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Countable lattice-ordered groups

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To Philip Hall – in memoriam†

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A lattice-ordered group is a group and a lattice such that the group operation distributes through the lattice operations (i.e. $f(g \vee h)k = fgk \vee fhk$ and dually). Lattice-ordered groups are torsion-free groups and distributive lattices. They further satisfy $f \wedge g = (f^{-1} \vee g^{-1})^{-1}$ and $f \vee g = (f^{-1} \wedge g^{-1})^{-1}$. Since the lattice is distributive, each lattice-ordered group word can be written in the form $\vee_A \wedge_B w_{\alpha\beta}$ where A and B are finite and each $w_{\alpha\beta}$ is a group word in $\{x_i : i \in I\}$. Unfortunately, even for free lattice-ordered groups, this form is not unique. We will use the prefix l- for maps between lattice-ordered groups that preserve both the group and lattice operations, and e for the identity element. A presentation $(x_i; r_j(\mathbf{x}) = e)_{i\in I,\ j\in J}$ is the quotient of the free lattice-ordered group F on $\{x_i : i \in I\}$ by the l-ideal (convex normal sublattice subgroup) generated by its subset $\{r_j(\mathbf{x}) : j \in J\}$. $\{x_i : i \in I\}$ is called a generating set and $\{r_j(\mathbf{x}) : j \in J\}$ a defining set of relations. If I and J are finite we have a finitely presented lattice-ordered group. If $|I| \leqslant \aleph_0$, we can effectively enumerate all lattice-ordered group words $r_1(\mathbf{x}), r_2(\mathbf{x}), \ldots$ in $\{x_i; i \in I\}$. If I is finite and J (for this enumeration) is a recursively enumerable set, we say that we have a recursively presented lattice-ordered group. Throughout $\mathbb{Z}$ denotes the group of integers and $\mathbb{R}$ the real line.

Our purpose in this paper is to prove the natural analogues of three theorems from combinatorial group theory (5), chapter IV, theorems 4·9, 3·1 and 3·5 – in particular, theorem C is a natural analogue of an unpublished theorem of Philip Hall (4).

Theorem A. *There is a* 2 *generator* 1 *relator lattice-ordered group l-isomorphic to a proper quotient of itself; i.e. it is non-Hopfian.*

Theorem B. *Every countable lattice-ordered group G can be l-embedded in a* 4 *generator lattice-ordered group that is finitely* (*recursively*) *presented if G is.*

Theorem C. *Every countable lattice-ordered group can be l-embedded in a* 9 *generator l-simple lattice-ordered group* (*i.e. no convex normal sublattice subgroups*).

Theorem 10A of (1) shows that every countable lattice-ordered group can be l-embedded in a 2 generator lattice-ordered group; however, that proof loses control of the defining relations. Theorem B maintains control of the defining relations at the cost of two extra generators. A lattice-ordered group satisfying $\forall x \forall y\ (x \leqslant y \text{ or } y \leqslant x)$ is called an o-group; as usual, $x \leqslant y$ is a shorthand for $x \wedge y = x$. Reilly (6) proved that every (countable) o-group can be l-embedded in a (countable) l-simple o-group..

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† I was at Downing College 1963–67 and took both of Philip Hall's Part III courses in 1967. They formed probably the most exciting, exhilarating, exacting and exhausting learning experience of my life.

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Theorems 12A and 12C of (1) show that every countable lattice-ordered group can be l-embedded in a countable l-simple (indeed simple) lattice-ordered group. The purpose of Theorem C is to put a finite bound on the number of generators; it is the natural analogue of Philip Hall's result for groups. Whereas the proofs in group theory rely heavily on free products with amalgamation or the Higman–Neumann–Neumann theorem (see (5)), these constructions are not valid for lattice-ordered groups ((1), theorem 10C and (2)) and we must resort to l-permutation groups and results (proved using l-permutation groups) whose group-theoretic analogues naturally belong to combinatorial group theory; recall that an l-permutation group (G, Ω) is a group G of order-preserving permutations of a linearly ordered set Ω that is a lattice-ordered group under the pointwise ordering: $\alpha(f \vee g) = \max\{\alpha f, \alpha g\}$, $\alpha(f \wedge g) = \min\{\alpha f, \alpha g\}$ for all $\alpha \in \Omega$ $(f, g \in G)$ – we require $f \vee g, f \wedge g \in G$ whenever $f, g \in G$. We will write $A(\Omega)$ for the lattice-ordered group of all order-preserving permutations of the linearly ordered set Ω, and $\operatorname{supp}(f) = \{\alpha \in \Omega : \alpha f \neq \alpha\}$ for the support of f. Every lattice-ordered group is an l-permutation group for some linearly ordered set Ω – (1) appendix 1.

The proofs of Theorems A–C become progressively harder – the first being an easy adaptation of (5), theorem IV·4·9.

We write $[x, y]$ for $x^{-1}y^{-1}xy$.

The reader should consult (1) for any undefined terms. We will need the following facts from (1); (i) comes from the proof of theorem 2E (see the remarks on page 58), (ii) from the proof of lemma 2.2.1. A *bump* is an order-preserving permutation whose unique extension to the Dedekind completion has only one interval of support.

Lemma 0 indicates the considerable 'local' freedom in the choice of cube roots and conjugators.

LEMMA 0. *Let Ω be a doubly homogeneous chain and $e < f, g \in A(\Omega)$.*

(i) *If $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 = \alpha_0 f$ in Ω and $h_i : [\alpha_i, \alpha_{i+1}] \cong [\alpha_{i+1}, \alpha_{i+2}]$ $(i = 0, 1)$, then there exists $h \in A(\Omega)$ such that $h^3 = f$ and $h|[\alpha_i, \alpha_{i+1}] = h_i$ $(i = 0, 1)$.*

(ii) *If $\alpha \in \operatorname{supp}(f)$, $\beta \in \operatorname{supp}(g)$ and f, g are bounded bumps the suprema (infima) of whose supports belong to the same $A(\Omega)$ orbit, any ordermorphism between $[\alpha, \alpha f]$ and $[\beta, \beta g]$ can be extended to an element $h \in A(\Omega)$ such that $h^{-1}fh = g$ and $\alpha f^n h = \beta g^n$ $(n \in \mathbb{Z})$. Moreover, we can assume that the support of h is contained in any open interval that properly includes the closure of the interval generated by* $\operatorname{supp}(f) \cup \operatorname{supp}(g)$.

Proof of Theorem A. Let $G = (x_0, x_1; x_1^{-1}x_0^2x_1 = x_0^3)$ and $\theta : G \to G$ be the l-homomorphism given by: $x_0\theta = x_0^2$ and $x_1\theta = x_1$. Since $(x_1^{-1}x_0^2x_1)\theta = x_1^{-1}x_0^4x_1 = x_0^6 = x_0^3\theta$, θ is well-defined. Moreover, θ is onto since $x_0 = x_1^{-1}x_0^2x_1 \cdot x_0^{-2} = [x_1, x_0^{-1}]\theta$. Hence $G/\operatorname{Ker}(\theta) \cong G\theta = G$. To complete the proof we must show that $\operatorname{Ker}(\theta) \neq \{e\}$. Let $e < f_0 \in A(\mathbb{R})$ have one interval of support (i.e. be a bump) and let $\alpha \in \operatorname{supp}(f_0)$. By lemma 0 (ii), there is $f_1 \in A(\mathbb{R})$ such that $f_1^{-1}f_0^2f_1 = f_0^3$ with $\alpha f_1 = \alpha$, $\alpha f_0 f_1 = \alpha f_0$ and $\alpha f_0^2 f_1 = \alpha f_0^3$ (f_0^2 and f_0^3 are bumps with the same support). Thus the sublattice subgroup of $A(\mathbb{R})$ generated by f_0 and f_1 is an l-homomorphic image of G. Now

$$\alpha f_1^{-1}f_0f_1f_0 = \alpha f_0^2 \neq \alpha f_0^3 = \alpha f_0 f_1^{-1}f_0f_1;$$

so $f_1^{-1}f_0f_1f_0 \neq f_0f_1^{-1}f_0f_1$. Therefore $[x_0, x_1^{-1}x_0x_1] \neq e$. But $= [x_0, x_1^{-1}x_0x_1]\theta\ [x_0^2, x_0^3] = e$. Consequently, $\operatorname{Ker}(\theta) \neq \{e\}$ as desired. |

The non-Hopfian 2 generator 1 relator group exhibited in (5), page 197, has the same presentation. However, that group is the quotient of the free group on x_0, x_1 by the normal subgroup generated by $x_1^{-1}x_0^2x_1x_0^{-3}$. We do not know if this group is lattice-

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orderable (it cannot be linearly ordered); even if it is, it may not be finitely presented as a lattice-ordered group.

We will use the following result which is easily extracted from the proof of theorem 10B of (1).

LEMMA 1. *Let H be a lattice-ordered group, and $\{f_\nu\colon \nu\in N\}$ and $\{g_\nu\colon \nu\in N\}$ be pairwise disjoint subsets of H with $e < f_\nu \leqslant g_\nu$ for all $\nu\in N$. Then H can be l-embedded in a lattice-ordered group containing an element t such that $t^{-1}f_\nu t = g_\nu$ for all $\nu\in N$.*

Proof of Theorem B. Let $G = (x_i; s_j(\mathbf{x}) = e)_{i\in I, j\in J}$ be countable. So we may assume $I, J \subseteq \omega = \{0, 1, 2, \ldots\}$ and $x_i \neq e$ all $i\in I$. We may also assume that each $x_i > e$ (replace any offending x_i by x_i^{-1} if $x_i < e$, by $x_i \vee e$ and $x_i^{-1} \vee e$ otherwise; the new generating set has cardinality no greater than $2|I|$). By (1), corollary 2 L, every countable lattice-ordered group can be l-embedded in $A(\mathbb{R})$. Thus G can be l-embedded in $A(\mathbb{R})$ and hence in the order-preserving permutations of the open interval $(0, 1)$; we will identify G with its image. Let $e < a_0\in A(\mathbb{R})$ be a bump with $0a_0 = 1$ and $\operatorname{supp}(a_0) \subseteq (-1, 2)$. Let $e < b_0\in A(\mathbb{R})$ with $(-1)b_0 = 2$. Let c_0 be the pointwise supremum of the pairwise disjoint set $\{b_0^{-n}a_0b_0^n\colon 0 < n\in\omega\}$. So $a_0 \wedge c_0 = e$, $b_0^{-1}a_0b_0 \leqslant c_0$ and $b_0^{-1}c_0b_0 \leqslant c_0 \leqslant b_0$. Hence G can be l-embedded in

$$H = (a, b, c, x_i; a \wedge c = e,\ b^{-1}ab \leqslant c,\ b^{-1}cb \leqslant c,\ c \leqslant b,\ x_i \leqslant a,\ s_j(\mathbf{x}) = e)_{i\in I, j\in J},$$

where again we use $x \leqslant y$ as a shorthand for $x \wedge y = x$. As in (1), lemma 13.1, it is easily seen that $b^{-m}ab^m \wedge b^{-n}ab^n = e$ in H if and only if m and n are distinct integers. By Lemma 1, H can be l-embedded in a lattice-ordered group containing an element d_0 such that $d_0^{-1}b^{-i}x_i\,b^id_0 = b^{-i}ab^i$ $(i\in I)$. Thus G can be l-embedded in

$$L = (a, b, c, d, x_i; a \wedge c = e,\ b^{-1}ab \leqslant c,\ b^{-1}cb \leqslant c,\ c \leqslant b,\ x_i \leqslant a,\ s_j(\mathbf{x}) = e,\ d^{-1}b^{-i}x_ib^id = b^{-i}ab^i)_{i\in I\, j\in J}.$$

But each $x_i = b^idb^{-i}ab^id^{-1}b^{-i}$ so L is generated by a, b, c, d. Moreover, if I is finite, L is finitely (recursively) presented if G is. |

Once Higman's Theorem for lattice-ordered groups (every recursively presented lattice-ordered group can be l-embedded in a finitely presented lattice-ordered group) is established, it will follow from Theorem B that there is a 4 generator finitely presented lattice-ordered group that contains as sublattice subgroups l-isomorphic copies of every recursively presented lattice-ordered group; i.e. there is a universal (for recursively presented lattice-ordered groups) finitely presented lattice-ordered group.

In order to prove Theorem C we first need a theorem which is of independent interest; it appears in (3) as the Messuage Lemma and is used to prove that there is no recursive algorithm to determine which finitely presented lattice-ordered groups have only one element and which do not. The theorem shows that given any finitely generated lattice-ordered group G and any word w in these generators, one can adjoin finitely many new generators and relations (dependent on w) so that the new lattice-ordered group contains an l-isomorphic copy of G if $w \neq e$ in G, and is trivial if $w = e$ in G, and do this in a uniform way. Let $|x| = x \vee x^{-1}$. Then $|x| \geqslant e$ and $|x| = e$ if and only if $x = e$ ((1), lemma 1.11.4).

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THEOREM D. *Let* $G = (x_i; s_j(\mathbf{x}) = e)_{i\in I, j\in J}$ *be a finitely generated lattice-ordered group and* w *a lattice-ordered group word in* $\{x_i : i \in I\}$. *Let*

$$G_w = (x_i, a_0, a_1, a_2, a_3, a_4; s_j(\mathbf{x}) = e,\ a_0^{-1}|w|\,a_0 = \vee\{|x_i| : i\in I\},\ a_1 \geqslant e,$$
$$a_2^{-1}|a_4|\,a_2 = |a_0|, |wa_4^{-1}| \wedge |a_4| = e,\ |a_2| \vee |a_3| \leqslant a_1^6,$$
$$a_3^{-1}a_1^3a_3 = (|a_0| \vee a_1^{-1}|a_0|\,a_1 \vee a_1^{-3}|a_0|\,a_1^3 \vee a_1^{-4}|a_0|\,a_1^4)^4)_{i\in I, j\in J}.$$

If $w = e$ *in* G, *then* $G_w = \{e\}$; *if* $w \neq e$ *in* G, G *can be* l*-embedded in* G_w.

The proof of Theorem D occurs in (3).

We can now prove Theorem C by a technique of C. F. Miller III.

So let C be a countable lattice-ordered group. By theorems 12A and C of (1), C can be l-embedded in a countable simple lattice-ordered group.

Indeed, we can obtain this fact directly from the techniques we have already used in the proof of Theorem B. (Let C be a countable lattice-ordered group. List all pairs $(f_0, g_0), (f_1, g_1), \ldots$ of elements of C for which $f_n, g_n \neq e$ $(n\in\omega)$. As in the proof of Theorem B, C can be l-embedded in

$$C^\dagger = (x_i, a, b, c, d; a \wedge c = e,\ b^{-1}ab \leqslant c,\ b^{-1}cb \leqslant c,\ c \leqslant b,\ d^{-1}b^{-i}x_ib^id = b^{-i}ab^i,$$
$$x_i \leqslant a,\ d^{-1}b^n\,|f_n|\,b^{-n}d = b^n\,|g_n|\,b^{-n},\ s_j(\mathbf{x}) = e)_{i\in I, j\in J, n\in\omega}.$$

Note that any two elements of C which strictly exceed e are conjugate in $C^\dagger$. Now let $C_0 = C, C_{m+1} = C_m{}^\dagger$ $(m\in\omega)$, and $H = \cup\{C_n : n\in\omega\}$. By construction, H is a countable l-simple lattice-ordered group.) Hence we may assume C itself is l-simple. By Theorem B, C can be l-embedded in a 4 generator lattice-ordered group G. Let $e \neq w\in C$. By Theorem D, G can be l-embedded in G_w. By Zorn's lemma, there is an l-ideal (normal convex sublattice subgroup) M of G_w maximal among those whose intersection with C is $\{e\}$. Thus C is naturally l-embedded in G_w/M (which has 9 generators). Let N be an l-ideal of G_w. If $N \cap C \neq \{e\}$, then $w\in N \cap C$ since C is l-simple. Thus $Nw = N$ in G_w/N. But in G_w/N all the defining relations of G_w hold. Therefore $G_w/N = \{e\}$ by Theorem D. Consequently, $N = G_w$. Hence G_w/M is l-simple. |

One might wonder at first if Theorem C can be strengthened to: every finitely (recursively) presented lattice-ordered group can be l-embedded in a finitely (recursively) presented l-simple one. The answer is no since it is easily seen (e.g. (5), p. 191):

LEMMA 2. *Every recursively presented* l*-simple lattice-ordered group has soluble word problem.*

Hence any recursively presented lattice-ordered group that can be l-embedded in a recursively presented l-simple lattice-ordered group must have soluble word problem (no claim is made that the recursive algorithm for solving the word problem can be explicitly given, merely that it exists). By (1), theorem 13A, there is a finitely presented lattice-ordered group which cannot be l-embedded in any recursively presented l-simple lattice-ordered group.

Since we cannot retain the niceness of the relations of the original lattice-ordered group, we might as well use (1), theorem 10A, in place of Theorem B in the proof of Theorem C. This will improve the theorem to:

Every countable lattice-ordered group can be l*-embedded in a 7 generator* l*-simple lattice-ordered group.*

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We close by posing the question: For which classes of lattice-ordered groups do analogues of Theorem A, B and C hold (e.g. for which varieties of lattice-ordered groups)?

A lattice-ordered group G is said to be *normal-valued* if $(f \vee e)(g \vee e) \leqslant (g \vee e)^2 (f \vee e)^2$ for all $f, g \in G$. The variety of all normal-valued lattice-ordered groups contains every proper variety of lattice-ordered groups (1), theorem 11D. By (1), lemma 11.6:

THEOREM E. *Any finitely generated l-simple normal valued lattice-ordered group is an o-group.*

Hence Theorem C fails for any class of normal-valued lattice-ordered groups that contains a countable lattice-ordered group that is not an o-group. In particular, it fails for any proper variety of lattice-ordered groups (apply (1), theorem 11D).

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