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GENERALIZED GREEN FUNCTIONS AND UNIPOTENT CLASSES FOR FINITE REDUCTIVE GROUPS, III

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Abstract. Lusztig's algorithm of computing generalized Green functions of reductive groups involves an ambiguity on certain scalars. In this paper, for reductive groups of classical type with arbitrary characteristic, we determine those scalars explicitly, and eliminate the ambiguity. Our results imply that all the generalized Green functions of classical type are computable.

§0. Introduction

This paper is a continuation of [S3, I and II]. Let G be a connected reductive group defined over a finite field $\mathbf{F}_q$ with Frobenius map $F : G \to G$, and let G_{uni} the unipotent variety of G. Let $\mathscr{N}_G$ be the set of pairs $(C, \mathscr{E})$, where C is a unipotent class in G, and $\mathscr{E}$ is a G-equivariant simple local system on C. Let L be a Levi subgroup of a parabolic subgroup of G, and let $(C_0, \mathscr{E}_0) \in \mathscr{N}_L$ be a cuspidal pair on L. We assume that L is F-stable, C is F-stable, and $F^*\mathscr{E}_0 \simeq \mathscr{E}_0$. For a fixed isomorphism $\varphi_0 : F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$, a generalized Green function $Q^G_{L,C_0,\mathscr{E}_0,\varphi_0}$ was defined in [L2, II], which is a G^F-invariant function on G^F_{uni} with values in $\bar{\mathbf{Q}}_l$. Generalized Green functions play a crucial role in the representation theory of a finite reductive group G^F, and it is very important to compute them explicitly since the computation of irreducible characters of G^F is essentially reduced to the computation of those generalized Green functions thanks to the theory of character sheaves [L2].

Let $\mathscr{N}_G^F$ be the set of $(C, \mathscr{E}) \in \mathscr{N}_G$ such that C is F-stable and $F^*\mathscr{E} \simeq \mathscr{E}$. For any $j = (C, \mathscr{E}) \in \mathscr{N}_G^F$, we choose an isomorphism $\psi : F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$. Note that ψ is unique up to scalar since $\mathscr{E}$ is simple. Let Y_j be the characteristic function on C^F defined by ψ, and we regard it as a function on G^F_{uni} by defining 0 outside of C^F. Then, $\{Y_j \mid j \in \mathscr{N}_G^F\}$ gives a basis of the $\bar{\mathbf{Q}}_l$-space of G^F-invariant functions on G^F_{uni}. In [L2, V], Lusztig gave an algorithm of expressing $Q^G_{L,C_0,\mathscr{E}_0,\varphi_0}$ as an explicit linear combination of the basis $\{Y_j \mid j \in \mathscr{N}_G^F\}$, where $\psi = \psi_j : F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ is determined from the choice of $\varphi_0 : F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$. (Here, φ_0 is chosen under a certain mild condition (see 1.2), and ψ_j is determined uniquely from φ_0 (see 1.3). But note that ψ_j is not directly computable.) Thus, the computation of the generalized Green function is reduced to the computation of those Y_j. Here, the function Y_j has a simple structure, up to scalar. If we choose a standard isomorphism $\psi_0 : F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$, the characteristic function Y_j^0 on C^F can be computed explicitly. There exists a scalar $\gamma_j \in \bar{\mathbf{Q}}_l^*$ such that $Y_j = \gamma_j Y_j^0$ for each j. Thus, the determination of generalized Green functions is reduced to the determination of those scalars γ_j for each $j \in \mathscr{N}_G^F$. This is a nontrivial problem

The problem of determining γ_j is closely related to the problem of finding a G^F-class in C^F (note that C^F has finitely many G^F-conjugacy classes) which has a good property with respect to the Frobenius action on a certain complex associated to $(L, C_0, \mathscr{E}_0)$. A priori such a good G^F-class in C^F depends on the choice of $(L, C_0, \mathscr{E}_0)$ since ψ_j is determined from $\varphi_0 : F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$. For example, in the case where $L = T$ is a maximal torus, $C_0 = \{1\}$, and

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$\mathscr{E}_0 = \bar{\mathbf{Q}}_l$ is the constant sheaf, $Q^G_{L,C_0,\mathscr{E}_0,\varphi_0}$ is nothing but the original Green function Q^G_T. In that case, assuming that ch $\mathbf{F}_q = p$ is a good prime, such a good G^F-class was found in [BS], [S1], [S2], called a split class in C^F (essentially, it is determined as a G^F-class for the adjoint group G), and the Green functions for exceptional groups were computed explicitly.

Those scalars γ_j were computed explicitly in [S3, I] in the case where $G = SL_n$ and F is of split type (for any p), F is of nonsplit type (p:large enough), and in [S3, II] in the case where $G = Sp_N$ or SO_N for any p. The main discovery there is that the split class for $(T, \{1\}, \bar{\mathbf{Q}}_l)$ behaves well for any choice of $(L, C_0, \mathscr{E}_0)$, namely, there exists a G^F-class in C^F which has a good property for any choice of $(L, C_0, \mathscr{E}_0)$, and the image of this class to the adjoint group coincides with the split class for $(T, \{1\}, \bar{\mathbf{Q}}_l)$ mentioned above. Such a class is called a split class in C^F.

Since the split class is conjecturally compatible with the isogeny, the existence of the split class is reduced to the case of simply connected groups. Sp_N or SL_n are simply connected, but SO_N is not. In that case, we need to consider the spin group Spin_N. In this paper, we prove that there exists a split element in Spin_N such that its image in SO_N is a split element in the sense of [S3, II]. We also prove the existence of the split elements in the case where $G = SL_n$ and F is of nonsplit type, for an arbitrary $p > 0$. We determine the scalars γ_j in those cases. Thus, the generalized Green functions can be computed in those cases. Since the determination of generalized Green functions is reduced to the case of simply connected, simple groups, our results imply that all the generalized Green functions are computable for any reductive group of classical type.

The main ingredients for the proof is the Frobenius version of the restriction theorem (1.8.1) proved in [S3, II]. This is especially useful in the case of SL_n of nonsplit type. In [S3, I], the existence of split elements was proved by the aid of the graded Hecke algebras, but in that case, we need to assume that p is large enough for applying the Jacobson–Morozov theorem in $p > 0$. The present proof using the restriction theorem is much simpler, and works well for any p.

§1. Generalized Green functions

1.1. Let G be a connected reductive group defined over a finite field $\mathbf{F}_q$ with Frobenius map F. Let $\mathbf{k}$ be the algebraic closure of $\mathbf{F}_q$ with ch $\mathbf{k} = p$. Let $\mathscr{M}_G$ be the set of triples $(L, C_0, \mathscr{E}_0)$, up to G-conjugacy, where L is a Levi subgroup of a parabolic subgroup of G, and $\mathscr{E}_0$ is a cuspidal local system on a unipotent class C_0 of L. As in (1.2.2) in [S3, I], one can define a semisimple perverse sheaf K on G associated to a triple $(L, C_0, \mathscr{E}_0) \in \mathscr{M}_G$. Let $\mathscr{W}_L = N_G(L)/L$. It is known that $\mathscr{W}_L$ is a Coxeter group, and $\mathrm{End}\, K \simeq \bar{\mathbf{Q}}_l[\mathscr{W}_L]$, K is decomposed as

$$K \simeq \bigoplus_{E \in \mathscr{W}_L^\wedge} E \otimes K_E, \tag{1.1.1}$$

where $K_E \simeq \mathrm{Hom}(E, K)$ is a simple perverse sheaf associated to $E \in \mathscr{W}_L^\wedge$.

Let G_{uni} be the unipotent variety of G, and let $\mathscr{N}_G$ the set of $(C, \mathscr{E})$, where C is a unipotent class of G, and $\mathscr{E}$ is an irreducible local system on C. Let Z_L be the center of L, and put $d = \dim Z_L$. It is known that $K[-d]|_{G_{\mathrm{uni}}}$ is a semisimple perverse sheaf on G_{uni},

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equipped with $\mathscr{W}_L$-action, and is decomposed as

$$K[-d]|_{G_{\mathrm{uni}}} \simeq \bigoplus_{(C,\mathscr{E})\in\mathscr{N}_G} V_{(C,\mathscr{E})}\otimes \mathrm{IC}(\overline{C},\mathscr{E})[\dim C], \tag{1.1.2}$$

where $V_{(C,\mathscr{E})}$ is an irreducible $\mathscr{W}_L$-module if it is nonzero. It follows from (1.1.1) and (1.1.2) that there exists a map $\mathscr{W}_L^\wedge \to \mathscr{N}_G$, $E\mapsto(C,\mathscr{E})$ such that

$$K_E|_{G_{\mathrm{uni}}} \simeq \mathrm{IC}(\overline{C},\mathscr{E})[\dim C + \dim Z_L] \tag{1.1.3}$$

and that $V_{(C,\mathscr{E})}\simeq E$. The generalized Springer correspondence shows that this map gives a bijection

$$\bigsqcup_{(L,C_0,\mathscr{E}_0)\in\mathscr{M}_G} \mathscr{W}_L^\wedge \to \mathscr{N}_G. \tag{1.1.4}$$

1.2. F acts naturally on the set $\mathscr{N}_G$ and $\mathscr{M}_G$ by $(C,\mathscr{E})\mapsto(F^{-1}(C),F^*\mathscr{E})$, $(L,C_0,\mathscr{E}_0)\mapsto(F^{-1}(L),F^{-1}(C_0),F^*\mathscr{E}_0)$. The map in (1.1.4) is compatible with this F-action. We choose $(L,C_0,\mathscr{E}_0)\in\mathscr{M}_G^F$. Here, we may assume that L is an F-stable Levi subgroup of an F-stable parabolic subgroup P of G, and that $F(C_0)=C_0, F^*\mathscr{E}_0\simeq\mathscr{E}_0$. Since $\mathscr{E}_0$ is a simple local system, the isomorphism $\varphi_0: F^*\mathscr{E}_0\xrightarrow{\sim}\mathscr{E}_0$ is unique up to scalar. We choose φ_0 by the condition that the induced map on the stalk of $\mathscr{E}_0$ at any point in C_0^F is of finite order. φ_0 induces a natural isomorphism $\varphi: F^*K\xrightarrow{\sim}K$. We consider the characteristic function $\chi_{K,\varphi}$ of K. The restriction of $\chi_{K,\varphi}$ on G^F_{uni} is the so-called generalized Green function $Q^G_{L,C_0,\mathscr{E}_0,\varphi_0}$ (cf. [L2, II]), which is a G^F-invariant function on G^F_{uni} with values in $\bar{\mathbf{Q}}_l$.

1.3. Let $(L,C_0,\mathscr{E}_0)\in\mathscr{M}_G^F$ be as in 1.2. Then, F acts naturally on $\mathscr{W}_L$, which induces a Coxeter group automorphism σ of order c. We consider the semidirect product $\widetilde{\mathscr{W}_L}=\langle\sigma\rangle\ltimes\mathscr{W}_L$. If an irreducible representation E of $\mathscr{W}_L$ is F-stable, namely, $F(E)\simeq E$ as $\mathscr{W}_L$-modules, E can be extended to a $\widetilde{\mathscr{W}_L}$-module in c different ways. For each $E\in(\mathscr{W}_L^\wedge)^F$, we choose an irreducible representation $\widetilde{E}$ of $\widetilde{\mathscr{W}_L}$, which is the preferred extension of E in the sense of [L2, IV, (17.2)]. Let $\sigma_E: E\xrightarrow{\sim}E$ be the action of σ on $\widetilde{E}$. If E is F-stable, then $E\otimes K_E$ is F-stable in the decomposition of K in (1.1.1), and there exists a unique isomorphism $\varphi_E: F^*K_E\xrightarrow{\sim}K_E$ such that $\sigma_E\otimes\varphi_E: F^*(E\otimes K_E)\xrightarrow{\sim}E\otimes K_E$ is the restriction of $\varphi: F^*K\xrightarrow{\sim}K$.

Let $(C,\mathscr{E})\in\mathscr{N}_G^F$ be any pair. By the generalized Springer correspondence, there exists a unique triple $(L,C_0,\mathscr{E}_0)\in\mathscr{M}_G^F$ such that $(C,\mathscr{E})$ corresponds to $E\in(\mathscr{W}_L^\wedge)^F$ under (1.1.4). Put

$$\begin{aligned} a_0 &= -\dim Z_L - \dim C,\\ r &= \dim G - \dim L + \dim(C_0\times Z_L). \end{aligned} \tag{1.3.1}$$

Then, we have

$$a_0 + r = (\dim G - \dim C) - (\dim L - \dim C_0). \tag{1.3.2}$$

By (1.1.3), $\mathscr{H}^{a_0}K_E|_C=\mathscr{E}$. We define an isomorphism $\psi: F^*\mathscr{E}\xrightarrow{\sim}\mathscr{E}$, so that $q^{(a_0+r)/2}\psi$ corresponds to the map $F^*(\mathscr{H}^{a_0}K_E)\xrightarrow{\sim}\mathscr{H}^{a_0}K_E$ induced from φ_E. For each $j=(C,\mathscr{E})\in$

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$\mathscr{N}_G^F$, we define a function Y_j on G^F_{uni} by

$$Y_j(g) = \begin{cases} \mathrm{Tr}\,(\psi, \mathscr{E}_g) & \text{if } g \in C^F, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.3}$$

1.4. The set $\{Y_j \mid j \in \mathscr{N}_G^F\}$ gives a basis of the $\bar{\mathbf{Q}}_l$-space of G^F-invariant functions of G^F_{uni}. In [L2, V], Lusztig gave a general algorithm of computing the generalized Green function as an explicit linear combination of various Y_j. Thus, the determination of the generalized Green functions is reduced to the determination of Y_j. Since $\mathscr{E}$ is a simple local system, the isomorphism $\psi : F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ is unique up to scalar. The typical isomorphism $\psi_0 : F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ among them is given as follows: choose $u \in C^F$, and put $A_G(u) = Z_G(u)/Z_G^0(u)$. Then, F acts naturally on $A_G(u)$, and the set of G-equivariant simple local systems $\mathscr{E}'$ on C such that $F^*\mathscr{E}' \simeq \mathscr{E}'$ is in bijection with the set of F-stable irreducible representations of $A_G(u)$. The correspondence is given, so that the stalk $\mathscr{E}'_u$ has the structure of the dual of the corresponding irreducible $A_G(u)$-module. Let ρ be the F-stable irreducible representation of $A_G(u)$ corresponding to $\mathscr{E}$. Let τ be the restriction of F on $A_G(u)$, and consider the semidirect product $\widetilde{A}_G(u) = \langle\tau\rangle \ltimes A_G(u)$. Then, ρ can be extended to an irreducible representation of $\widetilde{A}_G(u)$. We choose an extension $\widetilde{\rho}$ of ρ. Here, we consider $\mathscr{E}'_u$ as an irreducible $\widetilde{A}_G(u)$-module corresponding to $\widetilde{\rho}^*$, where $\widetilde{\rho}^*$ is the dual of $\widetilde{\rho}$.

We define $\psi_0 : F^*\mathscr{E} \xrightarrow{\sim} \mathscr{E}$ by the condition that the induced map on $\mathscr{E}_u$ coincides with the action of τ, and define a function $Y_j^0 : G^F{}_{\mathrm{uni}} \to \bar{\mathbf{Q}}_l$ in a similar way as Y_j, but by replacing ψ by ψ_0. Note that the set of G^F-conjugacy classes in C^F is in bijection with the set of $A_G(u)$-conjugacy classes in the coset $A_G(u)\tau \subset \widetilde{A}_G(u)$. We denote by u_a the G^F-class corresponding to the class $a\tau \in A_G(u)\tau$. Then, Y_j^0 is explicitly given as follows:

$$Y_j^0(g) = \begin{cases} \mathrm{Tr}\,(a\tau, \widetilde{\rho}^*) & \text{if } g \text{ is } G^F\text{-conjugate to } u_a, \\ 0 & \text{if } g \notin C^F. \end{cases} \tag{1.4.1}$$

Since ψ can be written as $\psi = \gamma\psi_0$ for some $\gamma \in \bar{\mathbf{Q}}_l^*$, we have $Y_j = \gamma Y_j^0$. Thus, the determination of Y_j is reduced to that of γ. We choose $u_0 \in C_0^F$, and consider $\widetilde{A}_L(u_0) = \langle\tau_0\rangle \ltimes A_L(u_0)$. Let ρ_0 be the irreducible representation of $A_L(u_0)$ corresponding to $\mathscr{E}_0$, and let $\widetilde{\rho}_0$ be its extension to $\widetilde{A}_L(u_0)$. We can define $\varphi_0 : F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$ by the condition that the map on $(\mathscr{E}_0)_{u_0}$ induced from φ_0 coincides with the action of τ_0 on $\widetilde{\rho}_0$. Thus, γ depends on the choice of $u_0, \widetilde{\rho}_0, u, \widetilde{\rho}$, and we denote it by $\gamma = \gamma(u_0, \widetilde{\rho}_0, u, \widetilde{\rho})$. In this paper, we shall prove the following result.

THEOREM 1.5. *Assume that G is a connected reductive group of classical type. For each F-stable unipotent class C of G, there exists $u^\bullet \in C^F$ satisfying the following property: for a triple $(L, C_0, \mathscr{E}_0) \in \mathscr{M}_G^F$, take $u_0^\bullet \in C_0^F$ as above (applied for L), and choose an extension $\widetilde{\rho}_0 \in \widetilde{A}_L(u_0^\bullet)^\wedge$, where ρ_0 corresponds to $\mathscr{E}_0$. For $(C, \mathscr{E}) \in \mathscr{N}_G^F$, take $u^\bullet \in C^F$, and let $\rho \in A_G(u^\bullet)^\wedge$ be such that $(C, \mathscr{E}) \leftrightarrow (u^\bullet, \rho)$. Then, for $(C, \mathscr{E})$ belonging to the series $(L, C_0, \mathscr{E}_0)$, there exists an extension $\widetilde{\rho}$ of ρ such that $\gamma(u_0^\bullet, \widetilde{\rho}_0, u^\bullet, \widetilde{\rho}) = 1$.*

In particular, for such a choice of $\varphi_0 : F^*\mathscr{E}_0 \xrightarrow{\sim} \mathscr{E}_0$, Y_j coincides with Y_j^0. Thus, the generalized Green functions $Q^G_{L, C_0, \mathscr{E}_0, \varphi_0}$ are explicitly computable for G.

1.6. $u^\bullet \in C^F$ in the theorem is called a split element, and the G^F-conjugacy class containing $u^\bullet$ is called the split class in C^F. Note that the split class is independent from

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the choice of $(L,C_0,\mathscr{E}_0)$. Also note that the split class is compatible with the isogeny. (In the case where G is an adjoint group, the split class is uniquely determined for a given C^F. In general, they are not unique; for example, see 3.2 in the case of spin groups, and 4.7 for special linear groups.)

The proof of the theorem is reduced to the case where G is simply connected, and simple. The existence of split elements (and the existence of $\widetilde{\rho}$ for the nonsplit case) was proved in [S3, I] for the case where (G,F) is SL_n of split type for any $p>0$, or SL_n of nonsplit type for sufficiently large p. In [S3, II], it was proved for $G=Sp_N$ or SO_N for any $p>0$. Thus, it remains to verify it for the case where G is the spin group Spin_N, or SL_n of nonsplit type for arbitrary p. In this paper, we discuss these two cases.

1.7. Let $(L,C_0,\mathscr{E}_0)\in\mathscr{M}_G$, and let P be a parabolic subgroup of G whose Levi subgroup is L. We consider another parabolic subgroup $Q\supset P$ of G with the Levi subgroup $M\supset L$. Then, $\mathscr{W}'_L=N_M(L)/L$ is in a natural way a subgroup of $\mathscr{W}_L$. Let $u\in G, u'\in M$ be unipotent elements. We put

$$Y_{u,u'}=\{x\in G\mid x^{-1}ux\in u'U_Q\}. \tag{1.7.1}$$

Then, $Z_G(u)\times Z_M(u')U_Q$ acts on $Y_{u,u'}$ by $(g,m):x\mapsto gxm^{-1}$ $(g\in Z_G(u), m\in Z_M(u')U_Q, x\in Y_{u,u'})$. Put

$$d_{u,u'}=\frac{1}{2}(\dim Z_G(u)+\dim Z_M(u'))+\dim U_Q. \tag{1.7.2}$$

It is known that $\dim Y_{u,u'}\le d_{u,u'}$. We denote by $X_{u,u'}$ the set of irreducible components of $Y_{u,u'}$ of dimension $d_{u,u'}$. Then, $A_G(u)\times A_M(u')$ acts on $X_{u,u'}$. We denote by $\varepsilon_{u,u'}$ the corresponding permutation representation of $A_G(u)\times A_M(u')$.

We denote by θ^G the inverse map of (1.1.4), that is,

$$\theta^G:\mathscr{N}_G\to\bigsqcup_{(L,C_0,\mathscr{E}_0)\in\mathscr{M}_G}\mathscr{W}_L^\wedge. \tag{1.7.3}$$

For each pair $(C,\mathscr{E})\in\mathscr{N}_G$, we choose (u,ρ), where $u\in C$, and $\rho\in A_G(u)^\wedge$ the irreducible representation corresponding to $\mathscr{E}$. We often write it as $(u,\rho)\leftrightarrow(C,\mathscr{E})$. We denote by $\theta^G_{u,\rho}\in\mathscr{W}_L^\wedge$ the representation E corresponding to $(C,\mathscr{E})$ under θ^G. The generalized Springer correspondence θ^M for M is defined similarly to (1.7.3). For each $(C',\mathscr{E}')\in\mathscr{N}_M$, we choose $(u',\rho)\leftrightarrow(C',\mathscr{E}')$. Then, $\theta^M_{u',\rho'}\in(\mathscr{W}'_L)^\wedge$. The restriction theorem [LS, 0.4, (4)] shows that

$$\langle\rho\otimes\rho'^*,\varepsilon_{u,u'}\rangle_{A_G(u)\times A_M(u')}=\langle\theta^G_{u,\rho}|_{\mathscr{W}'_L},\theta^M_{u',\rho'}\rangle_{\mathscr{W}'_L} \tag{1.7.4}$$

if (u,ρ) and (u',ρ') belong to the same class $(L,C_0,\mathscr{E}_0)$, where ρ'^* is the dual representation of ρ'. The left-hand side is equal to zero if (u,ρ) and (u',ρ') belong to a different class. Moreover, every irreducible representation of $A_G(u)\times A_M(u')$ which occurs in $\varepsilon_{u,u'}$ is obtained in this way.

1.8. We assume that $L\subset P, M\subset Q$ are both F-stable. We consider $Y_{u,u'}$ in (1.7.1), and assume that u,u' are F-stable. Then, F acts naturally on $Y_{u,u'}$, and acts on $X_{u,u'}$ as a permutation of irreducible components. We have an action of F on $\varepsilon_{u,u'}$. Put $A(u,u')=A_G(u)\times A_M(u')$, then F acts diagonally on $A(u,u')$. We define $\widetilde{A}(u,u')=\langle\tau_*\rangle\ltimes A(u,u')$, where $\langle\tau_*\rangle$ is the infinite cyclic group generated by τ_*, and τ_* acts on $A(u,u')$ via F. Now, $\varepsilon_{u,u'}$ turns out to be an $\widetilde{A}(u,u')$-module, which we denote by $\widetilde{\varepsilon}_{u,u'}$. In turn, if we choose

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extensions $\widetilde{\rho},\widetilde{\rho}'$ of ρ,ρ', respectively, they define an extension $\widetilde{\rho\otimes\rho'^*}$ of $\rho\otimes\rho'^*$ to irreducible $\widetilde{A}(u,u')$-module, where τ_* acts on $\rho\otimes\rho'^*$ as $\tau\otimes\tau'$ for $\widetilde{A}_G(u)=\langle\tau\rangle\ltimes A_G(u), \widetilde{A}_M(u')=\langle\tau'\rangle\ltimes A_M(u')$.

On the other hand, for each $(C,\mathscr{E})\in\mathscr{N}_G^F$ corresponding to (u,ρ), we consider the map $\psi_0: F^*\mathscr{E}\xrightarrow{\sim}\mathscr{E}$ constructed in 1.4, which we denote by $\psi_{u,\rho}$. $\psi_{u,\rho}$ can be extended to $\widetilde{\psi}_{u,\rho}: F^*\mathrm{IC}(\overline{C},\mathscr{E})\xrightarrow{\sim}\mathrm{IC}(\overline{C},\mathscr{E})$. By comparing this with the isomorphism $\varphi: F^*K\xrightarrow{\sim}K$ given in 1.2, we obtain an isomorphism $\sigma_{(u,\rho)}: V_{(u,\rho)}\xrightarrow{\sim}V_{(u,\rho)}$ such that the restriction of φ on $V_{(u,\rho)}\otimes F^*\mathrm{IC}(\overline{C},\mathscr{E})[\dim C]$ coincides with $\sigma_{(u,\rho)}\otimes\widetilde{\psi}_{(u,\rho)}$. (Here, we write $V_{(C,\mathscr{E})}$ as $V_{(u,\rho)}$.) Applying this to the reductive group M, and $(C',\mathscr{E}')\in\mathscr{N}_M^F$, we obtain the map $\sigma_{(u',\rho')}: V_{(u',\rho')}\xrightarrow{\sim}V_{(u',\rho')}$, where $(u',\rho')\leftrightarrow(C',\mathscr{E}')$. Here, $V_{(u,\rho)}$ is an irreducible $\mathscr{W}_L$-module, and $V_{(u',\rho')}$ is an irreducible $\mathscr{W}_L'$-module. We can decompose $V_{(u,\rho)}$ as

$$V_{(u,\rho)}=\bigoplus_{V_{(u',\rho')}\in(\mathscr{W}_L')^\wedge} M_{\rho,\rho'}\otimes V_{(u',\rho')},$$

where $M_{\rho,\rho'}=\mathrm{Hom}_{\mathscr{W}_L'}(V_{(u',\rho')}, V_{(u,\rho)}|_{\mathscr{W}_L'})$ is the multiplicity space for $V_{(u',\rho')}$. Thus, we can define an isomorphism $\sigma_{\rho,\rho'}: M_{\rho,\rho'}\xrightarrow{\sim}M_{\rho,\rho'}$ such that the restriction of $\sigma_{(u,\rho)}$ on $M_{\rho,\rho'}\otimes V_{(u',\rho')}$ coincides with $\sigma_{\rho,\rho'}\otimes\sigma_{(u',\rho')}$.

A variant of the restriction theorem (1.7.4) was proved in [S3, II, Cor. 1.9]. Under the notation above, the following formula holds.

$$\mathrm{Tr}\,(\sigma_{\rho,\rho'}, M_{\rho,\rho'})=q^{-(\dim C-\dim C')/2+\dim U_Q}\langle\widetilde{\varepsilon}_{u,u'}, \widetilde{\rho\otimes\rho'^*}\rangle_{A(u,u')\tau_*}, \tag{1.8.1}$$

where for representations V_1,V_2 of $\widetilde{A}(u,u')$, $\langle V_1,V_2\rangle_{A(u,u')\tau_*}$ is defined as

$$\langle V_1,V_2\rangle_{A(u,u')\tau_*}=|A(u,u')|^{-1}\sum_{a\in A(u,u')}\mathrm{Tr}\,(a\tau_*,V_1)\,\mathrm{Tr}\,((a\tau_*)^{-1},V_2).$$

We note that the following fact [S3, II, Lem. 1.11] is useful for the determination of γ, combined with (1.8.1).

(1.8.2) Suppose that $q^{-(a_0+r)/2}\sigma_{(u,\rho)}$ makes $\mathscr{W}_L$-module $V_{(u,\rho)}$ to the preferred extension to $\widetilde{\mathscr{W}_L}$. Then, we have $\gamma=1$. In particular, in the case where F acts trivially on $\mathscr{W}_L$, if $\sigma_{(u,\rho)}$ is $q^{(a_0+r)/2}$ times identity, then $\gamma=1$.

1.9. Let Z_G be the center of G. For each $\chi\in Z_G^\wedge$, we denote by $\mathscr{N}_\chi$ the set of $(C,\mathscr{E})\in\mathscr{N}_G$ such that Z_G acts on $\mathscr{E}$ according to χ. We denote by $A_G(u)_\chi^\wedge$ the set of $\rho\in A_G(u)^\wedge$ such that Z_G acts on ρ according to χ. If $(C,\mathscr{E})\leftrightarrow(u,\rho)$, then $(C,\mathscr{E})\in\mathscr{N}_\chi$ if and only if $\rho\in A_G(u)_\chi^\wedge$. Moreover, we denote by $\mathscr{M}_\chi$ the set of $(L,C_0,\mathscr{E}_0)\in\mathscr{M}_G$ such that Z_G acts on $\mathscr{E}_0$ according to χ. We have a partition

$$\mathscr{N}_G=\bigsqcup_{\chi\in Z_G^\wedge}\mathscr{N}_\chi,\quad A_G(u)^\wedge=\bigsqcup_{\chi\in Z_G^\wedge}A_G(u)_\chi^\wedge,\quad \mathscr{M}_G=\bigsqcup_{\chi\in Z_G^\wedge}\mathscr{M}_\chi.$$

Then, the generalized Springer correspondence θ^G is compatible with the action of Z_G, namely, for each $\chi\in Z_G^\wedge$, θ^G restricts to a bijection

$$\theta^\chi:\mathscr{N}_\chi\to\bigsqcup_{(L,C_0,\mathscr{E}_0)\in\mathscr{M}_\chi}\mathscr{W}_L^\wedge. \tag{1.9.1}$$

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§2. Preliminaries on spin groups

2.1. First, we review the definition of spin groups. Let $\mathbf{k}$ be an algebraically closed field with $\mathrm{ch}\,\mathbf{k} \neq 2$. Let V be a vector space over $\mathbf{k}$ with dimension $N \geq 3$, endowed with a nondegenerate symmetric bilinear form $(\ ,\ )$ on V. Let $C(V)$ be the Clifford algebra associated to the form $(\ ,\ )$. It is provided with an embedding $V \subset C(V)$, and the product $v \cdot v'$ for $v, v' \in V$ satisfies the relation $v \cdot v' + v' \cdot v = 2(v, v')$. Let $C^+(V)$ (resp. $C^-(V)$) be the subspace of $C(V)$ spanned by products of an even number (resp. an odd number) of elements in V. Then, $C(V) = C^+(V) \oplus C^-(V)$, and $C^+(V)$ is the subalgebra of $C(V)$. The spin group $\mathrm{Spin}(V)$ is the subgroup of the units of $C^+(V)$ consisting of all products $v_1 \cdots v_a$ with a : even, where $v_i \in V$ satisfy $(v_i, v_i) = 1$. This is a closed subgroup of units in $C^+(V)$. If $x \in \mathrm{Spin}(V)$, the map $v \mapsto xvx^{-1}$ leaves V invariant, and defines an element $\beta(x)$ of $SO(V)$; if $x = v_1 \cdots v_a$ as above, then $\beta(x) = \beta(v_1) \cdots \beta(v_a)$, where $\beta(v_i)(v) = -v + 2(v, v_i)v_i$ is (-1) times the reflection with respect to v_i. Thus, we have a homomorphism $\beta : \mathrm{Spin}(V) \to SO(V)$. If we fix a basis, and identify $SO(V)$ with SO_N, then we also write $\mathrm{Spin}(V)$ as Spin_N. β is the simply connected covering of $SO(V)$. We denote $\mathrm{Spin}(V)$ as G and $SO(V)$ as $\overline{G}$.

Let Z be the center of G. If N is odd, and Z has order 2, it is generated by $\varepsilon = (-1)$ times the unit element in $C(V)$. If N is even, and Z has order 4, it is generated by ε and $\omega = v_1 \cdots v_N$, where $v_1, \ldots, v_N$ are orthonormal basis of V. We have $\omega^2 = \varepsilon^{N/2}$. Hence, Z is cyclic of order 4 generated by ω if $N \equiv 2 \pmod 4$, and $Z \simeq \langle\omega\rangle \times \langle\varepsilon\rangle$ is a product of two cyclic groups of order 2 if $N \equiv 0 \pmod 4$. Put $Z_0 = \mathrm{Ker}\,\beta$. Then, $Z_0 = \{1, \varepsilon\}$ is of order 2.

2.2. We consider the $\mathbf{F}_q$-structure of G. Assume that V is defined over $\mathbf{F}_q$ with Frobenius map $F : V \to V$, and that the form $(\ ,\ )$ is compatible with F-action, namely, $F(v, w) = (F(v), F(w))$. Then, $C(V)$ has a natural $\mathbf{F}_q$-structure, which induces an $\mathbf{F}_q$-structure of G. We have the corresponding Frobenius map $F : G \to G$. $\overline{G}$ has also a natural $\mathbf{F}_q$-structure with Frobenius map F, and the map $\beta : G \to \overline{G}$ is F-equivariant. Here, F acts trivially on $Z_0 = \mathrm{Ker}\,\beta$. Thus, F acts trivially on Z if N is odd. Assume that $N \equiv 0 \pmod 4$. The quadratic form on V associated to $(\ ,\ )$ is given for $x = \sum_i x_i e_i$

$$(x, x) = \begin{cases} x_1^2 + \cdots + x_N^2 & \text{split case,} \\ x_1^2 + \cdots + x_{N-1}^2 + \delta x_N^2 & \text{nonsplit case,} \end{cases}$$

where $e_1, \ldots, e_N$ is a basis of V such that $F(e_i) = e_i$, and $\delta \in \mathbf{F}_q - \mathbf{F}_q^2$. Thus, in the split case, $v_i = e_i$ gives an orthonormal basis of V, stable by F. This implies that $\omega = v_1 \cdots v_N$ is F-stable. In the nonsplit case, $v_1, \ldots, v_N$ gives an orthonormal basis of V, where $v_i = e_i$ for $i = 1, \ldots, N-1$, and $v_N = \delta^{1/2} e_N$ with $\delta^{1/2} \notin \mathbf{F}_q$. Here, we have $F(\delta^{1/2}) = -\delta^{1/2}$, and so $F(v_N) = -v_N$. This implies that $F(\omega) = -\omega$. Summing up the above argument, we have:

(2.2.1) Assume that N is odd, or $N \equiv 2 \pmod 4$. Then, F acts trivially on Z. Assume that $N \equiv 0 \pmod 4$. If (G, F) is of split type, then F acts trivially on Z, while if (G, F) is of nonsplit type, then $F(\omega) = -\omega$.

2.3. Under the generalized Springer correspondence θ^χ in (1.9.1), if $\chi|_{Z_0} = 1$, θ^χ is essentially the same as the generalized Springer correspondence for SO_N. Now, assume that $\chi|_{Z_0} \neq 1$, namely that $\chi(\varepsilon) = -1$. It is known by [L1, 14] that $\mathscr{M}_\chi$ is in bijection with integers $d \in \mathbf{Z}$ such that $d \equiv N \pmod 4$ and $d(2d-1) < N$. For such d, there exists a unique triple $(L, C_0, \mathscr{E}_0) \in \mathscr{M}_\chi$ with L of type $B_{(d-1)(2d+1)/2} + A_1 + A_1 + \cdots$ if N is odd,

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and of type $D_{d(2d-1)/2}+A_1+A_1+\cdots$ if N is even, where the number of factors of A_1 is $(N-d(2d-1))/4$. In those cases, $\mathscr{W}_L \simeq W_{(N-d(2d-1))/4}$, where W_n is the Weyl group of type B_n. Thus, θ^χ gives a bijection

$$\theta^\chi : \mathscr{N}_\chi \xrightarrow{\sim} \bigsqcup_{\substack{d\in\mathbf{Z}\\ 4|N-d}} W^\wedge_{(N-d(2d-1))/4}. \tag{2.3.1}$$

2.4. $\beta : G \to \overline{G}$ gives a bijection $G_{\mathrm{uni}} \xrightarrow{\sim} \overline{G}_{\mathrm{uni}}$, and induces a bijection between the unipotent classes in G and those in $\overline{G}$, which is compatible with F-action. For $u \in G_{\mathrm{uni}}$, $\beta(u)$ determines a partition of N, by taking the Jordan block of $\beta(u)$, which we denote by $\lambda(u)$. Here, we write a partition λ of N as $\lambda=(\lambda_1,\dots,\lambda_k)$ with $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_k$ with $\sum_{i=1}^k \lambda_i = N$. We also write λ as $\lambda=(1^{m_1},2^{m_2},\dots)$. Note that the set of unipotent classes in O_N are in bijection with the set $\widetilde{X}_N$ of partitions λ of N such that m_i is even for even i, by considering their Jordan types. If all the m_i are even, the class in O_N splits into two classes in SO_N, otherwise any class C in O_N gives a single SO_N-class $C\cap SO_N$.

Put

$$X_N = \{\lambda \in \widetilde{X}_N \mid m_i \le 1 \text{ for odd } i\}. \tag{2.4.1}$$

For an integer $m \in \mathbf{Z}$, put

$$d(m) = \begin{cases} 0 & \text{if } m : \text{even}\ , \\ 1 & \text{if } m \equiv 1 \pmod 4, \\ -1 & \text{if } m \equiv -1 \pmod 4. \end{cases} \tag{2.4.2}$$

For $\lambda \in X_N$, we define an integer $d(\lambda)$ by $d(\lambda)=\sum_i d(\lambda_i)$. The following result was proved in [L1, Prop. 14.4] and [LS, Cor. 4.10].

PROPOSITION 2.5. *Assume that $\chi \in Z^\wedge$ is such that $\chi(\varepsilon)=-1$. Then, there exists a bijection between $\mathscr{N}_\chi$ and X_N with the following properties:*

(i) *Let $(C,\mathscr{E}) \in \mathscr{N}_\chi$ with $(u,\rho) \leftrightarrow (C,\mathscr{E})$. Then, $\lambda(u) \in X_N$. Under the correspondence (2.3.1), $(C,\mathscr{E})$ is mapped to the factor of d, where $d=d(\lambda)$ for $\lambda=\lambda(u)$.*

(ii) *Let $I=\{1\le i\le k \mid \lambda_i : \text{odd}\}$. Define an integer r by*

$$r = \begin{cases} (|I|-1)/2 & \text{if } |I| \text{ is odd}, \\ (|I|-2)/2 & \text{if } |I| \text{ is even } >0, \\ 0 & \text{if } |I|=0. \end{cases}$$

Then, ρ is a unique irreducible representation of $A_G(u)$ of dimension 2^r such that Z acts on ρ by χ. (More precisely, there exists a unique irreducible representation of dimension 2^r on which ε acts as -1 if $|I|$ is odd, and there exists exactly two irreducible representations of dimension 2^r on which ε acts as -1 if $|I|$ is even.)

2.6. Take $(u,\rho) \leftrightarrow (C,\mathscr{E}) \in \mathscr{N}_\chi$ as in Proposition 2.5. Following [L1, 14.2], we shall determine the structure of $A_G(u)$, and construct the representation $\rho \in A_G(u)^\wedge$ explicitly. Put $\beta(u)=\overline{u} \in \overline{G}$. Then, we have $\beta^{-1}(Z_{\overline{G}}(\overline{u}))=Z_G(u)$. β induces a natural surjective map $A_G(u) \to A_{\overline{G}}(\overline{u})$. In the case where $\lambda \in X_N$, $\varepsilon \notin Z_G^0(u)$, and $\beta^{-1}(Z_{\overline{G}}(\overline{u}))$ consists of two connected components. Thus, $A_G(u)$ is a central extension of $A_{\overline{G}}(\overline{u})$ with a kernel of order 2.

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This central extension can be described as follows. Let I be the set as in Proposition 2.5. We can write $V=\bigoplus_{i\in I}V_i\oplus V'$, the orthogonal direct sum decomposition into $\overline{u}$-stable subspaces, where $\dim V_i=\lambda_i$ for $i\in I$. For each $i\in I$, we consider an orthonormal basis $v_1^i,\dots,v_h^i$ of V_i (here we put $h=\lambda_i$). Put $x_i=v_1^iv_2^i\cdots v_h^i\in C(V)$ (note that h is odd). Those x_i satisfy the relation

$$x_i^2=\varepsilon^{\lambda_i(\lambda_i-1)/2},\quad x_ix_{i'}=\varepsilon x_{i'}x_i. \tag{2.6.1}$$

In fact, it follows from the defining relations that $(v_a^i)^2=(v_a^i,v_a^i)=1$, $v_a^iv_{a'}^i+v_{a'}^iv_a^i=2(v_a^i,v_{a'}^i)=0$, namely, $v_a^iv_{a'}^i=-v_{a'}^iv_a^i$ for $a\neq a'$. Then, we have, for $i\neq i'$,

$$\begin{aligned}x_i^2&=(-1)^{h-1}(v_2^i\dots v_h^i)(v_2^i\dots v_h^i)=\cdots=(-1)^{h(h-1)/2},\\ x_ix_{i'}&=(-1)^hv_1^{i'}(v_1^i\cdots v_h^i)(v_2^{i'}\cdots v_{h'}^{i'})=\cdots=(-1)^{hh'}(v_1^{i'}\cdots v_{h'}^{i'})(v_1^i\cdots v_h^i),\end{aligned} \tag{2.6.2}$$

where $h'=\lambda_{i'}$. Since h,h' are odd, we obtain (2.6.1).

Let $\widehat{\Gamma}$ be the subgroup of the group of units of $C(V)$ generated by x_i $(i\in I)$. Let Γ be the subgroup of $\widehat{\Gamma}$ consisting of elements which are products of an even number of generators x_i. Then, $\Gamma\subset Z_G(u)$, and the map $\Gamma\to Z_G(u)/Z_G^0(u)$ gives an isomorphism. The central extension $Z_G(u)/Z_G^0(u)\to Z_{\overline{G}}(\overline{u})/Z_{\overline{G}}^0(\overline{u})$ can be described by $\Gamma\to\Gamma/\{1,\varepsilon\}$. (Note that $\varepsilon=x_ix_{i'}(x_{i'}x_i)^{-1}\in\Gamma$.)

Let $C(V_I)$ be the Clifford algebra over $\bar{\mathbf{Q}}_l$ associated to the vector space $V_I\simeq\bar{\mathbf{Q}}_l^{|I|}$ and the quadratic form $\sum_{i\in I}(-1)^{\lambda_i(\lambda_i-1)/2}X_i^2$ with variables $X_i(i\in I)$, and $C^+(V_I)$ its $+$-part. Let $\bar{\mathbf{Q}}_l[\Gamma]$ be the group algebra of Γ and $\bar{\mathbf{Q}}_l[\Gamma]/\langle 1+\varepsilon\rangle$ the quotient algebra of $\bar{\mathbf{Q}}_l[\Gamma]$ by the two-sided ideal generated by $1+\varepsilon$. Then, $\bar{\mathbf{Q}}_l[\Gamma]/\langle 1+\varepsilon\rangle\simeq C^+(V_I)$. It follows that there exists an algebra homomorphism $f:\bar{\mathbf{Q}}_l[\Gamma]\to C^+(V_I)$. It is known (cf. [L1, 14.3]) that, $C^+(V_I)$ is a simple algebra in the case where $|I|$ is odd with $\dim C^+(V_I)=2^{|I|-1}$, and is the direct sum of two simple components of the same dimension in the case where $|I|$ is even with $\dim C^+(V_I)=2^{|I|-2}$. Thus, if $|I|$ is odd, f gives rise to an irreducible representation $\rho:\Gamma\to GL(V_\rho)$, where $C^+(V_I)=\mathrm{End}(V_\rho)$, and if $|I|$ is even, it gives rise to two irreducible representations ρ,ρ' such that $C^+(V_I)\simeq\mathrm{End}(V_\rho)\oplus\mathrm{End}(V_{\rho'})$. Here, $\dim\rho=2^{(|I|-1)/2}$ if $|I|$ is odd, and $\dim\rho=\dim\rho'=2^{(|I|-2)/2}$ if $|I|$ is even.

Remark 2.7. We take $\chi\in Z^\wedge$ such that $\chi(\varepsilon)=1$. Note that in this case χ is F-stable. As mentioned in 2.3, the generalized Springer correspondence θ^χ is essentially the same as the case of SO_N. By [S3, II], we already know that Theorem 1.5 holds for SO_N. Thus, for $(C,\mathscr{E})\in\mathscr{N}_\chi^F$, there exists a split element $v^\bullet\in\beta(C)^F$. By Proposition 2.5, the Jordan type λ of $\beta(C)$ is not contained in X_N. Since β gives a bijection $C^F\xrightarrow{\sim}\beta(C)^F$, one can find $u\in C^F$ such that $\beta(u)=v^\bullet$. In this case, it is known by [L1, 14.3] that $A_G(u)\simeq A_{\overline{G}}(v^\bullet)$. This isomorphism is compatible with F; thus, $A_G(u)$ is an elementary abelian two group with trivial F-action. We define a split element as $u=u^\bullet\in C^F$, then the G^F-class of $u^\bullet$ is uniquely determined by the split class in $\beta(C)^F$. Theorem 1.5 holds for such $\chi\in Z^\wedge$. Thus, the verification of the theorem is reduced to the case where $\chi(\varepsilon)=-1$, and to the determination of the split class in C^F for C of Jordan type $\lambda\in X_N$.

Remark 2.8. Assume that F is of nonsplit type. Then, N is even, and Z is generated by ε and ω. We consider the set $\mathscr{N}_\chi$ in Proposition 2.5. Then, $\chi(\varepsilon)=-1$. Since F is nonsplit, we have $F(\omega)=-\omega$ by (2.2.1). Thus,

$$\chi(F(\omega))=\chi(-\omega)=\chi(\varepsilon)\chi(\omega)=-\chi(\omega).$$

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This shows that χ is not F-stable, and so $\mathcal{N}_\chi^F = \emptyset$. In the nonsplit case, we do not need to consider the situation as in Proposition 2.5, and so Theorem 1.5 automatically holds.

§3. Proof of Theorem 1.5—the case of spin groups

3.1. By Remark 2.8, we may consider the case where F is of split type. So hereafter we assume that F is split. Following [S3, II], we review the definition of split elements in SO_N. Assume that $\lambda \in X_N$, and write it as $\lambda = (\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k)$.

(a) Assume that λ_j is odd, and put $h = \lambda_j$. We consider the vector space V_j over $\mathbf{k}$ of $\dim V_j = h$ with basis $e_1^j, \ldots, e_h^j$. Put

$$\delta_j = \delta_j(\lambda) = (\lambda_j - 1)/2 + j. \tag{3.1.1}$$

We define a nondegenerate symmetric bilinear form f_j on V_j by

$$f_j(e_{h-a+1}^j, e_a^j) = (-1)^{\delta_j - a} \qquad (1 \le a \le h), \tag{3.1.2}$$

and $f_j = 0$ for all other pairs of bases. We define a nilpotent transformation x_j on V_j by $x_j(e_a^j) = e_{a-1}^j$ with the convention $e_0^j = 0$. Then, $x_j \in \mathfrak{g}_j$, where $\mathfrak{g}_j = \operatorname{Lie} SO(V_j, f_j)$.

(b) Assume that $\lambda_j = \lambda_{j+1}$ is even. Put $h = \lambda_j$. We consider a vector space V_j of dimension $2h = 2\lambda_j$ with basis $e_1^j, \ldots, e_h^j, e_1^{j+1}, \ldots, e_h^{j+1}$. We define a nondegenerate symmetric bilinear form f_j on V_j by

$$f_j(e_{h-a+1}^j, e_a^{j+1}) = (-1)^{a-1} \qquad (1 \le a \le h), \tag{3.1.3}$$

and $f_j = 0$ for all other pairs of bases. We define a nilpotent transformation x_j on V_j by $x_j(e_a^j) = e_{a-1}^j$, $x_j(e_a^{j+1}) = e_{a-1}^{j+1}$, with the convention $e_0^j = e_0^{j+1} = 0$. Then, $x_j \in \mathfrak{g}_j$, where $\mathfrak{g}_j = \operatorname{Lie} SO(V_j, f_j)$.

Let $V = \bigoplus_j V_j$, where the sum is taken all j such that λ_j is odd, and a half of j such that λ_j is even (note that $\sharp\{j \mid \lambda_j = h\}$ is even for even h). We define a symmetric bilinear form $f = \bigoplus_j f_j$ on V. Then, $x = \sum_j x_j \in \mathfrak{g} = \bigoplus_j \mathfrak{g}_j$, where we can regard $\mathfrak{g} = \operatorname{Lie} SO(V, f)$. If we assume that $\{e_a^j\}$ is an $\mathbf{F}_q$-basis of V, then f gives an F-invariant symmetric bilinear form on V, which we simply denote by $(\ ,\ )$, and also write $SO(V)$ as $\overline{G}$. x determines the $\overline{G}^F$-conjugacy class of unipotent elements in $\overline{G}^F$, which is nothing but the split class in $\overline{C}^F$ for the unipotent class $\overline{C}$ in $\overline{G}$ with Jordan type λ. We denote the split class by $\overline{C}^\bullet \subset \overline{C}^F$. We choose a split unipotent element $v = v^\bullet \in \overline{C}^\bullet$.

3.2. Let C be the unipotent class in G with Jordan type $\lambda \in X_N$. Then, β induces a bijection $\beta : C^F \xrightarrow{\sim} \overline{C}^F$. We choose $u \in C^F$ such that $\beta(u) = \overline{u} = v$. But in contrast to the case where $\lambda \notin X_N$ (see Remark 2.7), the set $\beta^{-1}(\overline{C}^\bullet)$ is not a single G^F-class, so the split class in $\overline{C}^F$ does not determine a unique G^F-class in C^F. We define the split classes in C^F as all the G^F-classes contained in $\beta^{-1}(\overline{C}^\bullet)$. In the following discussion, we shall show that those G^F-classes satisfy the property of the split class.

We consider the action of F on $A_G(u)$. β induces a surjective map $\widetilde{\beta} : A_G(u) \to A_{\overline{G}}(\overline{u})$ with $\operatorname{Ker} \widetilde{\beta} = \{1, \varepsilon\}$. Here, $\widetilde{\beta}$ is F-equivariant, and we know that F acts trivially on $A_{\overline{G}}(\overline{u})$. For any $x \in A_G(u)$, $\widetilde{\beta}^{-1}(\widetilde{\beta}(x)) = \{x, \varepsilon x\}$, and so $F(x) = x$ or εx. Thus, F^2 acts trivially on $A_G(u)$.

We determine the action of F on $A_G(u)$ explicitly. By the discussion in 2.6, $A_G(u)$ is constructed from the orthonormal basis for V_j with λ_j: odd. So we need to convert the basis

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in V_j to an orthonormal basis. We consider the following situation. Let M be a vector space of $\dim M = h$ (h: odd) with basis $e_1, \dots, e_h$, endowed with nondegenerate symmetric bilinear form $(e_a, e_{h-a+1}) = (-1)^{c+a}$ for some fixed integer c. Let $\zeta \in \mathbf{k}$ be such that $\zeta^2 = -1$. Thus, $F(\zeta) = \zeta$ if $q \equiv 1 \pmod 4$ and $F(\zeta) = -\zeta$ if $q \equiv -1 \pmod 4$. The following lemma can be checked by a direct computation.

LEMMA 3.3. *Put $\gamma = (-1)^{c+1}$ and $h = 2m+1$. Assume that $F : M \to M$ is a Frobenius map with $F(e_i) = e_i$.*

(i) *We define elements $v'_1, \dots, v'_h$ of M by*

$$\begin{aligned} &v'_1 = e_1 + \frac{\gamma}{2} e_h, \qquad v'_2 = e_2 - \frac{\gamma}{2} e_{h-1}, \quad v'_3 = e_3 + \frac{\gamma}{2} e_{h-2}, \dots \\ &v'_h = e_1 - \frac{\gamma}{2} e_h, \qquad v'_{h-1} = e_2 + \frac{\gamma}{2} e_{h-1}, \quad v'_{h-2} = e_3 - \frac{\gamma}{2} e_{h-2}, \dots \\ &v'_{m+1} = e_{m+1}. \end{aligned} \tag{3.3.1}$$

Then, $(v'_1, v'_1) = (v'_2, v'_2) = \cdots = (v'_m, v'_m) = 1$, $(v'_{m+1}, v'_{m+1}) = (-1)^{m+1+c}$, and $(v'_h, v'_h) = (v'_{h-1}, v'_{h-1}) = \cdots = (v'_{m+2}, v'_{m+2}) = -1$. Moreover, $F(v'_a) = v'_a$ for $1 \le a \le h$.

(ii) *Define $v_1, \dots, v_h \in M$ by $v_1 = v'_1, v_2 = v'_2, \dots, v_m = v'_m$, $v_h = \zeta v'_h, v_{h-1} = \zeta v'_{h-1}, \dots v_{m+2} = \zeta v'_{m+2}$, and $v_{m+1} = v'_{m+1}$ (resp. $\zeta v'_{m+1}$) if $m+1+c$ is even (resp. odd). Then, $v_1, \dots, v_h$ gives an orthonormal basis of M. Moreover, $F(v_i) = v_i$ for any i if $q \equiv 1 \pmod 4$. If $q \equiv -1 \pmod 4$, then $F(v_1) = v_1, \dots, F(v_m) = v_m$, $F(v_h) = -v_h, F(v_{h-1}) = -v_{h-1}, \dots, F(v_{m+2}) = -v_{m+2}$, and $F(v_{m+1}) = (-1)^{m+1+c} v_{m+1}$.*

3.4. Let V_j be as in 3.1 for $\lambda_j = h$: odd. Write $h = 2m+1$. Then, $\delta_j = m+j$. By applying Lemma 3.3 for $M = V_j$ with $c = m+j$, one can find an orthonormal basis $v_1^j, \dots, v_h^j$ of V_j. Put $x_j = v_1^j \cdots v_h^j \in C(V)$. The action of F on the basis $\{v_1^j, \dots, v_h^j\}$ is given as in the lemma. In particular, $F(x_j) = x_j$ if $q \equiv 1 \pmod 4$. While if $q \equiv -1 \pmod 4$, we have

$$F(x_j) = (-1)^{m+1+\delta_j} (-1)^m x_j = (-1)^{(\lambda_j - 1)/2 + 1 + j} x_j. \tag{3.4.1}$$

By the discussion in 2.6, $A_G(u)$ is isomorphic to the subgroup of the group of units in $C^+(V)$ generated by an even number of products x_j such that λ_j is odd. Thus, we have the following lemma.

LEMMA 3.5. *If $q \equiv 1 \pmod 4$, then F acts trivially on $A_G(u)$. While if $q \equiv -1 \pmod 4$, then F acts nontrivially on $A_G(u)$.*

3.6. We apply the restriction formula (1.7.4) for the following case. Let M be the Levi subgroup of a parabolic subgroup Q of G such that $\beta(M) \simeq SL_2 \times SO_{N-4}$. Take $(u, \rho) \leftrightarrow (C, \mathscr{E}) \in (\mathscr{N}_G)_\chi$, and $(u', \rho') \leftrightarrow (C', \mathscr{E}') \in (\mathscr{N}_M)_\chi$. Then, u' satisfies the properties:

(i) The projection of $\beta(u')$ on SL_2 is regular unipotent.
(ii) The projection of $\beta(u')$ on SO_{N-4} corresponds to the unipotent elements of Jordan type in X_{N-4} under the correspondence $(\mathscr{N}_{\mathrm{Spin}_{N-4}})_\chi \simeq X_{N-4}$.

Put $\beta(M) = \overline{M}$ and $\beta(Q) = \overline{Q}$. Let $p : \overline{Q} \to \overline{M}$ be the natural projection. Put $\lambda = \lambda(u) \in X_N$ and $\lambda(u') = \lambda' \in X_{N-4}$. Assume that $\rho \otimes \rho'^*$ appears in $\varepsilon_{u,u'}$. Then, it is known by [LS, Lemmas 4.5 and 4.8], and only the following five cases occur; there exists an integer i such that one of the following conditions is satisfied.

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(I) λ_i is odd, $\lambda_i > \lambda_{i-1}+4$, and

$$\lambda'_j = \begin{cases} \lambda_j - 4 & \text{if } j = i, \\ \lambda_j & \text{otherwise.} \end{cases}$$

(II) $\lambda_i = \lambda_{i+1} \geq \lambda_{i-1}+2$ and

$$\lambda'_j = \begin{cases} \lambda_j - 2 & \text{if } j = i, i+1, \\ \lambda_j & \text{otherwise.} \end{cases}$$

(III) $\lambda_i = \lambda_{i+1} \geq \lambda_{i-1}+4$ and

$$\lambda'_j = \begin{cases} \lambda_j - 3 & \text{if } j = i, \\ \lambda_j - 1 & \text{if } j = i+1, \\ \lambda_j & \text{otherwise.} \end{cases}$$

(IV) $\lambda_{i+1} - 2 = \lambda_i \geq \lambda_{i-1}+1$ and

$$\lambda'_j = \begin{cases} \lambda_j - 1 & \text{if } j = i, \\ \lambda_j - 3 & \text{if } j = i+1, \\ \lambda_j & \text{otherwise.} \end{cases}$$

(V) $\lambda_{i+2} = \lambda_{i+1} = \lambda_i + 1$ and

$$\lambda'_j = \begin{cases} \lambda_j - 1 & \text{if } j = i, i+2, \\ \lambda_j - 2 & \text{if } j = i+1, \\ \lambda_j & \text{otherwise.} \end{cases}$$

3.7. Assume that $(u,\rho) \leftrightarrow (C,\mathscr{E}) \in (\mathscr{N}_G)_\chi$. We choose a split element $u \in C^F$, and assume that F acts nontrivially on $A_G(u)$. Since F^2 acts trivially on $A_G(u)$ (see 3.2), the order of τ is equal to 2. There exist two extensions $\widetilde{\rho}, \widetilde{\rho}'$ of ρ to $\widetilde{A}_G(u)$. We need to fix an appropriate extension $\widetilde{\rho}$ of ρ, called the split extension of ρ. Assume given $(u',\rho') \in (\mathscr{N}_M)_\chi$ as in 3.6, and choose a split element $u' \in C'^F$. Recall that $(u_0,\rho_0) \leftrightarrow (C_0,\mathscr{E}_0)$ for the cuspidal pair in L. We assume that the split extensions $\widetilde{\rho}_0$ of ρ_0 and $\widetilde{\rho}'$ of ρ' are already determined. We consider $\widetilde{\varepsilon}_{u,u'}$ and its subspace $\rho \otimes \rho'^*$. τ' acts on $\widetilde{\rho}'$ via T'. If we fix an extension $\widetilde{\rho}$ of ρ, where τ acts as T on $\widetilde{\rho}$, then τ_* act as $T_* = T \otimes T'$ with $T_*^2 = 1$. τ acts as $-T$ for $\widetilde{\rho}'$. Since $\rho \otimes \rho'^*$ is F-stable, the action of F on $\rho \otimes \rho'^*$ coincides with a scalar times T_*. If we can show that F^2 acts trivially on $\rho \otimes \rho'^*$, then by replacing $\widetilde{\rho}$ by $\widetilde{\rho}'$ if necessary, we can find an extension $\widetilde{\rho}$ of ρ such that F acts as T_* on $\widetilde{\rho \otimes \rho'^*}$. We call such $\widetilde{\rho}$ the split extension of ρ. By (1.8.2), we have the following lemma.

LEMMA 3.8. *Assume that F acts nontrivially on $A_G(u)$. If F^2 acts trivially on the subspace $\rho \otimes \rho'^*$ of $\varepsilon_{u,u'}$, then for the split extension $\widetilde{\rho}$ of ρ, we have $\gamma = 1$.*

3.9. Recall that $Y_{u,u'} = \{g \in G \mid g^{-1}ug \in u'U_Q\}$, and C' the unipotent class in M containing u'. Put

$$\mathcal{Q}_{u,C'} = \{gQ \in G/Q \mid g^{-1}ug \in C'U_Q\}. \tag{3.9.1}$$

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We have a natural map $q : Y_{u,u'} \to \mathcal{Q}_{u,C'}$. $Z_G(u)$ acts on $\mathcal{Q}_{u,C'}$ from the left, and the map q is $Z_G(u)$-equivariant. q induces an isomorphism

$$Y_{u,u'}/Z_M(u')U_Q \xrightarrow{\sim} \mathcal{Q}_{u,C'}. \tag{3.9.2}$$

If $u \in C^F$, $u' \in C'^F$, then $Y_{u,u'}, \mathcal{Q}_{u,C'}$ are defined over $\mathbf{F}_q$, and the map q is F-equivariant. Put $\overline{Q} = \beta(Q)$. We have $\beta(Y_{u,u'}) = \overline{Y}_{\overline{u},\overline{u}'}$, which is the variety defined with respect to $\overline{G}$ and $\overline{M}$. Then $\beta : Y_{u,u'} \to \overline{Y}_{\overline{u},\overline{u}'}$ gives a double covering. The variety $\overline{\mathcal{Q}}_{\overline{u},\overline{C}'}$ is defined similarly to $\mathcal{Q}_{u,C'}$, and a formula analogous to (3.9.2) holds for $\overline{Y}_{\overline{u},\overline{u}'}$. We show the following lemma.

LEMMA 3.10. *Assume that F acts trivially on $A_G(u)$ and on $A_M(u')$. Assume that $\overline{\mathcal{Q}}_{\overline{u},\overline{C}'}$ has finitely many $Z_{\overline{G}}(\overline{u})$-orbits, and each $Z_{\overline{G}}(\overline{u})$-orbit contains an element $\overline{y}\overline{Q}$ such that $\overline{y} \in \overline{Y}^F_{\overline{u},\overline{u}'}$. Then, F acts trivially on $\varepsilon_{u,u'}$.*

Proof. Let $\overline{y}\overline{Q} \in \overline{\mathcal{Q}}_{\overline{u},\overline{C}'}$ be such that $\overline{y} \in \overline{Y}^F_{\overline{u},\overline{u}'}$. Then, there exists $y \in G^F$ such that $yQ = \overline{y}\overline{Q}$ and that $y^{-1}uy \in u'_1 U_Q$ for $u'_1 \in C'^F$. This implies that $\overline{y}^{-1}\overline{u}\overline{y} \in \overline{u}'_1 U_{\overline{Q}}$, and $\overline{u}'_1$ is $\overline{M}^F$-conjugate to $\overline{u}'$. Hence, $\overline{u}'_1$ is a split element in $\overline{C}'$, and so u'_1 is a split element in C'. We have $y \in Y^F_{u,u'_1}$, and $\overline{y} \in \overline{Y}^F_{\overline{u},\overline{u}'_1}$. By replacing u' by u'_1 if necessary, we may assume that $y \in Y^F_{u,u'}$. Let R be the $Z_{\overline{G}}(\overline{u})$-orbit of $\overline{y}\overline{Q}$ in $\overline{\mathcal{Q}}_{\overline{u},\overline{C}'}$. Then, $q^{-1}R = Z_{\overline{G}}(\overline{u})\overline{y}Z_{\overline{M}}(\overline{u}')U_{\overline{Q}}$. Since $\varepsilon = -1 \in Z_G(u)$, we can write as $\beta^{-1}(q^{-1}R) = Z_G(u)yZ_M(u')U_Q$, with $y \in Y^F_{u,u'}$. Then,

$$\beta^{-1}(q^{-1}R) = \bigsqcup_{\alpha \in A_G(u), \beta \in A_M(u')} Z^0_G(u)x_\alpha y x'_\beta Z^0_M(u')U_Q,$$

where x_α, x'_β are representatives of α, β in $Z_G(u)$, $Z_M(u')$, respectively. Hence, the irreducible components of $\beta^{-1}(q^{-1}R)$ are given by $Z^0_G(u)x_\alpha y x'_\beta Z_M(u')$. Since F acts trivially on $A_G(u), A_M(u')$, all those irreducible components are F-stable. Thus, F acts trivially on $\varepsilon_{u,u'}$. □

PROPOSITION 3.11. *Let $u \in C, u' \in C'$ with $\lambda(u) = \lambda$, $\lambda' = \lambda(u')$. Assume that the type of (λ, λ') is given as in (I), (II), (IV), or (V) in 3.6. Let $u \in C^F$ be a split element such that F acts trivially on $A_G(u)$. Then, there exists a split element $u' \in C'^F$ such that F acts trivially on $\varepsilon_{u,u'}$. (For the case (III), see Remark 3.13 below.)*

Proof. We can identify $\mathcal{Q} = G/Q$ with the variety $\mathscr{F}$ of all two-dimensional totally isotropic subspaces of V. Then, $\mathcal{Q}_{u,C'}$ can be identified with the subvariety $\mathscr{F}_{u,C'}$ of $\mathscr{F}$ consisting of all $E \in \mathscr{F}$ such that

(i) E is $\overline{u}$-stable.
(ii) $\overline{u}|_E \neq 1$.
(iii) The Jordan type of $\overline{u}|_{E^\perp/E}$ is λ'.

Let $u \in C^F$ be a split element such that F acts trivially on $A_G(u)$. In each case of (I), (II), (IV), or (V) in 3.6, we shall find a split element $u' \in C'^F$ satisfying the assumption in Lemma 3.10. Then, the proposition follows from the lemma.

Case I. Let V_i be the subspace of V corresponding to λ_i, and $e_1, \dots, e_h$ be the F-stable basis of V_i (with $h = \lambda_i$) such that $xe_a = e_{a-1}$ for $x \in \mathfrak{g}^F$ as in 3.1 (a). Put $E = \langle e_1, e_2 \rangle$. It is easy to see that $\mathscr{F}_{u,C'} = \{E\}$. $SL(E) \times SO(E^\perp/E)$ is $\overline{G}^F$-conjugate to $\overline{M}$. Thus, there exists $\overline{u}' \in \overline{M}^F$ such that $\overline{u}'$ is $\overline{G}^F$-conjugate to $(\overline{u}|_E, \overline{u}|_{E^\perp/E})$. Then, $u' = \beta^{-1}(\overline{u}')$ is a split

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element in C'^F. Since $A_G(u) \simeq A_M(u')$ with F-action, F acts trivially on $A_M(u')$. One can choose $\overline{y} \in \overline{G}^F$ such that $\overline{y}\overline{Q}$ corresponds to E. Then, $\overline{y} \in \overline{Y}^F_{\overline{u},\overline{u}'}$. Thus, the assumption of Lemma 3.10 is satisfied.

Case II. Let V_i be as in 3.1(b). We write the basis $e^i_1,\dots,e^i_h,e^{i+1}_1,\dots,e^{i+1}_h$ of V_i as $e_1,\dots,e_h,f_1,\dots,f_h$, where $h=\lambda_i=\lambda_{i+1}$. Under the decomposition $V=\bigoplus_k V_k$ in 3.1, we define a subspace M_i of V as the direct sum of V_k such that $\lambda_k=h$. Let $x\in\mathfrak{g}$ be as in 3.1. Take $E\in\mathscr{F}_{u,C'}$. Then, $\dim(E\cap\operatorname{Ker}x)=1$. We choose $0\neq v\in E\cap\operatorname{Ker}x$. v is G_1-conjugate to $e_1\in V_i$, where G_1 is a subgroup of $Z_{\overline{G}}(\overline{u})$ which is isomorphic to $Sp(M_i)$. So assume that $e_1\in E\cap\operatorname{Ker}x$. Then, the condition on λ' implies that $E\subset V_i$. We see that E is of the form $E=\langle e_1,e_2+\alpha f_1\rangle$ with $\alpha\in\mathbf{k}$. It follows that

$$\mathscr{F}_{u,C'}=\bigcup_{g\in G_1} g\{\langle e_1,e_2+\alpha f_1\rangle \mid \alpha\in\mathbf{k}\}. \tag{3.11.1}$$

We choose $E=\langle e_1,e_2\rangle$. Then, $E\in\mathscr{F}^F_{u,C'}$. There exists $\overline{u}'\in\overline{M}^F$ such that $\overline{u}'$ is $\overline{G}^F$-conjugate to $(\overline{u}|_E,\overline{u}|_{E^\perp/E})$. Then, $u'=\beta^{-1}(\overline{u}')\in C'^F$ is a split element. Since $A_G(u)\simeq A_M(u')$ with F-action, F acts trivially on $A_M(u')$. It is easy to see that $\langle e_1,e_2+\alpha f_1\rangle$ is $Z_{\overline{G}}(\overline{u})$-conjugate to E. Thus, $\mathscr{F}_{u,C'}$ coincides with the $Z_{\overline{G}}(\overline{u})$-orbit of E. There exists $\overline{y}\in\overline{Y}^F_{\overline{u},\overline{u}'}$ such that $\overline{y}\overline{Q}$ corresponds to E. The assumption of Lemma 3.10 is satisfied.

Case IV. Put $M_i=V_i\oplus V_{i+1}$, where $\dim V_{i+1}=h$ and $\dim V_i=h-2$. Write the bases of V_{i+1},V_i in 3.1 as $e_1,\dots,e_h$ for V_{i+1}, and $e'_1,\dots,e'_{h-2}$ for V_i. Take $E\subset\mathscr{F}_{u,C'}$. Then, E must be in M_i. Take $0\neq v\in M_i\cap\operatorname{Ker}x$. The condition for λ' implies that $v=e_1$ up to scalar. Thus, we may assume that $v=e_1$. E can be written as $E=\langle e_1,e_2+\alpha e'_1\rangle$ with $\alpha\in\mathbf{k}$. The condition that $x|_{E^\perp/E}$ has type λ' is given by

$$(e_2+\alpha e'_1,e_{h-1}+\alpha e'_{h-2})=0. \tag{3.11.2}$$

This implies that $(e_2,e_{h-1})+\alpha^2(e'_1,e'_{h-2})=0$. Under the notation of (3.1.1), we have $\delta_i=\delta_{i+1}$. Thus, by (3.1.2), $(e_2,e_{h-1})=-(e'_1,e'_{h-2})$, and so $\alpha=\pm1$. We have

$$\mathscr{F}_{u,C'}=\{E=\langle e_1,e_2+e'_1\rangle,E'=\langle e_1,e_2-e'_1\rangle\ \}. \tag{3.11.3}$$

In particular, both of E,E' are F-stable. As in Case I, we can define $u'\in C'^F$ by using $(\overline{u}|_E,\overline{u}|_{E^\perp/E})$. Then, u' is a split element. $A_M(u')$ is obtained from $A_G(u)$ by removing x_i and x_{i+1}, under the notation in 2.6. Thus, F acts trivially on $A_M(u')$. As in the previous cases, we can find $\overline{y}\in\overline{Y}^F_{\overline{u},\overline{u}'}$ such that $\overline{y}\overline{Q}$ corresponds to E. If we choose $\overline{y}'\in\overline{G}^F$ such that $\overline{y}'\overline{Q}$ corresponds to E', then $\overline{y}'^{-1}\overline{u}\overline{y}'\in\overline{u}''U_{\overline{Q}}$ for some $\overline{u}''\in\overline{C}'^F$. But then, $\overline{u}''$ is also a split element in $\overline{C}''^F$, and $\overline{u}''$ is $\overline{M}^F$-conjugate to $\overline{u}'$. By replacing $\overline{y}'$ appropriately if necessary, we find $\overline{y}'\in\overline{Y}^F_{\overline{u},\overline{u}'}$ such that $\overline{y}'\overline{Q}$ corresponds to E'. Thus, the assumption of Lemma 3.10 is satisfied.

Case V. Put $h=\lambda_{i+1}=\lambda_{i+2}$, with h : even. Let M_{i+1} be the subspace of V which is the direct sum of V_k such that $\lambda_k=h$. Then, $V_{i+1}\subset M_{i+1}$. We write the basis of V_{i+1} given in 3.1 (b) as $e_1,\dots,e_h,f_1,\dots,f_h$, and write the basis of V_i in 3.1 (a) as $e'_1,\dots,e'_{h-1}$. Take $E\in\mathscr{F}_{u,C'}$. Then, $E\subset M_{i+1}\oplus V_i$. Take $0\neq v\in E\cap\operatorname{Ker}x$. Similarly to Case II, by taking the G_1-conjugate, we may assume that $v\in V_{i+1}\oplus V_i$, where G_1 is the subgroup of $Z_{\overline{G}}(\overline{u})$ which is isomorphic to $Sp(M_{i+1})$. Then, $E\subset V_{i+1}\oplus V_i$. It follows from the condition for λ' that v

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must be in V_{i+1}. Then, under the conjugation by $Z_{\overline{G}}(\overline{u})$, we may assume that $v=e_1$. E is given as $E=\langle e_1, e_2+z\rangle$ with $z\in(V_{i+1}\oplus V_i)\cap \mathrm{Ker}\, x$. So we write $z=\alpha f_1+\beta e'_1$. Then, the condition $x|_{E^\perp/E}$ has type λ' is given by

$$(e_2+\alpha f_1+\beta e'_1, e_h+\alpha f_{h-1}+\beta e'_{h-1})=0. \tag{3.11.4}$$

This implies that $\alpha(e_2,f_{h-1})+\alpha(f_1,e_h)+\beta^2(e'_1,e'_{h-1})=0$. Since $(e_2,f_{h-1})=(f_1,e_h)=-1$ by 3.1 (b), we have

$$\alpha=\frac{\beta^2(e'_1,e'_{h-1})}{2}. \tag{3.11.5}$$

Thus, for any $\beta\in\mathbf{k}$, there exists a unique $E_\beta=\langle e_1,e_2+z\rangle\in\mathscr{F}_{u,C'}$ given as above. In particular, if we put $E=\langle e_1,e_2\rangle$, then $E\in\mathscr{F}_{u,C'}$ and E_β is $Z_{\overline{G}}(\overline{u})$-conjugate to E. Thus, $Z_{\overline{G}}(\overline{u})$ acts transitively on $\overline{\mathcal{Q}}_{\overline{u},\overline{C}'}$. Here, $E\in\mathscr{F}^F_{u,C'}$, and $(\overline{u}|_E,\overline{u}|_{E^\perp/E})$ determines $u'\in C'^F$. There exists $\overline{y}\in\overline{Y}^F_{\overline{u},\overline{u}'}$ such that $\overline{y}\overline{Q}$ corresponds to E. Note our process to obtain u' from u corresponds to the procedure,

$$(\lambda_i,\lambda_{i+1},\lambda_{i+2})\mapsto(\lambda_i,\lambda_{i+1}-2,\lambda_{i+1}-2)\mapsto(\lambda_{i+1}-2,\lambda_{i+2}-2,\lambda_i),$$

where the second one is a rearrangement of the order of the parts. Here, the first one is $(\lambda_i,\lambda_{i+1},\lambda_{i+2})=(h-1,h,h)$, and the last one is $(\lambda'_i,\lambda'_{i+1},\lambda'_{i+2})=(h-2,h-2,h-1)$. We have $\delta_i=(\lambda_i-2)/2+i$ by (3.1.1). Then, $\delta'_{i+2}=(\lambda'_{i+2}-1)/2+(i+2)\equiv\delta_i \pmod 2$. It follows that $u'\in C'^F$ is a split element. Since $A_G(u)\simeq A_M(u')$ with F-action, F acts trivially on $A_M(u')$. Thus, the assumption of Lemma 3.10 is satisfied. □

Remark 3.12. In the statement in Proposition 3.11, the case (III) is not used in the later discussion, so we omitted it. Actually, in this case, the structure of $\mathscr{F}_{u,C'}$ is more complicated, and the splitness of $u'\in C'^F$ is not deduced from the split element $u\in C^F$.

We are now ready to prove Theorem 1.5 in the case of spin groups. Note that split elements are defined in 3.2, and split extensions are defined in 3.7.

Theorem 3.13. *Assume that $G=\mathrm{Spin}_N$. For each $(C,\mathscr{E})\in\mathscr{N}_G^F$, let $u^\bullet\in C^F$ be a split element, and assume that $(C,\mathscr{E})\leftrightarrow(u^\bullet,\rho)$. For each $(L,C_0,\mathscr{E}_0)\in\mathscr{M}_G^F$, let $u_0^\bullet\in C_0^F$ be a split element as above (applied for L), and assume that $(C_0,\mathscr{E}_0)\leftrightarrow(u_0^\bullet,\rho_0)$. Let $\widetilde{\rho}_0\in\widetilde{A}_L(u_0^\bullet)^\wedge$ be an extension of ρ_0, and for each $(C,\mathscr{E})$ belonging to the series $(L,C_0,\mathscr{E}_0)$, let $\widetilde{\rho}\in\widetilde{A}_G(u^\bullet)^\wedge$ be the split extension of ρ associated to $\widetilde{\rho}_0$. Then, $\gamma(u_0^\bullet,\widetilde{\rho}_0,u^\bullet,\widetilde{\rho})=1$.*

Proof. By Proposition 2.5, we may assume that $(C,\mathscr{E})\in(\mathscr{N}_G)^F_\chi$ for $\chi\in Z^\wedge$ such that $\chi(\varepsilon)=-1$. Then, the Jordan type of $\overline{C}$ is $\lambda\in X_N$. By Remark 2.8, $\mathscr{N}^F_\chi=\emptyset$ if F is of nonsplit type. Thus, we may assume that F is of split type. We define a split element $u=u^\bullet\in C^F$ as in 3.1 and 3.2. Assume that $N\geq 7$, and consider an F-stable Levi subgroup M of G such that $\beta(M)\simeq SL_2\times SO_{N-4}$. Take $(C',\mathscr{E}')\in(\mathscr{N}_M)_\chi$, and assume that $(C,\mathscr{E})\leftrightarrow(u,\rho)$, and $(C',\mathscr{E}')\leftrightarrow(u',\rho')$, where $\lambda=\lambda(u),\lambda'=\lambda(u')$. Then, the condition that $\rho\otimes\rho'^*$ appears in $\varepsilon_{u,u'}$ is exactly that the pair (λ,λ') satisfies the condition (I) $\sim$ (V) in 3.6. Now, for a given $u\in C$, one can find $u'\in C'$ such that (λ,λ') satisfies one of the conditions (I), (II), (IV), or (V). (Note that the case (III) is not necessary, since if the condition (III) is applicable for λ_i, then already the condition (II) can be applied for λ_i.) We choose a split element $u'\in C'^F$. Then, by Proposition 3.11, F acts trivially on $\varepsilon_{u,u'}$, and also F acts trivially on $A_M(u')$. Then, by (1.8.1), $\sigma_{\rho,\rho'}:M_{\rho,\rho'}\to M_{\rho,\rho'}$ is the scalar multiplication by

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$q^{-(\dim C-\dim C')/2+\dim U_Q}$. By induction, we may assume that $\sigma_{(u',\rho')}$ acts on $V_{(u',\rho')}$ as a scalar multiplication by $q^{(a'_0+r')/2}$, where a'_0, r' are defined similarly to a_0, r by replacing G by M. It follows that $\sigma_{(u,\rho)}$ acts on $V_{(u,\rho)}$ as a scalar multiplication by $q^{(a_0+r)/2}$, and we conclude that $\gamma = 1$ by (1.8.2).

It remains to consider the case where $N \leq 6$. If $N = 5$ or 6, (u,ρ) is a cuspidal pair by [LS, Cor. 4.9]. Thus, we have $\gamma = 1$. So we assume that $N = 4$ or 5. First, consider the case where $N = 5$. In this case, $X_5 = \{(5), (122)\}$, and $\mathcal{N}_\chi$ corresponds to $(L, C_0, \mathcal{E}_0) \in \mathcal{M}_\chi$, where L is a Levi subgroup of $G = \mathrm{Spin}_5$ such that $L \simeq SL_2$, and for $(u_0,\rho_0) \leftrightarrow (C_0, \mathcal{E}_0)$, u_0 is a regular unipotent element of L, and ρ_0 is a nontrivial character of $A_L(u_0) = Z_L \simeq \mathbf{Z}/2\mathbf{Z}$. Thus, if we regard $X_1 = \{(1)\}$ as the set corresponding to u_0, we have a bijection $X_1 \simeq (\mathcal{N}_L)_\chi$, and (1.8.1) holds for this situation. In particular, the verification of the theorem for the case $N = 5$ is reduced to the case $N = 1$, namely the case where $G = SL_2$. In this case, the assertion is already known by [S3, I, §3], and actually, it is easy to check directly. If $N = 4$, then $X_4 = \{(22), (13)\}$, and $\mathcal{N}_\chi$ corresponds to $(L, C_0, \mathcal{E}_0) \in \mathcal{M}_\chi$, where L is a Levi subgroup of $G = \mathrm{Spin}_4$ such that $L \simeq SL_2$, (u_0,ρ_0) is the same as above. In this case, we can regard $X_0 = \{(-)\}$ as the set corresponding to u_0, we have a bijection $X_0 \simeq (\mathcal{N}_L)_\chi$, and (1.8.1) holds. Thus, the theorem holds also for the case where $N = 4$. We have proved the theorem in the case where F acts trivially on $Z_G(u)$.

Next, we assume that F acts nontrivially on $A_G(u)$. But then, F^2 acts trivially on it. Thus, the theorem holds if we replace F by F^2. By Lemma 3.8, we can find a split extension $\widetilde{\rho}$ of ρ for each split element $u \in C^F$. Thus, the theorem holds for this case. The theorem is now proved. □

§4. Proof of Theorem 1.5—the case of special linear groups

4.1. In this section, we assume that $G = SL_n$, and $\mathrm{ch}\,\mathbf{k}$ is arbitrary. Let p be the characteristic exponent of $\mathbf{k}$, namely, write $p = \mathrm{ch}\,\mathbf{k}$ if $\mathrm{ch}\,\mathbf{k} > 0$, and $p = 1$ if $\mathrm{ch}\,\mathbf{k} = 0$. Let $F = \sigma F_0$ be a twisted Frobenius map on G, where F_0 is the standard Frobenius map, and σ is the graph automorphism defined by $\sigma(g) = \dot{w}_0{}^t g^{-1} \dot{w}_0{}^{-1}$, where $\dot{w}_0 \in G$ is the permutation matrix corresponding to the longest element $w_0 \in W = S_n$.

Let Z be the center of G. Then, $Z = \{\alpha I_n \mid \alpha \in \mathbf{k}, \alpha^n = 1\}$ is the cyclic group of order n', where n' is the largest divisor of n which is prime to p. Let $u = u_\lambda \in G$ be a unipotent element with Jordan type λ of n, and we denote by C_λ the unipotent class in G containing u_λ. We write $\lambda = (\lambda_1, \lambda_2, \dots \lambda_r) = (1^{m_1}, 2^{m_2}, \dots)$, where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r$, and put $I = \{i \in \mathbf{Z}_{\geq 1} \mid m_i \neq 0\}$. Let $\{v_{k,j} \mid 1 \leq k \leq r, 1 \leq j \leq \lambda_k\}$ be the Jordan basis of u such that $(u-1)v_{k,j} = v_{k,j-1}$ with the convention $v_{k,0} = 0$. For $i \in I$, we define a subspace V_i of V by $V_i = \langle v_{k,j} \mid \lambda_k = i \rangle$. We have a decomposition $V = \bigoplus_{i \in I} V_i$ into u-stable subspaces, where the Jordan type of $u|_{V_i}$ is (i^{m_i}). Let V_i^0 be the subspace of V_i generated by $\{v_{k,i} \mid \lambda_k = i\}$. (Note that the notation V_i here is different from that in 3.1.) The Levi part M_u of $Z_G(u)$ is isomorphic to the subgroup of $\prod_i GL(V_i^0)$ given by

$$M_u = \{g = (g_i) \in \prod_{i \in I} GL(V_i^0) \mid \prod_{i \in I} (\det g_i)^i = 1\}. \tag{4.1.1}$$

Let i' be the p'-part of i. Then, the condition in the right-hand side of (4.1.1) is equivalent to the condition that $\prod_i (\det g_i)^{i'} = 1$. Let n'_λ be the greatest common divisor of $\{i' \mid i \in I\}$,

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and write $i' = i''n'_\lambda$. Thus, we have

$$\prod_i (\det g_i)^{i'} = \left(\prod_i (\det g_i)^{i''}\right)^{n'_\lambda}.$$

It follows that the connected component M_u^0 of M_u is given by

$$M_u^0 = \{g = (g_i) \in \prod_{i\in I} GL(V_i^0) \mid \prod_{i\in I} (\det g_i)^{i''} = 1\}, \tag{4.1.2}$$

and $A_G(u) \simeq M_u/M_u^0$ is a cyclic group of order n'_λ. Note that n'_λ coincides with the greatest common divisor of $\lambda_1, \dots, \lambda_r, n'$.

4.2. Let $\chi \in Z^\wedge$, and consider $A_G(u)^\wedge_\chi$. We have a natural map $\mathbf{Z}/n'\mathbf{Z} \to \mathbf{Z}/n'_\lambda\mathbf{Z}$, and $\rho \in A_G(u)^\wedge_\chi$ is equivalent to the condition that χ is the lift of ρ. If $\chi \in Z^\wedge$ is of order d, then ρ is of order d also, and d is a divisor of n'_λ. For a given χ, there exists such a ρ uniquely. Thus, we have

$$|A_G(u)^\wedge_\chi| = \begin{cases} 1 & \text{if } d \text{ divides each } \lambda_i, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.1}$$

4.3. It is known by [L1] that in the case where $G = SL_n$, $(C, \mathscr{E})$ is a cuspidal pair for G if and only if C is the regular unipotent class, and $\rho \in A_G(u)^\wedge = Z^\wedge$ has order n for $(u,\rho) \leftrightarrow (C,\mathscr{E})$. Thus, for $\chi \in Z^\wedge$ of order n, there exists exactly one cuspidal pair $(C_u, \mathscr{E}_\rho)$ pertaining χ, where u : regular unipotent, $\rho \in A_G(u)^\wedge_\chi$.

The generalized Springer correspondence is described as follows. Fix $\chi \in Z^\wedge$. Then, there exists a unique Levi subgroup L which has a cuspidal pair $(C_0, \mathscr{E}_0) \in (\mathscr{N}_L)_\chi$, and this cuspidal pair is also unique. Assume that χ is of order d. Then, $L = S(GL_d \times GL_d \times \cdots \times GL_d)$ with n/d-factors. $u_0 \in C_0$ is a regular unipotent element in L, and $\rho_0 \in A_L(u_0)^\wedge$ corresponding to $\mathscr{E}_0$ is given as follows. We have a natural map $Z_{SL_d} \to Z_L(u_0)/Z_L^0(u_0)$, which gives an isomorphism $Z_{SL_d} \simeq A_L(u_0)$. Thus, $A_L(u_0)$ is the cyclic group of order d', and (u_0, ρ_0) is uniquely determined from the cuspidal pair of SL_d pertaining χ. In this case, $N_G(L)/L \simeq S_{n/d}$, the cyclic group of degree n/d. Thus, the generalized Springer correspondence pertaining χ is given as follows:

$$\mathscr{N}_\chi = \{(u,\rho) \mid \lambda_i : \text{divisible by } d, \rho \in A_G(u)^\wedge_\chi\} \leftrightarrow S_{n/d}^\wedge, \tag{4.3.1}$$

where $(u,\rho) \leftrightarrow (C_\lambda, \mathscr{E})$. (Here, ρ is determined uniquely from u by (4.2.1).)

By [LS, Prop. 5.2], this correspondence is given by $u_\lambda \mapsto E_\mu \in S_{n/d}^\wedge$, where $\mu = \lambda/d = (\lambda_1/d, \lambda_2/d, \dots) \in \mathscr{P}_{n/d}$, and E_μ is the irreducible representation of $S_{n/d}$ corresponding to μ.

4.4. We consider $(L, C_0, \mathscr{E}_0) \in \mathscr{M}_G$, where L is of type $A_{d-1} + A_{d-1} + \cdots$. Then, L is F-stable, and the regular unipotent class C is F-stable. The center Z of G is the set of scalar matrices, hence the action of F_0 and of F on Z are given by, for $z \in Z$, $F_0(z) = z^q$, $F(z) = z^{-q}$. Thus, the actions of F_0, F on $Z^\wedge$ are given by $\chi \mapsto \chi^q$, $\chi \mapsto \chi^{-q}$. Hence, χ is F_0-stable (resp. F-stable) if and only if $d|q-1$ (resp. $d|q+1$). Now, take $u_0 \in C_0^F$. Then, F acts on $A_L(u_0)$. We have $A_L(u_0) \simeq Z_{SL_d}$, and the action of F is compatible with this isomorphism. It follows that if $\chi \in Z^\wedge$ is F-stable, then the corresponding $\mathscr{E}_0$ is F-stable.

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We have $N_G(L)/L \simeq S_{n/d}$, and the action of F on $N_G(L)/L$ corresponds to the σ-action on $S_{n/d}$, namely the conjugation action by w_0.

We shall apply the restriction formula (1.8.1) for $(L, C_0, \mathscr{E}_0)$. Let Q be the F-stable parabolic subgroup of G, and let M be its F-stable Levi subgroup such that $M \simeq S(GL_d \times GL_{n-2d} \times GL_d)$. Take $(C, \mathscr{E}) \in \mathscr{N}_G$ and $(C', \mathscr{E}') \in \mathscr{N}_M$, and $u \in C, u' \in C'$. Here, u is of type λ and u' is of type λ'. We consider $Y_{u,u'}$, and $\mathcal{Q}_{u,C'}$ as in 3.9. Put $x = u - 1$. Then, $\mathcal{Q}_{u,C'}$ can be identified with

$$\mathscr{F}_{u,\lambda'} = \{(W \subset W') \mid \dim W = d, \dim W' = n-d, W, W' \text{: } x\text{-stable}, \\ x|_{W'/W} \text{ : type } \lambda', x|_W, x_{V/W'} \text{: regular nilpotent}\}. \tag{4.4.1}$$

4.5. Since $d|\lambda_i$ for any i, we put $\mu = \lambda/d \in \mathscr{P}_{n/d}$, and similarly, $\mu' = \lambda'/d \in \mathscr{P}_{n/d-2}$. Then, μ' is obtained from μ by removing two nodes from the corresponding Young diagram.

For the pair (λ, λ'), there are three possibilities; we write $\mu : \mu_1 \le \mu_2 \le \cdots, \mu' : \mu'_1 \le \mu'_2 \le \cdots$.

(I) $\mu_{i-1} + 2 \le \mu_i$, and $\mu'_i = \mu_i - 2$.
(II) $\mu_{i-1} + 1 \le \mu_i = \mu_{i+1}$, and $\mu'_i = \mu'_{i+1} = \mu_i - 1$.
(III) $\mu_{i-1} + 1 \le \mu_i$ and $\mu_{j-1} + 1 \le \mu_j$ for $i \ne j$, and $\mu'_i = \mu_i - 1, \mu'_j = \mu_j - 1$.

Let $E_\mu \in S^\wedge_{n/d}$ and $E_{\mu'} \in S^\wedge_{n/d-2}$. Then,

$$\langle E_\mu|_{S_{n/d-2}}, E_{\mu'} \rangle_{S_{n/d-2}} = \begin{cases} 1 & \text{for the case (I) or (II)}, \\ 2 & \text{for the case (III)}. \end{cases} \tag{4.5.1}$$

We fix μ, and consider the possibility for the choice of μ'. If there exists i such that $\mu_{i-1} \le \mu_i - 2$, we can apply (I). If there exists i such that $\mu_{i-1} + 1 \le \mu_i = \mu_{i+1}$, we can apply (II). Thus, for (III), we have only to consider the case where $\mu_i = \mu_{i-1} + 1$ for any i. But in this case, we can choose $\mu'_i = \mu_i - 1, \mu'_{i+1} = \mu_{i+1} - 1$, namely, we can choose $i \ne j$ in (III) as $j = i + 1$. Moreover, all the rows have multiplicity one.

LEMMA 4.6. *Let $\lambda = d\mu \in \mathscr{P}_n, \lambda' = d\mu' \in \mathscr{P}_{n-2d}$ be as in 4.5.*

(i) *Assume that (λ, λ') is in the case (I) or (II) in 4.5. Then, $Z_G(u)$ acts transitively on $\mathscr{F}_{u,\lambda'}$.*
(ii) *Assume that (λ, λ') is in the case (III) in 4.5. Let $\nu \in \mathscr{P}_{n-d}$ be the partition obtained from λ by replacing λ_i by $\nu_i = \lambda_i - d$, and $\nu' \in \mathscr{P}_{n-d}$ by replacing λ_j by $\nu'_j = \lambda_j - d$. Put*

$$\mathscr{F}^\nu_{u,\lambda'} = \{(W \subset W') \in \mathscr{F}_{u,\lambda'} \mid u|_{V/W} \text{ : type } \nu\}, \\ \mathscr{F}^{\nu'}_{u,\lambda''} = \{(W \subset W') \in \mathscr{F}_{u,\lambda'} \mid u|_{V/W} \text{ : type } \nu'\}.$$

Then, $\mathscr{F}_{u,\lambda'} = \mathscr{F}^\nu_{u,\lambda'} \sqcup \mathscr{F}^{\nu'}_{u,\lambda'}$, and $Z_G(u)$ acts transitively on $\mathscr{F}^\nu_{u,\lambda'}$ and on $\mathscr{F}^{\nu'}_{u,\lambda'}$. In particular, $\mathscr{F}_{u,\lambda'}$ has two $Z_G(u)$-orbits.

Proof. Assume that (λ, λ') is in the case (I) or (II). Let $\mu'' \in \mathscr{P}_{n/d-1}$ be the partition obtained from μ by replacing μ_i by $\mu''_i = \mu_i - 1$. Put $\lambda'' = d\mu''$. Put

$$\mathscr{F}^1_{u,\lambda''} = \{W \subset V \mid \dim W = d, W : x\text{-stable}, \\ x|_W \text{ : regular nilpotent}, x|_{V/W} : \text{ type } \lambda''\}.$$

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We have a surjective map $\varphi : \mathscr{F}_{u,\lambda'} \to \mathscr{F}^1_{u,\lambda''}$, $(W \subset W') \mapsto W$. Then, φ is $Z_G(u)$-equivariant. It is clear that $Z_G(u)$ acts transitively on $\mathscr{F}^1_{u,\lambda''}$ (note that we may replace $Z_G(u)$ by $Z_{GL_n}(u)$, and it is enough to consider the case where $d = 1$). We choose $W \in \mathscr{F}^1_{u,\lambda''}$, and put $\overline{V} = V/W$. u acts on $\overline{V}$, and put $\overline{u} = u|_{\overline{V}}$. Then, the fiber $\varphi^{-1}(W)$ is isomorphic to the variety of subspaces $\overline{W}' \subset \overline{V}$ such that $\overline{W}'$ is $\overline{u}$-stable, $\overline{u}|_{\overline{W}'}$ is of type λ', and $\overline{u}|_{\overline{V}/\overline{W}'}$ is regular unipotent. But this variety is the same type as $\mathscr{F}^1_{u,\lambda''}$, if we consider the dual space $\overline{V}^*$ of $\overline{V}$ and a $\overline{u}$-stable subspace $\overline{W}^\perp \subset \overline{V}^*$. In particular, $Z_{\overline{G}}(\overline{u})$ acts transitively on $\varphi^{-1}(W)$, where $\overline{G} = SL(\overline{V})$. Thus, $Z_G(u)$ acts transitively on $\mathscr{F}_{u,\lambda'}$.

The case (λ, λ') in the case (III) is dealt similarly. □

4.7. Let V be an n-dimensional vector space over $\mathbf{k}$, with standard basis $e_1, \dots, e_n$. We denote by V_0 the $\mathbf{F}_{q^2}$-subspace of V generated by $\{e_i\}$. We consider the sesquilinear form $\Psi(v, w) : V_0 \times V_0 \to \mathbf{F}_{q^2}$, that is, $\Psi(v, w) = {}^t v A \overline{w}$, where we write ${}^t v = (v_1, \dots, v_n) \in \mathbf{k}^n$, $\overline{w} = {}^t(w_1^q, \dots, w_n^q)$, and A is the Hermitian matrix, that is, ${}^t A = \overline{A}$. Then, A is unitary congruent to the identity matrix [G, Th. 10.3], that is, there exists $B \in GL_n(\mathbf{F}_{q^2})$ such that $A = {}^t B \cdot \overline{B}$. It follows that any sesquilinear form Ψ is unitary congruent to the form $(v, w) = \sum_{i=1}^n v_i w^q_{n-i}$.

Let $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathscr{P}_n$, and let $C = C_\lambda$ be the corresponding unipotent class. Let V_0 be the $\mathbf{F}_{q^2}$-subspace of V generated by the Jordan basis $\{v_{k,j}\}$ of $u \in C^{F^2}$. We define a sesquilinear form $(\ ,\ )$ on V_0 by

$$(v_{k,j}, v_{k',j'}) = \begin{cases} (-1)^{j+a_k} & \text{if } k = k', j + j' = \lambda_k, \\ 0 & \text{otherwise,} \end{cases} \tag{4.7.1}$$

where $a_k = \pm 1$. Then, $u \in C^{F^2}$ leaves the form $(\ ,\ )$ invariant, and so $u \in C^F$. We call u a split element in C^F. Thus, split elements are determined up to GL^F_n-conjugates.

Let (λ, λ') be as in 4.5. We choose $u \in C^F_\lambda$ a split element. Then, $\mathscr{F}_{u,\lambda'}$ and $\mathscr{F}^\nu_{u,\lambda'}, \mathscr{F}^{\nu'}_{u,\lambda'}$ in Lemma 4.6 are F-stable. We show the following lemma.

LEMMA 4.8. *Let $u \in C^F_\lambda$ be a split element.*

(i) *Assume that (λ, λ') is in the case (I) or (II) of 4.5. Then, there exists $E \in \mathscr{F}^F_{u,\lambda'}$, a split element $u' \in C'^F$, and $y \in Y^F_{u,u'}$ such that yQ corresponds to E under the identification $\mathcal{Q}_{u,C'} \simeq \mathscr{F}_{u,\lambda'}$.*

(ii) *Assume that (λ, λ') is in the case (III) with $j = i + 1$. Then, there exist a split element $u' \in C'^F$, $E \in (\mathscr{F}^\nu_{u,\lambda'})^F$, $E' \in (\mathscr{F}^{\nu'}_{u,\lambda'})^F$, and $y, y' \in Y^F_{u,u'}$ such that yQ (resp. $y'Q$) corresponds to E (resp. E').*

Proof. First consider the case (I) in 4.5. Let W_0 be the subspace of V_0 spanned by $v_{i,1}, \dots, v_{i,d}$. Then, W_0 is u-stable and $u|_{W_0}$ is regular unipotent. We consider the flag $(W_0 \subset W_0^\perp)$ in V_0. Then, $u|_{W_0^\perp/W_0}$ is of type λ' and $u|_{V_0/W_0^\perp}$ is regular unipotent. Thus, if we put $W = \mathbf{k} \otimes_{\mathbf{F}_{q^2}} W_0, W' = \mathbf{k} \otimes_{\mathbf{F}_{q^2}} W_0^\perp$, the flag $E = (W \subset W')$ is contained in $\mathscr{F}_{u,\lambda'}$, and is F-stable. Moreover, $u|_{W_0^\perp/W_0}$ is a split element of type λ' on $W_0^\perp/W_0$. Here, M is GL^F_n-conjugate to $S(GL(W) \times GL(W'/W) \times GL(V/W'))$. Thus, there exists a split element $u' \in C'^F$ such that u' is GL^F_n-conjugate to $(u|_W, u|_{W'/W}, u|_{V/W'})$. This implies that there exists $y \in Y^F_{u,u'}$ such that yQ corresponds to E under the isomorphism $\mathcal{Q}_{u,C'} \simeq \mathscr{F}_{u,\lambda'}$.

Next, consider the case (II) in 4.5. Put $\lambda_i = \lambda_{i+1} = h$, and $x = u - 1$. Let U_0 be the subspace of V_0 spanned by $v_{i,h}, v_{i+1,h}$. We define a form $\langle\ ,\ \rangle$ on U_0 by $\langle v, w \rangle = (v, x^{h-1} w)$.

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Then, $\langle v,w\rangle$ is a nondegenerate sesquilinear form on U_0. We choose $0\neq v\in U_0$ such that $\langle v,v\rangle=0$. Let W_0 be the subspace of V_0 spanned by $x^{h-1}v,x^{h-2}v,\dots,x^{h-d}v$. Then, W_0 is a u-stable subspace of V_0 with dimension d, and $u|_{W_0}$ is regular unipotent. We have $u|_{W_0^{\perp}/W_0}$ is of type λ', and $u|_{V_0/W_0^{\perp}}$ is regular unipotent. Thus, if we define the corresponding subspace W,W' in V as before, we see that $E=(W\subset W')\in\mathscr{F}^F_{u,\lambda'}$. Since $u|_{W_0^{\perp}/W_0}$ is a split element on $W_0^{\perp}/W_0$, one can find a split element $u'\in C'^F$, and $y\in Y^F_{u,u'}$ such that yQ corresponds to E, similarly to the previous case.

Finally, consider the case (III) of 4.5. Put $h=\lambda_i$. Write $v_{i,1},\dots,v_{i,h}$ as $e_1,\dots,e_h$, and write $v_{i+1,1},\dots,v_{i+1,h+d}$ as $e'_1,\dots,e'_{h+d}$. Let W_0 be the subspace of V_0 spanned by $\alpha e_1+e'_1,\dots,\alpha e_d+e'_d$ for $\alpha\in\mathbf{F}_{q^2}$. Then, W_0 is u-stable, $u|_{W_0}$ is regular unipotent, $u|_{W_0^{\perp}/W_0}$ is of type λ', and $u|_{V_0/W_0^{\perp}}$ is regular unipotent. Here, note that $u|_{V_0/W_0}$ is of type ν (resp. ν') if $\alpha\neq 0$ (resp. $\alpha=0$). Thus, if we define W,W' the corresponding subspaces of V for $\alpha\neq 0$, we see that $E=(W\subset W')\in(\mathscr{F}^{\nu}_{u,\lambda'})^F$, whereas if W,W' are defined from W_0 with $\alpha=0$, then $E'=(W\subset W')\in(\mathscr{F}^{\nu'}_{u,\lambda'})^F$. Hence, the assertion can be proved similarly. The lemma is proved. □

4.9. Assume that (λ,λ') is in the case (I) or (II) of 4.5. We choose a split element $u\in C^F_{\lambda}$. Then, by Lemma 4.8, there exists a split element $u'\in C'^F$ and $y\in Y^F_{u,u'}$. By Lemma 4.6, $Z_G(u)$ acts transitively on $\mathcal{Q}_{u,C'}$. Thus, $I_0=Z^0_G(u)yZ^0_M(u')U_Q$ is an F-stable irreducible component of $Y_{u,u'}$, and all other irreducible components of $Y_{u,u'}$ are obtained by the action of $A_G(u)\times A_M(u')$. Let A'_0 be the stabilizer of I_0 in $A_G(u)\times A_M(u')$, and A_0 be the quotient group $(A_G(u)\times A_M(u'))/A'_0$ (note that $A_G(u)\times A_M(u')$ is abelian). Then, $X_{u,u'}$ is in bijection with A_0, and $\varepsilon_{u,u'}$ is isomorphic to $\bar{\mathbf{Q}}_l[A_0]$ as A_0-modules. Note that F acts naturally on A_0, and the isomorphism $\varepsilon_{u,u'}\simeq\bar{\mathbf{Q}}_l[A_0]$ is compatible with F-actions. Let ρ_1 be the irreducible character of A_0 corresponding to $\rho\otimes\rho'^*$. We assume that ρ and ρ' are F-stable. Then, ρ_1 is F-stable. Now, the subspace $\rho\otimes\rho'^*$ of $\varepsilon_{u,u'}$ coincides with the subspace of $\bar{\mathbf{Q}}_l[A_0]$ on which A_0 acts via ρ_1. Hence, this is a one-dimensional subspace spanned by

$$z_{\rho_1}=\sum_{a\in A_0}\rho_1(a)^{-1}a\in\bar{\mathbf{Q}}_l[A_0].$$

Since ρ_1 is F-stable, we see that F acts trivially on $\rho\otimes\rho'^*\subset\varepsilon_{u,u'}$.

4.10. Next, assume that (λ,λ') is in the case (III). We follow the notation in the proof of Lemma 4.8. By Lemma 4.6(ii), we have a partition $\mathscr{F}_{u,\lambda'}=\mathscr{F}^{\nu}_{u,\lambda'}\sqcup\mathscr{F}^{\nu'}_{u,\lambda'}$. For $E=(W\subset W')\in\mathscr{F}_{u,\lambda'}$, we have $\dim(W\cap\mathrm{Ker}\,x)=1$, and $W\cap\mathrm{Ker}\,x\subset\langle e_1,e'_1\rangle$. Thus, we have a map $\mathscr{F}_{u,\lambda'}\to\mathbf{P}(\langle e_1,e'_1\rangle),(W\subset W')\mapsto W\cap\mathrm{Ker}\,x$. Then, $W\cap\mathrm{Ker}\,x=\langle v\rangle$, where $v=\alpha e_1+e'_1$ for some $\alpha\in\mathbf{F}_{q^2}$ (note that $E\notin\mathscr{F}_{u,\lambda'}$ if $v=e_1$). Here, $E\in\mathscr{F}^{\nu}_{u,\lambda'}$ if $\alpha\neq 0$, and $E\in\mathscr{F}^{\nu'}_{u,\lambda'}$ if $\alpha=0$. It follows that $\mathscr{F}^{\nu}_{u,\lambda'}$ is open in $\mathscr{F}_{u,\lambda'}$. Let $\mathcal{Q}^{\nu}_{u,C'}$ (resp. $\mathcal{Q}^{\nu'}_{u,C'}$) be the variety corresponding to $\mathscr{F}^{\nu}_{u,\lambda'}$ (resp. $\mathscr{F}^{\nu'}_{u,\lambda'}$), and put $Y^{\nu}_{u,u'}=q^{-1}(\mathcal{Q}^{\nu}_{u,C'})$ (resp. $Y^{\nu'}_{u,u'}=q^{-1}(\mathcal{Q}^{\nu'}_{u,C'})$), where $q:Y_{u,u'}\to\mathcal{Q}_{u,C'}$ is the map defined as in 3.10. We have $Y_{u,u'}=Y^{\nu}_{u,u'}\sqcup Y^{\nu'}_{u,u'}$, and $Y^{\nu}_{u,u'}$ is open in $Y_{u,u'}$. Hence, $\dim Y^{\nu}_{u,u'}=d_{u,u'}$. $Y^{\nu}_{u,u'}$ is stable under the action of $Z_G(u)\times Z_M(u')U_Q$. If we put $X^{\nu}_{u,u'}$ the set of irreducible components of dimension $d_{u,u'}$, then $X^{\nu}_{u,u'}$ is an $A_G(u)\times A_M(u')$-stable (nonempty) subset of $X_{u,u'}$. Thus, we have a subrepresentation $\varepsilon^{\nu}_{u,u'}$ of $\varepsilon_{u,u'}$. We consider the representation $\rho\otimes\rho'^*$ appearing in $\varepsilon_{u,u'}$. By 4.5(III), the multiplicity of $\rho\otimes\rho'^*$ in $\varepsilon_{u,u'}$ is equal to 2. Since $A_G(u)\times A_M(u')$ acts transitively on the

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set $X^{\nu}_{u,u'}$, a similar argument as in 4.9 shows that $\rho\otimes\rho'^{*}$ appears in $\varepsilon^{\nu}_{u,u'}$ with multiplicity one. In particular, $X^{\nu}_{u,u'}\neq X_{u,u'}$, and we see that $\dim Y^{\nu'}_{u,u'}=d_{u,u'}$. If we define $X^{\nu'}_{u,u'}$ as the set of irreducible components of dimension $d_{u,u'}$, we have $X_{u,u'}=X^{\nu}_{u,u'}\sqcup X^{\nu'}_{u,u'}$. Thus, $\varepsilon_{u,u'}$ is decomposed as $\varepsilon_{u,u'}=\varepsilon^{\nu}_{u,u'}\oplus\varepsilon^{\nu'}_{u,u'}$, where $\rho\otimes\rho'^{*}$ appears with multiplicity one in either of them. By applying the argument in 4.9, we see that F acts trivially on $\rho\otimes\rho'^{*}\subset\varepsilon^{\nu}_{u,u'}$, and on $\rho\otimes\rho'^{*}\subset\varepsilon^{\nu'}_{u,u'}$. Hence, F acts trivially on the (two-dimensional) isotypic subspace of $\rho\otimes\rho'^{*}$ in $\varepsilon_{u,u'}$.

Summing up the discussion in 4.9 and 4.10, we have the following proposition.

PROPOSITION 4.11. *Let (λ,λ') be as in 4.5. For a split element $u\in C^F_{\lambda}$, there exists a split element $u'\in C^F_{\lambda'}$ such that F acts trivially on the isotypic subspace of $\rho\otimes\rho'^{*}$ in $\varepsilon_{u,u'}$.*

4.12. For each $(C,\mathscr{E})\in\mathscr{N}^F_G$, we choose a split element $u=u^{\bullet}\in C^F$. Let $\rho\in A_G(u)^{\wedge}$ be an F-stable irreducible representation of $A_G(u)$ corresponding to $\mathscr{E}$. Note that $A_G(u)$ is abelian, and ρ is one-dimensional. As a split extension $\widetilde{\rho}$ of ρ, we choose a trivial extension such that τ acts trivially on the representation space of ρ.

Let $\chi\in Z^{\wedge}$ be an F-stable irreducible character of order d. We consider the correspondence $\mathscr{N}_{\chi}\simeq S^{\wedge}_{n/d}$ as in (4.3.1) associated to $(L,C_0,\mathscr{E}_0)\in\mathscr{M}_{\chi}$. We are now ready to prove Theorem 1.5 for SL_n. Note that split elements are defined in 4.7.

THEOREM 4.13. *Assume that $G=SL_n$, with F: nonsplit type. For each $(C,\mathscr{E})\in\mathscr{N}^F_{\chi}$, let $u^{\bullet}\in C^F$ be a split element, and assume that $(C,\mathscr{E})\leftrightarrow(u^{\bullet},\rho)$. For each $(L,C_0,\mathscr{E}_0)\in\mathscr{M}^F_G$, let $u_0^{\bullet}\in C_0^F$ be a split element as above (applied for L), and assume that $(C_0,\mathscr{E}_0)\leftrightarrow(u_0^{\bullet},\rho_0)$. Then, for each $(C,\mathscr{E})$ belonging to the series $(L,C_0,\mathscr{E}_0)$, we have $\gamma(u_0^{\bullet},\widetilde{\rho}_0,u^{\bullet},\widetilde{\rho})=1$, where $\widetilde{\rho}_0\in\widetilde{A}_L(u_0^{\bullet})^{\wedge}$ (resp. $\widetilde{\rho}\in\widetilde{A}_G(u^{\bullet})^{\wedge}$) is the split extension of ρ_0 (resp. ρ).*

Proof. Take $(u,\rho)\leftrightarrow(C,\mathscr{E})\in(\mathscr{N}_G)^F_{\chi}$ with $C=C_{\lambda}$. We consider an F-stable Levi subgroup M of G as in 4.4. Then, there exists $\lambda'\in\mathscr{P}_{n-2d}$ such that (λ,λ') is in the case (I) $\sim$ (III) in 4.5. Take $(u',\rho')\leftrightarrow(C',\mathscr{E}')\in(\mathscr{N}_M)^F_{\chi}$. By the restriction formula, $\rho\otimes\rho'^{*}$ appears in the representation $\varepsilon_{u,u'}$. We assume that $u\in C^F$ is a split element. Then, by Proposition 4.11, there exists a split element $u'\in C'^F$ such that F acts trivially on the isotypic subspace of $\rho\otimes\rho'^{*}$ in $\varepsilon_{u,u'}$. Then, by (1.8.1), $\sigma_{\rho,\rho'}:M_{\rho,\rho'}\to M_{\rho,\rho'}$ is a scalar multiplication by $q^{-(\dim C-\dim C')/2+\dim U_Q}$. By induction on n, we may assume that $q^{-(a_0'+r')/2}\sigma_{(u',\rho')}$ makes the irreducible $S_{n/d-2}$ module $V_{(u',\rho')}$ to the preferred extension to $\widetilde{S}_{n/d-2}$. Then, $q^{-(a_0+r)/2}\sigma_{(u,\rho)}$ makes the irreducible $S_{n/d}$-module $V_{(u,\rho)}$ to the preferred extension to $\widetilde{S}_{n/d}$. This implies, by (1.8.2), that $\gamma(u,\widetilde{\rho},u_0,\widetilde{\rho}_0)=1$. It remains to consider the case where $n=d$, $(u,\rho)\leftrightarrow(C_{\lambda},\mathscr{E})\in\mathscr{N}_{\chi}$. But in this case, $\lambda=(d)$ and u is regular unipotent, $\rho=\chi\in A_G(u)^{\wedge}=Z^{\wedge}$. Then $(C,\mathscr{E})$ is a cuspidal pair, and we have $\gamma=1$ automatically. The theorem is proved. ∎

REMARK 4.14. In the proof of Theorem 4.13, we have assumed that F is of nonsplit type. But the argument there works well for the split case, under a suitable modification, and it gives a proof of Theorem 1.5 for the split case. Of course, in the split case, one can use $M\simeq S(GL_d\times GL_{n-d})$, which makes the argument simpler. In any case, if we use the restriction formula (1.8.1), the discussion turns out to be much simpler compared with that used in [S3, I].

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