z. We define the relation η_{s+1} as the $\varnothing^{(n)}$-r.e. equivalence relation generated by the binary relation

$$\eta_s \cup \{\langle a^s_{c(e,x)}, \Phi_e^{\varnothing^{(n)}}(x)\rangle : e, x \in \mathbb{N} \,\&\, \Phi_e^{\varnothing^{(n)}}(x)\downarrow\}.$$

Using the recursion theorem and its uniformity, effectively in s, we choose a strictly increasing recursive sequence $\{e_i\}_{i\in\mathbb{N}}$ such that

$$\Phi_{e_i}^{\varnothing^{(n)}}(0) = a^s_{c(e_i,0)} \tag{10}$$

for each i. Then for all i, we let

$$a_i^{s+1} = a^s_{c(e_i,0)}. \tag{11}$$

Thus, condition (ii) is met for $u = s+1$. Since condition (iii) is met for $u = s$ and condition (10) is met for each i, condition (iii) is met for $u = s+1$ as well. Next, we define

$$\nu_{s+1}(x) = \nu_s(x)$$

for each $x \in \operatorname{dom}\nu_s$. Hence, condition (i) is met for $u = s+1$. Further, for every $x \notin \operatorname{dom}\nu_s$, we define

$$\nu_{s+1}(x) = \begin{cases} \nu_s(\Phi_e^{\varnothing^{(n)}}(y)), & \text{if } \langle x, a^s_{c(e,y)}\rangle \in \eta_{s+1} \,\&\, \Phi_e^{\varnothing^{(n)}}(y)\downarrow \in \operatorname{dom}\nu_s, \\ \uparrow, & \text{if } \exists i\,[\langle x, a_i^{s+1}\rangle \in \eta_{s+1}], \\ \beta(z), & \text{in other cases.} \end{cases}$$

Let us show that this definition is correct. Let e, j, v, and w be arbitrary integers such that

$$\Phi_e^{\varnothing^{(n)}}(v)\downarrow \in \operatorname{dom}\nu_s \,\&\, \Phi_j^{\varnothing^{(n)}}(w)\downarrow \in \operatorname{dom}\nu_s \,\&\, \langle x, a^s_{c(e,y)}\rangle \in \eta_{s+1} \,\&\, \langle x, a^s_{c(j,w)}\rangle \in \eta_{s+1}.$$

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Thus, $\langle a^s_{c(e,v)}, a^s_{c(j,w)}\rangle \in \eta_{s+1}$. Assume that $\nu_s(\Phi_e^{\varnothing^{(n)}}(v)) \neq \nu_s(\Phi_j^{\varnothing^{(n)}}(w))$. Then

$$\langle \Phi_e^{\varnothing^{(n)}}(v), \Phi_j^{\varnothing^{(n)}}(w)\rangle \notin \eta_s.$$

By the definition of equivalence η_{s+1}, we have that $\langle a^s_{c(e,v)}, a^s_{c(j,w)}\rangle \notin \eta_{s+1}$. It follows from this contradiction that the values of ν_{s+1} on elements $x \notin \operatorname{dom} \nu_s$ are defined correctly. Now, it is not hard to see that conditions (iv) and (v) are met for $u = s+1$.

By the definition of ν_{s+1}, for all e and x with $a^s_{c(e,x)} \in \operatorname{dom} \nu_{s+1}$, we have

$$\nu_{s+1}\left(a^s_{c(e,x)}\right) = \begin{cases} \nu_{s+1}(\Phi_e^{\varnothing^{(n)}}(x)), & \text{if } \Phi_e^{\varnothing^{(n)}}(x)\downarrow, \\ \beta(z), & \text{if } \Phi_e^{\varnothing^{(n)}}(x)\uparrow. \end{cases}$$

In addition, since for every i, conditions (10) and (11) are met, having met condition (v) for each u, we obtain that

$$\nu\left(a^s_{c(e,x)}\right) = \begin{cases} \nu(\Phi_e^{\varnothing^{(n)}}(x)), & \text{if } \Phi_e^{\varnothing^{(n)}}(x)\downarrow, \\ \beta(z), & \text{if } \Phi_e^{\varnothing^{(n)}}(x)\uparrow, \end{cases}$$

for all e and x. Hence, N_z will be met with the recursive $f : \langle e, x\rangle \mapsto a^s_{c(e,x)}$.

2. $s = 3z+1$ for some z. In this case, we satisfy the requirement P_z. Let y be the least element of the set $\bigcup_i [a^s_i]_{\eta_s}$. Fix j such that $y \in [a^s_j]_{\eta_s}$ and define

$$\nu_{s+1}(x) = \alpha(z) \tag{12}$$

for each $x \in \bigcup_{k\leqslant j} [a^s_k]_{\eta_s}$. For every $x \in \operatorname{dom} \nu_s$, we define $\nu_{s+1}(x) = \nu_s(x)$. Let η_{s+1} be the $\varnothing^{(n)}$-r.e. equivalence relation, generated by the binary relation

$$\eta_s \cup \left\{\langle x, y\rangle \in \mathbb{N}\times\mathbb{N} : x, y \in \operatorname{dom} \nu_{s+1} \setminus \operatorname{dom} \nu_s\right\}. \tag{13}$$

Now, for every i, we let

$$a^{s+1}_i = a^s_{i+j+1}.$$

It is not hard to see that in this case, conditions (i)–(v) are met for $u = s+1$ and P_z is met as well.

3. $s = 3z+2$ for some z. In this case, we satisfy the requirement R_z. Let $F_0, \ldots, F_k$ be all η_s-equivalence classes such that for all $i \leqslant k$ and $x \in F_i$, at least one of the assignments (12), (14) (see below) has been performed at one of the previous stages (the construction will guarantee that the tuple of such classes is always finite).

Now, we define some Turing functional Ψ assuming, by the recursion theorem, that an index r for which

$$\Psi = \Phi_r$$

is given. If $\Phi_z^{\varnothing^{(m)}}(r, l)\downarrow$ for each $l \leqslant k+1$ and there exist $p < q \leqslant k+1$ such that

$$\left\langle \Phi_z^{\varnothing^{(m)}}(r, p), \Phi_z^{\varnothing^{(m)}}(r, q)\right\rangle \in \eta_s,$$

then we define

$$\Psi^{\varnothing^{(m)}}(p) = a^s_0,\ \Psi^{\varnothing^{(m)}}(q) = a^s_1,$$

$$\nu_{s+1}(x) = \alpha(t) \tag{14}$$

for all $t = 0, 1$ and $x \in [a^s_t]_{\eta_s}$. Thus, we will have that

$$\nu\left(\Phi_z^{\varnothing^{(m)}}(r, p)\right) = \nu(\Phi_z^{\varnothing^{(m)}}(r, q)),$$

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but

$$\nu\left(\Psi^{\varnothing^{(m)}}(p)\right) = \alpha(0) \neq \alpha(1) = \nu\left(\Psi^{\varnothing^{(m)}}(q)\right).$$

Hence, R_z will be met. For every i, we let

$$a_i^{s+1} = a_{i+2}^s.$$

Now, we suppose that $\Phi_z^{\varnothing^{(m)}}(r,l)\downarrow$ for each $l \leqslant k+1$ and

$$\left\langle \Phi_z^{\varnothing^{(m)}}(r,p), \Phi_z^{\varnothing^{(m)}}(r,q)\right\rangle \notin \eta_s$$

for all $p < q \leqslant k+1$. Let $d \leqslant k+1$ be the least integer such that

$$\Phi_z^{\varnothing^{(m)}}(r,d) \notin \bigcup_{b\leqslant k} F_b.$$

If $\Phi_z^{\varnothing^{(m)}}(r,d) \notin \operatorname{dom} \nu_s$, then we fix j with $\Phi_z^{\varnothing^{(m)}}(r,d) \in [a_j^s]_{\eta_s}$ and define

$$\nu_{s+1}(x) = \beta(0)$$

for each $x \in \bigcup_{i\leqslant j} [a_i^s]_{\eta_s}$. For every i, we let

$$a_i^{s+1} = a_{i+j+1}^s.$$

Note that if $\Phi_z^{\varnothing^{(m)}}(r,d) \in \operatorname{dom} \nu_s$, then since $\Phi_z^{\varnothing^{(m)}}(r,d) \notin \bigcup_{b\leqslant k} F_b$, we have

$$\nu_s\left(\Phi_z^{\varnothing^{(m)}}(r,d)\right) \in \mathcal{B}.$$

Thus, $\Psi^{\varnothing^{(m)}}(d)\uparrow$ and $\nu_{s+1}(\Phi_z^{\varnothing^{(m)}}(r,d)) \notin \mathscr{A}\setminus\mathcal{B}$. Therefore, the requirement R_z will be met. Further, we define η_{s+1} by (13) and $\nu_{s+1}(x) = \nu_s(x)$ for each $x \in \operatorname{dom}\nu_s$. It is not hard to see that in this case, conditions (i)–(v) are met for $u = s+1$.

This completes the description of the construction and the proof of the theorem. □

Corollary 4. *Let A be an arbitrary oracle and let S be a nonempty at most countable set. Then for every nonempty subset S_0 of S there exists a numbering ν of S that for any $w \in S_0$ is Selivanov A-complete with respect to w and for no $z \in S\setminus S_0$ is A-complete with respect to z.*

Proof. Let us consider the computable family $\mathscr{A}$ defined as follows:

$$\mathscr{A} = \left\{\{y\} : y \in \mathbb{N} \,\&\, y < |S|\right\}.$$

Let $\mathcal{B}$ be a computable subfamily of $\mathscr{A}$ such that $|\mathcal{B}| = |S_0|$ and $|\mathscr{A}\setminus\mathcal{B}| = |S\setminus S_0|$. By the relativization of Theorem 5, there exists an A'-computable numbering μ of $\mathscr{A}$ that for any $X \in \mathcal{B}$ is Selivanov A-complete with respect to X and for no $Y \in \mathscr{A}\setminus\mathcal{B}$ is A-complete with respect to Y.

Now, we choose an injective sequence $\{a_y\}_{y<|S|}$ of elements of S such that

$$S = \{a_y : y < |S|\},\ S_0 = \{a_y : \{y\} \in \mathcal{B}\}.$$

Then the numbering ν is defined by

$$\nu(x) = a_y \Leftrightarrow \mu(x) = \{y\},\ x, y \in \mathbb{N},$$

satisfies the statement of the corollary. □

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References

Badaev, S. A. and Goncharov, S. S. (2014). Generalized computable universal numberings. *Algebra and Logic* **53** (5) 355–364.
Badaev, S. A., Goncharov, S. S. and Sorbi, A. (2003). *Completeness and Universality of Arithmetical Numberings*, Boston, MA, Springer US, 11–44.
Badaev, S. A., Goncharov, S. S. and Sorbi, A. (2008). Some remarks on completion of numberings. *Siberian Mathematical Journal* **49** (5) 780–783.
Denisov, S. D. and Lavrov, I. A. (1970). Complete numerations with infinitely many singular elements. *Algebra and Logic* **9** (5) 301–304.
Ershov, Y. L. (1968). On a hierarchy of sets, II. *Algebra and Logic* **7** (4) 212–232.
Ershov, Y. L. (1969a). Completely enumerated sets. *Siberian Mathematical Journal* **10** (5) 773–784.
Ershov, Y. L. (1969b). *Theory of Numberings 1*. Novosibirsk State University. Novosibirsk (In Russian).
Ershov, Y. L. (1970). On inseparable pairs. *Algebra and Logic* **9** (6) 396–399.
Ershov, Y. L. (1973). Theorie der numerierungen I. *Mathematical Logic Quarterly* **19** (19-25) 289–388.
Ershov, Y. L. (1977). *Theory of Numberings*. Nauka. Moscow (In Russian).
Ershov, Y. L. (1999). Theory of numberings. In: Griffor, E. R. (ed.) *Handbook of Computability Theory*, vol. 140 of Studies in Logic and the Foundations of Mathematics, New York, Elsevier, 473–503.
Faizrahmanov, M. K. (2023a). Numberings, c.e. oracles, and fixed points. *Computability* **12** (3) 271–282.
Faizrahmanov, M. K. (2023b). Positive reducibilities, extreme numberings, and completeness. *Siberian Advances in Mathematics* **33** (3) 204–213.
Golov, A. and Terwijn, S. A. (2022). Fixpoints and relative precompleteness. *Computability* **11** (2) 135–146.
Goncharov, S. S. and Sorbi, A. (1997). Generalized computable numerations and nontrivial Rogers semilattices. *Algebra and Logic* **36** (6) 359–369.
Jain, S. and Nessel, J. (2001). Some independence results for control structures in complete numberings. *Journal of Symbolic Logic* **66** (1) 357–382.
Khisamiev, Z. G. (2006). Subfamilies of special elements of complete numberings. *Algebra and Logic* **45** (6) 431–434.
Lachlan, A. H. (1964). Standard classes of recursively enumerable sets. *Mathematical Logic Quarterly* **10** (2-3) 23–42.
Mal'cev, A. I. (1963). Sets with complete numberings. *Algebra and Logic* **2** (2) 4–29. (In Russian).
Muchnik, A. A. (1963). Isomorphism of systems of recursively enumerable sets with effective properties. Translated by E.Mendelson. Transl., Ser. 2, *American Mathematical Society* **23** 7–13.
Myhill, J. (1955). Creative sets. *Mathematical Logic Quarterly* **1** (2) 97–108.
Rogers, H. (1958). Gödel numberings of partial recursive functions. *Journal of Symbolic Logic* **23** (3) 331–341.
Selivanov, V. L. (1988). Index sets of quotient objects of the post numeration. *Algebra and Logic* **27** (3) 215–224.
Selivanov, V. L. (2021a). Non-collapse of the effective Wadge hierarchy. In: De Mol, L., Weiermann, A., Manea, F. and Fernández-Duque, D. (eds.) *Connecting with Computability*, Cham, Springer International Publishing, 407–416.
Selivanov, V. L. (2021b). Precomplete numberings. *Journal of Mathematical Sciences* **256** (1) 96–124.
Smullyan, R. M. (1961). *Theory of Formal Systems*. (AM-47). Princeton, Princeton University Press.
Soare, R. I. (2016). *Turing Computability. Theory and Applications of Computability*, Berlin, Heidelberg, Springer-Verlag.
Terwijn, S. A. (2024). *Fixed Point Theorems in Computability Theory*, Cham, Springer Nature Switzerland, 214–224.

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