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Crystallization in forced flows: the Saffman–Taylor problem

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We consider the two-dimensional problem of crystal growth in a forced flow. A dendrite is placed in a Hele-Shaw cell with insulating walls and grows due to undercooling. We neglect the surface energy in the Gibbs–Thomson relation. The problem is formulated in terms of analytic functions similarly to closely related work on the viscous fingering problem of Saffman and Taylor. We derive a solvability condition for the existence of a steady-state needle-like solidification front in the limit of small Peclet number, $\mathrm{Pe} = V_\infty\, l/a$, where V_∞ is the characteristic velocity of the melt, $2l$ is the channel width, and a is the thermal diffusivity of the liquid. The velocity of the crystallization front is directly proportional to the hydrodynamic velocity V_∞ and undercooling, while the dendrite width l_d does not depend upon the physical parameters, and indeed, $l_d = l$.

1 Introduction

Developments in two-dimensional free-boundary problems [2, 4, 6, 13, 16, 18] point to the emergence of an asymptotic theory of pattern selection in realistic models of crystal growth, viscous fingering and the like. Ivantsov's dendrite [14] and the Saffman–Taylor finger [22, 24] become the basis for establishment of a mathematical strategy that enables one to make asymptotically accurate estimates of the singular effects of the capillary perturbation. This paper is aimed at the extension of the Saffman–Taylor problem to the two-dimensional solidification/melting of dendrites in a forced hydrodynamic flow. In setting up the model we closely follow Cummings *et al.* [7]. The physical system consists of a freezing cell, where ice can grow under a temperature drop [8, 9, 11]. Due to specific molecular interactions, a thin liquid film forms between the crystal boundary and the cell walls [8, 25]. So, the physical system is seemingly kindred to the ordinary Hele-Shaw problem of bubble displacement [22]. Indeed, if the fluid flow were absent, the problem of dendritic growth is reduced to the Saffman–Taylor problem of bubble displacement, in which the heat flow plays the role of displacing fluid while the dendrite is treated as a bubble. For this diffusion controlled growth the selection principle has been pointed out by several authors [1, 3–6, 10, 12, 15, 16, 23, 26], who showed that either surface energy or kinetic term in the Gibbs–Thomson relation could be responsible for selection of a unique dendritic shape. However, even for small Peclet numbers, when convective heat transfer is dominated by conduction, the fluid flow plays a key role in redistributing the heat flux over the growing interface. As a result, a crystal growing in a stream differs in

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shape from a freely solidifying dendrite. The main difficulty in extending the theory of Cummins *et al.* [7] to the case under consideration is the calculation of a closed solution of the heat exchange problem which we call the *Boussinesq problem* [17]. We solve the Boussinesq problem by the Wiener–Hopf technique, and discuss the case of small Peclet number in detail. The latter allows us to obtain an explicit solution for a steadily growing dendrite.

2 The model

We take the complex variable $z' = x' + iy'$ associated with co-ordinates (x', y') in the plane of the crystal. If the temperature at infinity, T_∞, is maintained below the melting temperature, T_p, the crystal grows upstream. Simultaneously, a fluid flow is imposed at infinity and the average velocity parallel to the plates of the Hele-Shaw cell satisfies Darcy's law with the permeability $k = b^2/12$, where b is the plate-spacing [2]. Because the fluid is incompressible, there is a complex potential $W' = \varphi' + i\psi'$, where φ' is the real velocity potential related to the pressure p and fluid viscosity μ as $\varphi' = -kp/\mu$, ψ' is the streamfunction. Normalizing co-ordinates by a half the channel width, l, the superheat $T - T_p$ by the undercooling $(T_\infty - T_p)$, and the complex potential by half the flow rate $V_\infty l$, we can write the mathematical model in dimensionless form as

$$\begin{gathered}
\nabla \cdot \mathbf{v} = 0, \qquad \mathbf{v} = \nabla\varphi, \qquad \mathrm{Pe}\mathbf{v} \cdot \nabla\theta = \triangle\theta, \quad z \in D(t); \\
\theta = 0, \quad -\frac{\partial\theta}{\partial n} = v_n, \quad \frac{\partial\varphi}{\partial n} = 0, \quad z \in \partial\Omega(t); \\
\mathbf{v} = (1, 0), \quad x \to -\infty; \\
\lim_{x\to-\infty} \theta(r) \to 1, \quad \frac{\partial\theta}{\partial y} = 0, \quad \frac{\partial\varphi}{\partial y} = 0, \quad y = \pm 1; \\
D(0) = D_0.
\end{gathered} \tag{1}$$

Here $\partial\Omega(t)$ is the solid–liquid interface, v_n is the outward normal velocity of dendrite crystallization (with respect to the liquid region $D(t)$), $t = t'\lambda(T_\infty - T_p)/l^2 L$ is the dimensionless time (positive, if we consider the melting case, and negative for crystallization), L is the volumetric latent heat, and λ is the thermal conductivity of the liquid. We introduce the Peclet number as $\mathrm{Pe} = V_\infty l/a$, where a is the thermal diffusivity of the liquid. It is assumed that the Stefan number $St = c|T_\infty - T_p|/L$ is small, where c is the volumetric heat capacity; hence the stationary temperature distribution is suitable for description of the temperature field. Since the problem is formulated in the melting/freezing timescale rather than the flow one, the normal hydrodynamic velocity at the crystal surface is negligibly small on that timescale. That is why the boundary condition $\partial\varphi/\partial n = 0$ is a reasonable approximation.

It is remarkable that the moving boundary problem (1) has a first integral establishing a connection between the total mass of the dendrite and the Peclet number. Such a relation can be immediately obtained by expressing the rate of mass change via the integral over the dendrite surface as

$$\frac{dM}{dt} = \frac{d}{dt}\int_D dv = \int_{\partial D} v_n d\Gamma, \tag{2}$$

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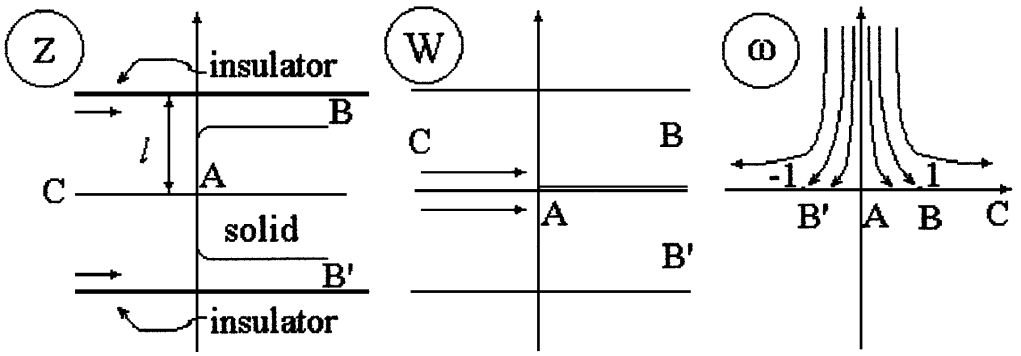


FIGURE 1. Flows in physical, complex potential and auxiliary planes, respectively.

where M is the dendrite mass. Taking into account the Stefan condition, one arrives at the relation

$$\frac{dM}{dt} = -\int_{\partial D} \frac{\partial \theta}{\partial n} d\Gamma = -Q, \tag{3}$$

where Q is the total heat flux. Though unknown in advance, the dependence of the total heat flux on the domain geometry can be traced by making use of the change of variables $z \longmapsto W$ in the heat transfer equation. Since the flow rate is known, the complex potential plane $W = \varphi + i\psi$ can be immediately constructed. Under the transformation $z \longmapsto W$, the dendrite collapses onto a slit along the positive φ-axis, but the insulating walls retain their shape and they are positioned at

$$-\infty \leqslant \varphi \leqslant \infty, \quad \psi = \pm 1. \tag{4}$$

It is important that the equation for the temperature field, θ, remains invariant,

$$\mathrm{Pe}\frac{\partial \theta}{\partial \varphi} = \left(\frac{\partial^2 \theta}{\partial \varphi^2} + \frac{\partial^2 \theta}{\partial \psi^2}\right). \tag{5}$$

The boundary conditions are rewritten as

$$\theta = 1, \qquad \varphi \to -\infty, \tag{6}$$

$$\theta = 0, \qquad 0 \leqslant \varphi \leqslant \infty, \quad \psi = 0, \tag{7}$$

$$\frac{\partial \theta}{\partial \psi} = 0, \qquad -\infty \leqslant \varphi \leqslant \infty, \quad \psi = \pm 1. \tag{8}$$

Equation (5) and boundary conditions (6)–(8) constitute the Boussinesq problem. One can see that the total heat flux Q remains invariant under the Boussinesq transformation as well, since

$$Q = \int_{\partial D} \left|\frac{\partial \theta}{\partial \psi}\right| \left|\frac{\partial \psi}{\partial n}\right| d\Gamma = \int_{\partial D} \left|\frac{\partial \theta}{\partial \psi}\right| \left|\frac{\partial \varphi}{\partial \Gamma}\right| d\Gamma = 2\int_0^{\infty} \left|\frac{\partial \theta}{\partial \psi}\right| d\varphi. \tag{9}$$

So, the flux does not depend upon the shape of crystal, yet it is a function of the Peclet number, $Q = Q(\mathrm{Pe})$. Later in the paper, we shall find an approximation to $Q(\mathrm{Pe})$ when Pe is small. Equation (3) can be integrated to give linear time-dependence of the dendrite

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mass. For a steadily moving finger-like dendrite with asymptotically parallel-sided walls, the change of mass rate is merely the crystallization speed, so that $U = Q(\mathrm{Pe})$. In the nonstationary case, (3) still holds and can be checked in experiments.

3 The hodograph method

We now restrict ourselves to *symmetric* dendrites and introduce an auxiliary plane $\omega = \xi + i\eta$ (see Figure 1) and an analytic function $z = f(\omega, t)$ mapping the auxiliary plane onto the region $D(t)$. The Stefan condition can be rewritten in terms of analytic functions, if we take into account the definition of the outward normal velocity of $\partial\Omega$,

$$v_n = \dot{x}(\xi)n_x + \dot{y}(\xi)n_y,$$

where n_x, n_y are the components of the unit vector normal to the interface. Making the transformation

$$W = -\frac{1}{\pi}\log(1-\omega^2), \tag{10}$$

we can write down the Stefan condition as (see Cummings *et al.* [7] for details)

$$\Re\left\{\frac{\partial f}{\partial t}\overline{i\frac{\partial f}{\partial \omega}}\right\} = -\left|\frac{\partial\theta}{\partial\psi}\right|\overline{\frac{\partial W}{\partial\omega}} = -\left|\frac{\partial\theta}{\partial\psi}\right|\frac{2\xi}{\pi\left(1-\xi^2\right)}. \tag{11}$$

In the context of the Hele-Shaw free boundary problem, (11) is called the Polubarinova-Galin equation [12]. In addition to the Polubarinova-Galin equation, the function $f(\omega, t)$ must be subject to the boundary condition imposed on the velocity at infinity,

$$\frac{\partial f}{\partial\omega} \sim \frac{\partial W}{\partial\omega} \sim -\frac{2}{\pi\omega}, \qquad \omega \to \infty. \tag{12}$$

One expects that the mapping function has similar singular behaviour at points $\xi = \pm 1$. Moreover, in the auxiliary plane we can imagine these flow singularities as two sinks with given discharge of unit strength. Finally, we should prescribe an initial condition. Thus, the free boundary problem is reduced to a nonlinear boundary value problem for analytic function $f(\omega, t)$, the solution of the Boussinesq problem (5)–(8) being included into the boundary condition (11).

4 The Boussinesq problem

It is convenient to introduce new variables and functions as

$$\Phi = \mathrm{Pe}\,\varphi/2, \quad \Psi = \mathrm{Pe}\,\psi/2, \quad \theta(\Phi, \Psi) = 1 + \exp(\Phi)u(\Phi, \Psi). \tag{13}$$

Then the Boussinesq problem is reduced to the following boundary value problem for the modified Helmholtz equation:

$$\frac{\partial^2 u}{\partial\Phi^2} + \frac{\partial^2 u}{\partial\Psi^2} - u = 0, \qquad -\infty < \Phi < \infty, \quad |\Psi| \leqslant \mathrm{Pe}/2, \tag{14}$$

$$u \to 0, \qquad \Phi \to \pm\infty,$$

$$u = -\exp(-\Phi), \qquad \Psi = 0, \qquad 0 \leqslant \Phi < \infty,$$

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$$\frac{\partial u}{\partial \Psi} = 0, \qquad \Psi = \pm \mathrm{Pe}/2, \qquad -\infty < \Phi < \infty.$$

This problem can be solved by the Wiener–Hopf technique [20] (see the Appendix), and, in the region $\Psi \geqslant 0, \Phi \geqslant 0$, the solution has the form

$$u(\Phi, \Psi) = -\mathrm{e}^{-\Phi} - 2\sum_{k=1}^{\infty} P_k \cos\left[(2k-1)\frac{\pi}{\mathrm{Pe}}\left(\Psi - \frac{\mathrm{Pe}}{2}\right)\right] \mathrm{e}^{-\gamma_k \Phi}, \tag{15}$$

where

$$\gamma_k = \sqrt{\frac{(2k-1)^2\pi^2}{\mathrm{Pe}^2} + 1}, \quad P_k = (-1)^{k+1}\frac{K_+(i\gamma_k)}{(k-1/2)\pi\gamma_k K_+(i)},$$

$$K_+(i\omega) = \prod_{n=1}^{\infty} \frac{(1 + b_{n-1/2}^2)^{1/2} + \omega b_{n-1/2}}{(1 + b_n^2)^{1/2} + \omega b_n}; \quad b_p = \frac{\mathrm{Pe}}{2p\pi}.$$

We are interested in the practically reasonable case of small Peclet numbers $\mathrm{Pe} \ll 1$ [8, 9, 11]. For the latter, taking the limit of (15) as the Peclet number tends to zero, one gets

$$\gamma_k \sim \frac{(2k-1)\pi}{\mathrm{Pe}}, \quad K_+(i) \sim 1,$$

$$K_+(i\gamma_k) \sim \frac{\sqrt{\pi}\Gamma(k+1/2)}{\Gamma(k)} = \frac{\pi(2k-1)!!}{2(2k-2)!!}.$$

Using the above relation and the asymptotic relation

$$\exp\left\{-\frac{\pi\Phi}{\mathrm{Pe}}\left(1 - \frac{\mathrm{Pe}}{\pi}\right)\right\} \sim \exp\left\{-\frac{\pi\Phi}{\mathrm{Pe}}\right\}, \quad \Phi \sim O(\mathrm{Pe}),$$

we arrive at the asymptotic expression for the heat flux on the slit,

$$\frac{\partial \theta}{\partial \Psi} \sim \frac{\partial u}{\partial \Psi} \sim 2\sum_{k=1}^{\infty} \frac{(2k-1)!!}{(2k-2)!!(2k-1)} \mathrm{e}^{-\gamma_k \Phi}$$

$$= \frac{2\mathrm{e}^{-\pi\Phi/\mathrm{Pe}}}{\sqrt{1 - \mathrm{e}^{-2\pi\Phi/\mathrm{Pe}}}}, \qquad \Psi = 0, \qquad 0 \leqslant \Phi < \infty. \tag{16}$$

Substituting the transformation (10) into (16), one gets

$$\frac{\partial \theta}{\partial \psi} = \frac{\mathrm{Pe}\sqrt{1-\xi^2}}{\xi}, \qquad -1 < \xi < 1. \tag{17}$$

5 Steady solution

The asymptotic solution (17) allows us to rewrite the Polubarinova–Galin equation (11) in the form

$$\Re\left\{i\frac{\partial f}{\partial t}\overline{\frac{\partial f}{\partial \omega}}\right\} = \begin{cases} (2\mathrm{Pe}/\pi)\left(1-\xi^2\right)^{-1/2}, & -1 < \xi < 1, \\ 0, & |\xi| > 1. \end{cases} \tag{18}$$

In this section we are looking for a travelling-wave solution, $f = Ut + F(\omega)$. It should be noted that the speed U is not equal to the hydrodynamic velocity of the oncoming stream, $v = 1$. So, one has to select the crystallization speed and the shape of the steadily

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moving dendrite by seeking a function F analytic in the upper half plane $\eta > 0$ and having logarithmic singularities at the points $\xi = \pm 1$, and at infinity as well. This function must obey the boundary condition

$$\Re\left\{U\overline{iF'}\right\} = \begin{cases} -\left(2\mathrm{Pe}/\pi\right)\left(1-\xi^2\right)^{-1/2}, & -1 < \xi < 1, \\ 0, & |\xi| > 1. \end{cases} \tag{19}$$

Thus, the nonlinear boundary condition (18) is reduced to the *linear Riemann–Hilbert* form (19) [19]. Actually, a family of travelling-wave solutions enables one to transfer a *functional* nonlinearity to a *parametric* one, the crystallization speed U being unknown in advance. In standard linear Riemann–Hilbert problems, no derivatives appear in the boundary condition [19]. Nevertheless, a trick similar to the ordinary Riemann–Hilbert technique can be used to get the solution for our case. The solution consists of two parts, with a particular solution as the first, and the solution of the homogeneous problem as the second. Scaling the crystallization speed U as $U = 2\mathrm{Pe}\beta/\pi$, one can choose the following particular solution:

$$F_{part} = \frac{1}{\beta}\log(\omega + \sqrt{\omega^2 - 1}) + i\gamma, \qquad \Im\gamma = 0. \tag{20}$$

Then the boundary condition for the homogeneous problem takes the form

$$\Re\left\{\overline{iF'_{gen}}\right\} = 0, \qquad -\infty < \xi < \infty. \tag{21}$$

Because of the logarithmic singularities at the points $\xi = \pm 1$ and at infinity, the solution to problem (21) is

$$F_{gen} = \alpha\log(\omega^2 - 1), \qquad \Im\alpha = 0. \tag{22}$$

Combining all of them and using Liouville's theorem applied to the function $F - F_{part} - F_{gen}$, the latter being extended into the lower half plane $\eta < 0$ by reflection, we have shown that the following three-parameter ansatz gives the unique travelling-wave solution:

$$f = \frac{1}{\beta}\log(\omega + \sqrt{\omega^2 - 1}) - \frac{2\beta\mathrm{Pe}\,t}{\pi} + \alpha\log(\omega^2 - 1) + i\gamma, \qquad \Im\alpha = \Im\gamma = 0. \tag{23}$$

The boundary condition at infinity and the two conditions imposed on the flow rate at the points $\xi = \pm 1$ result in three linear equations for the unknown constants α, β, and γ. The solutions to the system are $\alpha = -1/2\pi$, $\beta = -\pi$ and $\gamma = 1$. Then the interface is described by the parametric equations $(0 < \xi < 1)$

$$x = 2\mathrm{Pe}\,t - \frac{1}{2\pi}\log(1-\xi^2), \qquad y = \pm\left[-\frac{1}{\pi}\arctan\frac{\sqrt{1-\xi^2}}{\xi} + \frac{1}{2}\right]. \tag{24}$$

Eliminating ξ, one obtains

$$y = \pm\left[-\frac{1}{\pi}\arctan\frac{\exp\left(-\pi[x - 2\mathrm{Pe}\,t]\right)}{\sqrt{\left(-\exp\left(-2\pi[x - 2\mathrm{Pe}\,t]\right) + 1\right)}} + \frac{1}{2}\right]. \tag{25}$$

If we put $\xi = \cos\vartheta$ in (24), (25) can be rewritten in the famous Saffman–Taylor form as

$$x = 2\mathrm{Pe}\,t - \frac{1}{\pi}\log(\cos\pi y), \qquad \pi y = \pm\left[-\vartheta + \frac{\pi}{2}\right]. \tag{26}$$

The key difference from the Saffman–Taylor viscous finger concerns the automatic

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selection of the parameters involved. Parametrized by the thickness of the deposited film, a one-parameter Saffman–Taylor family predicts very many fingers, while in practice those observed have a width close to one-half that of the channel [22]. Precisely, only one physical condition, total mass balance, is available to specify the physical parameters; this is no length scale other than the channel width. However, the latter is the product of the velocity and layer thickness at infinity, and cannot be split to give the finger width and superficial velocity separately. Some additional argument must be invoked to select the solution, for example, the effect of surface tension or some other physical mechanisms [1, 3–6, 10, 12, 15, 15, 16, 23, 26]. In the case under consideration, apart from the mass balance, we have an additional equation, the total balance of heat. Integrating the heat transfer equation over the melt volume, one obtains from the left-hand side of (1)

$$\mathrm{Pe}\int \mathbf{V}\cdot\nabla\theta dv = 2\mathrm{Pe}+\mathrm{Pe}\int (V_x\theta)\big|_{x\to\infty}dy+\mathrm{Pe}\int_{\partial D} V_n\theta d\Gamma = 2\mathrm{Pe}+\mathrm{Pe}\int (V_x\theta)\big|_{x\to\infty}dy, \quad (27)$$

As the Peclet number diminishes, the second term on the right-hand side of (27) can be neglected (see (15)), and we arrive at the first integral as

$$\frac{dM}{dt} = 2\mathrm{Pe}. \quad (28)$$

The equality $Q = 2\mathrm{Pe}$ can be checked by direct substitution of (16) into (9). Since for a steadily moving finger-like dendrite the change of mass rate is the crystallization speed, we have $U = 2\mathrm{Pe}$.

Knowledge of the crystallization speed removes the uncertainty in (19). The remaining boundary conditions allow one to select the unique solution. Actually, the left-hand side of (19) does change as the Peclet number increases, and the right-hand side should be modified as well. Therefore, the selection principle is affected by the magnitude of Peclet number, but the finger of width $1/2$ is selected in the limit $\mathrm{Pe}\to 0$.

In the Hele-Shaw problem the speed U cannot be prescribed in advance, but, once U is given, the corresponding Polubarinova–Galin equation has a unique solution [12]. So, the principal difference between Hele-Shaw and dendritic growth problems is in the extra balance condition for the latter, and the selection in the latter problem is not coincidental.

6 Conclusions

We have showed that the crystallization speed depends upon the flow rate linearly. Moreover, there are no hydrodynamic parameters that influence the shape of dendrite growing in a channel. As the Peclet number tends to zero (motionless melt), our solution breaks down and cannot be matched to the Saffman–Taylor finger. Actually, the shape of the finger coincides with the 'one half' Saffman–Taylor finger, but the crystallization speed becomes undefined within the model. To resolve this paradox, in the heat transfer equation one has to take into account the term proportional to the Stefan number which is responsible for change of total heat capacity. The latter problem bears a close resemblance to the Hele-Shaw problem and, in principle, an approximation with respect to the Stefan number results in the Saffman–Taylor solution (see Brener & Melnikov [4] and Ben Amar & Brener [1]). Some selection is needed and up to now it appears that a particular solution could be picked out by either surface energy or a kinetic coefficient in Gibbs–

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Thomson relation. However, the above analysis shows that the only the Saffman–Taylor finger of width one half can be used to continue the Hele-Shaw solution to the case of crystallization in forced flow. Perturbing the Hele-Shaw model by including convective heat transfer and considering the limiting solution as $\mathrm{Pe}\to 0$, we arrive at the same universal one half Saffman–Taylor finger. This result, like the perturbation analyses for small surface tension, suggests that there is a 'structural' reason for the selection of the one-half finger and that it is an attractor.

Unfortunately, the above solution cannot be continued to the limit of large Peclet numbers. As shown in the Appendix, at large Peclet numbers the Boussinesq problem gives, as the principal term, the solution for unconfined flow. For the latter we know [7] that the Ivantsov-like dendrite emerges as the steady solution. Recalling that the both the Saffman–Taylor and Ivantsov solutions have one-parametric freedom, we observe the paradoxical situation when two limiting solutions cannot be specified in the model framework, while the intermediate asymptote is selected uniquely.

Of course, the Ivantsov dendrite cannot be considered as the solution to our problem, yet it is reasonable approximation of the crystal tip. Most likely, the tip radius in the Ivantsov solution can be found from the "outer" solution, which is still affected by the insulating walls. Namely, it might be expected that the tip radius can be found from the critical Peclet number. However, this question is open for discussion and is beyond the scope of our paper. Nevertheless, we do believe that convection is a reasonable mechanism capable of selecting the inherent morphological pattern in the range of small Stefan numbers. In reality though, small natural convection could be a competitive factor compared to surface effects.

Appendix

By symmetry, the Boussinesq problem (11) needs be considered only in the region $(-\infty,+\infty)\times[0,\mathrm{Pe}]$, provided that the boundary condition at the line $\Psi=0$ is determined by introducing two unknown functions $f_{\pm}(\Phi)$ as

$$u(\Phi,\Psi\to+0)=f_{-}(\Phi)+H(\Phi)\exp(-\Phi),$$

$$\frac{\partial u}{\partial\Psi}=H(-\Phi)f_{+}(\Phi),\qquad -\infty<\Phi<\infty,\quad \Psi\to+0,$$

where $H(\Phi)$ is the Heaviside function. Using the Fourier transform

$$U(\Psi,s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(\Phi,\Psi)\exp(is\Phi)\,d\Phi,\quad -\infty<s<\infty,$$

the Boussinesq problem is reduced to a boundary value problem for analytic functions $F_{\pm}(s)$, which are the Fourier images of $f_{\pm}(\Phi)$. The functions $F_{\pm}(s)$ are analytic functions regularly decreasing at infinity in the upper ('+'-sign) or lower ('−'-sign) half-planes of complex variable $s=R+iS$. On the real axis, $S=0$, these functions must be matched by the boundary condition

$$-\frac{\coth(\mathrm{Pe}\sqrt{s^2+1}/2)}{\sqrt{s^2+1}}F_{+}(s)=F_{-}(s)+\frac{i}{\sqrt{2\pi}(s+i)}.\qquad(A1)$$

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Applying the Wiener–Hopf technique [20], we get the solution to problem (A1) as

$$u(\Phi,\Psi)=\frac{\mathrm{Pe}}{2\pi K_+(i)}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{\cosh[(\Psi-\mathrm{Pe}/2)\sqrt{s^2+1}]}{\sinh[\mathrm{Pe}\sqrt{s^2+1}]K_+(s)\sqrt{s^2+1}}\mathrm{e}^{-is\Phi}\,ds, \tag{A2}$$

where K_+ is defined in equation (15). The contour integral around a large semicircle in the lower half-plane can be calculated using the residue theorem and we arrive at (15).

To check the solution's validity, we examine the asymptotic case $\mathrm{Pe}\gg 1$. As seen from the integral equation equivalent to our problem (the derivation is done by Kornev & Mukhamadullina [17]),

$$\int_0^\infty \Lambda(\xi-\Phi)\left(\frac{\partial u}{\partial\Psi}\right)\bigg|_{\Psi=0}\mathrm{e}^{\xi}\,d\xi=e^{-\Phi}, \tag{A3}$$

$$\Lambda(\xi-\Phi)=\frac{1}{\pi}\left\{K_0(|\xi-\Phi|)+2\sum_{k=1}^{\infty}K_0\left(\sqrt{(\xi-\Phi)^2+\mathrm{Pe}^2k^2/4}\right)\right\};$$

the principal term corresponds to the heat exchange problem for a single isothermal slit and homogeneous flow. The other slits do not contribute significantly to the heat flux at large Peclet numbers. The solution for a single slit has been obtained by several authors (see the discussion in Cummings *et al.* [7]). The objective now is to reconstruct the exact solution by using (15).

First, in the range $n\leqslant k$, where $k\ll\sqrt{\mathrm{Pe}}$, the expression

$$\frac{K_+(i\gamma_k)}{K_+(i)}=\prod_{n=1}^{\infty}M_n,\quad M_n=\frac{[(1+b_{n-1/2}^2)^{1/2}+\gamma_k b_{n-1/2}][(1+b_n^2)^{1/2}+b_n]}{[(1+b_n^2)^{1/2}+\gamma_k b_n][(1+b_{n-1/2}^2)^{1/2}+b_{n-1/2}]},$$

can be simplified by putting $M_n\sim 1$, so that we get

$$\frac{K_+(i\gamma_k)}{K_+(i)}\sim\prod_{n=k}^{\infty}M_n. \tag{A4}$$

Further simplification is achieved by expanding $b_{n-1/2}$ in powers of $1/n$, provided that the following inequalities hold: $n\geqslant k(1\ll k\ll\mathrm{Pe})$. Combining all the principal terms, we arrive at the relations

$$K_+(i\omega)=e^T,\quad T=\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{2n}\Lambda_n(\omega)\right)\sim\sum_{n=1}^{\infty}\frac{1}{2n}\Lambda_n(\omega), \tag{A5}$$

where

$$\Lambda_n(\omega)=\frac{b_n^2}{b_n^2+1}\left(1+\frac{\omega}{b_n(\sqrt{1+b_n^2}+\omega b_n)}\right).$$

Using the Euler–Maclaurin summation rule [21], we replace the sum in (A5) by the integral as

$$\begin{aligned}T&=\frac{1}{2}\int_{2\pi/\mathrm{Pe}}^{\infty}\frac{dy}{y\,(1+y^2)}+\frac{\omega}{2}\int_0^{\infty}\frac{y\,dy}{(1+y^2)(\sqrt{1+y^2}+\omega)}+O\left(\frac{1}{k}\right)\\&=\log\frac{\mathrm{Pe}}{2\pi}+\frac{\log(1+\omega)}{2}+O\left(\frac{1}{k}\right).\end{aligned} \tag{A6}$$

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Substituting (A5) and A(6) into the formula for the heat flux, one gets

$$\frac{\partial u}{\partial \Psi}(\Phi,+0) = -\frac{2\sqrt{2}}{\mathrm{Pe}}\sum_{k=1}^{\infty}\frac{\sqrt{1+\gamma_k}}{\gamma_k}\mathrm{e}^{-\Phi\gamma_k}\left(1+O\left(\frac{1}{k\mathrm{Pe}}\right)\right), \qquad \Phi \geqslant 0,$$

$$\gamma_k = \sqrt{1+\frac{\pi^2(2k-1)^2}{\mathrm{Pe}^2}}. \tag{A7}$$

Again using the Euler–Maclaurin summation rule, we retrieve the exact solution for a single slit:

$$\frac{\partial u}{\partial \Psi}(\Phi,+0) = \frac{\sqrt{2}}{\pi}\int_1^{\infty}\frac{\mathrm{e}^{-\Phi u}\,du}{\sqrt{u-1}} + O\left(\frac{1}{\mathrm{Pe}}\right)$$

$$= \sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{-\Phi}}{\sqrt{\Phi}} + O\left(\frac{1}{\mathrm{Pe}}\right). \tag{A8}$$

Iterating the integral equation (A3) we are persuaded that the next corrections are exponentially small, namely:

1. If $0 \leqslant \Phi \ll \sqrt{\mathrm{Pe}}$

$$\frac{\partial u}{\partial \Psi}(\Phi,+0) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{\Phi}} + \frac{2^{3/2}\Gamma(1/4)\mathrm{e}^{-\mathrm{Pe}/2}}{(\pi)^{3/2}\mathrm{Pe}^{1/4}}\left(1+O\left(\frac{1}{\mathrm{Pe}}\right)\right);$$

2. If $\Phi \gg \sqrt{\mathrm{Pe}}$

$$\frac{\partial u}{\partial \Psi}(\Phi,+0) = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{\Phi}} + \frac{4\sqrt{2}\mathrm{e}^{-\mathrm{Pe}/2}}{\pi\mathrm{Pe}^{1/2}}\frac{1}{\sqrt{\Phi}}\left(1+O\left(\frac{1}{\mathrm{Pe}}\right)\right).$$

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