J. Plasma Physics (1998), *vol.* 59, *part* 4, *pp.* 629–637. Printed in the United Kingdom

© 1998 Cambridge University Press

Van Kampen modes revisited

J. P. DOUGHERTY

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

(Received 8 September 1997 and in revised form 15 October 1997)

To Professor Radu Balescu in honour of his 65th birthday

The Van Kampen mode discussion of plasma oscillations is briefly reviewed alongside theories of non-equilbrium statistical mechanics and attention is drawn to illuminating connections between them.

1. Introduction

It is a pleasure to greet Radu. He has been a key contributor to the intense development of theoretical plasma physics that started in the late 1950s. As Editor of the *Journal of Plasma Physics* during 1967–94, I was delighted that he and his co-authors chose to publish many papers in the Journal, and was grateful to him for help in other ways. Although he is known to readers of this Journal for his work in plasma physics, he has also published extensively in the broader subject of non-equilibrium statistical mechanics (NESM), of which theoretical plasma physics forms a specialized sub-discipline. Particularly notable is his vast treatise *Equilibrium and Non-Equilibrium Statistical Mechanics* (Balescu, 1975). This was a natural outcome of his position as a member of the well-known research group in statistical mechanics at the Université Libre, Brussels, led for many years by Professor Ilya Prigogine.

In the present paper I wish to draw attention to a rather classical topic in plasma physics (though one that is still the subject of research: see Tracy *et al.* 1996), and its connection with recent work on NESM. The connection is not that of a mere special case or application, but of mathematical method and philosophy.

2. Non-equilibrium statistical mechanics

Our discussion is restricted to classical (rather than quantum) NESM. We consider the formal link between microscopic and macroscopic dynamics. At the microscopic level one has the orbit $X(t)$ in Γ space, where X specifies completely the positions and momenta of all the particles. To connect this with the macroscopic description, it is widely agreed that one must introduce a probability density $\rho(X,t)$. Two logically distinct reasons are offered:

(i) the individual particle orbits, which are known if $X(t)$ is known, are far too numerous and detailed to comprehend or to compute, and they cannot be observed by the macroscopic experimentalist;

(ii) molecular systems are such that the evolution equations for $X(t)$ exhibit instability with respect to the initial data.

https://doi.org/10.1017/S0022377898006618 Published online by Cambridge University Press

In this note, ρ is to be interpreted in the microcanonical picture, and the energy is the only constant of the motion.

For an isolated system, $X(t)$ is determined, for both future and past, by its value at one instant, a property that is inherited by $\rho(X,t)$, and implemented by Liouville's equation, symbolically

$$\dot{\rho} = L\rho, \tag{1}$$

where L is the Liouvillian operator and is invertible. The equation has a formal solution operator $\exp(Lt)$, which constitutes the group generated by L.

At this point methods of functional analysis are introduced. We must regard ρ as belonging to a suitably chosen function space E, and consider the spectral properties of L. Suppose at first that E is Hilbert space. L always has an eigenvalue 0, with an eigenfunction that represents thermal equilibrium. (The eigenvalue is simple in the case of an ergodic system.) The fact that the underlying motion is measure-preserving ensures that the rest of the spectrum must be purely imaginary, consistent with the reversibility of such systems. That, however, does not accord with macroscopic observation, namely the approach to equilibrium. But if the microscopic system has a sufficiently chaotic character (namely 'mixing', as suggested by the instability with respect to the initial data), the rest of the spectrum is a continuous one (in fact, absolutely continuous in the known examples). That precludes purely oscillating contributions, but still does not provide approach to equilibrium.

The eventual goal is to understand macroscopic evolution, and in particular to show how evolution equations can be derived. To the plasma physicist this means such things as the Boltzmann equation, the Vlasov equation, the Lenard–Balescu equation (Balescu 1960) and MHD equations of various degrees of sophistication (Balescu 1988*a*, *b*). The contents of the previous paragraph suggest how to start, but further apparatus will be needed, and the situation so far looks unpromising owing to the point about irreversibility.

3. Hierarchy equations and closure

Plasma physicists will be familiar with the standard way to deal with this situation in their own context. The 1-particle distribution $f_1(\mathbf{x}, \mathbf{v}, t)$ is accepted as a suitable level of description for each species. It is obtained from ρ by integrating over all the particles but one (variously called projection or marginalization). The remaining variables, associated with one particle, are then re-interpreted as 'free'. We attempt to reduce Liouville's equation by a similar projection, but find that f_1 does not satisfy an autonomous evolution equation owing to the presence of a term involving the 2-particle distribution f_2. And an attempted equation for f_2 similarly involves f_3, etc., forming the so-called BBGKY hierarchy. 'Closure' is made possible by associating the successive steps with a perturbation expansion, using the plasma parameter $(nh^3)^{-1}$ (n = particle density, $h = (\epsilon_0 kT/ne^2)^{1/2}$ = Debye length) as the small quantity. Such an expansion is a generalization to NESM of the Mayer cluster expansion of equilibrium theory. In the lowest approximation ρ is a product of identical factors f_1, with f_1 itself satisfying the Vlasov equation; there is no irreversibility. The next approximation uses a representation of the form

$$f_2(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t) = f_1(\mathbf{x}_1, \mathbf{v}_1, t) f_1(\mathbf{x}_2, \mathbf{v}_2, t) + g(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, t), \tag{2}$$

https://doi.org/10.1017/S0022377898006618 Published online by Cambridge University Press

where g expresses the correlations, and is small in the sense of the perturbations. Further assumptions are, however, required to obtain the evolution equation for f_1 alone, for, as things stand, one would require initial data for g. To eliminate g, one invokes 'Bogoliubov's hypothesis', which attributes a more rapid time scale to g, enabling g to be replaced by an adiabatic value. Some rather intricate calculations then lead to the Lenard–Balescu equation. For Radu's own methodology one may refer to his earlier book (Balescu 1963), published soon after his work appeared; it gives a good flavour of the subject at that stage. By whichever approach was used, irreversibility has entered here, although somewhat obscurely. These derivations may be regarded as the counterpart of those developed (using brilliant intuition) by Boltzmann in assuming the collisions (for molecules with short-range interactions) to be instantaneous and uncorrelated. The Boltzmann development can be presented as an instance of BBGKY theory but with a different small parameter.

4. Subdynamics

This may be regarded as an attempted generalization or abstraction of the ideas of the previous section. We leave open the choice of the macroscopic variable, simply writing it as $a(y,t)$, where y denotes new free variables (for example, in our previous case it reads $f_1(\mathbf{x},\mathbf{v},t)$, so y consists of the variables of the 1-particle phase space). In the general case a belongs to some function space over y, which we denote by A. Then the reduction in description is represented by a map $M: E \to A$, which is implemented by integration over a kernel $M(y,X)$:

$$a(y,t) = \int_\Gamma M(y,X)\,\rho(X,t)\,d\Omega. \tag{3}$$

Thus a is simply the expectation value of the quantity prescribed by $M(y,X)$. The 'discarding of information' is represented by a split $\rho = \rho_1 + \rho_2$ where $\rho_1 \in E_1$ is 'wanted' and $\rho_2 \in E_2$ is 'unwanted', and such that $M\rho_2 = 0$. Formally we have a decomposition $E = E_1 \oplus E_2$. Many authors also use the notation $\rho_1 = P\rho$, $\rho_2 = Q\rho$, where P and Q are projection operators. (Some authors use the words 'accessible' and 'inaccessible' in place of our 'wanted' and 'unwanted'.) One can now partition (1) in the form

$$\begin{matrix}\dot\rho_1\\ \dot\rho_2\end{matrix} = \begin{matrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{matrix} \begin{matrix}\rho_1\\ \rho_2\end{matrix}. \tag{4}$$

There then follows a formal calculation, which has been repeated in many publications so is omitted here. One attempts to form an evolution equation for ρ_1 by eliminating ρ_2. In the usual presentation, the Laplace transform is applied at the start, leading to the Laplace transform of the equation about to be quoted, but another version of the derivation leads directly to

$$\frac{\partial}{\partial t} - L_{11}(t)\ \ \rho_1(t) = D(t)\,\rho_2(0) + \int_0^t \psi(t-t')\,\rho_1(t')\,dt'. \tag{5}$$

This provides a Volterra integrodifferential equation for the evolution of ρ_1 in the interval $(0,t)$, including a memory term. Here D and ψ are complicated operators obtained from the parts of L. Equation (5) suffers from the same

https://doi.org/10.1017/S0022377898006618 Published online by Cambridge University Press

difficulty as we noticed in the BBGKY hierarchy: the initial value for ρ_2 has not been eliminated, so we do not yet have an equation for ρ_1 alone. Much research has been conducted into the homogeneous version of (5), or its Laplace transform. At first, the analogy with BBGKY theory might suggest that the kernel ψ should, in viable cases, by rapidly decaying, so that only the recent past of $\rho_1(t')$ is influential on this evolution. This would lead to an evolution equation of the form that the physicist would expect, being approximately Markovian and also introducing irreversibility. However, so long as one works in the Hilbert space model for E, the spectrum of (5) is contained in the spectrum of (1), and the origin of irreversible macroscopic equations remains unexplained.

5. Balescu's subdynamics

Subdynamics originated in the paper of Prigogine *et al.* (1969). In the next few years, Radu Balescu published a large number of papers on his own research in this field, and much of this material was gathered together into his book of 1975. His approach differed somewhat from that outlined in our previous section. As appropriate for a plasma physicist, he adopted $f_1(\mathbf{x}, \mathbf{v}, t)$ as the 'wanted' description throughout. He regarded subdynamics as a mechanism for replacing the pair (ρ_1, ρ_2) by a new one $(\tilde{\rho}_1, \tilde{\rho}_2)$ such that $\tilde{\rho}_1$ still represents the wanted data, but Liouville's equation has separated into uncoupled equations for $\tilde{\rho}_1$ and $\tilde{\rho}_2$ by a new decomposition of E. The development proceeds (as in his 1963 book) by constructing the perturbation series using diagrammatic methods together with summations and renormalizations as in high-energy physics. A large body of results is obtained, not restricted to plasma physics or even classical dynamics. Perhaps a wise pragmatism discouraged Radu from pursuing the analytic questions discussed here, for he notes in his Preface that 'the mathematicians will certainly feel unhappy'; however, the plasma physicists can find here the foundations for their subject.

6. Van Kampen modes and Landau damping

We now return to plasma physics. We recall the standard theory of electrostatic waves in a uniform infinite plasma in one dimension, as recounted in many textbooks, including Balescu (1963). The positive ions form a uniform fixed background, while the electrons are described in the Vlasov picture, by the 1-particle distribution functions, now called $f(x, v, t)$. This is linearized in the form $f(x, v, t) = f_0(v) + g(x, v, t)$, where f_0 is the unperturbed distribution, say Maxwellian, and g now means the small perturbation. The spatial dependence is assumed to take the form $\propto \exp(ikx)$, generality being recovered by the Fourier transform. The linearized Vlasov equations are

$$\frac{\partial g}{\partial t} + ikvg = -\frac{q}{m} E \frac{df_0}{dv}, \tag{6}$$

with Poisson's equation

$$\epsilon_0 \, ikE = q \int_{-\infty}^{\infty} g(v, t) \, dv \tag{7}$$

(q and m are the charge and mass). The dependent variables are $E(t)$ and

https://doi.org/10.1017/S0022377898006618 Published online by Cambridge University Press

$g(v,t)$, which are complex-valued Fourier components. The analogy with sub-dynamics is this: the experimentalist may be supposed to have equipment enabling the observation of E, at least for some set of values of (x,t). So E, which by (7) is just an integral of g over v, will be regarded as the 'wanted' data corresponding to ρ_1, while the remaining data contained in $g(v)$ are supposed to be beyond the reach of observation, and correspond to the 'unwanted' ρ_2.

Readers of this Journal hardly need reminding that the handling of (6) and (7) involves a difficulty if there are (as is usual) particles contained in $f_0(v)$ that are in resonance with wavelike fields. In his famous paper, Landau (1946) provided a solution by taking the Laplace transforms of (6) and (7), supposing that initial data $g(v,0)$ are supplied. Results are obtained from the usual inversion theorem for the transform, entailing the apparently innocuous assumption that the functions occurring are capable of the necessary analytic continuation. That is certainly the case if f_0 is Maxwellian and under mild smoothness conditions for the initial data. The results show the surprising phenomenon that $E(t)$ decays exponentially, although the 'unwanted' part of $g(v,t)$ does not do so, but becomes ever more pathological with t. The result can be re-expressed as a dispersion relation having complex solutions. This led to a vigorous debate as to the origin of irreversibility given that the original equations (even after linearization) were clearly reversible.

As a contribution to that debate, Van Kampen (1955) offered a new approach that is closer to the way physicists think, looking for normal modes of oscillation of the system. To do this, eliminate E between (6) and (7) and seek solutions of the resulting homogeneous equation that are $\propto \exp(-iwt)$. This gives a singular (and formally non-self-adjoint) integral equation. It has the property that (generally) no solutions exist if g is restricted to being in the usual classes of smooth functions, but if the class of functions is suitably enlarged to permit δ functions, a solution can exist for *any* (ω, k), i.e. there is no dispersion relation. But, after setting up a suitable apparatus of eigenfunctions of the adjoint equation, orthogonality and completeness, one can eventually recover Landau's results for the initial value problem. A more formal investigation of this work was published by Case (1959).

This approach raises new questions of interpretation. Do these singular eigenfunctions have a physical meaning? Perhaps it is sufficient to say that they are only used in the special superpositions that are selected to construct a $g(v,0)$ that is smooth (at $t=0$), and that remains smooth, although pathological, for $t>0$. And how is the exponential decay in $E(t)$ to be explained? Mathematically it results from an application of the Riemann–Lebesgue lemma applied to the integral for $E(t)$, in turn depending on g having been smooth. Incidentally, the same methodology applies equally to solving the equations in $t<0$, and leads again to decay of E as $t\to-\infty$. However, it is hard to elaborate these points convincingly in the present context, since (6) and (7) are already of limited validity physically owing to the linearization.

It should be noted that the above summary of this work omits modifications that are required in the case where the plasma is unstable, as happens when f_0 is two-humped. This is important in plasma physics, but does not affect the point I wish to make in this note. It is treated in Case (1959).

Both the Landau and Van Kampen approaches are presented in Balescu (1963, Chaps 4 and 5).

https://doi.org/10.1017/S0022377898006618 Published online by Cambridge University Press

Yet another approach to this problem, using spectral deformation, may be found in Crawford and Hislop (1989). Similar situations exist in other applications; see for example Strogatz *et al.* (1992).

7. Further consideration of NESM

It is time to discuss the connection between the two strands of thought we have been reviewing. In NESM, the starting point is Liouville's equation, (1). This is linear and exact, and in the Hilbert space formulation it has a continuous spectrum occupying the imaginary axis (after subtracting out the equilibrium solution). It is also of almost unimaginable complexity, representing around 10^{23} interacting particles. The purpose of the investigation is to develop evolution equations for much reduced descriptions $a(y,t)$ as in (3), and these are expected to be such that $a(y)$ is smooth in some sense with respect to y. Our Sec. 4 indicates the beginning of a programme for handling such a situation.

In the plasma physics problem sketched in Sec. 6 we find a somewhat similar mathematical situation, based on (6) and (7), with the imaginary axis again forming a continuous spectrum. There is an analogous concept of wanted and unwanted data. Although the problem can be handled by Landau's original transform method, the Van Kampen approach shows how the enlargement of the function space allows each point of the continuous spectrum to become an eigenvalue, with an explicit expression for the eigenfunction. It then shows how these combine to give a decaying contribution to $E(t)$ although not to $g(v,t)$ as a whole. The decay is contingent on some degree of smoothness of the initial data.

The version of subdynamics under discussion in the 1970s may be regarded as the analogue of the Landau solution of the plasma wave problem. Its defect is that there is little prospect of being able to prove the existence of the operators D and ψ or of establishing that they can be continued analytically as required. Thus it remains as a formal scheme incapable of mathematical justification. Nevertheless, if subjected to the machinery of perturbations, it can reproduce earlier results that were based originally on *ad hoc* assumptions and methods.

To make further progress, recent research has resorted to the replacement of genuine Hamiltonian dynamics with much simplified models. This idea is of course ancient and honourable in statistical physics, as for example in the Ising model and all its ramifications. In most cases, the models used for NESM operate with discrete time, i.e. maps rather than flows, and in very simple phase spaces, such as the unit interval or unit square. Examples include the Bernoulli map and the well-known baker map, together with generalizations of these. Some of the maps that have been studied are non-invertible, so are not fully representative of the real problem. A paper by Hasegawa and Saphir (1992) provides a good sample of this work. The Bernoulli and baker maps are analysed in great detail. Explicit expressions are given for the right- and left-eigenfunctions of the map (i.e. the discrete-time analogue of the Liouvillian). Some or all of these eigenfunctions are not contained in the original set (e.g. L_2) of functions, so the space of functions must be enlarged to include singular functions, thus following the example set in the Van Kampen theory of plasma oscillations.

https://doi.org/10.1017/S0022377898006618 Published online by Cambridge University Press

The following is an outline of the way a theory for NESM might be constructed, emphasizing the analogy with the theory of Van Kampen modes. We work with continuous time, so return to (1) and (3). We now adopt a bra–ket notation

$$|\dot{\rho}\rangle = L|\rho\rangle, \quad \rho \in E, \quad |a\rangle = M|\rho\rangle, \quad |a\rangle \in A. \tag{8}$$

Here E denotes a space of Lebesgue-integrable functions in which the theory is originally conceived; in this space the spectrum of L is the imaginary axis. If we extract the equilibrium value of E, that spectrum is purely continuous. We now create a new, much larger space $\hat{E}$ in which E is densely embedded, and in which the action of L can be extended as a densely defined operator $\hat{L}: \hat{E} \to \hat{E}$. This can be done in such a way that every point on the spectrum is now an eigenvalue, so possessing a (space of) eigenfunctions. (This is at the expense of $\hat{E}$ being non-separable). We write a typical eigenvalue as $-i\omega$. These functions are the *right-eigenfunctions* of $\hat{L}$ and will be written as $|r(\omega)\rangle$; they span a dense subspace of $\hat{E}$.

In this new setting, $\hat{L}$ is not self-adjoint, and we consider its adjoint $\hat{L}^*$. Its spectrum is the same as that of $\hat{L}$, and we can similarly construct the left-eigenvectors $\langle l(\omega)| \in \hat{E}^*$ with similar properties. Additional labels are needed to account for degeneracy of these eigenvalues, and the resulting additional summations will be omitted here.

One would expect there to be an orthogonality relation of the form

$$\langle l(\omega)|\, r(\omega)\rangle = 0 \quad \text{for} \quad \omega' \neq \omega, \tag{9}$$

and this, together with a consideration of the (spaces of) eigenfunctions with $\omega' = \omega$, would lead to the construction of orthonormal sets $\{|r(\omega)\rangle\}, \{\langle l(\omega)|\}$. This step involves some technical difficulties, which we leave aside in this note. (The nature of these difficulties in the plasma wave context is revealed in Case (1959) and Tracy *et al.* (1996).) Equipped with this, we can write down a solution of an initial-value problem in the form

$$|\rho(t)\rangle = \int_{-\infty}^{\infty} |r(\omega)\rangle \langle l(\omega)|\rho(0)\rangle\, e^{-i\omega t}\, d\omega. \tag{10}$$

If we now restrict $|\rho(0)\rangle$ to belong to E, the inner product of (10) becomes a smooth function of ω. The factor $|r(\omega\rangle$ still remains, with the integration over ω leaving $|\rho(t)\rangle \in E$, as must be the case by construction.

Next we apply (10) to compute $|a(t)\rangle$ using (8):

$$|a(t)\rangle = \int_{-\infty}^{\infty} M|\, r(\omega)\rangle \langle l(\omega)|\rho(0)\rangle\, e^{-i\omega t}\, d\omega. \tag{11}$$

This introduces a further contraction of M with $|r(\omega)\rangle$. If $M \in E$ in the sense of its dependence on X, this leaves the integrand in the form of a continuous function of ω multiplying $\exp(-i\omega t)$, and we can apply the Riemann–Lebesgue lemma to establish that $|a(t)\rangle$ would approach equilibrium as $t \to \infty$.

To investigate the evolution of $|a(t)\rangle$ for $t > 0$ in more detail, one would like to deform the contour of (11) into $\mathrm{Im}(\omega) < 0$. It may be that the integrand is incapable of the required analytic continuation, but if it is capable then one may distort the contour to capture contributions from the singularities, in the

https://doi.org/10.1017/S0022377898006618 Published online by Cambridge University Press

manner of the Landau calculation. This gives (in principle!) the required formulae for the approach to equilibrium at the macroscopic level prescribed by M.

The analytic function so created from (11) is regular for $\mathrm{Im}(\omega) > 0$ but has singularities in $\mathrm{Im}(\omega) < 0$. If one asks for the evolution for $t < 0$, the same methodology applies but the above statement about the domain of regularity is reversed. So the function has two sheets, which intersect on the real axis. This behaviour exactly parallels that of the plasma oscillation problem for the Maxwellian plasma, where the analytic function having that behaviour is the well-known plasma dispersion function.

It may be asked whether the integral in (10) can be so treated, to give a formalism that would apply to all choices of M. The indications are that this is not so, since the analytic continuation is not possible.

8. Commentary

We emphasize that (10) provides the unique solution for $\rho(t)$ (both for $t > 0$ and $t < 0$) implied by $\rho(0)$; this includes the possibility that ρ is distributional, or even a single orbit. Equation (11) is likewise a single evolution of $a(t)$, arising from the underlying solution (10). The fact that the behaviour of $a(t)$ appears to change abruptly at $t = 0$ does not imply any discontinuity at $t = 0$ but is a consequence of the restriction to smooth observables and values, which will characterize macroscopic evolution. This picture is therefore one in which macroscopically the system tends to thermal equilibrium at $t \to \pm\infty$, and a large fluctuation arises at around $t = 0$ then decays again. That is not to say that ρ itself tends to equilibrium (in our case the microcanonical distribution) as $t \to \pm\infty$, merely that it cannot be distinguished from equilibrium at the level of observation fixed by M.

This supports the view that irreversibility arises from preparation or intervention at the start of an observation, say at $t = 0$, combined with the fact that the observation will extend into the future and not the past. For $t > 0$ the evolution of $a(t)$ will be the same whether it arose from intervention or a fluctuation. This explanation is discussed (among others) by Peierls (1979, Sec. 3.8), although he does not refer to the point about smoothness. Further discussion involves the physiological and cosmological arrows of time, but cannot be continued here.

A further question is raised by the treatment of (11) by analytic continuation. This leads to contributions arising from the singularities. These are not necessarily poles, but, taking that outcome as an example, there will in general be too large a space of solutions available for a unique evolution to be imposed by $a(0)$. This is again very conveniently illustrated by the plasma wave problem: the Landau dispersion relation has an infinite sequence of complex solutions in p (the Laplace transform parameter). One of these alone would be adequate to solve the initial value problem in which $E(0)$ (for fixed k) is given. To choose the least damped one seems a reasonable, but not unique, procedure. More generally, in NESM itself, the same situation arises. This can be seen from (11), since $\rho(0)$ is not uniquely fixed by $a(0)$. Attempts to deal with this lead back to the manipulations outlined in Sec. 4, where the elimination of ρ_2, had it succeeded, would have led to a unique evolution. On the other hand,

https://doi.org/10.1017/S0022377898006618 Published online by Cambridge University Press

contributions at individual poles have unphysical aspects, so there is a difficulty concerning the natural idea of truncating the sum over them.

As a closing note, readers may wish to consult Radu's very recent new book (Balescu 1997) to learn his latest opinions on all aspects of NESM. We are all grateful for his achievements and his contributions to plasma physics.

References

Balescu, R. 1960 *Phys. Fluids* **3**, 52–63.

Balescu, R. 1963 *The Statistical Mechanics of Charged Particles*. Interscience, New York.

Balescu, R. 1975 *Equilibrium and Non-Equilibrium Statistical Mechanics*. Wiley-Interscience, New York.

Balescu, R. 1988*a* *Transport Processes in Plasmas*. Vol. 1: *Classical Transport*. North-Holland, Amsterdam.

Balescu, R. 1988*b* *Transport Processes in Plasmas*. Vol. 2: *Neoclassical Transport*. North-Holland, Amsterdam.

Balescu, R. 1997 *Statistical Dynamics: Matter Out of Equilibrium*. Imperial College Press, London.

Case, K. M. 1959 *Ann. Phys. (NY)* **7**, 349–364.

Crawford, J. D. and Hislop, P. 1989 *Ann. Phys. (NY)* **189**, 265–317.

Hasegawa, H. H. and Saphir, W. C. 1992 *Phys. Rev.* **A46**, 7401–7423.

Landau, L. D. 1946 *J. Phys. USSR* **10**, 25.

Peierls, R. F. 1979 *Surprises in Physics*. Princeton University Press.

Prigogine, I., George, C. and Henin, F. 1969 *Physica* **45**, 418–434.

Strogatz, S. H., Mirollo, R. E. and Matthews, P. C. 1992 *Phys. Rev. Lett.* **68**, 2730–2733.

Tracy, E. R., Brizard, A. J. and Kaufman, A. N. 1996 *J. Plasma Phys.* **55**, 449–486.

Van Kampen, N. G. 1955 *Physica* **21**, 949–963.

https://doi.org/10.1017/S0022377898006618 Published online by Cambridge University Press