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SURVEY

Open problems and questions about geodesics

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Abstract. The paper surveys open problems and questions related to geodesics defined by Riemannian, Finsler, semi-Riemannian and magnetic structures on manifolds. It is an extended report on problem sessions held during the International Workshop on Geodesics in August 2010 at the Chern Institute of Mathematics in Tianjin.

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1. *Notation and definitions*

In this paper, M is a connected C^∞ manifold and $\widetilde{M}$ is its universal cover. We consider various structures on M that create geodesics: Riemannian (later, also semi-Riemannian) metric g, Finsler metric F and magnetic structure ω. The unit tangent bundle defined by such a structure is denoted by SM and ϕ_t is the geodesic flow on SM. The geodesic defined by a vector $v \in SM$ is denoted by γ_v; it is parameterized by arc length unless otherwise specified.

The free loop space ΛM is the set of all piecewise smooth mappings from the circle S^1 to the manifold equipped with the natural topology. When M has a Riemannian metric (or other suitable structure) we can consider the subspace $\Lambda_T M$ of loops with length $\leq T$. Similarly, given two points $p, q \in M$, we denote by $\Omega(p, q)$ the space of piecewise smooth paths from p to q and by $\Omega_T(p, q)$ the subspace of paths with length at most T.

We define the energy functional on ΛM by

$$c \mapsto \int_{S^1} \|\dot{c}(s)\|^2 \, d\lambda(s),$$

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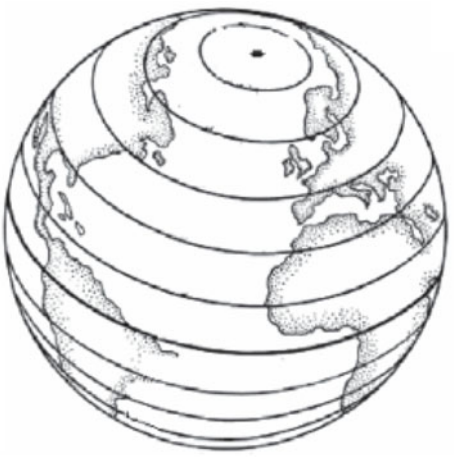


FIGURE 1. Birkhoff's mini-max argument.

where $\|\cdot\|$ is the norm induced by a Riemannian metric or other suitable structure and λ is the Lebesgue measure on S^1 normalized to be a probability measure. The critical points of this functional are the closed geodesics for the metric (parameterized at the constant speed $\mathrm{Length}(c)$). In the case of $\Omega(p, q)$, the circle S^1 is replaced by the interval $[0, 1]$.

2. *Closed geodesics*

Given a Riemannian manifold, does there exist a closed geodesic? If yes, how many (geometrically different) closed geodesics must exist? Existence is known if the manifold is closed. For surfaces (and certain manifolds of higher dimension, e.g. S^n), the result is essentially due to Birkhoff [**Bir1927**, Ch. V]. He used two arguments. The first is a variational argument in the free loop space ΛM. From the subspace of ΛM consisting of all loops homotopic to a given loop, one selects a sequence of loops whose lengths converge to the infimum of the length function on the subspace. If the loops from this sequence lie in a compact region of the manifold (for example, if the manifold itself is compact), then by the Arzelà–Ascoli theorem there exists a loop where the infimum is achieved; if the infimum is not zero, this loop is automatically a non-trivial closed geodesic. These arguments show the existence of closed geodesics in many cases, in particular, for compact manifolds with non-trivial fundamental group.

If the infimum of the length of the loops homotopic to a given loop is zero, Birkhoff suggested another procedure (actually, a trick) to prove the existence of closed geodesics, which is explained, for example, in [**Bir1927**, §§6, 7 of Ch. V]. This trick is nowadays called the *Birkhoff minimax procedure*, and it was used, for example, to show the existence of closed geodesics on any Riemannian sphere with dimension ≥ 2.

Let us recall this procedure in the simplest case, when the manifold is S^2. We consider the foliation of the sphere without two points (north and south poles) as in Figure 1: if we think of the standard embedding in $\mathbb{R}^3$, the fibers are the intersections of the sphere with the planes $\{x_3 = \text{const}\}$.

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Now apply a curve-shortening procedure to every curve of this foliation; there are many options for such a procedure. We need the evolution of the curves in this procedure to depend continuously on the curve. We obtain a sequence F_i of the foliations of the sphere without two points into curves. For each foliation F_i of the sequence, let γ_i be a leaf of maximal length. Because of topology, the lengths of the circles γ_i are bounded from below by a certain positive number. By the Arzelà–Ascoli theorem, the sequence of curves γ_i has a convergent subsequence. The limit of this subsequence is a stable point of the shortening procedure, and is therefore a geodesic.

For arbitrary closed manifolds, the existence of a closed geodesic is due to Lyusternik and Fet [**LF1951**, **Fet1952**]. They considered the energy functional on the loop space ΛM and showed that the topology of ΛM is complicated enough so that the energy functional must have critical points with non-zero energy (which are non-trivial closed geodesics). Lyusternik and Schnirelmann [**LS1929a**, **LS1929b**] had earlier used a related argument to show that any Riemannian metric on S^2 has at least three simple closed geodesics.

How many closed geodesics must exist on a closed manifold? For surfaces, the answer is known: there are always infinitely many geometrically different closed geodesics. This is easily proved using Birkhoff's first argument when the fundamental group is infinite. The remaining cases of the sphere and the projective plane were settled by [**Ban1993**, **Fra1992**]. In higher dimensions, Rademacher has shown that a closed manifold with a generic Riemannian metric admits infinitely many geometrically different closed geodesics [**Rad1989**].

The main approach to proving the existence of infinitely many closed geodesics has been to apply Morse theory to the energy functional on the free loop space ΛM. The critical points of this functional are precisely the closed geodesics. But it has to be remembered that each closed geodesic can be traversed an arbitrary number of times. It is thus important to distinguish geometrically different geodesics from repetitions of the same geodesic. This distinction is difficult to make and so far, despite some published claims, the existence of infinitely many closed geodesics on a general compact Riemannian manifold has not been proven. Nor have the resources of the Morse theory approach been fully exhausted.

One can modify these classical questions in different directions; this will be done below.

2.1. *Riemannian metrics on spheres.* Except in dimension two, all that is currently known for a general Riemannian metric on a sphere are the results that hold for all compact Riemannian manifolds. *Is there an extension of Lyusternik and Schnirelmann's result to higher dimensions, in particular, to* S^3?

2.2. *Finsler metrics on* S^2. The arguments of Fet (and Morse) can be adapted to the Finsler setting: one can show the existence of at least one closed geodesic on every compact manifold. Moreover, if the Finsler metric is reversible, or if the manifold is a surface other than the sphere or the projective plane, one can show the existence of infinitely many geometrically different closed geodesics.

The following example constructed in [**Kat1973**] shows that the number of closed geodesics on a 2-sphere with an irreversible metric can be two.

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Consider the sphere S^2 with the standard metric g_0 of constant curvature one and the Killing field V induced by the one-parameter group of rotations about the north and south poles. Define a one parameter family of functions on T^*M by

$$F_\alpha(\xi) = \sqrt{g_0^*(\xi, \xi)} + \alpha\xi(V),$$

where g_0^* is the dual metric of g_0. For small enough $\alpha \in \mathbb{R}$, there is a Randers–Finsler metric, F_α such that

$$\|v\|_\alpha = F_\alpha \circ L_\alpha^{-1}(v),$$

where $L_\alpha : T^*M \to T^M$ is the Legendre transformation induced by the function $\frac{1}{2}F_\alpha^2$. If V is scaled so that its integral curves have the same period as the geodesics of g_0, then g_α has precisely two closed geodesics for each irrational α. They are the equator (the unique great circle tangential to V) traversed in both directions. See [**Zil1982**, **Rad2004**] for more details.

This example can be generalized to higher dimensions: one obtains a Finsler metric on S^n with precisely $2[(n+1)/2]$ distinct prime closed geodesics; see [**Kat1973**].

Bangert and Long in [**BL2010**, **Lon2006**] proved that there are always at least two distinct prime closed geodesics for every irreversible Finsler metric on S^2.

Conjecture 2.2.1. (Long, Bangert, Problem 15 from [**Álv2006**]) Every irreversible Finsler metric on S^2 has either exactly two or infinitely many distinct prime closed geodesics.

There exist results supporting this conjecture. In particular, Hofer, Wysocki and Zehnder in [**HWZ2003**] studied Reeb orbits on contact S^3. Their result can be projected down to S^2, and implies that the total number of distinct prime closed geodesics for a bumpy Finsler metric on S^2 is either two or infinite, provided the stable and unstable manifolds of every hyperbolic closed geodesics intersect transversally (see also [**HP2008**, **Rad2016**]).

The closed geodesics in Katok's example are elliptic.

Conjecture 2.2.2. (Long) The existence of one hyperbolic prime closed geodesic on a Finsler S^2 implies the existence of infinitely many distinct prime closed geodesics.

Conjecture 2.2.3. (Long) Every Finsler S^2 has at least one elliptic prime closed geodesic.

The conjecture agrees with a result of Long and Wang who proved that there are always at least two elliptic prime closed geodesics on every irreversible Finsler S^2, if the total number of prime closed geodesics is finite [**LW2008**]. The conjecture does not contradict [**Grj1979**], where an example of a metric such that all closed simple geodesics are hyperbolic is constructed. Indeed, for certain metrics on the sphere (and even for the metric of certain ellipsoids), most prime closed geodesics are not simple.

2.3. *Of complete Riemannian metrics with finite volume.*

Question 2.3.1. (Bangert) Does every complete Riemannian manifold with finite volume have at least one closed geodesic?

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The question was answered affirmatively for dimension two. Moreover, in dimension two, a complete Riemannian manifold of finite volume even has infinitely many geometrically different geodesics [**Ban1980**]. The argument that was used in the proof is based on the Birkhoff minimax procedure, which we recalled in the beginning of §2, and does not work in dimensions ≥ 3. One can even hope to construct counterexamples in the class of Liouville-integrable geodesic flows. In this case, most orbits of the geodesic flow are rational or irrational windings on the Liouville tori; they are closed, if all of the corresponding frequencies are rational. Since there are essentially $(n-1)$ frequencies in dimension n, one can hope that, if $n > 2$, it would be possible to ensure that there is always at least one irrational frequency. Initial attempts to find a counterexample on $T^2 \times \mathbb{R}$ with the metric of the form $a(r)\, d\phi^2 + b(r)\, d\psi^2 + dr^2$ were, however, unsuccessful.

A more difficult problem would be to prove *that on any complete Riemannian manifold of finite volume there exist infinitely many closed geodesics*.

Questions of this nature are also interesting in the realm of Finsler geometry. One would expect the results for reversible Finsler metrics to be very similar to those for Riemannian metrics. For irreversible Finsler metrics, Katok's example shows that there are compact Finsler manifolds with only finitely many closed geodesics, but it is still possible that all non-compact Finsler manifolds with finite volume might have infinitely many closed geodesics. On the other hand, the following question is completely open.

Question 2.3.2. (Bangert) Does there exist an irreversible Finsler metric of finite volume on $\mathbb{R} \times S^1$ with no closed geodesics?

2.4. *Of magnetic flows on closed surfaces.* It is known that the trajectory of a charged particle in the presence of magnetic forces (= 'magnetic geodesic') is described by a Hamiltonian system with the 'kinetic' Hamiltonian of the form $\sum_{i,j} p_i p_j g^{ij}$ on T^*M with the symplectic form $dp \wedge dx + \pi^*\omega$, where ω is a closed (but not necessarily exact) form on M and $\pi : T^*M \to M$ is the canonical projection.

We assume that our surface M^2 is closed.

Question 2.4.1. (Paternain) Is there at least one closed magnetic geodesic in every positive energy level?

The only known case in which the answer is negative is when ω is the area form for a hyperbolic surface and the magnetic flow is the horocycle flow. On the other hand, Ginzburg showed that there is always a closed orbit if the magnetic field is weak enough [**Gin1989**, **Gin1996**].

If ω is exact, the affirmative answer for all closed orientable surfaces and all energy levels was obtained in [**CMP2004**], which is partially based on [**Tai1992**]. Actually, by [**AMP2015**, **AMMP2017**], in this case there are at least two closed geodesics on almost every energy level. We give some details.

If the energy is greater than Mañé's critical value c_u, then the magnetic flow is conjugate to a Finsler geodesic flow, which has ≥ 2 closed orbits in the case of S^2 (by [**BL2010**]) and infinitely many closed orbits if the genus of the surface is positive (at least two for each free homotopy class). The lower bound of two is sharp because of the Katok example mentioned above.

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For energy less than c_u, the results about more than one closed geodesic are only for almost every energy level. Contreras obtained at least two closed geodesics [**Con2006**]. In [**AMP2015**] this was improved to at least three and, in fact, infinitely many under a non-degeneracy assumption, which was removed in [**AMMP2017**].

In the case of magnetic monopoles (i.e., when ω is not exact), the answer is affirmative if the energy level is contact; this is an immediate corollary of the proof of the Weinstein conjecture in dimension three by Taubes [**Tau2007**]. On the other hand, Benedetti has given examples of magnetic monopoles on S^2 with energy levels that are not contact; see [**Ben2016**, Proposition 4.9]. On S^2, if the form ω is nowhere vanishing, the existence of a closed magnetic geodesic on an energy level for a sufficiently big energy is established in [**AB2016**, Corollary 4.6]; see also [**Gin1989**, **Gin1996**].

2.5. *Kropina metrics with non-integrable distribution.* A Kropina metric on a Riemannian manifold (M, g) is the function on the tangent bundle given by $F(\xi) = (g(\xi, \xi))/(\omega(\xi))$, where ω is a nowhere vanishing 1-form. We will assume that $\omega \wedge d\omega$ nowhere vanishes; this assumption implies that the distribution $\mathrm{Ker}(\omega)$ is not integrable. A Kropina metric is not a Finsler metric, since it is not defined on the kernel of ω, but still the function F is one-homogeneous and strictly convex at $\xi \notin \mathrm{Ker}(\omega)$; one can use it to define the distance on M and geodesics on M.

Recent interest in Kropina metrics is due to their relations to the so-called chains in Cauchy–Riemann geometry (see [**CMMM2019**]).

Question 2.5.1. Does every Kropina metric on a closed manifold have a closed geodesic?

If the fundamental group is non-trivial, the answer is positive and the proof is essentially the same as in the Riemannian case. But the Birkhoff–Morse–Fett approaches that were used to prove the existence of closed geodesics for simply connected Riemannian manifolds cannot be generalized directly, in particular, because of the existence of examples of Kropina metrics whose geodesics of length one lie in an arbitrarily small neighborhood.

As a simple special case of Question 2.5.1, we suggest considering $M = S^3 = \mathrm{Spin}(3)$ equipped with arbitrary g and a 1-form ω that is homogeneous with respect to the left action of Spin(3).

3. *Path and loop spaces*

As noted in the previous section, one of the main approaches to proving the existence of closed geodesics is to use topological complexity of the loop space ΛM to force the existence of critical points of the energy functional. Loops with length $\leq T$ correspond to critical points in $\Lambda_T M$. Similarly, geodesics joining two points p and q can be studied by investigating the path spaces $\Omega(p, q)$ or $\Omega_T(p, q)$. The homology of these spaces has been much studied.

3.1. *Sums of the Betti numbers.* Let p, q be points in a Riemannian manifold. The space $\Omega_T(p, q)$ of paths from p to q with length $\leq T$ has the homotopy type of a finite

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complex (see, e.g., [**Mil1963**]), and hence the sum of its Betti numbers is finite for each T. The same is true for the space Λ_{TM} of loops with length at most T.

Question 3.1.1. (Paternain) How do the sums of the Betti numbers for Λ_{TM} and $\Omega_T(p, q)$ grow as $T \to \infty$? Does the growth depend on the metric?

It was shown by Gromov, in [**Gro2001**], that, if M is simply connected, there is a constant C such that the sum of the Betti numbers of Λ_{CN} is at least $\sum_{i=0}^{N} b_i(\Lambda)$.

Question 3.1.2. (Gromov) Does the number of closed geodesics with length $\leq T$ grow exponentially as $T \to \infty$ if the Betti numbers of the loop space grow exponentially?

From [**Gro1978**], it follows that the answer is positive for generic metrics.

Here is a potentially interesting example. Consider $f : S^3 \times S^3 \to S^3 \times S^3$ such that the induced action on $H_3(S^3 \times S^3)$ is hyperbolic. Let M be the mapping torus for f: i.e., $S^3 \times S^3 \times [0, 1]$ with $(x, 1)$ identified with $(f(x), 0)$ for each $x \in S^3 \times S^3$. Then $\pi_1(M) = Z$ and the universal cover $\widetilde{M}$ is homotopy equivalent to $S^3 \times S^3$. Hence the Betti numbers of the loop space grow polynomially. On the other hand, the hyperbolic action of f on H_3 gives hope for exponential growth of the sum of the Betti numbers of $\Omega_T(p, q)$.

3.2. *Stability of minimax levels (communicated by Nancy Hingston).* Let M be a compact manifold with a Riemannian (or Finsler) metric g. Let G be a finitely generated abelian group. Given a non-trivial homology class $X \in H_*(\Lambda M; G)$, the critical (minimax) level of X is

$$\begin{aligned} \operatorname{cr} X &= \inf\{a : X \in \text{Image } H_*(\Lambda_a M; G)\} \\ &= \inf_{x \in X} \sup_{\gamma \in \text{Image } x} \text{Length}(\gamma). \end{aligned}$$

Here $\Lambda_a M$ is the subset of the free loop space ΛM consisting of loops whose length is at most a. The second definition is the minimax definition: the singular chain x ranges over all representatives of the homology class X, and γ ranges over all the points in the image of x, which are loops in M.

Question 3.2.1. (Hingston) Do there exist a metric on S^n and a homology class $X \in H_*(\Lambda M; \mathbb{Z})$ with

$$0 < \operatorname{cr}(mX) < \operatorname{cr}(X)$$

for some $m \in \mathbb{N}$?

The special case of this question, when the dimension is two and m is odd, was answered negatively in [**CL2014**, Corollary 1.3].

Given a metric g on M and a finitely generated abelian group G, the *global mean frequency* is defined as

$$\alpha_{g,G} = \lim_{\deg X \to \infty} \frac{\deg X}{\operatorname{cr} X}, \tag{3.1}$$

where the limit is taken over all non-trivial homology classes $X \in H_*(\Lambda S^n; G)$.

The resonance theorem from [**HR2013**] says that if M is a sphere and G is a field, then the limit (3.1) exists. It is clear, in this case, that $\alpha_{g,G}$ depends on the metric g. But does

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it really depend on the field G? The degree of X does not depend on anything but X. But what about the critical level cr X? Does cr X depend on the coefficients?

Let us note that, for the spheres, the non-trivial homology groups of the free loop space (with integer coefficients) are all $\mathbb{Z}$ or $\mathbb{Z}_2 =: \mathbb{Z}/2\mathbb{Z}$. (For odd spheres they are all $\mathbb{Z}$.) Let us look at the case where $X \in H_k(\Lambda; \mathbb{Z}) = \mathbb{Z}$. For each $m \in \mathbb{N}$, there is a critical level

$$\mathrm{cr}(mX) = \inf\{a : mX \in \mathrm{Image}\ H_*(\Lambda_a M)\}.$$

If $j, m \in \mathbb{N}$, then, clearly (since Image $H_*(\Lambda_a)$ is an additive subgroup of $H_*(\Lambda)$),

$$\mathrm{cr}(jmX) \leq \mathrm{cr}(mX).$$

But can there be strict inequality? Here is a little 'example' to show how this would affect the global mean frequency: suppose it were the case that there were real numbers $a < b < c < d$ with

$$\mathrm{cr}(mX) = \left\{ \begin{array}{c} d \text{ if } \gcd(m, 3) = \gcd(m, 7) = 1, \\ c \text{ if } 3|m \text{ but } \gcd(m, 7) = 1, \\ b \text{ if } 7|m \text{ but } \gcd(m, 3) = 1, \\ a \text{ if } 21|m. \end{array} \right\}$$

Conjecture 3.2.2. (Hingston)

$$\begin{aligned} \alpha_{g,G} &= a \quad \text{if } G = \mathbb{Q}, \\ \alpha_{g,G} &= b \quad \text{if } G = \mathbb{Z}_3, \\ \alpha_{g,G} &= c \quad \text{if } G = \mathbb{Z}_7, \\ \alpha_{g,G} &= a \quad \text{if } G = \mathbb{Z}_p,\ p \neq 3, 7. \end{aligned}$$

4. *Curvature conditions and hyperbolicity of the geodesic flow*

Many results about geodesics and the geodesic flow assume that all sectional curvatures are negative or one of the following increasingly weaker properties:

(1) all sectional curvatures are non-positive;
(2) no focal points;
(3) no conjugate points.

These properties can be characterized by the behavior of Jacobi fields.

(0) Negative curvature: the length of any (non-trivial) Jacobi field orthogonal to a geodesic is a strictly convex function.
(1) Non-positive curvature: the length of any Jacobi field is a convex function.
(2) No focal points: the length of an initially vanishing Jacobi field is a non-decreasing function along a geodesic ray.
(3) No conjugate points: a (non-trivial) Jacobi field can vanish at most once.

The no focal point property is equivalent to convexity of spheres in the universal cover. Most interesting results about manifolds with non-positive curvature extend readily to manifolds with no focal points.

For a compact manifold, negative curvature implies that the geodesic flow is uniformly hyperbolic, in other words, an Anosov flow. This means that there is a $D\phi_t$-invariant splitting of the tangent bundle of the unit tangent bundle SM,

$$\mathrm{TSM} = E^s \oplus E^0 \oplus E^u,$$

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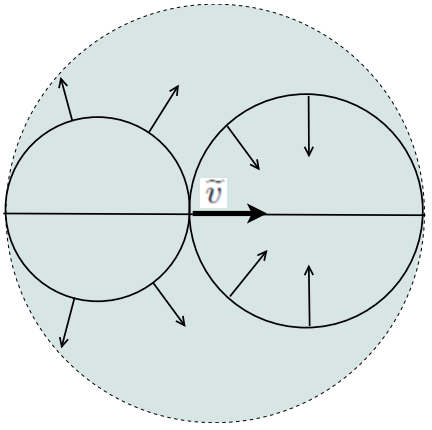


FIGURE 2. Stable and unstable manifolds.

in which E^0 is the one-dimensional subbundle tangential to the orbits of the geodesic flow, and there are constants $C \geq 1$ and $\lambda > 0$ such that, for any $t \geq 0$ and any vectors $\xi \in E^s$ and $\eta \in E^u$,

$$\|D\phi_t(\xi)\| \leq Ce^{-\lambda t}\|\xi\| \quad \text{and} \quad \|D\phi_{-t}(\eta)\| \leq Ce^{-\lambda t}\|\eta\|. \tag{4.1}$$

(Here we have in mind the usual Sasaki metric; the same property would also hold for any equivalent metric with different constants C and λ.) This splitting is Hölder continuous, but usually not smooth.

The bundles E^s and E^u for an Anosov geodesic flow are integrable; their integral foliations are usually denoted by W^s and W^u. The lifts to the universal cover of the leaves of W^s and W^u are closely related to horospheres. If $\widetilde{v}$ is the lift to $S\widetilde{M}$ of $v \in SM$, the lifts to $S\widetilde{M}$ of $W^s(v)$ and $W^u(v)$ are formed by the unit vectors that are normal to the appropriate horospheres orthogonal to $\widetilde{v}$ and are on the same side of the horosphere as $\widetilde{v}$; see Figure 2.

4.1. *Relations between these concepts.* The notions introduced above are related as follows.

$$\begin{array}{ccccc} \text{negative curvature} & \Longrightarrow & \text{non-positive curvature} & \Longrightarrow & \text{no focal points} \\ \Downarrow & & & & \Downarrow \\ \text{Anosov geodesic flow} & & \Longrightarrow & & \text{no conjugate points} \end{array}$$

That Anosov geodesic flow implies no conjugate points is proved in part B of [**Mañ1987**].

The class of compact manifolds that support metrics with variable negative curvature is much larger than the class that support hyperbolic metrics (of constant negative curvature). The earliest examples of manifolds that support variable but not constant negative curvature were given by Mostow–Siu [**MS1980**] and Gromov–Thurston [**GT1987**]. Recent work of Ontaneda has vastly increased the supply of examples [**Ont2011**, **Ont2014**].

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Conjecture 4.1.1. (Klingenberg) If a closed manifold admits a metric with Anosov geodesic flow, then it admits a metric with negative sectional curvature.

All known examples of metrics with Anosov geodesic flow are perturbations of metrics with negative curvature. It is not difficult to show using the uniformization theorem and the Thurston geometerization theorem that the conjecture is true in dimensions two and three. Klingenberg [**Kli1974**] showed that seven properties of Riemannian manifolds with negative curvature extend to those with Anosov geodesic flow. One of these is Preissman's theorem [**Pre1943**] that the fundamental group of a manifold with negative sectional curvatures cannot contain a copy of $\mathbb{Z} \times \mathbb{Z}$.

Several examples of manifolds that admit metrics of non-positive curvature but cannot support a metric of negative curvature (or with Anosov geodesic flow) can be found in the introduction to [**BBE1985**]. These examples have a copy of $\mathbb{Z} \times \mathbb{Z}$ in their fundamental group. The simplest of them is due to Heintze who took two cusped hyperbolic 3-manifolds, cut off the cusps, glued the two pieces together along their boundary tori and then smoothed out the metric to obtain a manifold with non-positive curvature.

Question 4.1.2. If (M, g) has no conjugate points, does M carry a metric with non-positive sectional curvature?

Again, it is not difficult to use the uniformization theorem to show that the answer is affirmative in dimension two. But the problem is still open, even in dimension three. See [**CS1986**, **Leb2002**, **IK2014**] for results of the nature that the fundamental groups of manifolds with no conjugate points share properties with those of non-positive curvature. Section 8 of [**IK2014**] contains a number of open problems of which we mention the following.

Question 4.1.3. Is the fundamental group of a closed manifold without conjugate points semihyperbolic?

Semihyperbolicity is a condition introduced by Alonso and Bridson [**AB1995**] to describe non-positive curvature in the large for an arbitrary metric space.

The geodesic flow is partially hyperbolic if there are a $D\phi_t$-invariant splitting

$$TSM = E^s \oplus E^c \oplus E^u$$

and constants $C \geq 1$ and $\lambda > \mu > 0$ such that (4.1) holds, and, in addition, for all t and all $\zeta \in E^c$,

$$C^{-1} e^{-\mu|t|} \|\zeta\| \leq \|D\phi_t(\zeta)\| \leq C e^{\mu|t|} \|\zeta\|.$$

Anosov geodesic flows are degenerate examples of partially hyperbolic geodesic flows. Genuine examples (with $\dim E^c \geq 2$) have been constructed by Carneiro and Pujals; see [**CP2011**, **CP2013**]. They deformed a higher-rank locally symmetric space of non-compact type. This answered a question raised by Herman in emails to many dynamicists.

5. *Negative curvature and hyperbolicity of the geodesic flow*

5.1. *Does the marked length spectrum determine the metric?* Suppose (M, g) is a closed Riemannian manifold. The marked length spectrum of M is the function that assigns

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to each free homotopy class of loops the infimum of the length of loops in the class (i.e., the length of the shortest closed geodesic lying in this class).

Conjecture 5.1.1. [**BK1985**] Two metrics with negative curvature on a compact manifold must be isometric if they have the same marked length spectrum.

Croke and Otal showed that this conjecture is true for metrics on surfaces [**Ota1990**, **Cro1990**]. Indeed, it is enough to assume that the metrics have non-positive curvature. The problem is open in higher dimensions. Hamenstädt showed that the geodesic flows of the two metrics must be C^0-conjugate, thereby reducing the problem to Conjecture 5.2.1 in the next subsection [**Ham1991**]. Guillarmou and Lefeuvre [**GL2018**] have shown that the conjecture is locally true in the sense that two close enough metrics with Anosov geodesic flow and the same length spectrum are isometric.

One obtains a natural interesting modification of this conjecture by replacing negative curvature by non-existence of conjugate points. On the other hand, some restriction on the metrics is necessary as the examples of Croke and Kleiner, which we describe in the next subsection, provide non-isometric metrics with the same marked length spectrum. It is also necessary to consider the marked length spectrum rather than the length spectrum. Vignéras gave examples of non-isometric hyperbolic surfaces that have the same set of lengths for their closed geodesics [**Vig1980**].

One can ask a similar question for non-manifolds. Consider a 2-dimensional (metrical) simplicial complex such that every simplex is hyperbolic with geodesic edges and such that the CAT(-1) condition holds on every vertex. Assume, in addition, that every edge is contained in at least two simplices.

Question 5.1.2. (Benjamin Schmidt) Does the marked length spectrum determine such a metric (in the class of all locally CAT(-1) metrics on this space, or in the class of all 2-dimensional (metrical) simplicial complexes with the above properties homeomorphic to the given complex)?

If the simplicial complex is (topologically) a manifold, the answer is positive and is due to [**HP1997**]. One can also ask the question in higher dimensions. An easier, but still interesting version of the question is when we assume that every edge is contained in at least three simplices.

The questions above are closely related to the boundary rigidity problem, which we now recall. Given a compact manifold M with smooth boundary N, a Riemannian metric g on M induces a non-negative real-valued function d on $N \times N$, where $d(p, q)$ is the distance in (M, g) between p and q. We call (M, g) boundary rigid if a Riemannian manifold g' on M that induces the same function on $N \times N$ must be isometric to g'.

Question 5.1.3. (Boundary rigidity problem) What conditions on M, N and g imply boundary rigidity?

Michel conjectured in [**Mic1981**] that (M, g) should be boundary rigid if it is *simple*. Simplicity means that M is simply connected and any two points of M are connected by a unique geodesic segment $\gamma_{p,q} : [0, 1] \to M$, and this unique geodesic has the property that $\gamma_{p,q}(t) \notin N$ if $0 < t < 1$. Michels conjecture is known to be true when

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(M, g) is a simple subspace of a Euclidean space [**Gro1983**], when $(M.g)$ is a simple subspace of a symmetric space with negative curvature [**BCG1995**], and when M is two dimensional [**PU2005**].

One can modify Question 5.1.3 by requiring that the other Riemannian metric g' in the definition of boundary rigid manifolds above also satisfies some additional assumption. Special cases of the last question were answered in [**Cro1990**, **CDS2000**, **SU2000**, **BI2013**].

One can also ask the boundary rigidity question for Finsler metrics; recent references with non-trivial results include [**CD2010**, **BI2010**].

5.2. *The conjugacy problem.* Two Riemannian manifolds (M_1, g_1) and (M_2, g_2) have C^k-conjugate geodesic flows if there is an invertible map $h : S_1^M \to S_2^M$ such that h and h^{-1} are C^k and $h \circ \phi_t^1 = \phi_t^2 \circ h$ for all t. Here ϕ_t^i is the geodesic flow for the metric g_i. The following is a long standing conjecture.

Conjecture 5.2.1. Compact Riemannian manifolds with negative curvature must be isometric if they have C^0-conjugate geodesic flows.

As with Conjecture 5.1.1, it may be possible to relax the hypothesis of negative curvature to no conjugate points, but some restriction on the metrics is required. Weinstein pointed out that all of the Zoll metrics on S^2 have conjugate geodesic flows. Using the explicit rotationally-symmetric examples of Zoll metrics (Tannery metrics in the terminology of [**Bes1978**, Ch. 4]), Croke and Kleiner constructed examples of different metrics on an arbitrary closed manifold such that their geodesic flows are conjugate and their marked length spectra coincide (see [**CK1994**]).

The conjecture is true in dimension two; indeed, it is enough to assume that one of the manifolds has non-positive curvature [**CFF1992**]. It follows from the theorem of Besson–Courtois–Gallot, discussed in the next subsection, that the conjecture holds if one of the metrics is locally symmetric. Croke and Kleiner showed that C^1 conjugacy implies that the volumes of the manifolds are the same with no restrictions on the metrics and implies isometry of the metrics if one of them has a global parallel vector (as is the case in a Riemannian product) [**CK1994**].

5.3. *Entropy and locally symmetric spaces with negative curvature.* Let h_{Liou} denote the entropy of the geodesic flow with respect to the Liouville measure, let h_{top} be its topological entropy and let h_{vol} be the volume entropy

$$h_{\text{vol}} = \lim_{r\to\infty} \frac{1}{r} \log \text{Vol}\, B_{\widetilde{M}}(p, r),$$

where $B_{\widetilde{M}}(p, r)$ is the ball of radius r around a point p in the universal cover $\widetilde{M}$. It is always true that $h_{\text{vol}} \le h_{\text{top}}$ and they are equal if there are no conjugate points, in particular, if the curvature is negative [**Man1979**, **FM1982**]. By the variational principle for entropy, $h_{\text{Liou}} \le h_{\text{top}}$ for any metric; these entropies are equal for a locally symmetric metric.

Katok [**Kat1982**] showed that if g is an arbitrary metric and g_0 is a metric of constant negative curvature on a surface of genus ≥ 2 such that $\text{area}(g) = \text{area}(g_0)$, then

$$h_{\text{Liou}}(g) \le h_{\text{Liou}}(g_0) = h_{\text{top}}(g_0) \le h_{\text{top}}(g)$$

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and both inequalities are strict unless g also has constant negative curvature. He explicitly formulated the following influential and still open conjecture.

Conjecture 5.3.1. [**Kat1982**] $h_{\text{top}} = h_{\text{Liou}}$ for a metric of negative curvature on a compact manifold if and only if it is locally symmetric.

Katok's arguments extend to higher dimensions provided the two metrics are conformally equivalent. Flaminio showed that Conjecture 5.3.1 holds for deformations of constant curvature metrics [**Fla1995**].

Katok's paper implicitly raised the questions of whether, in higher dimensions, h_{top} is minimized and h_{Liou} is maximized (among metrics of negative curvature of fixed volume on a given manifold) by the locally symmetric metrics. It is now known that the locally symmetric metrics are critical points for both entropies [**KKW1991**], but the locally symmetric spaces do not maximize h_{Liou} except in the two-dimensional case [**Fla1995**]. The topological entropy, however, is minimized.

Gromov [**Gro1983**] conjectured that if $f : (M, g) \to (M_0, g_0)$ is a continuous map of degree $d \neq 0$ between compact connected oriented n-dimensional Riemannian manifolds and M_0 has constant negative curvature, then

$$h_{\text{vol}}(g)^n \operatorname{Vol}(M, g) \geq |d| h_{\text{vol}}(g_0)^n \operatorname{Vol}(M_0, g_0)$$

with equality if and only if g also has constant negative curvature and f is homotopic to a d-sheeted covering. The quantity h^n Vol has the advantage of being invariant under homothetic rescaling of the manifold; this obviates the assumption that the two manifolds have the same volume.

THEOREM 5.3.2. [**BCG1995**] *The above conjecture of Gromov is true even when* (M_0, g_0) *is allowed to be a locally symmetric space of negative curvature.*

We refer the reader to [**BCG1995**, §9] and the survey [**Ebe2001**] for a list of the many corollaries of this remarkable theorem.

5.4. *Regularity of the Anosov structure.* An extensive list of results about the regularity of the Anosov splitting for the geodesic flow in negative curvature (and for other Anosov systems) can be found in the introduction to [**Has1994**]. In dimension two, or if the curvature is 1/4-pinched, the splitting is always at least C^1. In higher dimensions, it is always Hölder continuous but typically not C^1. It is natural to ask the following question.

Question 5.4.1. Does there exist a closed Riemannian manifold of negative curvature such that E^s and E^u are C^1 smooth, but the dimension is > 2 and the metric is not 1/4-pinched?

Another motivation for this question is the question of Pollicott in the next subsection.

A program to construct open sets of metrics whose geodesic flows have Anosov splitting that have low regularity of the Anosov splitting on large subsets of the unit tangent bundle is explained in [**HW1999**, §4]; see, in particular, Proposition 12. The idea is to perturb a suitable base example. This base example must have directions in its unstable manifolds with widely different expansion rates. Unfortunately, there are no known examples of geodesic flows that are suitable bases for the perturbation argument; see the discussion

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at the end of [**HW1999**, §4]. Along all geodesics in complex hyperbolic space, one has Lyapunov exponents of one and two corresponding to parallel families of planes with curvature -1 and -4 respectively.

Question 5.4.2. Is there a metric of negative curvature for which the ratio of largest positive Lyapunov exponent to smallest positive Lyapunov exponent is greater than two for typical geodesics (not just exceptional closed geodesics)?

The Anosov splitting is C^∞ only if and only the metric is locally symmetric. This follows from [**BFL1992**, **BCG1995**]. For surfaces, Hurder and Katok [**HK1990**] showed that the splitting must be C^∞ if one of the stable or unstable foliations is $C^{1+o(x \log |x|)}$. In higher dimensions, it is expected that the splitting must be C^∞ if it is C^2, but this question still seems to be open.

5.5. *How many closed geodesics of length $\leq T$ exist? (Communicated by Pollicott).* Let (M, g) be a closed Riemannian manifold with negative sectional curvatures. It is known [**Mar2004**] that the number $N(T)$ of closed geodesics of length $\leq T$ grows asymptotically as e^{hT}/hT.

Conjecture 5.5.1. $N(T) = (1 + O(e^{-\varepsilon T})) \int_2^{e^{hT}} (du/\log u)$.

The conjecture is true for all metrics such that the stable and unstable bundles E^s and E^u are C^1 smooth, which is the case for surfaces and for $1/4$ pinched metrics. Question 5.4.1 above asks whether there are other examples.

5.6. *Infinitely many simple closed geodesics.*

Question 5.6.1. (Miller [**Mil2001**], Reid) Are there infinitely many simple closed geodesics in every hyperbolic complete three-manifold of finite volume?

The answer is positive for surfaces and estimates for the growth of the number of simple closed geodesics with length $\leq L$ are in [**Riv2001**, **Mir2008**]. One can ask the same question for manifolds of variable negative curvature in any dimension. The answer is positive for generic metrics of negative curvature.

Reid [**Rei1993**] has examples of (arithmetic) hyperbolic manifolds of finite volume in which every closed geodesic is simple.

6. *Non-positive curvature and non-uniform hyperbolicity of the geodesic flow*

Recall that the rank of a vector v in such a manifold is the dimension of the space of Jacobi fields along the geodesic γ_v that are covariantly constant. It is easily seen that rank is upper semicontinuous. All vectors have rank ≥ 1, since the velocity vector field is a covariantly constant Jacobi field. The rank of the manifold is the minimum rank of a vector. The set of vectors of minimum rank is obviously open and is known to be dense [**Bal1982**]. The definitions generalize the classical notion of rank for locally symmetric spaces of non-compact type.

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6.1. *Ergodicity of geodesic flows.*

Question 6.1.1. Is the geodesic flow of a metric of non-positive curvature on a closed surface of genus ≥ 2 ergodic with respect to the Liouville measure?

It is known that the flow is ergodic on the set of geodesics passing through points where the curvature is negative [**Pes1977**]. The complement of this set consists of vectors tangential to zero curvature geodesics, i.e., geodesics along which the Gaussian curvature is always zero. It is not known whether this set must have measure zero. There is an analogous question in higher dimensions: *is the geodesic flow of a closed rank one manifold of non-positive curvature ergodic*? Here it is known that the flow is ergodic on the set of rank one vectors [**BB1982**, **Bur1983**], but it is not known if the complementary set (of higher-rank vectors) must have measure zero.

Several papers published in the 1980s (notably, [**Bur1983**, **BB1982**, **BBE1985**]) stated that this problem had been solved. These claims were based on an incorrect proof that the set of higher-rank vectors must have measure zero.

The measure considered above is the Liouville measure. There is a (unique) measure of maximal entropy for the geodesic flow of a rank one manifold. It was constructed by Knieper, who proved that it is ergodic [**Kni1998**].

6.2. *Zero curvature geodesics and flat strips.* Consider a closed surface of genus ≥ 2.

Question 6.2.1. (Burns) Is there a C^∞ (or at least C^k, $k \geq 3$) metric for which there exists a non-closed geodesic along which the Gaussian curvature is everywhere zero?

The question is of particular interest when the metric has non-positive curvature, since a negative answer, in this case, would give a positive answer to Question 6.1.1.

If we assume that the curvature is only C^0, then a metric with such a geodesic can be constructed. Each end of the geodesic is asymptotic to a closed geodesic. But a geodesic along which the curvature is zero cannot spiral into a closed geodesic if the curvature is C^2; see [**Rug1998**]. Wu [**Wu2013**] has recently given a negative answer to the question under the assumption that the subset of the surface where the curvature is negative has finitely many components; he also shows under this hypothesis that only finitely many free homotopy classes can contain zero curvature closed geodesics.

In a translation surface, the number of free homotopy classes that contain closed geodesics with length $\leq L$ grows quadratically with L.

Question 6.2.2. (Knieper) Consider a closed surface of genus ≥ 2 with a metric of non-positive curvature. Let $Z(L)$ be the number of free homotopy classes containing closed geodesics with length $\leq L$ along which the curvature is everywhere zero. Is there a quadratic upper bound for $Z(L)$, i.e., is there a constant $C \geq 0$ such that

$$Z(L) \leq C \cdot L^2?$$

A negative answer to Question 6.2.1 in the case of non-positive curvature would imply that $Z(L)$ is bounded.

A flat strip is a totally geodesic isometric immersion of the Riemannian product of $\mathbb{R}$ with an interval. Zero curvature geodesics can be viewed as flat strips with infinitesimal

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width. It is known that all flat strips (with positive width) in compact surfaces with non-positive curvature must close up in the $\mathbb{R}$-direction; in other words, they are really immersions of the product of S^1 with an interval. Furthermore, for each $\delta > 0$, there are only finitely many flat strips with width $\geq\delta$. Proofs can be found in a preprint of Cao and Xavier [**CX**]. The main idea is that if two flat strips of width δ cross each other at a very shallow angle, then their intersection contains a long rectangle with width close to 2δ.

It is possible that there are only finitely many flat strips. This is true when the set where the curvature is negative has finitely many components [**Wu2013**], but the only effort to prove it in general [**RodHer2003**] was unsuccessful.

Remark. Question 6.2.1 still makes sense even if the curvature of the manifold is not restricted to be non-positive. It might generalize also to higher dimensions. What can be said about a geodesic along which all sectional curvatures are zero? Or, less stringently, the sectional curvatures of planes containing the tangent vector?

6.3. *Flats in rank one manifolds with non-positive curvature.* A flat is a flat strip of infinite width: in other words, a totally geodesic isometric immersion of the Euclidean plane. Eberlein asked whether a compact rank one manifold that contains a flat must contain a closed flat, i.e., a totally geodesic isometric immersion of a flat torus. Bangert and Schroeder gave an affirmative answer for real analytic metrics [**BS1991**]. The problem is open for C^∞ metrics.

6.4. *Besson–Courtois–Gallot.* Does their rigidity theorem from [**BCG1995**], Theorem 5.3.2 in this paper, generalize to higher-rank symmetric spaces of non-compact type?

Connell and Farb [**CF2003a**, **CF2003b**] extended the barycenter method, which plays a vital role in [**BCG1995**], and proved that the theorem holds for a product in which each factor is a symmetric space with negative curvature and dimension ≥ 3. We also refer the reader to their extensive survey in [**CF2003c**].

A recent preprint of Merlin [**Mer2016**] uses a calibration argument to show that $h^4_{\mathrm{vol}}\,\mathrm{Vol}$ is minimized by the locally symmetric metric on a compact quotient of the product of two hyperbolic planes.

7. *Manifolds without conjugate points (rigidity conjectures)*

There are many rigidity results for manifolds with non-positive curvature that might extend to manifolds with no conjugate points. One still has the basic setting of a universal cover homeomorphic to $\mathbb{R}^n$ in which any two points are joined by a unique geodesic. However, the convexity of the length of Jacobi fields, which is the basis of many arguments used in non-positive curvature, is no longer available.

Two major results of this nature are the following theorems.

THEOREM 7.0.1. *A Riemannian metric with no conjugate points on a torus is flat.*

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THEOREM 7.0.2. *Let g be a complete Riemannian metric without conjugate points on the plane $\mathbb{R}^2$. Then, for every point p,*

$$\liminf_{r\to\infty} \frac{\text{area } B(p, r)}{\pi r^2} \geq 1,$$

with equality if and only if g is flat.

Theorems 7.0.1 and 7.0.2 are easy for manifolds with non-positive curvature but require subtle arguments in the context of no conjugate points.

The two-dimensional case of Theorem 7.0.1 was proved by Hopf [**Hop1948**] and the general case by Burago and Ivanov [**BI1994**]. The Lorentzian analog of Theorem 7.0.1 is false; two-dimensional counterexamples are constructed in [**BM2013**]. The Finsler analog is also false as there are examples of Finsler metrics without conjugate points on the torus that are not the Minkowski metric (see e.g. [**Bus1955**, §33]).

Question 7.0.3. (Communicated by S. Ivanov) Is the geodesic flow of a Finsler metric on the torus conjugate to that of the (locally) Minkowski metric on the torus?

All smooth examples we are aware of have geodesic flows that are smoothly conjugate to that of the locally Minkowski metric, but, generally, it is not known whether the velocity vectors of geodesics with irrational rotation numbers are dense in the tangent bundle.

Theorem 7.0.2 is a recent result of Bangert and Emmerich [**BE2013**], which greatly improved on earlier results in [**BK1991**, **BE2011**].

Bangert and Emmerich's work was motivated by a question, asked by Bangert and Burns–Knieper, which is answered by the following theorem, the proof of which is based on Theorem 7.0.2.

THEOREM 7.0.4. [**BE2013**] *Consider a complete Riemannian metric without conjugate points on the cylinder $\mathbb{R} \times S^1$. Assume that the ends spread sublinearly, i.e.,*

$$\lim_{d(p,p_0)\to\infty} \frac{l(p)}{\text{dist}(p,\ p_0)} = 0,$$

where $l(p)$ is the length of the shortest geodesic loop based at p. Then the metric is flat.

7.1. *Divergence of geodesics.* Let α and β be two geodesics in a complete simply connected Riemannian manifold without conjugate points.

Question 7.1.1. Suppose $\alpha(0) = \beta(0)$. Does $\text{dist}(\alpha(t), \beta(t)) \to \infty$ as $t \to \infty$?

The answer is positive in dimension two [**Gre1954**]. The question is open in higher dimensions; the proof in [**Gre1956**] is incorrect. The answer to the corresponding question about Finsler metrics is negative. The Teichmüller metric on Teichmüller space has no conjugate points, but Masur showed that there exist two rays through the same point which do not diverge [**Mas1975**].

The following is a closely related question.

Question 7.1.2. Suppose $\text{dist}(\alpha(t), \beta(t)) \to 0$ as $t \to -\infty$. Does $\text{dist}(\alpha(t), \beta(t)) \to \infty$ as $t \to \infty$?

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Without positive answers to these questions there would seem to be little hope for a satisfactory analog of the sphere at infinity, which plays a prominent role in the case of non-positive curvature.

7.2. *Parallel postulate questions.* The flat strip theorem is another basic tool in the study of manifolds with non-positive curvature. It states that 'parallel geodesics' must bound a flat strip. More precisely, if α and β are two geodesics in a simply connected manifold with non-positive curvature such that $\mathrm{dist}(\alpha(t), \beta(t))$ is bounded for all $t \in \mathbb{R}$, then the two geodesics are the edges of a totally geodesic isometric immersion of the Riemannian product of $\mathbb{R}$ with an interval.

The flat strip theorem fails for manifolds with no conjugate points (although it does extend fairly easily to manifolds without focal points). Counter examples have been given in [**Bur1992**] and by Kleiner (unpublished). Kleiner's example has a copy of $\mathbb{Z} \times \mathbb{Z}$ in its fundamental group, but it does not contain a corresponding flat torus. He perturbs the Heintze example [**BBE1985**, Example 4], which we described in §4.1. The following basic question appears to be open in general (although [**Esc1977**, Proposition 4] suggests an affirmative answer under some extra hypotheses).

Question 7.2.1. Must homotopic closed geodesics in a manifold of dimension ≥ 3 with no conjugate points be homotopic through closed geodesics?

In dimension two, the answer is positive. This follows from a modification of the Birkhoff minimax procedure, recalled in §2, combined with the results/methods of [**Mor1924**], as we now explain.

Consider two homotopic closed geodesics γ_0 and γ_1. They have the same length by [**Mor1924**, Theorem 12], which we denote by L_0 and their lifts to the universal cover bound a periodic strip S by [**Mor1924**, §19]. Next, take a continuous bijection $h : S^1 \times [0, 1] \to M$ such that $S^1 \times \{0\} = \gamma_0$ and $h(S^1 \times \{1\}) = \gamma_1$. Then apply to each curve $h(S^1 \times \{t\})$ a shortening procedure; in dimension two one can organize the shortening procedure such that the lift of the projection to the universal cover does not leave the strip S.

The natural modification of the Birkhoff minimax procedure proves the existence of a closed geodesic homotopic to γ_0 whose lift to the universal cover lies in the strip S; this geodesic also has length L_0. Arguing as in [**Mor1924**, §19], we obtain that the lift of this geodesic cuts the strip into two substrips, S_1 and S_2. Recursively applying the same arguments, we obtain that the strip S has an everywhere dense set of non-intersecting geodesics such that any of them cuts the strip into two substrips. Passing to the limit, we get a foliation of the strip by lifts of closed geodesics and hence the desired homotopy of γ_0 to γ_1 through closed geodesics.

Note that, for dimensions ≥ 3, part of the argument still works (in particular, homotopic closed geodesics in a closed manifold with no conjugate points must have the same length) and one can show that γ_0 and γ_1 belong to the same connected component of the set of closed curves of length equal to L_0, but it is not clear why the set of closed geodesics homotopic to γ_0 is path-connected.

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One can still hope for a version of the flat strip theorem in manifolds with no conjugate points in which there is a large enough family of 'parallel' geodesics. One would then hope to find a totally geodesic isometric immersion of the Euclidean plane. Rigidity might hold in the large, even though the examples above show that it breaks down locally.

The simplest question of this type asks if a Riemannian plane satisfying Euclid's fifth postulate must be flat. Consider the plane $\mathbb{R}^2$ with a complete Riemannian metric. Assume that, for every geodesic and for every point not on the geodesic, there exists precisely one non-trivial geodesic that passes through the point and does not intersect the geodesic. This assumption implies that the metric has no conjugate points.

Question 7.2.2. [**BK1991**] Must a metric satisfying this version of the parallel postulate be flat?

The question looks like a question in the synthetic geometry, but it is not, since we do not require *a priori* that the other axioms of the Euclidean geometry are fulfilled (for example, the congruence axioms).

7.2.1. *Higher-rank rigidity.*

Conjecture 7.2.3. (Spatzier) Let (M, g) be a closed symmetric space of non-compact type and of higher rank. Then the only metrics on M with no conjugate points are homothetic rescalings of g.

7.3. *Is $M \times S$ with no conjugate points a direct product?* Consider the product of a closed surface M^2 of genus ≥ 2 with the circle S. Let g be a Riemannian metric on $M \times S$ with no conjugate points.

Conjecture 7.3.1. (Burago–Kleiner) The metric on the $\mathbb{Z}$-cover corresponding to the S^1 factor is a direct product: $(M \times \mathbb{R}, g) = (M, g_1) \times (\mathbb{R}, dt^2)$.

One can, of course, make more general conjectures, e.g., about metrics without conjugate points on products, or on manifolds which admit non-positively curved metrics with higher rank, but the above conjecture with $M \times S^1$ seems to be the easiest one (and is probably still hard to prove).

7.4. *Magnetic geodesics without conjugate points.* One can generalize the notion of 'conjugate points' for magnetic geodesics (even for arbitrary natural Hamiltonian systems.)

Question 7.4.1. (Paternain) Consider a magnetic flow on a closed surface of genus ≥ 2 and an energy level such that the topological entropy vanishes. Suppose that the magnetic geodesics on this level do not have conjugate points. Is the system locally symmetric in the sense that the metric has constant curvature and the magnetic form is a constant multiple of the volume form?

Many of the earlier questions in this section can also be asked about magnetic geodesics lying on a certain (possibly, sufficiently high) energy level. Let us mention [**Bia2000**], where a natural analog of Theorem 7.0.1 was proved under the assumption that the metric is conformally flat, and it was conjectured that this assumption is not essential.

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8. *Beyond no conjugate points*

8.1. *Ergodic geodesic flows.* Does every closed manifold (with dimension ≥ 2) admit a metric (Riemannian or Finsler) with ergodic geodesic flow?

This is known for surfaces [**Don1988II**], for 3-manifolds [**Kat1994**], for product manifolds in which the factors have dimension ≤ 3 [**BG1994**] and for spheres [**BG**]. Donnay and Pugh have shown that any embedded surface can be perturbed in the C^0 topology to an embedded surface whose geodesic flow is ergodic [**DP2004**]. These constructions all make essential use of the focusing caps introduced by Donnay in [**Don1988I**].

The general problem is still wide open despite some reports of its solution ([**Ber2000**, p. 87] and [**Ber2003**, §10.9]).

Question 8.1.1. Is there a Riemannian metric on a closed manifold with everywhere positive sectional curvatures and ergodic geodesic flow?

Metrics close to the standard metric on S^2 would be especially interesting. The best result in this direction is due to Burago and Ivanov [**BI2016**]. They exhibit smooth Finsler metrics with positive measure theoretic entropy that are small perturbations of the round metric on S^4.

8.2. *The measure of transitive and recurrent sets.* Let (M^2, g) be a closed surface with ergodic geodesic flow. Then every tangent vector lies in one of the following sets:

$$T_b := \{v \in SM \mid \text{any lift of } \gamma_v \text{ stays in a bounded subset of } \widetilde{M}\};$$
$$T_p := \{v \in SM \mid \text{any lift of } \gamma_v \text{ is unbounded, and approaches infinity properly}\};$$
$$T_i := TM \backslash (T_b \cup T_p).$$

All three sets are measurable and invariant under the geodesic flow. By ergodicity, one of them has full measure.

Question 8.2.1. (Benjamin Schmidt) Which of these sets has full measure?

The expected answer depends on the genus of the surface: on the torus, the set T_i should have full measure, and on the surfaces of higher genus, T_p should have full measure.

In this context, it is natural to consider only minimizing geodesics: if one shows that the set of minimizing geodesics has non-zero measure, then it has full measure by ergodicity of the geodesic flow. However, there exist examples of metrics on the torus such that the set of minimizing geodesics has small Hausdorff dimension.

The question seems to make sense in any dimension.

8.3. *Generic metrics.* We denote by $\mathcal{G}^k(M)$ the space of all smooth Riemannian metrics on M with the C^k-topology. Riemannian metrics are sections of the bundle S^2T^*M. Our topology is the weak C^k-topology on the space of embeddings of M to S^2T^*M (see, e.g., [**Hir1976**, Ch. II]). The definition can be extended to metrics of all signatures and to Finsler metrics.

We use C^k-topology, either locally or on closed manifolds, where we may assume that the atlas is finite and the gluing functions have all derivatives bounded. Then the

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convergence in the topology means the following: a sequence $g^{(1)}, g^{(2)}, \ldots$ of metrics converges to a metric g in the topology C^k if, in local charts, the entries $g_{ij}^{(m)}$ (viewed as local functions in this chart) and their partial derivatives $\partial g_{ij}^{(m)}/\partial x_\ell$, $\partial^2 g_{ij}^{(m)}/\partial x_\ell \partial x_s, \ldots$ up to order k converge to those of g_{ij}.

Question 8.3.1. Assume that M is closed. Do the metrics with positive entropy form a C^k-dense subset of $\mathcal{G}(M)$?

This question can be asked for any k. For $k = 2$, it was recently answered positively in all dimensions in [**Con2010**] (see also [**CP2002**]). For $k = \infty$ and dimension two, it was answered positively in [**KW2002**].

Question 8.3.2. Assume that M is closed. If $\dim(M) \geq 3$, do the metrics for which the geodesic flow is transitive (i.e., has an orbit whose closure is SM) form a dense subset of $\mathcal{G}(M)$?

This question is closely related to the famous Arnold diffusion conjecture. If $\dim(M) = 2$ and k is large enough, the answer is negative if we take a metric with integrable geodesic flow such that certain Liouville tori are irrational and any small perturbation of the metric has non-transitive geodesic flow by the KAM theory.

Question 8.3.3. Assume that M is closed. Is there a dense subset of metrics in $\mathcal{G}$ for which the tangent vectors to the closed geodesics are dense in SM?

It is known that this is the case for metrics with non-uniformly hyperbolic geodesic flows. Note that it is relatively easy to construct (see, e.g., [**Wei1970**]) a metric on any manifold such that a certain non-empty open subset of SM contains no vectors tangential to a closed geodesic.

8.4. *Density in the manifold.* In the previous section, we discussed the unit tangent bundle SM. In this section, we ask similar questions about M itself, and we will not assume that the metrics are generic.

Question 8.4.1. Is the union of the closed geodesics always dense in the manifold?

Question 8.4.2. Does every metric on a compact surface of positive genus have a geodesic that is dense in the surface?

8.5. *Finsler metrics with positive flag curvature.* Flag curvature is the Finslerian analog of sectional curvature. The following were shown in [**AZ1988**]: any closed Finsler surface with constant flag curvature zero is locally Minkowskian; any sufficiently smooth closed Finsler surface with constant negative curvature is Riemannian (there are non-Riemannian examples which are not sufficiently smooth); and any connected closed Finsler surface with constant positive flag curvature is homeomorphic to S^2 or $\mathbb{R}P^2$. The result for negative flag curvature also follows from [**Fou1997**] and is true in all dimensions.

Problem 8.5.1. Describe Finsler metrics on S^2 of flag curvature one.

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If such a metric is reversible, then it is the round Riemannian metric [**Bry2006**]. The Katok examples described in §2.2 have constant flag curvature one. All projectively flat metrics of constant flag curvature on S^2 are described in [**Bry1997**]. Bryant has also constructed examples that are not projectively flat [**Bry1996**, **Bry2002**]. Unlike Katok's example, all of the examples in Bryant's papers are Zoll, i.e., all of their closed geodesics are closed and of the same length.

Even though the Finsler metrics of constant flag curvature one on S^2 are not fully classified, the possible behavior of their geodesic flows is understood. It is shown in [**BFIMZ2019**] that the geodesic flow of a metric of constant curvature one is conjugated to that of the Katok metric. This implies that one of the following two possibilities must occur.

(i) Precisely two geodesics are closed. These two geodesics are a simple closed curve parameterized in opposite directions. This curve is tangential to a non-trivial Killing vector field which vanishes precisely at two points. All of the non-closed geodesics have the same asymptotic rotation number ρ around the fixed points of the Killing field. There are distinct irrational numbers α_1 and α_2 such that the lengths of the two closed geodesics are $\alpha_1\pi$ and $\alpha_2\pi$.

(ii) All geodesics are closed. There are an even integer k and positive rational numbers β_1 and β_2 with $1/\beta_1 + 1/\beta_2 = 1$ such that at most two geodesics have length different from $k\pi$ and the lengths of the exceptional geodesics are $\beta_1\pi$ and $\beta_2\pi$. In the case $\beta_1 = \beta_2 = 2$, all geodesics are closed and have the same length 2π; we will call such metrics *Zoll metrics*.

Moreover, the Finsler metrics satisfying (i) necessary admit a Killing vector field and are quite well understood. Applying the Zermelo transformation to such metrics (see, e.g., [**FM2018**]), one obtains non-Zoll metrics satisfying (ii). Moreover, any such metric (of flag curvature one on S^2 and satisfying (ii)) admitting a Killing vector field can be obtained by this procedure. Unfortunately, no other examples of non-Zoll metrics satisfying (ii) are known.

Problem 8.5.2. Construct examples of Finsler metrics of constant curvature on S^2 that are not Zoll and such that they admit no Killing vector field.

For recent progress in this direction, see [**LM2018**].

In a closed (even complete) Riemannian manifold M of positive sectional curvature, two closed totally geodesic submanifolds such that the sum of their dimensions is at least the dimension of M must intersect [**Fra1961**].

Question 8.5.3. Does Frankel's theorem remain true for Finsler manifolds? More precisely, must two closed totally geodesic submanifolds of a closed Finsler manifold with positive flag curvature intersect if the sum of their dimensions is at least the dimension of the ambient manifold?

In dimension two, the result was established in [**BFIMZ2019**]. A positive answer to this question was claimed in [**KP2000**], but the proof has a serious flaw, which is explained in [**BFIMZ2019**]. Note that there are Finsler surfaces with positive flag curvature that contain non-intersecting closed geodesics (see [**BFIMZ2019**, **Rad2016**]). This is possible

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because a curve must have geodesic parameterizations in *both* directions in order to be totally geodesic; non-reversible Finsler metrics can therefore have simple closed geodesics that are not totally geodesic submanifolds.

The simplest, still open, version of Question 8.5.3 is the following question.

Question 8.5.4. Let (M^3, F) be a closed three-dimensional reversible Finsler manifold of positive flag curvature and let N be a closed totally geodesic two-dimensional submanifold. Is it true that each closed geodesic intersects this submanifold or lies on it?

8.6. *Gaidukov in higher dimensions.* We consider a Riemannian metric on an oriented closed surface M of genus ≥ 1. Let Γ be a non-trivial free homotopy class and let p be a point in M. A theorem of Gaidukov [**Gai1966**] says that there exist a closed geodesic $\gamma : \mathbb{R} \to M$ in Γ and a ray $\beta : [0, \infty) \to M$ with $\beta(0) = p$ such that $\operatorname{dist}(\beta(t), \gamma(t)) \to 0$ as $t \to \infty$. As explained in [**BP1986**], Gajdukov's results follow easily from the classical results of [**Mor1924**, **Hed1932**].

Problem 8.6.1. (Benjamin Schmidt) Generalize this statement to higher dimensions.

Gaidukov's proof is profoundly two dimensional and cannot be generalized. But investigations in the direction of Problem 8.6.1 are important and plentiful in Aubry–Mather theory and Arnold diffusion. Instead of a single closed geodesic, one should consider a minimizing set, namely, the support of a minimal measure (see, e.g., [**Mat1991**, **CI1999**, **Mat2004**, **Ber2010**]).

8.7. *Systolic and diastolic inequalities for surfaces (communicated by Guth, Rotman and Sabourau).* Let (M, g) be a compact Riemannian surface. The inequalities in question compare the length of certain short closed geodesics with the square root of the area of (M, g). Recall that the systole $\operatorname{sys}(M, g)$ is the least length of a non-trivial closed geodesic. Two other geometrically meaningful constants can be defined by the minimax procedure

$$L(M, g) = \inf_f \max_{t \in \mathbb{R}} F[f^{-1}(t)],$$

in which the infimum is taken over all (Morse) functions $f : M \to \mathbb{R}$ and the functional F is either (a) the total length of $f^{-1}(t)$ or (b) the length of its longest component. In both cases, $L(M, g)$ is realized as the length of a certain union of closed geodesics. In case (a), $L(M, g)$ is one definition of the diastole $\operatorname{dias}(M, g)$ of the Riemannian surface (at least two different notions of diastole have been studied; see [**BS2010**]).

By a result originally due to [**Cro1988**] and later improved in [**NR2002**, **Sab2004**, **Rot2006**], for every Riemannian metric g on the sphere S^2, one has

$$\operatorname{sys}(S^2, g) \leq \sqrt{32}\sqrt{\operatorname{area}(S^2, g)}.$$

Actually, in [**Cro1988**], it was suggested that the constant $\sqrt{32}$ in the above inequality can be replaced by $\sqrt{2\sqrt{3}}$. The following example due to [**Cro1988**] shows that one cannot go below $\sqrt{2\sqrt{3}}$: take two congruent equilateral triangles and glue them along their boundaries. The example is not smooth and suggests the study of systolic inequalities

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on the space of smooth metrics with conical singularities. On the space of such metrics, let us consider the function

$$\sigma(g) = \frac{\text{area}(S^2, g)}{\text{sys}(S^2, g)^2}.$$

This function has many nice properties: for example, it is Lipschitz with respect to the appropriate distance on the space of metrics. By [**Cro1988**], the function has a positive minimum, and the natural conjecture is that the minimum is attained on the sphere constructed from two equilateral triangles as, described above.

Other critical points of this function are also interesting: Balacheff showed that the round metric on the sphere is a critical point [**Bal2006**]. Moreover, by [**ÁB2014**], Zoll Finsler manifolds (M, F) of all dimensions satisfy the following alternative: for any Finsler deformation F_t with $F_0 = F$, either the function $t \mapsto \sigma(F_t)$ has a strict local minimum at $t = 0$, or the family F_t is tangential up to every order to the space of Zoll metrics. In dimension two the result is further improved in [**ABHS2017**, **ABHS2018**], where it is shown that both in the Riemannian and in the Finslerian cases the function $\sigma(F_t)$ defined above has a strict local minimum at $t = 0$ if F is a Zoll metric.

Note, though, that in the papers [**ÁB2014**, **ABHS2017**, **ABHS2018**] the metrics and the deformation are assumed to be smooth (at least C^3). In view of the Croke's example, it may be natural to consider also non-smooth metrics and non-smooth deformations.

Question 8.7.1. (Babenko) Does there exist, in the space of metrics with conical singularities, a (continuous or smooth) family of metrics g_t such that g_0 is the round metric of the sphere, g_1 is the metric from the example above and $\sigma(g_t)$ is a decreasing function of t?

The following is a simpler version of this question.

Question 8.7.2. (Babenko) Does every Riemannian metric on S^2 with area 4π that is close enough to the round metric have a closed geodesic with length $\leq 2\pi$? In other words, is it true that $\sigma \geq 1/\pi$ for metrics close to the round metric?

The answer depends on what is meant by 'close'. If 'close' means C^2-close, then the positive answer follows from [**ABHS2017**]. In that paper, the condition 'close' is given explicitly in terms of a pinching condition on the curvature. It is proved that, in the neighborhood of a Zoll metric consisting of metrics such that this pinching condition is satisfied, the equality $\sigma = 1/\pi$ holds if and only if the metric is Zoll.

A corollary of one of the main results in [**ABHS2018**] generalizes this to Finsler metrics and a C^3-neighborhood of the Zoll metrics: more precisely, $\sigma \geq 1/\pi$ in a small C^3-neighborhood of all Zoll Finsler metrics on S^2, with equality if and only if the metric is Zoll.

So a related question is as follows: under which topologies is the round metric a local minimizer of $\sigma(F_t)$ or $\sigma(g_t)$? Does this remain true for weaker topologies than C^2, e.g., C^1 or Gromov–Hausdorff?

For surfaces of higher genus, there exists a constant C such that

$$\text{sys}(M, g) \leq C \frac{\log(\text{genus}(M))}{\sqrt{\text{genus}(M)}} \text{area}(M, g); \tag{8.1}$$

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the dependence on the genus of M in this inequality is sharp. In [**Ber2003**, Note 7.2.12], inequality (8.1) is attributed to Gromov and it is explained from which papers, results and ideas of Gromov (8.1) follows. Section 7.2.1 of [**Ber2003**] is a very nice survey on systolic inequalities for surfaces.

For the diastole defined by (a), Balacheff and Sabourau showed in [**BS2010**] that there is a constant C such that

$$\operatorname{dias}(M, g) \leq C\sqrt{\operatorname{genus}(M)}\sqrt{\operatorname{area}(M, g)};$$

the dependence of this inequality on the genus is again optimal.

Question 8.7.3. (Guth) Is there a constant C such that the invariant defined by (b) above is bounded from above by $C\sqrt{\operatorname{area}(M, g)}$?

An affirmative answer to Guth's question would mean that the three quantities under consideration all depend on the genus in different ways, and are therefore measuring different features of the surface. A positive answer would also show that one can always find a pants decomposition of a closed Riemannian surface of genus g into $3g - 3$ disjoint closed geodesics of length at most $C\sqrt{\operatorname{area}(M)}$. This would precisely give the optimal Bers' constant for a genus g surface. Even for hyperbolic surfaces, this question is still open (see [**Bus1992**] for partial results).

The question above also makes sense in higher dimensions. In this case, $f^{-1}(t)$ would be an $(n-1)$-complex, where n is the dimension of the manifold, and F would measure its total volume or the volume of its largest component.

8.8. *Questions related to the systole in higher dimensions.*

Question 8.8.1. ([**Gro2001**, Question 4.11]) Does a Riemannian metric on a real projective space with the same volume as the canonical metric have a closed geodesic with length $\leq \pi$?

In dimension two, an affirmative answer is in [**Gro2001**, Proposition 4.10].

Question 8.8.2. (Álvarez Paiva) Can there be Riemannian metrics on S^3 or $S^2 \times S^1$ with unit volume all of whose closed geodesics are long? More specifically, does every metric on these spaces have a closed geodesic with length $\leq 10^{24}$ if the volume is one?

8.9. *Metrics such that all geodesics are closed.* This is a classical topic—the first examples go back at least to [**Zol1903**]; see [**Bes1978**] for details. The book [**Bes1978**] is still up to date, and many problems/questions listed in it (in particular, in Ch. 0 §D) are still open. We suggest the following as the most interesting question from their list.

Question 8.9.1. Let M be a closed manifold not covered by a sphere. Let g be a metric on M for which all geodesics are closed. Is (M, g) a CROSS (= compact rank one symmetric) manifold (with the standard metric)?

Of course, variants of this question can be asked about Finsler and Lorentzian metrics. In the Finsler setting, one may ask, for example, to describe all Finsler metrics on $CP(n)$ such that their geodesics are geodesics of the standard (Fubini–Study) metric on $CP(n)$.

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Of course, the answer in the Finslerian case is expected to be much more difficult; for example, by [**Bry1996**], on the 2-sphere there exists a two-parameter family of projectively flat Finsler metrics on the sphere, such that the flag curvature is $+1$.

In the Lorenzian setting, one can ask to construct all manifolds such that all light-like geodesics are closed (see e.g., [**MS2013**, **Suh2013b**]).

An easier version of the question above is the following conjecture sometimes attributed to Berger.

Conjecture 8.9.2. Let (M, g) be a closed simply connected manifold such that all geodesics are closed. Then all geodesics have the same length.

The conjecture is proved when M is the sphere of dimension $\neq 3$ in [**GG1981**, **RW2015**].

9. *Lorentzian metrics and metrics of arbitrary signature*

9.1. *Closed geodesics.* Most of the questions we asked about the Riemannian and Finsler metric can be modified such that they are also interesting in the semi-Riemannian metrics (i.e., of arbitrary signature) and, in particular, when the signature is Lorentzian. It appears, though, that the answers in the Lorentzian case are sometimes very different from those in the Riemannian case. Many methods that were effectively used in the Riemannian case, for example the variational methods, do not work in the case of general signature.

Let us consider, as an example, the question analogous to the one we considered in §2: how many geometrically different closed geodesics must there be for a Lorenzian metric on a closed manifold.

First, let us note that there are two possible natural notions of closed geodesic in the Lorentzian setting: one may define a closed geodesic as an immersion $\gamma : S^1 \to M$ such that $\nabla_{\dot\gamma}\dot\gamma = 0$, or as a curve $\gamma : S^1 \to M$ that can be locally reparameterized in such a way that it satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = 0$.

In the Riemannian case, these definitions are essentially equivalent, since this reparameterization can always be made 'global'. If the signature is indefinite, it is easy to construct examples of an embedding $\gamma : S^1 \to M$ such that, locally, γ can be reparameterized so that it becomes an affinely parameterized geodesic, but globally such reparameterization is impossible: the velocity vector of the geodesic after returning to the same point is proportional to but not equal to the initial velocity vector. Of course, this is possible only if the velocity vector is light-like.

We will follow most publications and define a closed geodesic as an embedding $\gamma : S^1 \to M$ such that $\nabla_{\dot\gamma}\dot\gamma = 0$.

To the best of our knowledge, the existence of a closed geodesic on a Lorentzian manifold is a quite complicated problem and nothing is known in dimensions ≥ 3. In dimension two, a closed orientable manifold with a metric of signature $(+, -)$ is homeomorphic to the torus. By [**Suh2013a**], every Lorentzian two-dimensional torus has at least two simple closed geodesics, one of which is definite, i.e., time-like or space-like. Moreover, explicit examples show the optimality of this claim.

We therefore ask the following question.

Question 9.1.1. Does every closed Lorentzian manifold have at least one closed geodesic?

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One can, of course, ask this question about manifolds of arbitrary signature and also about complete (note that, in the Lorentzian setting, there are many different non-equivalent notions of completeness) manifolds of finite volume.

See the introduction to the paper [**FJP2011**] for a list of known results about the existence of closed time-like geodesics under additional assumptions.

9.2. *Light-like geodesics of semi-Riemannian metrics.* Let M be a closed manifold with semi-Riemannian metric g of indefinite signature.

Question 9.2.1. Can there exist a complete semi-Riemannian metric g and a non-trivial 1-form η on a closed manifold such that, for every light-like geodesic $\gamma(t)$, the function $\eta(\dot\gamma(t))$ grows linearly in both directions: i.e., for every light-like geodesic, there exist $C_1 \neq 0$, C_2 such that $\eta(\dot\gamma(t)) = C_1 \cdot t + C_2$?

A negative answer to this question would give an easy proof of the semi-Riemannian version of the projective Lichnerowicz–Obata conjecture.

Let a connected Lie group G act on a closed manifold (of dimension at least two) by projective transformations (diffeomorphisms that take geodesics to geodesics without necessarily preserving the parameterization). Then G acts by isometries, or g has constant sectional curvature.

If g is Riemannian, the conjecture was proved in [**Mat2005**, **Mat2007**]. For metrics of Lorentzian signature, the conjecture was proved in dimension two in [**Mat2012b**] and for all other dimensions in [**BMR2015**]. The proof is complicated, and does not generalize for other signatures. In the semi-Riemannian case, the following argument gives a proof for closed manifolds provided that the answer to Question 9.2.1 above is positive.

It is known (see, for example, [**Mat2007**]) that a 1-form η_i generates a one-parameter group of projective transformations if and only if

$$\eta_{i,jk} + \eta_{j,ik} - \frac{2}{(n+1)}\eta^{\ell}_{,\ell k} g_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}$$

(for a certain 1-form λ_i). We take a light line geodesic $\gamma(t)$, multiply the above equation by $\dot\gamma^i\dot\gamma^j\dot\gamma^k$ (and sum with respect to repeating indexes) at every point $\gamma(t)$ of the geodesic. The terms with the metric g disappear since $g_{ij}\dot\gamma^i\dot\gamma^j = 0$, so we obtain the equation $(d^2/dt^2)\eta(\dot\gamma(t)) = 0$, which implies that $\eta(\dot\gamma(t)) = t \cdot C_1 + C_2$. A negative answer to the question above implies that $C_1 = 0$. Hence $\eta(\dot\gamma(t))$ is constant, which, in turn, implies that $\eta_{i,j} + \eta_{j,i}$ is proportional to g_{ij}. Then the covector field η_i generates a one-parameter group of conformal transformations. Finally, the proof of the conjecture follows from a classical observation of Weyl [**Wey1921**] that every transformation that is projective and conformal is a homothety.

A version of the question above is *whether, for a complete semi-Riemannian metric on a closed manifold M, the tangent vector of almost every geodesic remains in a bounded set of TM*. A positive answer to this question immediately implies that the answer to the initial question is negative, thereby proving the projective Lichnerowicz–Obata conjecture on closed manifolds.

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9.3. *Completeness of closed manifolds of arbitrary signature (communicated by H. Baum).* It is well known that a closed Riemannian manifold is geodesically complete, in the sense that every geodesic $\gamma : (a, b) \to M$ can be extended to a geodesic $\tilde{\gamma} : \mathbb{R} \to M$ such that $\tilde{\gamma}|_{(a,b)} = \gamma$. It is also well known that, for any indefinite signature, there exist metrics on closed manifolds that are not geodesically complete.

Question 9.3.1. What geometric assumptions imply that a metric (possibly, of a fixed signature) on a closed manifold is geodesically complete?

We, of course, are interested in geometric assumptions that are easy to check or which are fulfilled for many interesting metrics.

Let us mention a few classical results. For compact homogeneous manifolds, geodesic completeness was established in [**Mar1972**]. For compact Lorentz manifolds of constant curvature, geodesic completeness was proved in [**Car1989**] (flat case) and [**Kli1996**] (general case).

We refer to the paper [**San2015**] for a list of interesting results on this topic and for a list of open questions from which we repeat here only the following.

Question 9.3.2. [**San2015**] Assume that a compact Lorentzian manifold is globally conformal to a manifold of constant curvature. Must it be geodesically complete?

Note also that, in the non-compact case, homogeneous manifolds of indefinite signature are not necessary geodesically complete (see, for example, [**San2015**, Example 2 in §4]). It is interesting to understand whether completeness of a homogeneous manifold can follow from algebraic properties of the isometry group.

10. *Integrability and ergodicity of geodesic flows on surfaces of higher genus*

10.1. *Metrics with integrable geodesic flow on surfaces of genus* ≥ 2.

Question 10.1.1. (Bangert) Does there exist a Riemannian metric on a closed surface of genus ≥ 2 whose geodesic flow is completely integrable?

The answer may depend on the smoothness of the metric and on what we understand by 'completely integrable': whether the integral is functionally independent of the Hamiltonian on an open everywhere dense subset, or whether we additionally assume that the subset has full measure.

Mañé showed that a Hamiltonian flow on surfaces is generically Anosov or has zero Liouville exponents [**Mañ1996**]. This suggests that an easier version of the above question would be the following.

Question 10.1.2. (Paternain) Is there a Riemannian metric on a closed surface of genus ≥ 2 whose geodesic flow has zero Liouville entropy?

Bangert and Paternain have outlined some ideas about what a Finsler metric with this property might look like; finding a Riemannian metric is certainly harder.

The following would be a special case of the Finsler version of the question above.

Question 10.1.3. Let F_1, F_2 be Finsler metrics on a closed surface of genus ≥ 2. Assume every (unparameterized) F_1-geodesic is an F_2-geodesic. Must F_1 be obtained from F_2 by multiplication by a constant and adding a closed form?

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One can also ask this question for arbitrary closed manifolds that can carry a hyperbolic metric (if F_1, F_2 are Riemannian, the answer is affirmative [**Mat2003a**]).

The previous question is related to the other questions in this section because of the following observation from [**MT1998**]: one can use the second metric to construct an integral of the geodesic flow of the first one. If both metrics are Riemannian, the integral is quadratic in velocities and the affirmative answer follows from [**Kol1983**]; see [**MT2000**].

Question 10.1.4. Does there exist a non-Riemannian Finsler metric satisfying the Landsberg condition on a surface of genus ≥ 2?

The Landsberg condition is defined in [**Bao2007**]. It implies the existence of an integral for the geodesic flow of the metric, which is the relation of this question with the present section. See [**GR2013**] for a negative answer to the question assuming non-existence of conjugate points. Note that the natural analog of Question 10.1.4 is completely open in all settings: locally, globally, in all dimensions, and is one of the most important problems in the Finsler geometry (see, e.g., [**Bao2007**]).

In the class of real-analytic metrics, the papers [**Lan2019**, **Pat1997**] answer positively Questions 10.1.3 and 10.1.4.

10.2. *Integrable geodesic flows with integrals of higher degree.*

Conjecture 10.2.1. [**BKF1995**] If the geodesic flow of a Riemannian metric on the torus T^2 admits an integral that is polynomial of degree three in the velocities, then the metric admits a Killing vector field.

A Killing vector field V allows us to construct an integral

$$I : TM \to \mathbb{R}, \quad I(\xi) := g(V, \xi)$$

that is evidently linear in velocities; its third power is then an integral cubic in velocities.

The motivation to study metrics admitting integrals polynomial in velocities comes from the following observation (which dates back at least to Darboux and Whittaker): if the geodesic flow admits an integral that is analytic in velocities, then each component of this integral that is homogeneous in velocities is also an integral. The natural idea is then to study the integrals that are polynomial in velocities of low degree.

By the result of Kolokoltsov [**Kol1983**], no metric on a surface of genus ≥ 2 admits an integral that is polynomial in velocities and is functionally independent of the energy integral. The state of the art if the surface is the sphere or the torus can be explained by Table 1.

In the table, 'Degree' means the smallest degree of a non-trivial integral polynomial in velocities. 'All is known' means that there exists an effective description and classification (which can be found in [**BMF1998**]).

A simpler version of the question is when we replace the geodesic flow in the question above by a Lagrangian system with the Lagrangian of the form $L(x, \xi) := K + U = \sum g_{ij}\xi^i\xi^j + U(x)$. In this case, we assume that the integral is a sum of polynomials of degrees three and one in velocities. The 'partial negative results' in the table above correspond to this case; moreover, in most cases, it is assumed that the metric g_{ij} is flat;

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TABLE 1. Existence of polynomial in velocity integrals.

	Sphere S^2	Torus T^2
Degree 1	All is known	All is known
Degree 2	All is known	All is known
Degree 3	Series of examples	Partial negative results
Degree 4	Series of examples	Partial negative results
Degree ≥ 5	Nothing is known	Nothing is known

see, for example, [**Bia1987**, **Mir2010**, **DKT2012**] (and [**Bia2010**] for results that do not require this assumption).

Similar conjectures could be posed for integrals of every degree. If the degree d is odd, the conjecture is that the existence of an integral that is polynomial of the degree d in velocities implies the existence of a Killing vector field. If the degree d is even, the conjecture is that the existence of an integral that is polynomial of the degree d in velocities implies the existence of an integral that is quadratic in velocities and not proportional to the energy integral. Note also that, by [**KM2016**], a C^2-generic metric does not admit a non-trivial integral which is polynomial in velocities.

Further open problems on integrable systems and integrable geodesic flows can be found in the recent survey [**BMMT2018**].

10.3. *Integrable billiards and the Birkhoff conjecture.* Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain bounded by a smooth curve. We consider the (internal) billiard problem. The ball (a point) moves with unit speed and no friction following a rectilinear path inside the domain. When the ball hits the boundary, it is reflected elastically according to the law of optical reflection: the angle of reflection equals the angle of incidence.

We ask whether there exists a non-constant smooth function on the tangent bundle which is constant along every trajectory. It is well known, see, e.g., [**Tab2005**], that a billiard in an ellipse is integrable.

Conjecture 10.3.1. If the billiard in Ω is integrable, then the boundary of Ω is an ellipse.

The special case of the conjecture in which the caustics of the billiard trajectories are assumed to foliate the whole of Ω except for one point was proved in [**Bia1993**]. Recently, essential progress is due to [**ASK2016**], where the conjecture was proved for tables bounded by small perturbations of ellipses of small eccentricity.

One can also assume that the integral is polynomial in velocities; non-trivial results in this direction include [**Bol1990**, **BM2017**].

See also [**Gut2012**] for generalizations and other problems related to billiards.

10.4. *Metrics such that one can explicitly find all geodesics.* Geodesics of a metric are solutions of a non-linear ordinary differential equation $\nabla_{\dot\gamma}\dot\gamma = 0$ which cannot be *explicitly* solved in most cases. There are only a few examples of two-dimensional metrics for which one can explicitly find *all* geodesics using elementary functions: they are metrics of constant curvature and Darboux-superintegrable metrics (see [**BMM2008**] for

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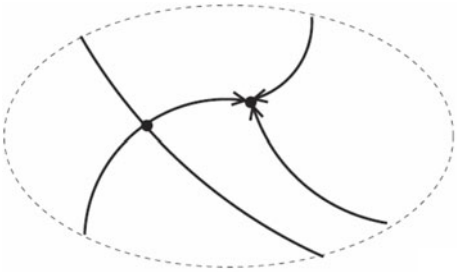


FIGURE 3. A fragment of a stationary net.

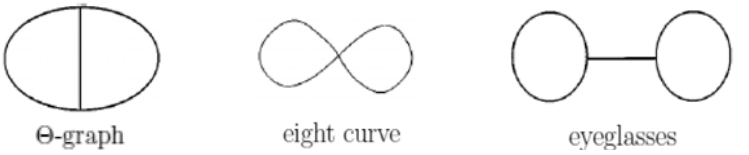


FIGURE 4. Three stationary nets.

a definition). An interesting problem is to construct other examples of metrics such that all geodesics are explicitly known. A related problem (asked recently in [**Tao2010**]) is to construct metrics with an explicitly given distance function.

11. *Stationary nets (communicated by Rotman)*
A graph G in a Riemannian surface (S, g) is called a *stationary net* if every edge is a geodesic and if at every vertex the sum of diverging unit vectors is zero. Vertices must have valence at least three.

11.1. *Stationary Θ-nets.* A Θ-graph is a graph that looks like the Greek letter 'Theta': two vertices connected by three edges (see Figure 3).

Question 11.1.1. [**HM1996**] Does every metric on S^2 admit a stationary Θ-graph?

Partial results in this direction were obtained in [**HM1996**]: they showed that, on every sphere of positive curvature, there exists a stationary Θ-net, a stationary eight curve net or a stationary eyeglasses net, see Figure 4.

11.2. *Density of stationary nets and of closed geodesics.* We call a stationary net non-trivial if it has at least one vertex of valency ≥ 3, in other words, if it is not a disjoint union of simple closed geodesics.

Question 11.2.1. (Gromov) Are non-trivial stationary nets dense on any closed Riemannian surface? In other words, for each non-empty open subset U, is there a stationary net intersecting U?

The answer is evidently positive for every surface of constant curvature. Indeed, in this case, tangent vectors to closed geodesics are dense in the unit tangent bundle and we can take a finite union of closed intersecting geodesics as a stationary net. On other manifolds,

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the answer is less trivial, for example, because in higher dimensions, periodic geodesics may not intersect.

12. *How to reconstruct a metric from its geodesics*

A (Riemannian or semi-Riemannian) metric allows one to construct geodesics. Every non-zero vector at every point is tangential to a unique curve from the family of geodesics. This section is dedicated to open problems related to the following questions: given a family of curves with this property, are they the geodesics for a metric; and, if there is such a metric, is it unique and how can it be reconstructed? These questions can also be posed for Finsler metrics. In this section, we assume that the dimension of the manifold is at least two.

Let us introduce some notation. By a *path structure* we will understand (following [**Tho1925**]) a family of curves $\gamma_\alpha(t)$ such that, for any point x and for any vector $v \in T_xM$, $v \neq 0$, there exists a unique curve γ from this family such that, for a certain t, we have $\gamma(t) = x$ and $\dot\gamma(t) \in \mathrm{span}(v)$. We think that the curves are smooth and smoothly depend on the parameters $\alpha = (\alpha_1, \dots, \alpha_{2n-2})$. We may insist that the parametrization of the curves stays fixed, or we may be willing to reparameterize them.

The first step is to find an affine connection for which the curves are the geodesics. The problem reduces to a system of linear equations (whether one can actually write down and solve them depends on the form in which the curves are given). We outline the main ideas.

First, consider the case in which the parametrization of the curves is fixed. If all the curves γ from this path structure are affinely parameterized geodesics of a connection $\nabla = (\Gamma^i_{jk})$, then, at any point $p \in M$, the Christoffel symbols satisfy the system of equations

$$\ddot\gamma(t)^i + \Gamma(\gamma(t))^i_{jk}\dot\gamma(t)^j\dot\gamma(t)^k = 0 \tag{12.1}$$

for all curves γ from the path structure such that $\gamma(t) = p$. This is a system of n linear equations in the $n^2(n+1)/2$ unknowns $\Gamma(p)^i_{jk}$. Consequently, $n(n+1)/2$ curves γ_α from the path structure give us a system of $n^2(n+1)/2$ linear equations in $n^2(n+1)/2$ unknowns $\Gamma(p)^i_{jk}$. It is an easy exercise to see that if the velocity vectors of the curves at the point p are in general position, then this system is uniquely solvable; by solving it we obtain the Christoffel symbols at this point. Clearly, these Christoffel symbols should satisfy equation (12.1) for all curves γ from the path structure (so generic path structures do not come from a connection). It depends on how the curves γ are given whether it is possibly to check this. For example, if all the curves are given by explicit formulas that depend algebraically on t and on α, then this is an algorithmically doable but computationally complicated task.

Note that the above considerations show that reconstruction of a symmetric affine connection from affinely parameterized geodesics is unique.

Let us now deal with the reconstruction of a connection from unparameterized curves. Our goal is to find a connection $\nabla = (\Gamma^i_{jk})$ for which each curve γ_α from our path structure, after an appropriate reparameterization, is a geodesic. In this case, a similar idea works. The analog of (12.1) is

$$\ddot\gamma(t)^i + \Gamma(\gamma(t))^i_{jk}\dot\gamma(t)^j\dot\gamma(t)^k = c\dot\gamma^i, \tag{12.2}$$

where the unknowns are $\Gamma(p)^i_{jk}$ and c (although we are interested only in $\Gamma(p)^i_{jk}$). For one

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curve γ containing $p \in M$, we have, therefore, n equations in $n^2(n+1)/2+1$ unknowns $\Gamma(p)^i_{jk}$ and c. For two curves γ_1, γ_2, we then obtain $2n$ equations in $n^2(n+1)/2+2$ unknowns $\Gamma(p)^i_{jk}$, c_1, c_2 and so on. We see that, for $k > (n^2(n+1))/(2(n-1))$ curves γ_α from the path structure (passing through the point p), we have more equations than unknowns; by solving this system (if it is solvable) we obtain a connection. See [**Mat2012a**, §2.1] for more details.

Note that (as was already known to [**LevCiv1896**, **Wey1921**]) the solution Γ^i_{jk} of (12.2), if it exists, is not unique: two connections $\nabla = (\Gamma^i_{jk})$ and $\bar{\nabla} = (\bar{\Gamma}^i_{jk})$ have the same unparameterized geodesics if and only if there exists a (0, 1)-tensorfield ϕ_i such that

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_{\ k}\phi_j + \delta^i_{\ j}\phi_k. \tag{12.3}$$

Thus, the freedom in reconstructing a connection is an arbitrary choice of a 1-form ϕ_i.

Connections $\nabla = (\Gamma^i_{jk})$ and $\bar{\nabla} = (\bar{\Gamma}^i_{jk})$ related by (12.3) are called *projectively equivalent*; projective equivalence of two connections means that they have the same geodesics considered as unparameterized curves.

Let us now touch on the question of whether/how one can reconstruct a metric (Riemannian or of arbitrary signature) from a path structure. As we explained above, it is relatively easy to reconstruct an affine connection (respectively, a class of projectively equivalent affine connections) from affinely (respectively, arbitrary) parameterized geodesics, so the actual question is how to reconstruct a metric from its affine connection (respectively, a class of projectively equivalent affine connections), when it is possible, and what is the freedom. Let us first discuss how/whether it is possible to reconstruct a metric parallel with respect to a given symmetric affine connection.

This question is well studied; see, for example, the answers of Bryant and Thurston in [**BT2011**]. A theoretical answer is as follows: the affine connection determines the holonomy group. The existence of a metric with a given affine connection is equivalent to the existence of a non-degenerate bilinear form preserved by the holonomy group (see, e.g., [**Sch1973**]). A practical test for the existence of the metric is as follows: the connection allows us to construct the curvature tensor $R^i_{\ jk\ell}$, and if the connection is the Levi-Civita connection of a metric, then this metric satisfies the equations

$$R^s_{\ jk\ell}g_{si} = -R^s_{\ ik\ell}g_{sj}, \quad R^s_{\ jk\ell}g_{si} = R^s_{\ \ell ij}g_{sk}. \tag{12.4}$$

These equations are, essentially, the algebraic symmetries of the Riemann curvature tensor: the first one corresponds to $R_{ijkm} = -R_{jikm}$, and the second corresponds to $R_{ijkm} = R_{kmij}$. One should view these equations as linear equations in the unknowns g_{ij}; the number of equations is bigger than the number of unknowns so it is expected that the system has no non-zero or non-degenerate solution (and, in this case, there exists no metric compatible with this connection). In many cases, the solution is unique (up to multiplication by a conformal coefficient) and, in this case, we already have the conformal class of the metrics. Now, having the conformal class of the metric, we have the conformal class of the volume form and it is easy to reconstruct the metric using the condition that the volume form is parallel, which immediately reduces to the condition that a certain 1-form is closed (see also [**MT2014**]).

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Note that if the equations (12.4) do not give enough information, one could consider their 'derivatives'

$$R^s{}_{jk\ell,m}g_{si} = -R^s{}_{ik\ell,m}g_{sj}, \quad R^s{}_{jk\ell,m}g_{si} = R^s{}_{\ell ij,m}g_{sk}$$

which gives us again a huge system of equations in the same unknowns g_{ij}. If necessary, one then considers higher-order derivatives until there is enough information. The general theory says that, in the analytic category, the existence of a non-degenerate solution of the resulting system of equations implies the existence of a metric whose Levi-Civita connection is the given one. In the non-analytic setting, however, there exist C^∞ counterexamples.

Let us now discuss the uniqueness of the reconstruction of a metric from its affinely parameterized geodesics, modulo the multiplication of the metric by a constant. A generic metric can be reconstructed from its affinely parameterized geodesics; this follows from the observation that, for a generic metric g, the system (12.4) has only the trivial solutions const g. So the question is: for which metrics is the reconstruction not unique, that is, which non-proportional metrics have the same Levi-Civita connection? Locally, the answer is known. For Riemannian metrics, it was understood already by Cartan and Eisenhart [**Eis1923**]. For metrics of arbitrary signature, the answer is in the recent papers [**Bou2015a**, **Bou2015b**]. Both results are a more-or-less explicit local description of all metrics having the same Levi-Civita connection; such metrics are called *affinely equivalent*.

In the Riemannian case, a global (i.e., when the manifold is closed or complete) analog of the result of Cartan and Eisenhart is due to [**DeR1952**], where it was proved that the universal cover is the direct product of Riemannian manifolds.

In the pseudo-Riemannian case, the situation is more complicated. Consider two affinely equivalent metrics g and $\tilde{g}$ and the (1,1) tensor A given by the condition $\tilde{g}(A(\cdot), \cdot) = g(\cdot, \cdot)$. It is self-adjoint with respect to both metrics and parallel with respect to the Levi-Civita connection. Its generalized eigenspaces therefore generate mutually orthogonal totally geodesic foliations, the sum of whose dimensions is the dimension of the manifold. Under the additional assumption that the manifold is geodesically complete, one can show (see, e.g., [**Wu1967**]) that the universal cover of the manifold splits as the direct product of the (universal covers of the) leaves of these foliations with the restriction of the metrics to them,

$$\tilde{M} = M_1 \times \cdots \times M_k, \quad g = g_{|M_1} + \cdots + g_{|M_k}, \quad \tilde{g} = \tilde{g}_{|M_1} + \cdots + \tilde{g}_{|M_k}.$$

The most interesting case is when A has only one eigenvalue and, since it is clearly a constant, one can assume that it is zero. In this case, all known examples on closed manifolds are when g admits light-like parallel vector fields, and A is built with the help of these vector fields. We therefore suggest, as an interesting unsolved problem, the following question.

Question 12.0.2. Suppose a closed manifold (M, g) admits a non-zero (1, 1)-tensor field that is self-adjoint with respect to g, parallel and nilpotent. Does this manifold or its double cover admit a non-zero light-like parallel vector field?

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Also, in the Finsler case, parameterized geodesics determine the connection. Note, however, that, in the Finsler geometry, the connection is non-linear and actually there are more than one popular and widely used connections. We will call two Finsler metrics *affinely equivalent* if any geodesic of the first metric (considered as a curve parameterized such that the length of the velocity vector is a constant) is a geodesic of the second metric; this is equivalent to the Berwald connections of the first and the second metric coinciding in the natural sense.

Problem 12.0.3. Describe all affinely equivalent Finsler metrics.

A standard example of affinely equivalent non-proportional Finsler metrics can be constructed in the class of Berwaldian metrics. One can also construct easy examples on a direct product of Finsler manifolds. The simplest version of the above question is to *construct non-Berwaldian examples of affinely equivalent non-proportional Finsler metrics which are not product metrics.*

The 'unparameterized' versions of these problems for Riemannian and pseudo-Riemannian metrics are the subject of the recent survey [**Mat2012a**]. Roughly speaking, the situation is similar to the one in the 'parameterized' case. Locally, a general strategy for reconstructing a metric is understood: the existence of a metric compatible with a path structure is equivalent to the existence of (non-degenerate) parallel sections of a certain tensor bundle [**EM2008**]. A connection on this tensor bundle is constructed by the projective structure constructed by the path structure. Thus, a theoretic answer is to see whether the holonomy group of this connection preserves a certain non-degenerate element of the fiber, and a practical method is to construct the curvatures of the connection and look for elements of the fiber compatible with the curvature. As an interesting open problem, we suggest the following.

Problem 12.0.4. Construct a system of scalar invariants of a projective structure that vanish if and only if there exists (locally, in a neighborhood of almost every point) a metric compatible with a given projective structure.

In dimension two, the problem was solved in [**BDE2009**], and the system of invariants is quite complicated—the simplest invariant has degree five in derivatives. It is possible that, in higher dimensions, the system of invariants could be easier in some ways, since, in this case, the system of partial differential equations corresponding to the existence of a metric for a projective structure has a higher degree of overdeterminacy. In particular, fewer differentiations are needed to construct the first obstruction to the existence of a metric class (see the recent paper [**DE2014**]).

This problem is, of course, much more difficult in the Finslerian case. In dimension ≥ 3, not every path structure is metrizable, and actually this result is micro-local (see [**Dou1941**]). This topic is one of the questions studied within the topic called *inverse problem of calculus of variations*.

Let us now consider dimension two. First, observe that if the Finsler metric is irreversible, then its forward and backward geodesics do not necessarily coincide. One therefore needs to extend the definition of the path structure: by *irreversible path structure* we will understand a family of curves $\gamma_\alpha(t)$ such that, for any point x and for any vector

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$v \in T_xM$, $v \neq 0$, there exists a unique curve γ from this family such that, for a certain t, we have $\gamma(t) = x$ and $\dot{\gamma}(t)$ is proportional to v with a positive coefficient. Path structures, in the sense of the initial definition from the beginning of this section, will be called *reversible path structures*.

An easy example of an irreversible path structure is provided by circles of radius one on the Euclidean plane $\mathbb{R}^2$ with clockwise orientation. This path structure is locally Finsler-metrizable, that is, there exists a Finsler metric such that all curves of the path structure are its geodesics.

It is known that, in dimension two, locally, each reversible path structure is metrizable [**ÁB2010**, Theorem B].

Question 12.0.5. In dimension two, is each irreversible path structure metrizable?

We actually ask this question locally. Globally, and on closed manifolds, the answer is negative. A simple example provides the torus $\mathbb{R}^2/\mathbb{Z}^2$ with the irreversible path structure from the example above when the curves from the path structure are circles of radius one. Indeed, in this case, not all pairs of points on the universal cover can be connected by a geodesic.

Let us now discuss the freedom in reconstructing the (Riemannian or pseudo-Riemannian) metric from its unparameterized geodesics; an equivalent question is how many different metrics can have the same geodesics considered as unparameterized curves. Of course, the metrics g and $\text{const} \cdot g$ have the same geodesics. In [**Mat2012a**], it was shown that, for a k depending on the dimension, a C^k-generic metric has the property that any projectively equivalent metric is proportional to it with a constant coefficient. In fact, as follows from [**KM2016**], in all dimensions $k = 2$ is enough.

There exist local (the first examples are due already to Lagrange, Beltrami and Dini) and global examples of non-proportional projectively equivalent metrics. Locally, in the Riemannian case, a complete description of projectively equivalent metrics is due to Levi-Civita [**LevCiv1896**] and in arbitrary signature is due to [**BM2015**]. Globally, in the Riemannian case, the situation is also fairly clear (see, e.g., [**MT1998**, **Mat2003a**, **Mat2003b**]), but if the metrics have arbitrary signature, virtually nothing is known. A general problem is to understand the topology of closed manifolds admitting non-proportional projectively equivalent metrics of indefinite signature and, as the simplest version of this problem, we suggest the following question.

Question 12.0.6. Can a three-dimensional sphere admit two non-proportional projectively equivalent metrics of Lorentzian signature?

In the Finslerian case, the freedom of reconstructing a metric from its geodesic can be much higher. For example, adding any sufficiently small closed 1-form to the Finsler metric does not change geodesics of this metric. This transformation of the metric was called a *trivial projective change* in [**Mat2013**].

Question 12.0.7. In dimension ≥ 3, for a generic Finsler metric F, is any metric projectively equivalent to F locally proportional to a trivial projective change of the metric?

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In dimension two, generic Finsler metrics admit projectively equivalent metrics which do not come from a trivial projective change.

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