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Non-linear angle-sum relations for polyhedral cones and polytopes

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Abstract. It is shown that the internal and external angles at the faces of a polyhedral cone satisfy various bilinear relations. The first two of these are related to the Gauss–Bonnet and Steiner parallel formulae for spherical polytopes, while the third is completely new. However, the proofs are basically combinatorial in nature, rather than differential geometric, as in the more classical treatments. These relations lead to inversion formulae, analogous to Euler-type relations, for certain functions defined on polytopes and polyhedral cones. As a result, various new relations involving quermassintegrals and Grassmann angles are found; there is also an application to lattice polytopes.

1. *Introduction.* Let K be a polyhedral cone in E^d ($d \geqslant 1$). In the sequel, we shall usually assume that K is pointed, with apex the origin o of E^d, and has non-empty interior; as we shall soon observe, this convenient restriction is unnecessary. For each pair of faces $F \subset G$ of K, let $\beta(F, G)$ and $\gamma(F, G)$ be the internal and external angles, respectively, of the cone G at its face F (see Grünbaum (8), Chapter 14 or (9); for definitions), normalized so that the total angle is 1. It is a natural convention that $\beta(F, F) = 1 = \gamma(F, F)$; we shall also write $\beta(F, G) = 0 = \gamma(F, G)$ if $F \nsubseteq G$. We shall prove the following theorems, which give bilinear relations between the internal and external angles.

THEOREM 1. $\Sigma_F \beta(o, F)\gamma(F, K) = 1$.

THEOREM 2. $\Sigma_F (-1)^{\dim F}\beta(o, F)\gamma(F, K) = 0$.

THEOREM 3. $\Sigma_F (-1)^{\dim F}\gamma(o, F)\beta(F, K) = 0$.

In each case, the sum extends over all faces F of the cone K, including $\{o\}$ and K itself. By example, in case $d = 3$, we shall show that there is no reasonable fourth relation of the same type.

The statement of Theorem 1, which was independently conjectured by Hadwiger (private communication), is equivalent to a particular case of the Steiner parallel formula for spherically convex sets, and together with Theorem 2 implies the Gauss–Bonnet formula for the Euler characteristic of a spherical polytope. We shall discuss this in more detail in section 4. We should emphasize here, however, that our point of view, being combinatorial in nature, is quite different from that of the classical differential geometric treatment. In any case, we are more interested in applications of our relations, and so we shall not dwell on these connexions.

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We shall give a direct, geometric proof of Theorem 1 in section 2. In section 3, we shall demonstrate the far from obvious fact that the three relations of the theorems are equivalent; the proof uses the incidence algebra of the lattice of faces of the cone K.

As consequences of Theorems 2 and 3, we shall obtain inversion formulae for certain pairs of functions defined on polytopes or polyhedral cones. Specifically:

THEOREM 7. *Let ϑ and ψ be two functions (which may be scalar or vector valued) defined on convex polytopes or polyhedral cones, and let P be a polytope or polyhedral cone, respectively. Then the following relations are equivalent*:

$$\psi(P) = \Sigma_F \gamma(F, P)\,\vartheta(F),$$
$$\vartheta(P) = \Sigma_F (-1)^{\dim P - \dim F} \beta(F, P)\,\psi(F),$$

where the sum extends over all (non-empty) faces of F of P.

Particular applications of Theorem 7 concern generalizations of various known relations involving angles. We show that Gram's theorem is just the first of a series of relations involving internal angles and quermassintegrals. In the same way, Grünbaum's Grassmann angle relations for polytopes extend to relations involving Grassmann angles and quermassintegrals; similar generalizations concern angles introduced by Larman and Mani. These results have spherical analogues, which include Grünbaum's Grassmann angle relations for polyhedral cones, and a result corresponding to Shephard's Euler-type relations for quermassintegrals. They also have analogues involving vector valued functions; in particular, we prove a vectorial analogue of Gram's theorem. Another application, to lattice polytopes, shows that Ehrhart's reciprocity law and a result of Macdonald are equivalent, and leads us to generalize conjectures of Wills and Hadwiger about the numbers of lattice points in convex bodies.

2. *Polar cones.* If K is a polyhedral cone in E^d, and C is a face of K (with, say, $o \in C$), we denote by $K \mid C$ the image of K under the orthogonal projection with kernel $\operatorname{aff} C$. It is clear that, for each pair of faces F and G of K containing C, we have

$$\beta(F, G) = \beta(F \mid C, G \mid C), \quad \gamma(F, G) = \gamma(F \mid C, G \mid C).$$

Clearly also, it is immaterial whether or not K is full-dimensional (see, for example, Grünbaum (8), 14·3 or (9)). So, we shall assume where appropriate that K is pointed with apex o, and that $\dim K = d$.

We recall that all angles are normalized so that the total angle is 1. We may do this as follows. Let G be any convex cone in E^d with apex o, and let $\alpha(G)$ denote the angle of G at o; that is, the proportion of the unit ball in $\operatorname{aff} G$ which belongs to G. Then:

LEMMA 1. $\alpha(G) = \int_G \exp(-\pi \|x\|^2)\,dx$, *where the measure is ordinary Lebesgue measure in* $\operatorname{aff} G$.

The proof is immediate, on noticing that the factor $\exp(-\pi \|x\|^2)$ induces the appropriate normalization.

LEMMA 2. *Let G_1 and G_2 be convex cones with apex o in orthogonal subspaces of E^d. Then $\alpha(G_1 \times G_2) = \alpha(G_1)\,\alpha(G_2)$.*

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For, each $x \in G_1 \times G_2$ can be expressed in the form $x = x_1 + x_2$, where $x_i \in G_i$ $(i = 1, 2)$, and so

$$\begin{aligned}\alpha(G_1 \times G_2) &= \int_{G_1 \times G_2} \exp(-\pi \|x\|^2)\, dx \\ &= \int_{G_1}\int_{G_2} \exp(-\pi(\|x_1\|^2 + \|x_2\|^2))\, dx_1 dx_2 \\ &= \alpha(G_1)\,\alpha(G_2),\end{aligned}$$

as claimed.

Now let K be a pointed polyhedral cone in E^d with apex o and non-empty interior. Let K^* be the polar cone

$$K^* = \{x^* \in E^d \mid \langle x, x^* \rangle \leqslant 0 \quad \text{for each} \quad x \in K\},$$

and for each face F of K, let

$$\hat{F} = \{x^* \in K^* \mid \langle x, x^* \rangle = 0 \quad \text{for each} \quad x \in F\}$$

be the corresponding face of K^*.

LEMMA 3. *The cones $F \times \hat{F}$ (F a face of K) have disjoint interiors, and cover E^d.*

For, let N denote the nearest-point map of K (xN is the nearest point of K to $x \in E^d$; see, for example, McMullen–Shephard (16), 1·4.) For each point $x \in E^d$, there is a unique face F of K (possibly K itself), such that $xN \in \operatorname{relint} F$. But then we see that $x \in \operatorname{relint} F \times F$, and has a unique representation of the form $x = xN + \hat{x}$, where $\hat{x} \in \hat{F}$. The assertions of the lemma follow at once.

Theorem 1 is an immediate consequence. For, we note that $\beta(o, F) = \alpha(F)$, $\gamma(F, K) = \alpha(\hat{F})$, and hence

$$\begin{aligned}\Sigma_F \beta(o, F)\,\gamma(F, K) &= \Sigma_F \alpha(F)\,\alpha(\hat{F}) \\ &= \Sigma_F \alpha(F \times \hat{F}) = 1,\end{aligned}$$

as required.

3. *The incidence algebra.* It is possible to give direct geometrical proofs of Theorems 2 and 3, analogous to that of Theorem 1. For example, to prove Theorem 2, we can consider the cones $(-F) \times \hat{F}$, with the same angles as $F \times \hat{F}$, whose contribution to covering E^d is counted as $(-1)^{\dim F}$. It is not too difficult to show that a general point of E^d has a covering by these cones totalling zero, and the theorem follows. (The proof of Theorem 3 is a little less straightforward.)

However, the theorems are proved more easily using the incidence algebra $\mathscr{A}(K)$ of the lattice of faces of K. We refer the reader to Rota (22) for further details (in the context of more general lattices), but we repeat the basic definitions here. The elements of $\mathscr{A}(K)$ are the real valued functions ξ, defined on ordered pairs of faces of K, such that $\xi(F, G) = 0$ if $F \nsubseteq G$. Two elements are added in the obvious way, and an associative multiplication is defined by

$$\xi \circ \eta(F, G) = \Sigma_J \xi(F, J)\,\eta(J, G).$$

Clearly, the identity element is the *delta function* δ, defined by $\delta(F, G) = 1$ or 0 as $F =$ or $\neq G$. It can be proved that an element $\xi \in \mathscr{A}(K)$ has an inverse if and only if $\xi(F, F) \neq 0$ for all faces F of K.

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In view of the remarks at the beginning of section 2, we see that for all faces F and G of K,

$$\Sigma_J \beta(F,J)\gamma(J,G) = \begin{cases} 1 & \text{if} \quad F \subseteq G, \\ 0 & \text{if} \quad F \nsubseteq G. \end{cases}$$

The right side defines the *zeta function* ζ of $\mathscr{A}(K)$ (Rota (22)); thus Theorem 1 can be expressed as

THEOREM 1′. $\beta \circ \gamma = \zeta$.

Now the inverse of the zeta function is the *Möbius function* μ (Rota (22)), whose definition for $\mathscr{A}(K)$ can be seen to be $\mu(F, G) = (-1)^{\dim G - \dim F}$, for $F \subseteq G$. The relation $\mu \circ \zeta = \delta$ just expresses, for each pair of faces $F \subseteq G$, the Euler relation between the numbers of faces of the cone $G \mid F$.

We define new angle functions $\bar{\beta}$ and $\bar{\gamma}$ by

$$\bar{\beta}(F,G) = (-1)^{\dim G - \dim F}\beta(F,G),$$
$$\bar{\gamma}(F,G) = (-1)^{\dim G - \dim F}\gamma(F,G).$$

We then have:

LEMMA 4. $\bar{\beta} = \mu \circ \beta$, $\bar{\gamma} = \gamma \circ \mu$.

In case $F = \{o\}$ and $G = K$, where K is a pointed cone in E^d with non-empty interior, the first relation is equivalent to

$$\Sigma_J (-1)^{\dim J}\beta(J,K) = (-1)^{\dim K}\beta(o,K),$$

which is a result of Sommerville (30) (see also Grünbaum (9)).

The result for pairs of faces of intermediate dimensions follows at once (compare, again, the remarks at the beginning of section 2). The second relation follows from the first, if we replace K by the polar cone K^* (compare Grünbaum (9), and the proof of Theorem 1).

We deduce at once from Theorem 1′ and Lemma 4:

THEOREM 2′. $\bar{\beta} \circ \gamma = \delta = \beta \circ \bar{\gamma}$.

For, $\bar{\beta} \circ \gamma = \mu \circ \beta \circ \gamma = \mu \circ \zeta = \delta$. The other relation is proved similarly, using $\delta = \zeta \circ \mu$.

Each of the relations of Theorem 2′ expresses

$$\Sigma_J (-1)^{\dim J}\beta(F,J)\gamma(J,G) = \begin{cases} (-1)^{\dim F} & \text{if} \quad F = G, \\ 0 & \text{if} \quad F \neq G, \end{cases}$$

which implies the statement of Theorem 2.

We next observe that Theorem 2′ (together with the remarks at the beginning of the section) shows that $\beta^{-1} = \bar{\gamma}$ (and $\gamma^{-1} = \bar{\beta}$). Hence:

THEOREM 3′. $\gamma \circ \bar{\beta} = \delta\,(= \bar{\gamma} \circ \beta)$.

Since $\gamma \circ \bar{\beta} = \gamma \circ \mu \circ \beta = \bar{\gamma} \circ \beta$, there is only one relation here. As above, it implies the statement of Theorem 3.

The facts that $\beta^{-1} = \bar{\gamma}$ and $\gamma^{-1} = \bar{\beta}$ lead to the further interesting results that we can express γ in terms of β alone, and vice-versa. For, we first observe that $\beta - \delta$ is

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nilpotent; indeed, $(\beta-\delta)^{\dim K+1}=0$ (compare the definition of σ_r below). We deduce at once that

$$\overline{\gamma}=\beta^{-1}=(\delta+(\beta-\delta))^{-1}$$
$$=\sum_{r=0}^{\dim K}(-1)^r(\beta-\delta)^r.$$

If we write $$\sigma_r(F,G)=\Sigma\,\beta(F,F_1)\,\beta(F_1,F_2)\dots\beta(F_{r-1},G),$$

where the sum is taken over all strictly increasing chains $F\subset F_1\subset F_2\subset\dots\subset F_{r-1}\subset G$ of length r (with $\sigma_0=\delta$), then we have

$$\overline{\gamma}=\sum_{r=0}^{\dim K}(-1)^r\sigma_r.$$

Since, clearly, $\sigma_r(F,G)=0$ unless $\dim G-\dim F\geqslant r$, and since

$$\overline{\gamma}(F,G)=(-1)^{\dim G-\dim F}\gamma(F,G),$$

we thus arrive at the following.

THEOREM 4. *If* $\dim G-\dim F=k$, *then* $\gamma(F,G)=\sum_{r=0}^{k}(-1)^{k-r}\sigma_r(F,G)$.

The above argument involves β and γ symmetrically; hence, or by considering the polar cone K^*, we find

THEOREM 5. *If* $\dim G-\dim F=k$, *then* $\beta(F,G)=\sum_{r=0}^{k}(-1)^{k-r}\tau_r(F,G)$, *where*

$$\tau_r(F,G)=\Sigma\,\gamma(F,F_1)\dots\gamma(F_{r-1},G),$$

the sum extending over all chains $F\subset F_1\subset\dots\subset F_{r-1}\subset G$ *of length* r.

4. *The Steiner parallel and Gauss–Bonnet formulae.* For all faces F and G of a polyhedral cone K, we write

$$\phi_r(F,G)=\sum_{J^r}\beta(F,J^r)\,\gamma(J^r,G),$$

where the sum extends over all r-faces J^r of K. If C is the face of apices of K (which we do not assume here to be pointed or d-dimensional), the quantities $\phi_r(K)=\phi_r(C,K)$ are spherical analogues of the quermassintegrals of K (or of $S\cap K$, where S is the unit sphere in E^d). Actually, comparison with sections 5 and 7 will show that they may better be thought of as analogues of the intrinsic volumes V_r, to be defined in section 5.

Theorems 1 and 2 can now be rephrased as $\Sigma_r\,\phi_r(K)=1$, and, if $K\neq C$, so that K is not a subspace, $\Sigma_r\,(-1)^r\phi_r(K)=0$. (If K is a subspace, the latter sum is $(-1)^{\dim K}$.) Since $\phi_0(K)=\phi_d(K^*)$ is the (normalized) spherical area of the polar set $S\cap K^*$ of $S\cap K$, which is the set of points of S at spherical distance at least $\frac{1}{2}\pi$ from $S\cap K$, these formulae give relationships between the spherical areas of $S\cap K$, $S\cap K^*$ and the quantities $\phi_1(K),\dots,\phi_{d-1}(K)$.

In the more classical treatments, whose proofs use differential geometrical techniques (we refer the reader to Allendoerffer-Weil (2), Herglotz (12), Allendoerffer (1) and Santaló (27); the results of the last paper are most easy to compare with ours), it

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is more usual to add and subtract the formulae. Since the Euler characteristic of $S \cap K$ for a convex cone K is

$$\chi(S \cap K) = \begin{cases} 0 & \text{if } K \text{ is a point,} \\ 1 & \text{if } \dim K > 0, \text{ and } K \text{ is not a subspace,} \\ 1-(-1)^{\dim K} & \text{if } K \text{ is a subspace} \end{cases}$$

($S \cap K$ is, respectively, empty, a ball (of some dimension) or a sphere), we immediately obtain:

$$\sum_{t \geqslant 0} \phi_{2t}(K) = 1 - \tfrac{1}{2}\chi(S \cap K),$$

$$2 \sum_{t \geqslant 0} \phi_{2t+1}(K) = \chi(S \cap K).$$

The first formula (involving ϕ_0) is usually called (a special case of the) *Steiner parallel formula*, while the second (giving $\chi(S \cap K)$) is the *Gauss–Bonnet formula*.

In fact, since the ϕ_r satisfy the *valuation property*

$$\phi_r(K_1 \cup K_2) + \phi_r(K_1 \cap K_2) = \phi_r(K_1) + \phi_r(K_2),$$

in case $K_1 \cup K_2$ is also a convex cone, the inclusion-exclusion principle (see Perles-Sallee (19), Rota (23)) enables us to extend the domain of definition of ϕ_r to any finite union of spherical polytopes (or the not necessarily convex cones they generate), and so to prove the Gauss–Bonnet formula in these cases also. (The continuity of the ϕ_r, which is more obvious from results quoted in section 7, actually enables us to extend the domain of definition of ϕ_r even further, but we now pass beyond our chosen purlieu.)

From the valuation property, we can also deduce the essential uniqueness of the relation of Theorem 2. Specifically, we have:

THEOREM 6. *If $\lambda_0, \lambda_1, \ldots, \lambda_d$ are real numbers, such that the relation*

$$\sum_{r=0}^{d} (-1)^r \lambda_r \phi_r(K) = 0$$

holds for all d-dimensional pointed polyhedral cones K, then $\lambda_0 = \lambda_1 = \ldots = \lambda_d$.

For, let K_1 and K_2 be any two d-dimensional pointed polyhedral cones, whose union is also pointed and convex, and whose intersection is $(d-1)$-dimensional. Then

$$\sum_{r=0}^{d-1} (-1)^r \lambda_r \phi_r(K_1 \cap K_2) = \sum_{r=0}^{d} (-1)^r \lambda_r \{\phi_r(K_1) + \phi_r(K_2) - \phi_r(K_1 \cup K_2)\}$$

$$= 0.$$

Since, by a suitable choice of K_1 and K_2, we can clearly take $K_1 \cap K_2$ to be an arbitrary $(d-1)$-dimensional cone, we deduce from the obvious inductive assumption (if $d \geqslant 2$; the case $d = 1$ is clearly trivial) that $\lambda_0 = \ldots = \lambda_{d-1}$. Consideration of any particular cone K at once yields the theorem.

5. *Applications to polytopal functions.* Theorems 1, 2 and 3 can be extended without difficulty to convex polytopes. The conventions concerning the empty face $\varnothing$ can be obtained by regarding the (infinite) right cylinder over a polytope P as a cone, whose apex, lying at infinity, corresponds to $\varnothing$. Thus we take $\gamma(\varnothing, F) = \frac{1}{2}$ (for $F \neq \varnothing$),

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$\beta(\varnothing, v) = \frac{1}{2}$ for vertices v of P, and $\beta(\varnothing, F) = 0$ if $\dim F \geqslant 1$. However, the additional relations obtained from Theorems 1, 2 and 3 merely express the well-known relation $\Sigma_v \gamma(v, P) = 1$, and the first part of Lemma 4 again.

There are similar extensions to more general polyhedral sets; we shall leave these to the imagination of the reader.

More fruitful is the consideration of various functions defined on polytopes which involve angles. If ψ is a function defined on polytopes, which may take scalar or vector values, we can compose ψ with an element ξ of the incidence algebra $\mathscr{A}(P)$ by defining

$$\psi \circ \xi(P) = \Sigma_F \psi(F)\, \xi(F, P).$$

(When we consider the obvious matrix representation of the incidence algebra, we see that this composition corresponds to multiplication of a row vector by a matrix on the right.) It is clear that, for all ξ and η in $\mathscr{A}(P)$, we have $\psi \circ (\xi \circ \eta) = (\psi \circ \xi) \circ \eta$. In all our applications, $\psi(\varnothing)$ will vanish, so the conventions involving $\varnothing$ introduced above will not be needed.

The theorem which yields the results of this section is just Theorem 7 translated into the language of the incidence algebra.

THEOREM 7′. *The relations* $\psi = \vartheta \circ \gamma$ *and* $\vartheta = \psi \circ \bar{\beta}$ *are equivalent.*

The proof is clear, since $\gamma \circ \bar{\beta} = \delta = \bar{\beta} \circ \gamma$.

There are a number of applications of Theorem 7; we illustrate by means of examples. In this section, we shall give two results, which are scalar and vector analogues.

Our first application concerns the quermassintegrals. There are several (not obviously equivalent) ways of defining these. If K is a compact convex set in E^d, B is the unit ball, and ρ is a non-negative variable, then the volume $V(K+\rho B)$ is a polynomial in ρ of degree d, which is written $V(K+\rho B) = \sum_{i=0}^{d} \binom{d}{i} W_i(K)\, \rho^i$. $W_i(K)$ is the ith *quermassintegral* of K; it also measures (with some suitable normalization) those i-dimensional flats of E^d which meet K, and so is also proportional to the mean $(d-i)$-dimensional volume of the orthogonal projections of K on to $(d-i)$-dimensional flats of E^d. (For further details, see Bonnesen-Fenchel (4) or Hadwiger (10).)

We prefer to take a different normalization, and consider instead the *intrinsic r-volume* $V_r(K)$ of K, defined by

$$\omega_{d-r} V_r(K) = \binom{d}{r} W_{d-r}(K),$$

where ω_{d-r} is the volume of the $(d-r)$-dimensional unit ball. The advantage of this normalization is that, if K is r-dimensional, then $V_r(K) = V(K)$ is the ordinary r-dimensional volume of K. ($V_0(K) = 1$ corresponds to the Euler characteristic, as will become clear below.)

In case P is a polytope, we can deduce from the expressions above that

$$V_r(P) = \Sigma_{F^r} \gamma(F^r, P)\, V(F^r),$$

the sum being taken over all r-faces F^r of P. If we define the function ξ_r by

$$\xi_r(P) = V(P) = V_r(P),$$

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the r-dimensional volume of P, if $\dim P = r$, and 0 otherwise, then $V_r = \xi_r \circ \gamma$. Our inversion formula, Theorem 7, then tells us that $\xi_r = V_r \circ \bar{\beta}$, or

THEOREM 8. $\Sigma_F (-1)^{\dim F} \beta(F, P) V_r(F) = (-1)^r \delta_{r, \dim P} V_r(P)$, *where* δ *is the Kronecker delta.*

The case $r = 0$ is just the well-known angle-sum relation of Gram (7):

$$\Sigma_F (-1)^{\dim F} \beta(F, P) = 0 \quad \text{for} \quad \dim P > 0.$$

For our second application, we turn to the vector valued analogues of the quermass-integrals and intrinsic volumes. Following Schneider (28), we define the *moment vector* (Zielvektor) $z(K)$ of K by

$$z(K) = \int_K x \, dx,$$

where the measure is Lebesgue measure in aff K, so that we calculate $z(K)$ intrinsically. Then $z(K) = V(K) g(K)$, where $g(K)$ is the centre of gravity of K. As with the volume, we have a polynomial expression $z(K + \rho B) = \sum_{i=0}^{d} \binom{d}{i} \rho^i q_i(K)$; the $q_i(K)$ are the *quermassvectors* of K. As with the quermassintegrals, we prefer to renormalize, and define the *intrinsic* $(r+1)$*-moment* $z_{r+1}(K)$ of K by $\omega_{d-r} z_{r+1}(K) = \binom{d}{r} q_{d-r}(K)$. (We choose the suffix $r+1$ rather than r, because z_{r+1} is homogeneous of degree $r+1$:

$$z_{r+1}(\lambda K) = \lambda^{r+1} z_{r+1}(K)$$

for $\lambda \geqslant 0$.) Again, the normalization is such that $z_{r+1}(K) = z(K)$ if $\dim K = r$.

Schneider (28) has shown that, in case P is a polytope,

$$z_{r+1}(P) = \Sigma_{F^r} \gamma(F^r, P) z(F^r),$$

where the sum extends over all the r-faces F^r of P. As with the intrinsic volumes, we immediately deduce the inverse formula:

THEOREM 9. $\Sigma_F (-1)^{\dim F} \beta(F, P) z_{r+1}(F) = (-1)^r \delta_{r, \dim P} z_{r+1}(P)$.

A particular case of Theorem 9 is of great interest. For, $z_1(P) = s(P)$ is the Steiner point of P, which is the vector analogue of the Euler characteristic (see Hadwiger-Schneider (11)). We see that we have obtained a vector analogue of Gram's theorem:

COROLLARY. *If* $\dim P > 0$, $\Sigma_F (-1)^{\dim F} \beta(F, P) s(F) = 0$.

Sallee (24) has defined an *Euler-type relation* to be one of the form

$$\Sigma_F (-1)^{\dim F} \phi(F) = \phi^*(P),$$

where $\phi^*(P) = \pm \phi(\pm P)$ (for some fixed choice of signs). In the language of the incidence algebra, we have $\phi^* = \phi \circ \nu$, where $\nu(F, G) = (-1)^{\dim F}$ if $F \subseteq G$, and 0 otherwise. Then $\nu^2 = \nu \circ \nu = \delta$ (this is just a concealed form of the relation $\mu \circ \zeta = \delta$; compare the remark after Theorem 7″ below), so that $\phi^{**} = (\phi^*)^* = \phi$. A number of examples are known of functions which satisfy Euler-type relations; the most general, concerning mixed volumes, are due to Shephard (29), while others can be found in Grünbaum (8). A particular case of Shephard's result is $V_r^* = (-1)^r V_r$, to which we shall refer again in section 7.

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A further alternative formulation of Theorem 7 draws a parallel between our relations and the Euler-type relations.

THEOREM 7″. *The following relations are equivalent*:

$$\psi(P) = \Sigma_F (-1)^{\dim F} \gamma(F, P)\, \vartheta(F),$$

$$\vartheta(P) = \Sigma_F (-1)^{\dim F} \beta(F, P)\, \psi(F).$$

That this is equivalent to Theorem 7 is clear; just replace $\vartheta(P)$ by $(-1)^{\dim P}\vartheta(P)$. We prefer the original formulation of Theorem 7, however, in spite of the greater symmetry of that of Theorem 7″, because this seems to be the form in which the inversion formulae naturally occur.

6. *Lattice polytopes.* The investigations of this paper were originally prompted by a wish to invert a certain function W, introduced by Wills (31) in his study of lattice polytopes. If K is a convex body, we define

$$W(K) = \sum_{r \geqslant 0} V_r(K).$$

An equivalent definition, which underlines the dimensional invariance of W (the same value is obtained if $W(K)$ is evaluated in $\operatorname{aff} K$ in case $\dim K < d$), is

$$W(K) = \int_{E^d} \exp(-\pi\rho^2(x, K))\, dx,$$

where $\rho(x, K)$ is the (euclidean) distance of x from K. Wills (31) has conjectured that the number $G(K)$ of lattice points (points of E^d with integer cartesian coordinates) satisfies $G(K) \leqslant W(K)$. The conjecture is easily proved in case $d \leqslant 2$, and has recently been proved for $d = 3$ (see Overhagen (18)). We remark that W is monotone, so that to verify the conjecture it is only necessary to consider polytopes.

In case P is a polytope, we have $W(P) = \Sigma_F \gamma(F, P)\, V(F)$, where $V(F)$ is the k-dimensional volume of F if $\dim F = k$. (Compare also with Lemma 2 for the integral definition of W.) It immediately follows from Theorem 7 (or Theorem 8) that

$$V(P) = \Sigma_F (-1)^{\dim P - \dim F} \beta(F, P)\, W(F),$$

and we have inverted W.

Bokowski (3) has remarked that W is the unique continuous (or monotone) rigid motion invariant valuation which agrees with G on unit lattice cubes of all dimensions. Now, if P is a lattice polytope, and n is a non-negative integer, we have polynomial expressions

$$G(nP) = \sum_{r \geqslant 0} G_r(P)\, n^r,$$

$$W(nP) = \sum_{r \geqslant 0} V_r(P)\, n^r.$$

(For the first, see Reeve (20), (21), Macdonald (14), (15) and Ehrhart (5), (6); the second follows directly from the definition.) Then the functions G_r and V_r are (integer) homogeneous valuations of degree r, and V_r is the unique continuous (or monotone) rigid

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motion invariant valuation which agrees with G_r on unit lattice cubes of all dimensions. We therefore wish to propose as a problem the following

CONJECTURE. *If P is a lattice polytope, then $G_r(P) \leqslant V_r(P)$ for all $r \geqslant 0$.*

Since $G_0(P) = V_0(P) = 1$ and $G_d(P) = V_d(P)$ (in E^d), the two extreme cases are trivial. We shall remark later that the case $r = d-1$ is also easy, but we shall not offer any further direct evidence in favour of the conjecture. Instead, we shall point out some suggestive correspondences between the functions G_r and V_r.

Firstly, Ehrhart (5) has shown that the number $G^0(P)$ of lattice points in the relative interior of the lattice polytope P satisfies

$$G^0(nP) = (-1)^{\dim P} \sum_{r \geqslant 0} G_r(P)\,(-n)^r.$$

(See also Macdonald (15).) This is the *reciprocity law*. Since $G(P) = \Sigma_F G^0(F)$, or $G = G^0 \circ \zeta$, we at once deduce the Euler-type relation for G_r:

$$G_r^*(P) = (-1)^r G_r(P).$$

(Compare with the relation $V_r^*(P) = (-1)^r V_r(P)$, mentioned in section 5.)

Reeve (20), (21) and Macdonald (14) introduced the angle function A, which associates with each lattice point in the lattice polytope P the internal angle of P at that point. Thus:

$$A(P) = \Sigma_F \beta(F, P)\, G^0(F),$$

or $A = G^0 \circ \beta$. Since $\beta \circ \gamma = \zeta$, we have $A \circ \gamma = G^0 \circ \zeta = G$, so that $A = G \circ \bar{\beta}$ also, or

$$A(P) = \Sigma_F (-1)^{\dim P - \dim F} \beta(F, P)\, G(F).$$

If we now replace P by nP, and use the reciprocity law for $G^0(nF)$, we obtain two polynomial expressions $A(nP) = \sum_{r \geqslant 0} A_r(P)\, n^r$, where

$$A_r(P) = \begin{cases} \Sigma_F (-1)^{r - \dim F} \beta(F, P)\, G_r(F), \\ \Sigma_F (-1)^{\dim P - \dim F} \beta(F, P)\, G_r(F). \end{cases}$$

We have thus given a new proof of the result of Macdonald (15): $A_r(P) = 0$ if $r \not\equiv \dim P \pmod 2$. (We also have $A_0(P) = 0$ for $\dim P > 0$, by Gram's theorem.)

Conversely, we can reverse the argument, using $G = A \circ \gamma$ and $G^0 = A \circ \bar{\gamma}$, to deduce Ehrhart's reciprocity law from Macdonald's result.

We have also obtained the second (partial) correspondence between G_r and V_r. For, we have seen that

$$\Sigma_F (-1)^{\dim F} \beta(F, P)\, G_r(F) = 0 \quad \text{if} \quad r \not\equiv \dim P \pmod 2.$$

The case $r = d-1$ of the Euler-type relation for G_r proved above implies that, if P is a d-dimensional lattice polytope,

$$G_{d-1}(P) = \tfrac{1}{2} \Sigma_{\text{facets } F} G_{d-1}(F).$$

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Since $G_{d-1}(F) = V_{d-1}(F)/\Delta(F)$, where $\Delta(F)$ is the determinant of the induced lattice in aff F, and $\Delta(F) \geqslant 1$, we have

$$G_{d-1}(P) \leqslant \tfrac{1}{2}\Sigma_{\text{facets } F} V_{d-1}(F) = V_{d-1}(P).$$

This fact is already clear from Ehrhart (6). McMullen–Wills (17) proved a stronger result in case $d = 3$; here it is equivalent to Hadwiger's conjecture (private communication) $G(\partial K) \leqslant W(\partial K)$, where ∂K is the boundary of K, and

$$W(\partial K) = 2\,\Sigma_{r \not\equiv \dim P \,(\mathrm{mod}\, 2)} V_r(K).$$

If P is a lattice polytope,

$$G(\partial P) = G(P) - G^0(P) = 2\,\Sigma_{r \not\equiv \dim P (\mathrm{mod}\, 2)}\, G_r(P),$$

so our conjecture would also imply that of Hadwiger.

We remark in conclusion that Reeve (20), (21) and Macdonald (14) use the fact that the coefficients $G_d(P)$ and $A_d(P)$ are just the volume of P to calculate the volume, given enough values of $G(nP)$ or $A(nP)$.

7. *Grassman angles for polytopes and cones.* It is natural to enquire whether the relations we have found can be generalized to the Grassmann angles of Grünbaum (9), or to the angles introduced by Larman–Mani (13). We shall see that they can, but this is because these angles can be subsumed under our bilinear angles.

Let K be a polyhedral cone in E^d, not necessarily d-dimensional, but whose subspace C of apices is distinct from K (so that K is not a subspace). Let c be a fixed point of C, for $1 \leqslant j \leqslant d-1$, let $\gamma^j(C, K)$ be the Grassmann angle of j-flats through c which support K, and let $\beta^j(C, K)$ be the complementary angle of those j-flats which meet relint K. To find expressions for β^j and γ^j in terms of the spherical quermassintegrals ϕ_r, we need a result of Santaló ((25); see also (27)).

LEMMA 5. *For* $1 \leqslant r \leqslant j$, $\int \phi_r(C \cap L^j, K \cap L^j)\, dL^j = \phi_{d+r-j}(C, K)$, *where the integral is over the Grassmannian of all* j*-flats* L^j *through a fixed point* c *of* C, *with the invariant measure normalized so that the total measure is* 1.

Lemma 5 and Theorem 1 give at once:

COROLLARY. *For* $j \geqslant 1$, $\int \phi_0(C \cap L^j, K \cap L^j)\, dL^j = \sum_{r \leqslant d-j} \phi_r(C, K)$.

We can now evaluate $\beta^j(C, K)$ (and hence $\gamma^j(C, K)$). For, if S is the unit sphere (centred at c), since K is not a subspace,

$$\begin{aligned}\beta^j(C, K) &= \int \chi(S \cap C \cap L^j, S \cap K \cap L^j)\, dL^j \\ &= \int \{2 \sum_{t \geqslant 0} \phi_{2t+1}(C \cap L^j, K \cap L^j)\}\, dL^j \\ &= 2 \sum_{t \geqslant 0} \phi_{d-j+2t+1}(C, K).\end{aligned}$$

It follows that

$$\begin{aligned}\gamma^j(C, K) &= 1 - \beta^j(C, K) \\ &= 2 \sum_{t \geqslant 0} \phi_{d-j-2t-1}(C, K).\end{aligned}$$

We also note that $\beta^j(K, K) = 1$ if $j + \dim K > d$, and 0 otherwise. If P is a polytope, and F a face of P, we write $\beta^j(F, P)$, $\gamma^j(F, P)$ with the obvious meanings.

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We are now in a position to prove the Grassmann angle relations of Grünbaum (9). We shall, in fact, obtain more general results. It is useful to prove first:

LEMMA 6. *Let P be a polytope. Then*

$$\Sigma_F (-1)^{\dim F} V_r(F)\, \phi_s(F, P) = (-1)^r \delta_{rs} V_r(P).$$

For, by Theorem 3,

$$\begin{aligned}\Sigma_F (-1)^{\dim F} V_r(F)\, \phi_s(F, P) &= \Sigma_F \Sigma_{G^r} \Sigma_{J^s} V(G^r)(-1)^{\dim F} \gamma(G^r, F)\, \beta(F, J^s)\, \gamma(J^s, P) \\ &= \Sigma_{G^r} \Sigma_{J^s} V(G^r)(-1)^r \delta(G^r, J^s)\, \gamma(J^s, P) \\ &= (-1)^r \delta_{rs} V_r(P),\end{aligned}$$

as claimed.

Our first result generalizes Grünbaum's Theorem 3·3.

THEOREM 10. *Let P be a d-polytope in E^d. Then, for $r \leqslant d-j-1$,*

$$\Sigma_F (-1)^{\dim F} \gamma^j(F, P)\, V_r(F) = (1-(-1)^{d-j})(-1)^r V_r(P).$$

For,

$$\begin{aligned}\Sigma_F (-1)^{\dim F} \gamma^j(F, P)\, V_r(F) &= 2 \sum_{t \geqslant 0} \Sigma_F (-1)^{\dim F} V_r(F)\, \phi_{d-j-2t-1}(F, P) \\ &= 2 \sum_{t \geqslant 0} (-1)^r \delta_{r, d-j-2t-1} V_r(P) \\ &= (1-(-1)^{d-j})(-1)^r V_r(P),\end{aligned}$$

as asserted.

Notice that, if we sum over all s in Lemma 6, we obtain the Euler-type relation $V_r^* = (-1)^r V_r$, due to Shephard (29), which we have previously mentioned. It is clear that we can also prove vector valued analogues of these results, involving the intrinsic moments z_{r+1} instead of the V_r.

For polyhedral cones, we have exactly analogous results. Let K be a d-dimensional polyhedral cone, with face of apices C. We first have the analogue of Lemma 6; the proof is so similar, we shall omit it.

LEMMA 7. $\Sigma_F (-1)^{\dim F} \phi_r(C, F)\, \phi_s(F, K) = (-1)^r \delta_{rs} \phi_r(C, K)$.

We then have the analogue of Theorem 10, whose proof we shall also omit.

THEOREM 11. *For $r \leqslant d-j-1$,*

$$\Sigma_F (-1)^{\dim F} \phi_r(C, F)\, \gamma^j(F, K) = (1-(-1)^{d-j})(-1)^r \phi_r(C, K).$$

If, on the other hand, we sum over s in Lemma 7, we obtain the 'Euler-type relation':

THEOREM 12. $\Sigma_F (-1)^{\dim F} \phi_r(C, F) = (-1)^r \phi_r(C, K)$.

Now, summing again over $r = d-j+2t+1$ $(t \geqslant 0)$, we obtain Grünbaum's Theorem 2·8:

THEOREM 13. *For $1 \leqslant j \leqslant d - \dim C - 1$,*

$$\Sigma_F (-1)^{\dim F} \beta^j(C, F) = (-1)^{d-j+1} \beta^j(C, K).$$

The reader will find it easy, starting from Lemma 7, and using the several relations between the ϕ_r, β^j and γ^j, to construct many more formulae of this kind. We could

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also use various results of Poincaré and Sommerville (see Grünbaum (8), 14.2.1) to prove Grünbaum's Theorems 2·9 and 3·4. However, the proofs involve standard arguments, which make no use of our angle-sum relations; as we have not been able to generalize the theorems, we shall not mention them.

Larman–Mani (13) also introduced some angle functions. In fact, their angle $\alpha_{k-1}(P, F)$ is just

$$\alpha_{k-1}(P, F) = \int \phi_0(F \cap L^k, P \cap L^k)\, dL^k = \sum_{s \leqslant d-k} \phi_s(F, P),$$

by the corollary to Lemma 5. In some ways, this is a more natural generalization of the ordinary external angle γ than Grünbaum's γ^j; $\alpha_{k-1}(P, F)$ measures the supporting half k-spaces of P at F. Their Theorem 2 then has the following generalization.

THEOREM 14.

$$\Sigma_F (-1)^{\dim F} \alpha_{k-1}(P, F)\, V_r(F) = \begin{cases} (-1)^r V_r(P) & \text{if} \quad r \leqslant d-k, \\ 0 & \text{otherwise.} \end{cases}$$

This follows immediately from Lemma 6 and the definition.

The complementary angle β_{k-1} of Larman–Mani (13) is

$$\beta_{k-1}(P, F) = 1 - \alpha_{k-1}(P, F) = \sum_{s \geqslant d-k+1} \phi_s(F, P),$$

which measures the half k-spaces at F which meet relint P, so that we have similarly:

THEOREM 15. $\Sigma_F (-1)^{\dim F} \beta_{k-1}(P, F)\, V_r(F) = \begin{cases} (-1)^r V_r(P) & \text{if} \quad r \geqslant d-k+1, \\ 0 & \text{otherwise.} \end{cases}$

8. *Concluding remarks.* In view of the statements of Theorems 1, 2 and 3, it might be conjectured that there is a fourth relation between the internal and external angles, of the form

$$\Sigma_F \gamma(o, F)\, \beta(F, K) = \lambda,$$

independent of K. That there is not may be seen as follows. Let K be a 3-dimensional polyhedral cone, with (normalized) dihedral angles $\theta_1, \ldots, \theta_k$ and face angles $\phi_1, \ldots, \phi_k$. Then the polar cone K^* has dihedral angles $\frac{1}{2} - \phi_1, \ldots, \frac{1}{2} - \phi_k$ and face angles $\frac{1}{2} - \theta_1, \ldots, \frac{1}{2} - \theta_k$, and so

$$\Sigma_F \gamma(o, F)\, \beta(F, K) = \{\tfrac{1}{2} \Sigma (\theta_i - \tfrac{1}{2}) + \tfrac{1}{2}\} + \{\tfrac{1}{2} \Sigma \theta_i\} + \{\tfrac{1}{2} \Sigma (\tfrac{1}{2} - \phi_i)\} + \{\tfrac{1}{2} \Sigma (-\phi_i) + \tfrac{1}{2}\},$$

which is clearly not constant. This should not, however, be unexpected when looked at from the point of view of the incidence algebra. For, the relation of Theorem 3′, which we saw could be written $\gamma \circ \mu \circ \beta = \delta$, clearly implies nothing about $\gamma \circ \beta$.

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