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Cocycles, Hausdorff measures and cross ratios

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Abstract. Let f be a flip-invariant Hölder continuous function on the unit tangent bundle T^1M of a closed negatively curved Riemannian manifold M. We show that conditionals on strong unstable manifolds of the Gibbs equilibrium state defined by f can be realized as Hausdorff measures. Moreover, cohomology classes of flip invariant cocycles are in one-to-one correspondence to cross ratios on the space of four pairwise distinct points of the ideal boundary of the universal covering $\tilde{M}$ of M.

0. *Introduction*

Let M be a closed Riemannian manifold of negative sectional curvature. Recall that the *geodesic flow* Φ^t is the smooth dynamical system on the unit tangent bundle T^1M of M generated by the *geodesic spray* X. There is a Hölder continuous Φ^t-invariant decomposition $TT^1M = \mathbb{R}X \oplus TW^{su} \oplus TW^{ss}$, where TW^{su} (respectively TW^{ss}) is the tangent bundle of the *strong unstable* (respectively the *strong stable*) foliation W^{su} (respectively W^{ss}).

Recall that a Hölder-continuous *additive cocycle* for the geodesic flow Φ^t is a Hölder continuous function $\zeta: T^1M \times \mathbb{R} \to \mathbb{R}$ which satisfies $\zeta(v, s+t) = \zeta(v, s) + \zeta(\Phi^s v, t)$ for all $v \in T^1M$ and $s, t \in \mathbb{R}$. Let $\mathcal{F}: T^1M \to T^1M$ be the *flip* $v \to \mathcal{F}v = -v$; the *flip* $\mathcal{F}\zeta$ of a cocycle ζ is the cocycle $\mathcal{F}\zeta(v, t) = \zeta(\mathcal{F}\Phi^t v, t)$. We call ζ *quasi-invariant under the flip* if and only if the cocycles ζ and $\mathcal{F}\zeta$ are cohomologous or, in other words, if and only if the cohomology class of ζ is invariant under $\mathcal{F}$.

Let $\partial\tilde{M}$ be the ideal boundary of the universal covering $\tilde{M}$ of M.

Definition. A *generalized cross ratio* is a Hölder continuous positive function Cr on the space of quadruples of pairwise distinct points in $\partial\tilde{M}$ with the following properties:

(1) Cr is invariant under the action of $\pi_1(M)$ on $(\partial\tilde{M})^4$;
(2) $Cr(\xi, \xi', \eta, \eta') = Cr(\xi', \xi, \eta, \eta')^{-1}$;
(3) $Cr(\xi, \xi', \eta, \eta') = Cr(\eta, \eta', \xi, \xi')$;
(4) $Cr(\xi, \xi', \eta, \eta')Cr(\xi', \xi'', \eta, \eta') = Cr(\xi, \xi'', \eta, \eta')$;
(5) $Cr(\xi, \xi', \eta, \eta')\, Cr(\xi', \eta, \xi, \eta')Cr(\eta, \xi, \xi', \eta') = 1$.

The usual cross ratio on the boundary of $\tilde{M}$ (see [**O2**, **B**]) is a generalized cross ratio in the above sense.

In §1 we show the following.

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THEOREM A. *Flip invariant cohomology classes of Hölder continuous cocycles on T^1M are in one-to-one correspondence to generalized cross ratios on $\partial\tilde{M}$.*

In this sense, the usual cross ratio on $\partial\tilde{M}$ corresponds to the cocycle $\zeta(v,t) = t$ on T^1M.

Let $\mathcal{M}$ be the space of Φ^t-invariant Borel probability measures on T^1M, and for $\nu \in \mathcal{M}$ denote by h_ν the entropy of ν. For every Hölder continuous function $f: T^1M \to \mathbb{R}$ the *pressure* $pr(f)$ of f is defined by $pr(f) = \sup\{h_\nu - \int f\,d\nu \mid \nu \in \mathcal{M}\}$. Notice that we do not use the standard sign convention for the pressure here. Rather, $pr(f)$ according to our definition corresponds to the pressure of the function $-f$ as defined in the standard literature.

The function f admits a unique *Gibbs equilibrium state* $\eta_f \in \mathcal{M}$ which is characterized by $h_{\eta_f} = \int f\,d\eta_f + pr(f)$. Call f *quasi-invariant under the flip* if for every periodic point v of Φ^t of period $\tau > 0$ we have $\int_0^\tau f(\Phi^s v)\,ds = \int_0^\tau f(\Phi^s(-v))\,ds$.

Let π be the canonical projection of the unit tangent bundle $T^1\tilde{M}$ of $\tilde{M}$ onto $\partial\tilde{M}$. For every $p \in \tilde{M}$ the restriction π_p of π to the fibre $T^1_p\tilde{M}$ at p of the fibration $T^1\tilde{M} \to \tilde{M}$ is a homeomorphism. Let $P: T^1\tilde{M} \to \tilde{M}$ be the canonical projection. The Gibbs equilibrium state η_f then induces for every $p \in \tilde{M}$ a finite Borel measure $\bar{\eta}_f^p$ on $\partial\tilde{M}$, equivariant under the action of $\pi_1(M)$ on $\tilde{M}$ and $\partial\tilde{M}$, in such a way that for every $v \in T^1\tilde{M}$ and $t > 0$ the Radon–Nikodym derivative at $\pi(v)$ of $\bar{\eta}_f^{P\Phi^t v}$ with respect to $\bar{\eta}_f^{Pv}$ just equals $e^{\int_0^t f(\Phi^s v)\,ds + tpr(f)}$ (see the construction of these measures in §2). The measure $\bar{\eta}_f^p$ then projects via π_p to a finite Borel measure η_f^p on $T^1_p\tilde{M}$, and by equivariance we obtain for every $x \in M$ a finite Borel measure η_f^x on T^1_xM.

Denote by DTM (respectively $DT\tilde{M}$) the smooth fibre bundle over M (respectively $\tilde{M}$) whose fibre DTM_x at $x \in M$ (respectively $DT\tilde{M}_x$ at $x \in \tilde{M}$) equals $T^1_xM \times T^1_xM$ (respectively $T^1_x\tilde{M} \times T^1_x\tilde{M}$). We call a function β on DTM *symmetric* if β is invariant under the natural involution $(v,w) \to (w,v)$. In §2 we show the following.

THEOREM B. *Let $f: T^1M \to \mathbb{R}$ be quasi-invariant under the flip and Hölder continuous. Then there is a Hölder continuous symmetric function $d_f: DTM \to [0,\infty)$ with the following properties:*

(1) *there is a number $\kappa > 0$ such that for every $x \in M$ the restriction of d_f^κ to DTM_x is a distance on T^1_xM defining the usual topology;*
(2) *for every $x \in M$ the measure η_f^x is the $1/\kappa$-dimensional spherical measure on T^1_xM induced by d_f^κ.*

§3 is devoted to a further investigation of the measures $\eta_f^p (p \in M)$ and of the Gibbs equilibrium state η_f.

Let W^s be the *stable foliation* on T^1M and recall that the lift to $TW^s = \mathbb{R}X \oplus TW^{ss}$ of the Riemannian metric on M defines a family λ^s of Lebesgue measures on the leaves of W^s. Moreover, let η_f^{su} be a family of measures on strong unstable manifolds which transform under the geodesic flow via $(d/dt)\eta_f^{su} \circ \Phi^t\,|_{t=0} = f + pr(f)$.

Let λ_M be the Lebesgue measure on M. We show the following.

THEOREM C. *The measure $\bar{\eta}_f$ on T^1M defined by $\int \psi\,d\bar{\eta}_f = \iint \psi\,d\eta_f^x\,d\lambda_M(x)$ satisfies $d\bar{\eta}_f = d\lambda^s \times d\eta_f^{su}$ with respect to a local product structure (up to a constant).*

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In the case when the function f is constant, Theorem C was earlier shown by Knieper [**Kn**].

1. *Cocycles and cross ratios*

In this section we consider a Hölder continuous additive cocycle for the geodesic flow Φ^t on the unit tangent bundle of a compact negatively curved manifold M, i.e. a Hölder continuous function $\zeta: T^1M \times \mathbb{R} \to \mathbb{R}$ which satisfies $\zeta(v, s+t) = \zeta(v, s)+\zeta(\Phi^s v, t)$ for all $v \in T^1M$ and all $s, t \in \mathbb{R}$. Let $\mathcal{F}: T^1M \to T^1M$ be the *flip* $v \to \mathcal{F}(v) = -v$. We can then define a new Hölder continuous cocycle $\mathcal{F}\zeta$ on T^1M by $\mathcal{F}\zeta(v, t) = \zeta(\mathcal{F}\Phi^t v, t)$. We require that the cocycles ζ and $\mathcal{F}\zeta$ are cohomologous. This is equivalent to the following: let $v \in T^1M$ be a periodic orbit of Φ^t of period $\tau > 0$; then $\zeta(v, \tau) = \zeta(-v, \tau)$.

Consider the lift of ζ to a cocycle on $T^1\tilde{M}$ which we denote again by ζ. The next lemma is a standard application of the fact that strong stable manifolds of T^1M contract exponentially fast under the geodesic flow.

LEMMA 1.1. *There is a uniformly Hölder continuous kernel $k_\zeta: \tilde{M} \times \tilde{M} \times \partial\tilde{M} \to \mathbb{R}$ which satisfies $k_\zeta(Pv, P\Phi^t v, \pi(v)) = \zeta(v, t)$ for all $v \in T^1\tilde{M}$ and all $t \in \mathbb{R}$. We have $k_\zeta(x, y, \xi) + k_\zeta(y, z, \xi) = k_\zeta(x, z, \xi)$ for all $x, y, z \in \tilde{M}$ and all $\xi \in \partial\tilde{M}$.*

Now let $\psi: T^1M \to \mathbb{R}$ be a Hölder continuous function which defines an equivalence for the cocycles ζ and $\mathcal{F}\zeta$. The lift of ψ to $T^1\tilde{M}$ (which we denote again by ψ) then satisfies $\psi(\Phi^t v) + k_\zeta(Pv, P\Phi^t v, \pi(-v)) - \psi(v) + k_\zeta(Pv, P\Phi^t v, \pi(v)) = 0$ for all $v \in T^1\tilde{M}$ and all $t \in \mathbb{R}$.

For $p \in \tilde{M}$ and $v \neq w \in T^1_p\tilde{M}$ there is up to reparametrization a unique geodesic γ in $\tilde{M}$ joining $\gamma(-\infty) = \pi(w)$ to $\gamma(\infty) = \pi(v)$. Since the curvature of $\tilde{M}$ is bounded from above by a negative constant there is a unique $t \in \mathbb{R}$ depending smoothly on p such that $\text{dist}(\gamma(t), p) = \text{dist}(\gamma(\mathbb{R}), p)$. Define $\alpha_\zeta(v, w) = \frac{1}{2}(\psi(\gamma'(t)) - k_\zeta(\gamma(t), p, \pi(v)) - k_\zeta(\gamma(t), p, \pi(w)))$. Identify $T^1\tilde{M}$ with the diagonal $\{(v, v) \mid v \in T^1\tilde{M}\}$ in $DT\tilde{M}$. Then we have the following.

LEMMA 1.2. *The function $\alpha_\zeta: DT\tilde{M} - T^1\tilde{M} \to \mathbb{R}$ is Hölder continuous and invariant under the action of $\pi_1(M)$ on $DT\tilde{M}$. Moreover, for $(v, w), (u, z) \in DT\tilde{M}$ with $u \in W^s(v)$ and $z \in W^s(w)$ we have $\alpha_\zeta(v, w) - \alpha_\zeta(u, z) = \frac{1}{2}[k_\zeta(Pv, Pu, \pi(v)) + k_\zeta(Pv, Pu, \pi(w))]$.*

Proof. Hölder continuity of the function α_ζ is an immediate consequence of Hölder continuity of ζ. To show the equality in the lemma let γ be a geodesic in $\tilde{M}$ and let $(v, w), (u, z) \in DT\tilde{M}$ be such that $\pi(v) = \pi(u) = \gamma(\infty)$ and $\pi(w) = \pi(z) = \gamma(-\infty)$. Write $x = Pv, y = Pu$ and assume that the projection of x to γ (respectively the projection of y to γ) equals $\gamma(0)$ (respectively $\gamma(\tau)$). By the definition of α_ζ we then obtain that

$$\begin{aligned}\alpha_\zeta(v, w) - \alpha_\zeta(u, z) \quad = \quad & \tfrac{1}{2}[\psi(\gamma'(0)) - k_\zeta(\gamma(0), x, \gamma(\infty)) - k_\zeta(\gamma(0), x, \gamma(-\infty)) \\ & -\psi(\gamma'(\tau)) + k_\zeta(\gamma(\tau), y, \gamma(\infty)) + k_\zeta(\gamma(\tau), y, \gamma(-\infty))].\end{aligned}$$

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Since ψ provides an equivalence between the cocycles ζ and $\mathcal{F}\zeta$ the right-hand side of this equation is equal to

$$\begin{aligned}&\tfrac{1}{2}[k_\zeta(\gamma(0),\gamma(\tau),\gamma(-\infty))+k_\zeta(\gamma(0),\gamma(\tau),\gamma(\infty))+k_\zeta(x,\gamma(0),\gamma(\infty))\\&\qquad+k_\zeta(x,\gamma(0),\gamma(-\infty))+k_\zeta(\gamma(\tau),y,\gamma(\infty))+k_\zeta(\gamma(\tau),y,\gamma(-\infty))]\\&=\quad\tfrac{1}{2}[k_\zeta(x,y,\gamma(\infty))+k_\zeta(x,y,\gamma(-\infty))]\end{aligned}$$

which shows the lemma. □

By the above, the function α_ζ projects to a Hölder continuous function on $DTM - T^1M$ which we denote by the same symbol.

Now let χ be a cocycle for Φ^t which is cohomologous to ζ, i.e. there is a Hölder continuous function $\varphi: T^1M \to \mathbb{R}$ such that $\chi(v,t) = \varphi(\Phi^t v) + \zeta(v,t) - \varphi(v)$ for all $v \in T^1M$ and all $t \in \mathbb{R}$. Then we have the following.

LEMMA 1.3. *$\alpha_\chi(v,w) = \alpha_\zeta(v,w) + \frac{1}{2}\varphi(v) + \frac{1}{2}\varphi(w)$ for all $(v,w) \in DTM - T^1M$.*

Proof. Since $k_\chi(Pv, P\Phi^t v, \pi(v)) = \varphi(\Phi^t v) + k_\zeta(Pv, P\Phi^t v, \pi(v)) - \varphi(v)$ the function $v \to \psi(v) + \varphi(v) + \varphi(-v)$ provides an equivalence between the cocycles χ and $\mathcal{F}\chi$. The lemma now follows immediately from the definitions of the functions $\alpha_\chi, \alpha_\zeta$. □

Let c be the cocycle defined by the constant function 1, i.e. $c(v,t) = t$ for all $v \in T^1M, t \in \mathbb{R}$. Recall that for $(v,w) \in DTM - T^1M$ the *Gromov product* $(v \mid w)$ is defined to be $(v \mid w) = \alpha_c$.

LEMMA 1.4. *For every Hölder continuous cocycle ζ for Φ^t as above, there are numbers $a, b > 0$ such that $a\alpha_\zeta(v,w) \le (v \mid w) + b$ for all $(v,w) \in DTM - T^1M$.*

Proof. By Lemma 1.2 and the arguments in the proof of Lemma 2.2 of [**H2**] it suffices to show that there are numbers $a, b > 0$ such that $k_\zeta(x,y,\zeta) \le a\,\mathrm{dist}(x,y) + b$ for all $x, y \in \tilde{M}$ and all $\xi \in \partial\tilde{M}$. Since k_ζ is uniformly Hölder continuous there is a number $c > 0$ such that $k_\zeta(x,y,\xi) \le c$ for all $x, y \in \tilde{M}$ with $\mathrm{dist}(x,y) \le 1$ and all $\xi \in \partial\tilde{M}$. From this and the formula in the statement of Lemma 1.1 we obtain the required estimate. □

For a quadruple (ξ, ξ', η, η') of four distinct points in $\partial\tilde{M}$ we now define the *cross ratio* $[\xi, \xi', \eta, \eta']_\zeta$ of ζ as follows.

Choose $x \in \tilde{M}$ and let $v, w, u, z \in T^1_x\tilde{M}$ be such that $\pi(v) = \xi, \pi(w) = \xi', \pi(u) = \eta$ and $\pi(z) = \eta'$. Define $[\xi, \xi', \eta, \eta']_\zeta = \exp(\alpha_\zeta(v,u) + \alpha_\zeta(w,z) - \alpha_\zeta(v,z) - \alpha_\zeta(w,u))$. By Lemma 1.2 this definition is independent of the choice of $x \in \tilde{M}$. Moreover, we have the following properties which are well known for the usual cross ratio (i.e. for the cross ratio of the cocycle c).

LEMMA 1.5.

(1) *$[\ \ ]_\zeta$ is a Hölder continuous positive function on the space of quadruples of four distinct points in $\partial\tilde{M}$ which is invariant under the action of $\pi_1(M)$.*
(2) *If the cocycles ζ, χ are cohomologous then $[\ \ ]_\zeta = [\ \ ]_\chi$.*
(3) $[\xi, \xi', \eta, \eta']_\zeta = [\xi', \xi, \eta, \eta']_\zeta^{-1}$.

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(4) $[\eta, \eta', \xi, \xi']_\zeta = [\xi, \xi', \eta, \eta']_\zeta$.
(5) $[\xi, \xi', \eta, \eta']_\zeta [\xi', \xi'', \eta, \eta']_\zeta = [\xi, \xi'', \eta, \eta']_\zeta$.
(6) $[\xi, \xi', \eta, \eta']_\zeta [\xi', \eta, \xi, \eta']_\zeta [\eta, \xi, \xi', \eta']_\zeta = 1$.

Proof. The lemma follows immediately from the definitions and Lemma 1.3. □

Lemma 1.5 shows that $[\quad]_\zeta$ is a generalized cross ratio in the sense of the definition in the introduction.

We want to show that generalized cross ratios are in one-to-one correspondence to cohomology classes of Hölder continuous flip invariant cocycles ζ on T^1M by mapping ζ to $[\quad]_\zeta$.

Let FM be the Stiefel manifold of orthonormal 2-frames in TM. A point in FM consists of an ordered pair (v, w) of orthogonal unit vectors in TM with the same base-point, and the first factor projection $Q: FM \to T^1M$ gives FM the structure of an $(n-2)$-sphere bundle over T^1M whose fibre FM_v at $v \in T^1M$ has a natural identification with the $(n-2)$-sphere of unit tangent vectors in $T^1_{Pv}M$ which are orthogonal to v.

The *frame flow* Ψ^t is a smooth dynamical system on FM defined by $\Psi^t(v, w) = (\Phi^t v, \|_t w)$, where $\|_t w$ is the unit tangent vector in $T^1_{P\Phi^t v}M$ which is the image of w under parallel transport along the geodesic $s \to P\Phi^s v$ $(s \in [0, t])$.

LEMMA 1.6. *Every generalized cross ratio determines a Hölder continuous additive cocycle for the frame flow on FM.*

Proof. Denote again by Ψ^t the frame flow on the Stiefel manifold $F\tilde{M}$ of orthonormal 2-frames in $T\tilde{M}$. Let Cr be a generalized cross ratio on $\partial\tilde{M}$ and for $(v, w) \in F\tilde{M}$ and $t \neq 0$ define $\chi((v, w), t) = \log Cr(\pi(-v), \pi(v), \pi(w), \pi(\|_t w))$. For $s, t \neq 0, s \neq -t$ we obtain from identity (4) in the definition of a generalized cross ratio that

$$\begin{aligned}\chi((v, w), s+t) &= \log Cr(\pi(-v), \pi(v), \pi(w), \pi(\|_s w)) \\ &\quad + \log Cr(\pi(-v), \pi(v), \pi(\|_s w), \pi(\|_{s+t} w)) \\ &= \chi((v, w), s) + \chi(\Psi^s(v, w), t).\end{aligned}$$

Thus the map $\chi: F\tilde{M} \times \mathbb{R} - \{0\} \to \mathbb{R}$ satisfies the equation of an additive cocycle. Since Cr is Hölder continuous we can extend χ to a Hölder continuous cocycle $\chi: F\tilde{M} \times \mathbb{R} \to \mathbb{R}$ by defining $\chi((v, w), 0) = 0$ for all $(v, w) \in F\tilde{M}$. Moreover, since Cr is invariant under the action of $\pi_1(M)$ the function χ is also invariant under the action of $\pi_1(M)$ on $F\tilde{M}$ and hence it projects to a Hölder continuous additive cocycle on FM. □

For $v \in T^1M$ now let λ_v be the normalized Lebesgue measure on the fibre FM_v of FM at v. Then we have the following.

LEMMA 1.7. *Let $\chi: FM \times \mathbb{R} \to \mathbb{R}$ be an additive cocycle for the frame flow on FM. Then the function $\zeta: T^1M \times \mathbb{R} \to \mathbb{R}, \zeta(v, t) = \int_{FM_v} \chi(z, t)\, d\lambda_v(z)$ is an additive cocycle for Φ^t.*

Proof. Since $\chi(z, s+t) = \chi(z, s) + \chi(\Psi^s z, t)$ for all $z \in FM$ and all $s, t \in \mathbb{R}$ we have $\zeta(v, s+t) = \int_{FM_v} (\chi(z, s) + \chi(\Psi^s z, t))\, d\lambda_v(z) = \zeta(v, s) + \zeta(\Phi^s v, t)$ since the restriction

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of Ψ^s to FM_v is an isometry of the fibre FM_v onto the fibre $FM_{\Phi^s v}$ of $FM \to T^1M$ and hence maps the Lebesgue measure λ_v on FM_v to the Lebesgue measure $\lambda_{\Phi^s v}$ on $FM_{\Phi^s v}$. □

Lemma 1.6 and Lemma 1.7 together show that every generalized cross ratio on $\partial\tilde{M}$ induces a Hölder continuous additive cocycle ζ on T^1M. This cocycle is in fact invariant under the flip $\mathcal{F}$.

LEMMA 1.8. *Let ζ be a cocycle defined as above by a generalized cross ratio Cr. Then $\zeta(v,t) = \zeta(-\Phi^t v, t)$ for all $v \in T^1M$ and all $t \in \mathbb{R}$.*

Proof. Recall that the fibre FM_v of FM at v has a natural identification with the unit sphere in the orthogonal complement of v in $T_{Pv}M$ and hence there is a natural smooth involution $\Lambda: FM \to FM$ such that $Q \circ \Lambda = \mathcal{F} \circ Q$. Moreover we have $\Lambda \circ \Psi^t \circ \Lambda \circ \Psi^t =$ Id for all $t \in \mathbb{R}$. This involution lifts to an involution on $F\tilde{M}$ which we denote again by Λ. Now $Cr(\xi, \xi', \eta, \eta') = Cr(\xi', \xi, \eta', \eta)$ and hence the definition of the cocycle χ on $F\tilde{M}$ induced by Cr shows that $\chi((v,w),t) = \log Cr(\pi(-v), \pi(v), \pi(w), \pi(\|_t w)) = \log Cr(\pi(v), \pi(-v), \pi(\|_t w), \pi(w)) = \chi(\Lambda(\Psi^t(v,w)), t)$ for all $(v,w) \in F\tilde{M}$ and all $t \in \mathbb{R}$. Integration of this identity over the fibres of the fibration $FM \to T^1M$ then yields the lemma. □

So far we have seen that every generalized cross ratio determines a Hölder continuous cocycle on T^1M. The cocycle in turn then defines a generalized cross ratio. The following proposition shows that these constructions are compatible:

PROPOSITION 1.9. *Let ζ be a Hölder continuous flip invariant additive cocycle. Then the cocycle χ derived from $[\ \ ]_\zeta$ is cohomologous to ζ.*

Proof. Let ζ be any Hölder continuous flip invariant additive cocycle and let χ be the cocycle derived as above from $[\ \ ]_\zeta$. Recall that two Hölder continuous cocycles α, β on T^1M are cohomologous if and only if we have $\alpha(v,\tau) = \beta(v,\tau)$ for every periodic point v of Φ^t of period $\tau > 0$. Let $v \in T^1M$ be such a point, let $\tilde{v} \in T^1\tilde{M}$ be a lift of v to $T^1\tilde{M}$ and let $\Psi \in \pi_1(M)$ be an isometry such that $d\Psi(\tilde{v}) = \Phi^\tau \tilde{v}$. The arguments of Otal [**O1**] (see also [**Bn**]) now show that for every $\xi \in \partial\tilde{M} - \{\pi(-\tilde{v}), \pi(\tilde{v})\}$ we have $[\pi(-\tilde{v}), \pi(\tilde{v}), \xi, \Psi(\xi)]_\zeta = \zeta(v,\tau)$. Namely, for $k \geq 1$ let $w_k \in T^1\tilde{M}$ be the unique vector which satisfies $Pw_k = P\tilde{v}$ and $\pi(w_k) = \Psi^k(\xi)$. Now the angle at $P\tilde{v}$ between $-\tilde{v}$ and w_k is bounded from below by a constant not depending on k and hence comparison with a plane of constant negative curvature shows that the distance between $P\tilde{v}$ and the geodesic in $\tilde{M}$ joining $\pi(-\tilde{v})$ to $\Psi^k(\xi)$ is bounded from above by a universal constant. Hölder continuity of the kernel k_ζ then implies that $|\alpha_\zeta(-\tilde{v}, w_k)| \leq c$ for a constant $c > 0$ not depending on k.

Similarly, the angle at $P\Phi^{k\tau}\tilde{v}$ of the triangle with vertices $\pi(-v)$, $P\Phi^{k\tau}\tilde{v}$, $\Psi^k(\xi)$ is independent of k and hence again by Hölder continuity of k_ζ and the definition of α_ζ there is a constant $\bar{c} > 0$ such that $|\alpha_\zeta(\tilde{v}, w_k) - \zeta(v, k\tau)| \leq \bar{c}$. Together we obtain that

$$\begin{aligned} &|\log[\pi(-\tilde{v}), \pi(\tilde{v}), \xi, \Psi(\xi)]_\zeta - \zeta(v,\tau)| \\ &= \frac{1}{k}|\log[\pi(-\tilde{v}), \pi(\tilde{v}), \xi, \Psi^k(\xi)]_\zeta - \zeta(v, k\tau)| \end{aligned}$$

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$$\leq \frac{1}{k}|\alpha_\zeta(\tilde{v}, w_k) - \zeta(v, k\tau)| + \frac{1}{k}|-\alpha_\zeta(-\tilde{v}, w_k) + \alpha_\zeta(\tilde{v}, w_0) - \alpha_\zeta(-\tilde{v}, w_0)| \leq \frac{1}{k} \cdot \text{constant}$$

This shows that $[\pi(-\tilde{v}), \pi(\tilde{v}), \xi, \Psi(\xi)]_\zeta = \zeta(v, \tau)$ for every $\xi \in \partial\tilde{M} - \{\pi(-\tilde{v}), \pi(\tilde{v})\}$.

For $k \geq 1$ let A_k be the isometry of $\tilde{v}^\perp \subset T_{P\tilde{v}}\tilde{M}$ which is obtained by translating $\tilde{v}^\perp$ parallel along the geodesic $s \to P\Phi^s\tilde{v}$ $(s \in [0, k\tau])$ and identifying $(\Phi^{k\tau}\tilde{v})^\perp \subset T_{P\Phi^{k\tau}\tilde{v}}\tilde{M}$ with $\tilde{v}^\perp$ via the restriction of $d\Psi^{-k}$. With $A_1 = A$ we then have $A_k = A^k$. In other words, $k \to A_k$ defines a homomorphism of $\mathbb{Z}$ into a one-parameter subgroup of $O(n-1)$. Since every one-parameter subgroup of the orthogonal group $O(n-1)$ comes back infinitely often to every neighborhood of Id this means that we can find a sequence $\{k_j\} \subset \mathbb{Z}$ such that $A_{k_j} \to \mathrm{Id}$ $(j \to \infty)$ in $O(n-1)$. By Hölder continuity of the cross ratio and the definition of the cocycle η derived from $[\ ]_\zeta$ this just means that $(1/k_j)(\eta(\tilde{v}, k_j\tau) - k_j\zeta(v, \tau)) = \eta(v, \tau) - \zeta(v, \tau) \to 0$ $(j \to \infty)$, and consequently that $\eta(v, \tau) = \zeta(v, \tau)$ for every periodic point v for Φ^t of period $\tau > 0$. This shows the proposition. □

Now let S be a compact negatively curved surface. The ideal boundary $\partial\tilde{S}$ of the universal covering $\tilde{S}$ of S is homeomorphic to the circle S^1 and hence an orientation for $\tilde{S}$ determines an ordering for triples of pairwise distinct points. In other words, if we fix an orientation for $\tilde{S}$, then an ordered pair of points $(\xi, \eta) \in \partial\tilde{S}$ determines a unique half-open interval $[\xi, \eta[$ in $\tilde{S} \sim S^1$. Call a quadruple $[\xi_1, \xi_2, \xi_3, \xi_4]$ of pairwise distinct points in $\partial\tilde{S}$ *ordered* if $\xi_{i+1} \in [\xi_i, \xi_{i+2}[$ (where we compute indices mod 4) for every $i \in \{1, \ldots, 4\}$.

Recall that a *geodesic current* on $\partial\tilde{S}$ is a $\pi_1(S)$-invariant locally finite Borel measure on the space $G\tilde{S} = (\partial\tilde{S} \times \partial\tilde{S} - \Delta)/\mathbb{Z}_2$ of unoriented geodesics. If I_1 and I_2 are disjoint intervals in $\partial\tilde{S}$, then we identify $I_1 \times I_2$ with the set of all geodesics in $\tilde{S}$ with one endpoint in I_1, the second endpoint in I_2. We call a $\pi_1(M)$-invariant signed measure on $G\tilde{S}$ a *signed current*. Thus a signed current is a formal difference $\mu_1 - \mu_2$ of two geodesic currents μ_1, μ_2.

LEMMA 1.10. *Let Cr be a generalized cross ratio on $\partial\tilde{S}$. Then there is a unique signed current μ on $G\tilde{S}$ which satisfies $\mu([\xi_1, \xi_2[\times[\xi_3, \xi_4[) = \log Cr(\xi_1, \xi_2, \xi_3, \xi_4)$ for every ordered quadruple $(\xi_1, \xi_2, \xi_3, \xi_4)$ of pairwise distinct points in $\partial\tilde{S}$.*

Proof. Let $(\xi_1, \xi_2, \xi_3, \xi_4)$ be an ordered quadruple of points in $\partial\tilde{S}$. Define $\mu[\xi_1, \xi_2[\times[\xi_3, \xi_4[= \log Cr(\xi_1, \xi_2, \xi_3, \xi_4)$. Then $\mu[\xi_1, \xi_2[\times[\xi_3, \xi_4[= \mu([\xi_3, \xi_4[\times[\xi_1, \xi_2[)$ and hence μ can be viewed as a function on subsets of $G\tilde{S}$. Now for $\xi \in [\xi_1, \xi_2[$ we have $\log Cr(\xi_1, \xi_2, \xi_3, \xi_4) = \log Cr(\xi_1, \xi, \xi_3, \xi_4) + \log Cr(\xi, \xi_2, \xi_3, \xi_4)$ and consequently μ is additive and hence induces a unique signed measure on $G\tilde{S}$. Since Cr is also invariant under $\pi_1(S)$, this measure is also invariant under $\pi_1(S)$. □

Remark.

(1) If Cr is such that $Cr(\xi_1, \xi_2, \xi_3, \xi_4) \geq 1$ for every ordered quadruple of points in $\partial\tilde{S}$, then the signed measure defined in Lemma 1.10 is in fact a geodesic current.

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(2) For μ as above and an ordered quadruple $\xi_1, \xi_2, \xi_3, \xi_4$ of pairwise distinct points in $\partial\tilde{S}$ we have $\mu[\xi_1, \xi_2] \times [\xi_3, \xi_4] = \mu[\xi_1, \xi_2[\times[\xi_3, \xi_4[= \mu]\xi_1, \xi_2[\times]\xi_3, \xi_4[$.

Again let M be an arbitrary compact Riemannian manifold of negative sectional curvature. We use Lemma 1.10 to derive the converse of Proposition 1.9.

PROPOSITION 1.11. *Let Cr be a generalized cross ratio on $\partial\tilde{M}$ and let ζ be the cocycle derived form Cr as above. Then $Cr = [\quad]_\zeta$.*

Proof. We again consider a cross ratio Cr on the boundary $\partial\tilde{S}$ of the universal covering $\tilde{S}$ of a compact surface S. Let μ be the signed measure on $\partial\tilde{S} \times \partial\tilde{S} - \Delta$ induced by Cr as in Lemma 1.10. Let $v \in T^1S$ be a point whose orbit under Φ^t in the forward and backward direction is dense in T^1S. The arguments of Otal [**O1**] show that then μ is completely determined by its value on subsets of the form $[\xi_1, \xi_2[\times[\xi_3, \xi_4[$, where $(\xi_1, \xi_2, \xi_3, \xi_4)$ is an ordered quadruple of points consisting of endpoints of lifts of the geodesic γ in S with initial velocity $\gamma'(0) = v$. Moreover, for any such quadruple the value $\mu[\xi_1, \xi_2[\times[\xi_3, \xi_4[- \mu[\xi_2, \xi_3[\times[\xi_4, \xi_1[$ is completely determined by the *intersection* of μ with the set of all closed geodesics in S, or equivalently by the cocycle ζ on T^1S derived from Cr.

Now

$$\begin{aligned}\mu[\xi_1, \xi_2[\times[\xi_3, \xi_4[- \mu[\xi_2, \xi_3[\times[\xi_4, \xi_1[&= \log Cr(\xi_1, \xi_2, \xi_3, \xi_4) - \log Cr(\xi_2, \xi_3, \xi_4, \xi_1)\\ &= \log Cr(\xi_1, \xi_2, \xi_3, \xi_4) + \log Cr(\xi_2, \xi_3, \xi_1, \xi_4)\\ &= -\log Cr(\xi_3, \xi_1, \xi_2, \xi_4)\end{aligned}$$

by the definition of a generalized cocycle on T^1S, and this shows that for every ordered quadruple $(\xi_1, \xi_2, \xi_3, \xi_4)$ of pairwise distinct points on $\partial\tilde{S}$ the value of $Cr(\xi_4, \xi_2, \xi_3, \xi_1)$ is completely determined by ζ.

Now let $(\xi_1, \xi_2, \xi_3, \xi_4)$ be any ordered quadruple of pairwise distinct points in $\partial\tilde{S}$ and choose $\eta \in]\xi_3, \xi_4[$. Then the quadruples $(\xi_3, \eta, \xi_4, \xi_1)$ and $(\xi_4, \xi_2, \xi_3, \eta)$ are ordered and $\log Cr(\xi_1, \xi_2, \xi_3, \xi_4) = \log Cr(\xi_1, \eta, \xi_3, \xi_4) + \log Cr(\eta, \xi_2, \xi_3, \xi_4)$. Consequently $Cr(\xi_1, \xi_2, \xi_3, \xi_4)$ is completely determined by ζ as well. In summary we conclude that $Cr = [\quad]_\zeta$.

Now if M is an arbitrary compact negatively curved manifold then again Cr is completely determined by its values on quadruples of endpoints of lifts of the same geodesic γ in M whose tangent is dense in T^1M in the forward and backward direction. Let $(\xi_1, \xi_2, \xi_3, \xi_4)$ be such a quadruple and choose an embedded circle $S^1 \subset \partial\tilde{M}$ through these points in such a way that on this circle the quadruple is ordered. Now Cr determines a signed measure on S^1, and the above arguments together with continuity of Cr on the space of pairwise distinct points in $\partial\tilde{M}$ imply that this measure and hence $Cr(\xi_1, \xi_2, \xi_3, \xi_4)$ is completely determined by ζ. In other words, we have $Cr = [\quad]_\zeta$. □

To summarize our above considerations, in this section we have constructed a bijection between the space of generalized cross ratios on $\partial\tilde{M}$ and the space of cohomology classes of Hölder continuous cocycles on T^1M which are invariant under the flip. This is just the statement of Theorem A from the introduction.

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2. *Gibbs and Hausdorff measures*

We continue to use the assumptions and notations from §1.

Let $f: T^1M \to \mathbb{R}$ be a Hölder continuous function on T^1M. Then f induces a unique Gibbs equilibrium state η_f which is a Φ^t-invariant Borel probability measure on T^1M. Moreover, f defines a family η_f^{su} of measures on the leaves of the strong unstable foliation, uniquely up to a constant, with the following properties:

(1) the measures η_f^{su} are locally finite and positive on open sets;

(2) canonical maps are absolutely continuous with respect to the measures η_f^{su}, with Hölder continuous Jacobian;

(3) $(d/dt)\eta_f^{su} \circ \Phi^t \mid_{t=0} = f + pr(f)$, where $pr(f) \in \mathbb{R}$ is the *pressure* of f.

From Property (2) we obtain the following: If B is a Borel-subset of a strong unstable manifold with $\eta_f^{su}(B) = 0$ then $\eta_f(\bigcup\{W^s(z) \mid z \in B\}) = 0$.

A particular Gibbs measure is the *normalized Lebesgue–Liouville measure* λ on T^1M. The measures λ^{su} for λ on strong unstable manifolds are contained in the Lebesgue measure class. Similarly, the *Bowen–Margulis measure* μ is the Gibbs equilibrium state of a constant function.

Lift the measures η_f^{su} to a family of locally finite Borel measures $\tilde{\eta}_f^{su}$ on the leaves of $W^{su} \subset TT^1\tilde{M}$.

For every $x \in \tilde{M}$ the projection $\pi : T^1\tilde{M} \to \partial\tilde{M}$ restricts to a homeomorphism π_x of $T_x^1\tilde{M}$ onto $\partial\tilde{M}$, and for every $v \in T_x^1\tilde{M}$ the restriction of $\pi_x^{-1} \circ \pi$ to $W^{su}(v)$ is a homeomorphism of $W^{su}(v)$ onto $T_x^1\tilde{M} - \{-v\}$. Thus the measure $\tilde{\eta}_f^{su}$ on $W^{su}(v)$ projects under $\pi_x^{-1} \circ \pi \mid_{W^{su}(v)}$ to a Borel measure η_f^v on $T_x^1\tilde{M}$ whose restriction to $T_x^1\tilde{M} - \{-v\}$ is locally finite.

The measures $\eta_f^v, \eta_f^w (v, w \in T_x^1\tilde{M})$ are absolutely continuous on $T_x^1\tilde{M} - \{-v, -w\}$, with continuous Radon–Nikodym derivative. More precisely, for $w \in T_x^1\tilde{M} - \{-v\}$ the Radon–Nikodym derivative $J_v(w)$ of η_f^w with respect to η_f^v at w is defined and the function $J_v : w \to J_v(w)$ is continuous on $T_x^1\tilde{M} - \{-v\}$. Thus we obtain a Borel measure η_f^x on $T_x^1\tilde{M}$ by defining $\eta_f^x = J_v\eta_f^v$. The restriction of η_f^x to $T_x^1\tilde{M} - \{-v\}$ is locally finite.

On the other hand, for every $w \in T_x^1\tilde{M} - \{-v\}$ we have $J_w\eta_f^w = \eta_f^x$ on $T_x^1\tilde{M} - \{-v, -w\}$ and hence the measure η_f^x is defined independent of the choice of $v \in T^1\tilde{M}$ and is finite.

For $x, y \in \tilde{M}$ the homeomorphism

$$\pi_y^{-1} \circ \pi_x : T_x^1\tilde{M} \to T_y^1\tilde{M}$$

is absolutely continuous with respect to the measures η_f^x, η_f^y. Its Jacobian at $v \in T_x^1\tilde{M}$ equals the Jacobian at v with respect to the measures $\tilde{\eta}_f^{su}$ on the strong unstable manifolds

$$W^{su}(v) \quad \text{and} \quad W^{su}(\pi_y^{-1} \circ \pi(v))$$

of the *canonical map* which assigns to

$$z \in W^{su}(v) - W^s(-\pi_y^{-1} \circ \pi(v))$$

the unique point

$$[\pi_y^{-1} \circ \pi(v), z]$$

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of the intersection of $W^s(z)$ with $W^{su}(\pi_y^{-1} \circ \pi(v))$. This shows that for fixed $v \in T^1_x\tilde{M}$ the Jacobian at v of the map $\pi_y^{-1} \circ \pi$ with respect to the measures η_f^x, η_f^y depends Hölder-continuously on $y \in \tilde{M}$. In particular, the assignment $x \in \tilde{M} \to \eta_f^x(T^1_x\tilde{M})$ is Hölder continuous. Now since the measures $\tilde{\eta}_f^{su}$ are equivariant under the action of the fundamental group $\pi_1(M)$ of M on $T^1\tilde{M}$ the same is true for the measures η_f^x $(x \in \tilde{M})$ and hence they induce for every $p \in M$ a finite measure η_f^p on T^1_pM. Thus we obtain the following.

LEMMA 2.1. *The function f induces for every $p \in M$ a finite Borel measure η_f^p on T^1_pM, uniquely up to a universal constant. The assignment $p \in M \to \eta_f^p(T^1_pM)$ is Hölder continuous.*

The purpose of this section is to characterize the measures η_f^p as Hausdorff measures with respect to a suitable distance, i.e. to show Theorem B from the introduction. For this recall that the function f defines a cocycle ζ on T^1M by $\zeta(v,t) = \int_0^t f(\Phi^s v)\,ds$. We assume that f is *quasi-invariant under the flip*, i.e. that the cocycles ζ and $\mathcal{F}\zeta$ are equivalent; this is the case if and only if we have $\int_0^\tau f(\Phi^s v)\,ds = \int_0^\tau f(-\Phi^s v)\,ds$ for every periodic point v for Φ^t of period $\tau > 0$. Moreover, replacing f by $f + pr(f)$ we may assume that the pressure of f vanishes.

Recall from §1 the definition of the function α_ζ on $DTM - T^1M$; we denote this function in what follows by α_f if ζ is the cocycle defined by f. First of all we have the following.

LEMMA 2.2. *If the entropy of η_f does not vanish then there are numbers $\varepsilon > 0, T_0 > 0$ such that $(1/T)\int_0^T f(\Phi^s v)\,ds \geq \varepsilon$ for all $v \in T^1M$ and all $T \geq T_0$.*

Proof. We argue by contradiction and assume that there is a sequence $(v_j)_j \subset T^1M$ and a sequence $(T_j)_j \subset \mathbb{R}$ such that $T_j \to \infty$ $(j \to \infty)$ and that $(1/T_j)\int_0^{T_j} f(\Phi^s v_j)\,ds \leq 1/j$. Define a Borel probability measure η_j on T^1M by $\int \varphi\,d\eta_j = (1/T_j)\int_0^{T_j} \varphi(\Phi^s v_j)\,ds$ and let η be a weak limit of the measures η_j as $j \to \infty$. Then η is invariant under the geodesic flow and $\int f\,d\eta \leq 0$. Since by our assumption the pressure of f vanishes, the entropy h_η of η has to vanish as well and in fact η equals the unique Gibbs equilibrium state of f. But this is a contradiction to our assumption that the entropy of η_f is non-zero. □

Recall from §1 the definition of the Gromov product $(\,|\,)$ on $DTM - T^1M$; we have $(v \mid w) = \alpha_1(v,w)$ where 1 stands for the constant function $f \equiv 1$.

Part (2) of the following lemma is well known for the constant function $f \equiv h$ (where $h > 0$ is the topological entropy of the geodesic flow); see [**GH**].

LEMMA 2.3. *Let f be as in Lemma 2.2.*

(1) *There are numbers $c > 0, d > 0$ such that $(v \mid w) \leq c\alpha_f(v,w) + d$ for all $(v,w) \in DTM - T^1M$.*
(2) *There is a constant $\kappa > 0$ such that $\alpha_f(v,w) \geq \min\{\alpha_f(v,z), \alpha_f(z,w)\} - \kappa$ for all $x \in M$ and all pairwise distinct points $v, w, z \in T^1_xM$.*

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Proof. Denote by k_f the kernel defined as in Lemma 1.1 by the cocycle induced by f. Lemma 2.2 and the fact that k_f is Hölder continuous imply that there are numbers $\varepsilon > 0, r > 0$ such that $k_f(Pv, P\Phi^t v, \pi(v)) \geq \varepsilon t - r$ for all $v \in T^1\tilde{M}$ and all $t \geq 0$. This together with the definition of α_f and the arguments in the proof of Lemma 2.3 of [**H2**] implies the first part of the lemma.

To show part (2), denote the unique geodesic in $\tilde{M}$ joining ζ to ξ for $\xi \neq \zeta \in \tilde{M} \cup \partial\tilde{M}$ by $[\xi, \zeta]$. Since $\tilde{M}$ is of bounded negative curvature there is a constant $\chi > 0$ and for every triple (ξ_1, ξ_2, ξ_3) of pairwise district points in $\partial\tilde{M}$ there is $p \in \tilde{M}$ with the following properties:

(a) $\operatorname{dist}(p, [\xi_i, \xi_{i+1}]) \leq \chi$;

(b) the angle at p between any two of the geodesics $[p, \xi_i]$ is at least $1/\chi$.

Now let $x \in \tilde{M}$ and let $v_i \in T^1_x\tilde{M}$ be such that $\pi(v_i) = \xi_i$. By the definition of α_f and the Hölder continuity of k_f there is a number $\rho_1 > 0$ such that

$$|\alpha_f(v_i, v_{i+1}) + \tfrac{1}{2}k_f(p, x, \xi_i) + \tfrac{1}{2}k_f(p, x, \xi_{i+1})| \leq \rho_1$$

($i = 1, 2, 3$, and the indices taken mod 3).

Let $\gamma\colon \mathbb{R} \to \tilde{M}$ be the unique geodesic in $\tilde{M}$ joining $\gamma(0) = p$ to $\gamma(\operatorname{dist}(p, x)) = x$. From the Hölder continuity of k_f there is a number $\rho_2 > 0$ such that $| k_f(x, p, \xi) + k_f(x, p, \gamma(+\infty)) | \leq \rho_2$ if the angle at p between γ and $[p, \xi_i]$ is at least $1/2\chi$. By (b) above we can now consider two cases.

(1) The angle at p between γ and $[p, \xi_i]$ is at least $1/2\chi$ for $i = 1, 2, 3$. Then $\alpha_f(v_i, v_{i+1}) \geq \alpha_f(v_j, v_{j+1}) - \rho_3$ for $i, j \in \{1, 2, 3\}$ where $\rho_3 > 0$ is a universal constant.

(2) There is $i \in \{1, 2, 3\}$ such that the angle between γ and $[p, \xi_i]$ at p is smaller than $1/2\chi$. By (b) above, i is unique and we may assume that $i = 1$. The above considerations then show that

$$|\alpha_f(v_2, v_3) - k_f(p, x, \gamma(\infty))| \leq \rho_1 + \rho_2.$$

To estimate $\alpha_f(v_1, v_2)$ and $\alpha_f(v_1, v_3)$ let θ be the Busemann function at ξ_1 normalized at p with gradient $\operatorname{grad}\theta$. Define $l = \sup\{t \leq \operatorname{dist}(p, x) \mid \angle(\gamma'(s), -\operatorname{grad}\theta(\gamma(s))) < 1/2\chi \text{ for all } s \in [0, t]\}$. Again by the Hölder continuity of k_f and the fact that $\tilde{M}$ is of bounded negative curvature we can find a universal constant $\rho_4 > 0$ such that $|\frac{1}{2}k_f(p, \gamma(\ell), \xi_1) + \frac{1}{2}k_f(p, \gamma(\ell), \xi_j)| \leq \rho_4$ for $j = 2, 3$. Together with the above estimate we obtain that

$$|\alpha_f(v_1, v_2) - k_f(\gamma(\ell), x, \gamma(\infty))| \leq \rho_5, \quad |\alpha_f(v_1, v_3) - k_f(\gamma(\ell), x, \gamma(\infty))| \leq \rho_5,$$

where again $\rho_5 > 0$ is a universal constant. This shows, in particular, that $|\alpha_f(v_1, v_2) - \alpha_f(v_1, v_3)| \leq 2\rho_5$.

On the other hand we have

$$\alpha_f(v_2, v_3) - k_f(\gamma(\ell), x, \gamma(\infty)) \geq k_f(p, \gamma(\ell), \gamma(\infty)) - \rho_1 - \rho_2.$$

From Lemma 2.2 we obtain the existence of a constant $\rho_6 > 0$ such that

$$k_f(\beta(0), \beta(t), \beta(\infty)) \geq -\rho_6$$

for every geodesic β in $\tilde{M}$ and every $t \geq 0$. This yields $\alpha_f(v_2, v_3) \geq \alpha_f(v_1, v_2) - \rho_1 - \rho_2 - \rho_5 - \rho_6$ and finishes the proof of part (2) of the lemma. □

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For $(v,w) \in DTM$ now write $d_f(v,w) = e^{-\alpha_f(v,w)}$. By Lemma 2.3 there is a number $a > 0$ such that for every $p \in M$ the function d_f^a is a 'distance' on T_p^1M. More precisely, if $\kappa > 0$ is as in Lemma 2.3, then for $a > 0$ and $p \in M$ the restriction of d_f^a to T_p^1M satisfies the 'quasi-ultrametric' inequality $d_f^a(v,z) \leq \exp(a\kappa)\max\{d_f^a(v,w), d_f^a(w,z)\}$ $(v,w,z \in T_p^1M)$. If $a > 0$ is sufficiently small so that $\exp(a\kappa) < \sqrt{2}$ then there is for every $p \in M$ a distance ρ_f^a on T_p^1M constructed canonically from d_f^a and such that $(3 - 2\exp(a\kappa))\rho_f^a(v,w) \leq d_f^a(v,w) \leq \rho_f^a(v,w)$ for all $v, w \in T_p^1M$ (see Chapter 7 of [**GH**]).

For $p \in M$, $v \in T_p^1M$ and $r > 0$ now let $B_f(v,r) = \{w \in T_p^1M \mid d_f(v,w) \leq r\}$ and define the *spherical measure* ζ^p of d_f on T_p^1M by $\zeta^p(A) = \sup_{i>0}\inf\{\sum_{j=1}^{\infty} r_j \mid r_j \leq 1/i$ and $A \subset \cup_{j=1}^{\infty} B_f(v_j, r_j)$ for some $v_j \in T_p^1M (j \geq 1)\}$. The arguments in the proof of Proposition 2.8 of [**H2**] then imply the following.

PROPOSITION 2.4. *Let $f: T^1M \to \mathbb{R}$ be Hölder continuous of pressure zero, quasi-invariant under the flip and assume that the entropy of η_f does not vanish. Then for every $p \in M$ the one-dimensional spherical measure on T_p^1M defined by d_f coincides with η_f^p up to a universal constant.*

Now the entropy of η_f just equals the Hausdorff dimension of the measure η_f^p with respect to the 'distance' $d_1: (v,w) \to d_1(v,w) = e^{-(v|w)}$ (see [**H1**]). On the other hand, by Lemma 1.4 and its proof we have $d_f \geq cd_1^b$ for constants $c > 0, b > 0$ only depending on the Hölder norm of f. From this we conclude that in fact Proposition 2.4 holds for every Hölder continuous function on T^1M of pressure zero which is quasi-invariant under the flip.

COROLLARY 2.5. *For every $c > 0, \alpha > 0$ there is a number $\varepsilon = \varepsilon(c,\alpha) > 0$ such that the entropy of η_f is at least ε for every function $f: T^1M \to \mathbb{R}$ whose Hölder norm with exponent α does not exceed c and which is quasi-invariant under the flip.*

Proof. For a Hölder continuous function f on T^1M denote by h_f the entropy of η_f. Let B be the set of all functions f on T^1M which are quasi-invariant under the flip and whose Hölder norm with exponent α does not exceed c. Clearly B is convex and hence connected and there is a constant $\bar{c} > 0$ such that the Hölder α-norm of $f + pr(f)$ does not exceed $\bar{c}$ for all $f \in B$.

Let $f \in B$ and assume that $h_f \neq 0$. Write $g = f + pr(f)$; by Lemma 1.4 there are constants $a, b > 0$ only depending on $\bar{c}$ such that $d_g \leq ad_1^b$. Now h_f equals the d_1-Hausdorff dimension of η_g^p and hence by Proposition 2.4 we have $h_f \geq 1/b$. In other words we have $B_0 = \{f \in B \mid h_f > 0\} = \{f \in B \mid h_f \geq 1/b\}$.

The assignment $f \to h_f$ is upper semi-continuous and consequently the set B_0 is open and closed in B. Since B_0 contains the constant functions from B and since B is connected this means that $B_0 = B$ and shows the corollary. □

Recall that two distances d_1, d_2 on a topological space X are called *uniformly k-quasi-conformally equivalent* for some $k \geq 1$ if the following is satisfied. For $x \in X$ denote by $B_i(x,r)$ the ball of radius $r > 0$ about x with respect to the distance d_i. Then there is for every $x \in X$ and every $r > 0$ a number $\rho(r) > 0$ such that $B_1(x, \rho(r)) \subset B_2(x,r) \subset B_1(x, k\rho(r))$.

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We continue to assume that $f: T^1M \to \mathbb{R}$ is Hölder continuous, of pressure zero and quasi-invariant under the flip.

LEMMA 2.6. *There is a constant $k > 0$ only depending on the Hölder norm of f such that d_f is uniformly k-quasi-conformally equivalent to d_1.*

Proof. Let $(v, w) \in DT\tilde{M} - T^1\tilde{M}$ and let y be the projection of x to the geodesic γ joining $\gamma(-\infty) = \pi(w)$ to $\gamma(\infty) = \pi(v)$. Let $t = \operatorname{dist}(x, y)$; since the triangle in $\tilde{M}$ with vertices $x, y, \pi(v)$ has a right angle at y, comparison with a space of constant negative curvature shows that there is a number $a > 0$ not depending on t such that $\operatorname{dist}(y, P\Phi^t v) \leq a$. Since the kernel k_f defined as in Lemma 1.1 by f is uniformly Hölder continuous we can thus find a number $b > 0$ not depending on v and w such that

$$|\tfrac{1}{2}k_f(y, x, \pi(v)) + \tfrac{1}{2}k_f(y, x, \pi(w)) - k_f(P\Phi^t v, x, \pi(v))| \leq b.$$

By the definition of d_f this just means that there is a number $c > 0$ only depending on the Hölder norm of f such that $c^{-1}d_f(v, w) \leq e^{-\int_0^t f(\Phi^s v)\,ds} \leq cd_f(v, w)$. On the other hand, the same argument applied to the constant function 1 yields $c^{-1}d_1(v, w) \leq e^{-t} \leq cd_1(v, w)$. From this the lemma easily follows. □

For $v \in T^1M$ and $r > 0$ denote as before the closed ball of radius r in $T^1_{Pv}M$ about v with respect to the 'distance' d_f by $B_f(v, r)$. Assume again that $pr(f) = 0$; then we can estimate the η_f^p-volume of d_f-balls in T^1_pM as follows.

LEMMA 2.7. *There is a constant $a > 0$ such that $a^{-1}\varepsilon \leq \eta_f^p(B_f(v, \varepsilon)) \leq a\varepsilon$ for all $p \in M$, $v \in T^1_pM$ and $\varepsilon \in (0, 1]$.*

Proof. We only show the estimate from below; the estimate from above follows in the same way. For $v \in T^1M$ and $t > 0$ write $\varphi(v, t) = e^{-\int_0^t f(\Phi^s v)\,ds}$ and let $B_1(v, r)$ be the ball of radius r about v with respect to the distance d_1.

By Lemma 2.6 and its proof there is a number $a > 0$ such that

$$\pi B_f(v, \varphi(v, t)) \supset \pi B_1(\Phi^t v, a)$$

for all $v \in T^1\tilde{M}$ and all $t > 0$.

Since Jacobians of the canonical maps are uniformly Hölder continuous we can find a number $b > 0$ such that $\eta_f^{Pw}(B_0(w, a)) \geq b$ for all $w \in T^1M$. On the other hand, the considerations in the proof of Lemma 2.6 show that there is a number $d > 0$ such that

$$|k_f(P\Phi^t v, Pv, \pi(v)) - k_f(P\Phi^t v, Pv, \pi(w))| \leq d$$

for all $v \in T^1\tilde{M}$, $t > 0$ and $w \in B_1(\Phi^t v, a)$. But $k_f(P\Phi^t v, Pv, \pi(w))$ is the logarithm of the Radon–Nikodym derivative at $\pi(w)$ of the projection of η_f^{Pv} with respect to the projection of $\eta_f^{P\Phi^t v}$ to $\partial\tilde{M}$ and consequently

$$\eta_f^{Pv}B_f(v, \varphi(v, t)) \geq \int_{B(\Phi^t v, a)} e^{-k_f(Pv, P\Phi^t v, \pi(w))}\,d\eta_f^{P\Phi^t v}(w) \geq \varphi(v, t)\,e^{-d}b$$

which gives the required estimate from below. □

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PROPOSITION 2.8. *Let $f, g: T^1M \to \mathbb{R}$ be Hölder continuous functions of pressure zero which are quasi-invariant under the flip. If the d_f-Hausdorff dimension of η_g^p equals* 1 *then $\eta_f = \eta_g$.*

Proof. Let f, g be as in the proposition and assume that the cocycles defined by f, g are not cohomologous. Then $\int f\, d\eta_g > \int g\, d\eta_g$ and hence by the Birkhoff ergodic theorem there is a number $\varepsilon > 0$ and for every $p \in M$ a subset A of T_p^1M of full η_g^p-measure such that $\frac{1}{T}\int_0^T f(\Phi^s v)\, ds \geq \frac{1}{T}\int_0^T g(\Phi^s v)\, ds + 2\varepsilon$ for every $v \in A$ for $T \geq T(v)$, where $T(v) > 0$ is a constant depending on v. The proof of Lemma 2.6 then shows that there is a number $r(v) < e^{-T(v)}$ such that $B_g(v, r) \subset B_f(v, r^{1+\varepsilon})$ for all $r \leq r(v)$.

By Lemma 2.3 there is for a given number $\tau > 1$ a universal constant $b = b(\tau) > 1$ such that $\hat{B}_g(v, r) = \cup\{B_g(w, \tau r) \mid w \in T_{P_v}^1M, B_g(w, \tau r) \cap B_g(v, r) \neq \emptyset\} \subset B_g(v, br)$ for every $v \in T^1M$ and every $r > 0$. Lemma 2.7 then shows that $\eta_g^p \hat{B}_g(v, r)/\eta_g^p B_g(v, r) \leq a^2 b < \infty$ for a universal constant $a > 0$ depending on g and every $r > 0$. Thus we obtain from Theorem 2.8.17 of [**F**] that the balls $B_g(v, r)$ form a η_g^p-Vitali relation. Hence for every $m \geq 1$ there are points $v_i^m \in A$, numbers $r_i^m \leq \min\{r(v_i), 1/m\}$ such that the balls $B_g(v_i^m, r_i^m)$ are pairwise disjoint and cover η_g^p-almost all of A.

Then $B = \cap_{m=1}^\infty \cup_i B_g(v_i^m, r_i^m)$ has full measure in (T_p^1M, η_g^p). Now $\eta_g^p(B_g(v_i^m, r_i^m)) \geq cr_i^m$ for a universal constant $c > 0$ implies that $\sum_i r_i^m \leq 1/c$ for all $m > 0$.

On the other hand, we have $B \subset \cup_i B_f(v_i^m, (r_i^m)^{1+\varepsilon})$ for all $m > 0$ and hence the d_f-Hausdorff dimension of B is not larger than $(1+\varepsilon)^{-1}$. In other words, the d_f-Hausdorff dimension of η_g^p is strictly smaller than 1 which shows the proposition. □

Call two functions f and g on T^1M *equivalent* if the cocycles they define are cohomologous. This defines an equivalence relation on the space of Hölder continuous functions. The equivalence class of f consists of all functions g whose integrals over all closed orbits of Φ^t coincide with the integrals of f.

Let f be of pressure zero and quasi-invariant under the flip. Ledrappier showed [**L2**] that f is equivalent to a function which is smooth along the leaves of the unstable foliation. Lemma 2.2 implies that, by replacing the resulting function by an equivalent one, we may assume that f is positive.

Write $f(v, t) = \int_0^t f(\Phi^s v)\, ds$ and define a cocycle $\alpha(v, t)$ by $\alpha(v, f(v, t)) = t$. Then $\Psi^t(v) = \Phi^{\alpha(v,t)}(v)$ defines a Hölder continuous time change of Φ^t which is smooth along the leaves of the unstable foliation (see [**P**]).

Even though it may not be possible to obtain a time change of class C^1 in this way, the resulting flow is Anosov in the following sense:

(1) for each $v \in T^1M$ the set $\{w \in T^1M \mid d(\Psi^t v, \Psi^t w) \to 0\,(t \to -\infty)\}$ is a smoothly embedded submanifold of $W^u(v)$ which can be realized as the graph of a smooth function on $W^{su}(v)$;

(2) for each $v \in T^1M$ the set $\{w \in T^1M \mid d(\Psi^t v, \Psi^t w) \to 0 (t \to \infty)\}$ is a Hölder continuous submanifold of $W^s(v)$ which can be realized as the graph of a Hölder continuous function on $W^{ss}(v)$.

Here d denotes any distance on T^1M defining the usual topology. For every Φ^t-invariant measure η on T^1M the measure $f\eta$ is invariant under Ψ^t. Now let η_f^u

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(respectively η_f^s) be a family of conditional measures on unstable (respectively stable) manifolds for the Gibbs measure η_f. Disintegration of η_f^u (respectively η_f^s) over flow boxes of uniform size then yields a family $\bar{\eta}^{su}$ (respectively $\bar{\eta}^{ss}$) of conditional measures for η_f on strong unstable (respectively strong stable) manifolds for Ψ^t. The measures $\bar{\eta}^{su}$ (respectively $\bar{\eta}^{ss}$) are quasi-invariant under the action of Ψ^t and for every periodic point $v \in T^1M$ of Ψ^t of period $\tau > 0$ the Jacobian of Ψ^τ with respect to the measures $\bar{\eta}^{su}$ (respectively $\bar{\eta}^{ss}$) at v equals e^τ (respectively $e^{-\tau}$). But this just means that $f\eta_f$ is the measure of maximal entropy for Ψ^t.

The arguments in [**H1**] then imply the following.

PROPOSITION 2.9. *Let $f, g: T^1M \to \mathbb{R}$ be Hölder continuous of pressure zero and quasi-invariant under the flip. Let Ψ^t be a time change of Φ^t defined as above by f. Then the d_f-Hausdorff dimension of η_g^p is the entropy of the Ψ^t-invariant measure on T^1M in the measure class of η_g.*

3. *Further properties of conditional measures of Gibbs states*

In this section we consider the Gibbs equilibrium state $\eta = \eta_f$ of a Hölder continuous function $f: T^1M \to \mathbb{R}$. In contrast to the previous sections we do not now require that f is quasi-invariant under the flip.

Recall from §2 that f induces a family η^{su} of locally finite Borel measures on strong unstable manifolds which project to a family η^p $(p \in M)$ of finite Borel measures on the fibres of the fibration $T^1M \to M$.

Let λ_M be the Lebesgue measure on M. We then obtain a finite Borel measure $\bar{\eta}$ on T^1M by defining $\bar{\eta}(A) = \int \eta^p(A \cap T_p^1M)\, d\lambda_M(p)$.

On the other hand, the measures η^{su} on the leaves of W^{su} and the Lebesgue measures λ^s on the leaves of W^s can be pasted together to form a finite Borel measure σ on T^1M which satisfies $d\sigma = d\lambda^s \times d\eta^{su}$ with respect to a local product structure. An explicit construction of this measure can be found in the paper [**L1**] of Ledrappier. In fact, $\bar{\eta}$ and σ coincide up to a constant.

PROPOSITION 3.1. *$d\bar{\eta} = d\lambda^s \times d\eta^{su}$ with respect to a local product structure (up to a constant).*

Proof. As above let σ be defined by $d\sigma = d\lambda^s \times d\eta^{su}$; we have to show that $\bar{\eta} = \sigma$. For notational convenience we will consider the lifts of $\bar{\eta}, \sigma, \eta^{su}$ and λ^s to $T^1\tilde{M}$ which we denote by the same symbols.

For $v \in T^1\tilde{M}$ and $\rho > 0$ define

$$C(v, \rho) = \{w \in T_{Pv}^1\tilde{M} \mid \angle(v, w) \leq \rho\}$$

and for $\delta > 0$ let

$$D(v, \delta, \rho) = \{w \in T^1\tilde{M} \mid \operatorname{dist}(Pv, Pw) \leq \delta, \pi(w) \in \pi C(v, \rho)\}.$$

Let $\varepsilon > 0$ and let d be the distance on $T^1\tilde{M}$ induced by the Sasaki metric.

Since the Jacobians of the canonical maps are Hölder continuous there is a number $\delta(\varepsilon) > 0$ such that for $v, w \in T^1\tilde{M}$ with $d(v, w) \leq 2\delta(\varepsilon)$ the Jacobian at v of the

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canonical map

$$z \in W^{su}(v) - W^s(-w) \to [w, z] \in W^{su}(w) \cap W^s(z)$$

with respect to the measures η^{su} is contained in the interval $[(1+\varepsilon)^{-1}, 1+\varepsilon]$ and, similarly, the Jacobian at v of the canonical map

$$u \in W^s(v) - W^u(-w) \to [u, w] \in W^s(w) \cap W^{su}(u)$$

with respect to the measures λ^s is contained in $[(1+\varepsilon)^{-1}, 1+\varepsilon]$ (where $W^u(-w)$ is the leaf of the unstable foliation containing $-w$).

Let $\tilde{\lambda}$ be the Lebesgue measure on $\tilde{M}$. Since the curvature of $\tilde{M}$ is uniformly bounded there is number $\beta \in (0, 1)$ such that

$$\tilde{\lambda}(B(x, \delta)) \leq (1+\varepsilon)\tilde{\lambda}(B(x, \beta\delta))$$

for all $x \in \tilde{M}$ and all $\delta \leq \delta(\varepsilon)$.

For $v \in T^1\tilde{M}$, $w \in W^s(v)$ and $\rho > 0$ define

$$A(v, w, \rho) = \{z \in W^{su}(w) \mid \pi(z) \in \pi(C(v, \rho))\}.$$

Since the projection $\pi : T^1\tilde{M} \to \partial\tilde{M}$ is Hölder continuous and equivariant under the action of $\pi_1(M)$ there is for every $\delta \leq \delta(\varepsilon)$ a number $\rho(\delta) \leq \delta(\varepsilon)$ such that $\text{dist}(Pz, Pw) \leq \delta - \delta\beta$ for all $z \in A(v, w, \rho(\delta))$ whenever $\text{dist}(Pv, Pw) \leq \delta$.

Now let $\delta \leq \delta(\varepsilon)$ and $\rho \leq \rho(\delta)$. For $p \in \tilde{M}$, $v \in T^1_p\tilde{M}$ and

$$y \in B(Pv, \delta), \quad w \in C(v, \rho)$$

the Jacobian at w of the map $\pi_y^{-1} \circ \pi_p$ with respect to the measures η^y, η^p is contained in the interval $[(1+\varepsilon)^{-1}, 1+\varepsilon]$. In particular, we have

$$\bar{\eta}(D(v, \delta, \rho)) \leq \tilde{\lambda}(B(Pv, \delta))\eta^p(C(v, \rho))(1+\varepsilon)$$

and

$$\eta^p(C(v, \rho)) \leq (1+\varepsilon)\eta^{su}(A(v, v, \rho)).$$

On the other hand, for every $w \in B^s(v, \beta\delta)$ the set $A(v, w, \rho)$ is contained in $D(v, \delta, \rho)$ and consequently

$$\begin{aligned} \sigma(D(v, \delta, \rho)) &\geq \sigma\Big(\bigcup_{w \in B^s(v, \beta\delta)} A(v, w, \rho)\Big) \geq \tilde{\lambda}(B(Pv, \beta\delta))\eta^{su}(A(v, v, \rho))(1+\varepsilon)^{-2} \\ &\geq (1+\varepsilon)^{-5}\bar{\eta}(D(v, \delta, \rho)). \end{aligned}$$

For Borel sets $A \subset \tilde{M}, C \subset \partial\tilde{M}$ now define

$$D(A, C) = \{w \in T^1\tilde{M} \mid Pw \in A, \pi(w) \in B\}.$$

Let $x \in \tilde{M}, \delta \leq \delta(\varepsilon)$ and let $U \subset \partial\tilde{M}$ be open. Since the balls

$$C(v, \rho) \quad (v \in T^1_x\tilde{M}, \rho > 0)$$

form a Vitali relation for every Radon measure on $T^1_x\tilde{M}$ there are

$$v_i \in T^1_x\tilde{M}, \quad \rho_i \leq \rho(\delta)$$

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such that the balls $C(v_i, \rho_i)$ $(i \geq 1)$ are pairwise disjoint, contained in $\pi_x^{-1}(U)$ and cover η^x-almost all of $\pi_x^{-1}(U)$. Then the sets $D(v_i, \delta, \rho_i)$ are pairwise disjoint $(i \geq 1)$ and contained in $D(\overline{B}(x, \delta), U)$ and

$$\begin{aligned}\bar{\eta}(D(\overline{B}(x,\delta),U)) &= \sum_{i=1}^{\infty} \bar{\eta}(D(v_i,\delta,\rho_i)) \leq \left(\sum_{i=1}^{\infty} \sigma(D(v_i,\delta,\rho_i))\right)(1+\varepsilon)^5 \\ &\leq (1+\varepsilon)^5 \sigma(D(\overline{B}(x,\delta),U)).\end{aligned}$$

Since $\bar{\eta}$ is a Radon measure we also obtain from this that

$$\bar{\eta}(D(\overline{B}(x,\delta),C)) \leq (1+\varepsilon)^5 \sigma(D(\overline{B}(x,\delta),C))$$

for every $x \in \tilde{M}$, $\delta \leq \delta(\varepsilon)$ and every Borel subset C of $\partial\tilde{M}$.

With the same argument we conclude that

$$\bar{\eta}(D(A,C)) \leq (1+\varepsilon)^5 \sigma(D(A,C))$$

for all Borel subsets $A \subset \tilde{M}, C \subset \partial\tilde{M}$. Since $\varepsilon > 0$ was arbitrarily chosen, we have in fact that $\eta(D(A, C)) \leq \sigma(D(A, C))$. Now the sets of the form $D(A, C)$ for $A \subset \tilde{M}$ open, $C \subset \partial\tilde{M}$ open, form a basis for the topology of $T^1\tilde{M}$ and consequently we have $\bar{\eta}(U) \leq \sigma(U)$ for every open subset U of $T^1\tilde{M}$ and hence $\bar{\eta} \leq \sigma$.

To show that $\sigma \leq \bar{\eta}$ we proceed in exactly the same way. Namely, with the above notation, for $\delta \leq \delta(\varepsilon)$ choose $\alpha(\delta) \leq \delta(\varepsilon)$ sufficiently small so that

$$D(v, \beta\delta, \alpha) \subset \bigcup_{w \in B^s(v,\delta)} A(v, w, \alpha)$$

for all $v \in T^1\tilde{M}$ and all $\alpha \leq \alpha(\delta)$.

The above estimates then imply that

$$\begin{aligned}\sigma(D(v,\beta\delta,\alpha)) &\leq \sigma\left(\bigcup_{w\in B^s(v,\delta)} A(v,w,\alpha)\right) \leq (1+\varepsilon)^3 \tilde{\lambda}(B(Pv,\delta))\eta^{Pv}(C(v,\alpha)) \\ &\leq (1+\varepsilon)^5 \bar{\eta}(D(v,\beta\delta,\alpha))\end{aligned}$$

for all $\alpha \leq \alpha(\delta)$ and consequently

$$\sigma(D(v,\delta,\alpha)) \leq (1+\varepsilon)^5 \bar{\eta}(D(v,\delta,\alpha))$$

whenever $\delta \leq \delta(\varepsilon)\beta$ and $\alpha \leq \alpha(\delta)$. As before, this implies that $\sigma \leq \bar{\eta}$ and finishes the proof of the proposition. □

Remark. In the case when f is a constant function, i.e. if η equals the Bowen–Margulis measure on T^1M, the above proposition was previously shown by Knieper [**Kn**].

Now recall that the measure η can be written uniquely in the form $d\eta = d\eta^{su} \times d\eta^{ss} \times dt$ with respect to a local product structure, where η^{ss} is a family of locally finite Borel measures on the leaves of the strong stable foliation and dt is the usual Lebesgue measure on the flow lines of the geodesic flow (compare [**L1**]). The measures η^{ss} are absolutely continuous with respect to the unstable foliation, with Hölder continuous

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Radon–Nikodym derivatives. We then obtain a family η^u of locally finite Borel measures on the leaves of $W^u \subset T^1M$ by defining $d\eta^u = dt \times d\eta^{su}$.

Let $v \in T^1M$ and let A be an open, relatively compact neighborhood of v in $W^u(v)$. The measure η^u on $W^u(v)$ then induces a probability measure η_A on A by defining $\eta_A(B) = \eta^u(B)/\eta^u(A)$ for every Borel subset B of A. Then we have the following.

LEMMA 3.2. *For every continuous function φ on T^1M the limit $\lim_{t\to\infty} \int_A \varphi \circ \Phi^t \, d\eta_A$ exists and equals $\int \varphi \, d\eta$.*

Proof. By subdividing A if necessary, it suffices to show the lemma for sets A which are contained in a subset C of T^1M with a local product structure.

In other words, we assume that $v \in A$ and that there is an open neighborhood B of v in $W^{ss}(v)$ and a homeomorphism $\wedge$ of $A \times B$ onto a neighborhood C of v in T^1M with the following properties:

(i) $\wedge(w, v) = w$ for all $w \in A$;
(ii) $\wedge(v, z) = z$ for all $z \in B$;
(iii) $\wedge(A \times \{z\})$ is contained in a leaf of W^u for all $z \in B$;
(iv) $\wedge(\{w\} \times B)$ is contained in a leaf of W^{ss} for all $w \in A$.

Now let φ be a continuous function on T^1M. By writing φ as a sum of its positive and negative parts we see that we may assume that φ is non-negative and satisfies $\int \varphi \, d\eta = \beta > 0$.

Let $\varepsilon \in (0, 1)$; we have to find a number $T_0 > 0$ such that

$$(1 - \varepsilon)\beta < \int_A (\varphi \circ \Phi^T) \, d\eta_A < (1 + \varepsilon)\beta$$

for all $T \geq T_0$.

For this let $\delta \in (0, \varepsilon/4)$ be sufficiently small so that $(1 + \delta)^{-3} > 1 - \varepsilon/2$ and $(1 + \delta)^3 < 1 + \varepsilon/2$. Recall that the map $\wedge$ is absolutely continuous with respect to the measure $\eta^u \times \eta^{ss}$ on $A \times B$ and the measure η on C, with Hölder continuous Jacobian, and that this Jacobian equals 1 on $A \times \{v\}$.

Thus, by choosing B small enough we may assume that the Jacobian of $\wedge$ in $A \times B$ is contained in $((1 + \delta)^{-1}, 1 + \delta)$. This means, in particular, that

$$(1 + \delta)^{-1}\eta^u(A)\eta^{ss}(B) < \eta(C) < (1 + \delta)\eta^u(A)\eta^{ss}(B). \qquad (*)$$

Now A and B are relatively compact subsets of $W^u(v)$, respectively $W^{ss}(v)$, and hence as $T \to \infty$ the diameter of the sets $\Phi^T \wedge (\{w\} \times B)$ tends to zero uniformly in $w \in A$. Since φ is continuous, hence uniformly continuous on T^1M this implies that there is a number $T_1 > 0$ such that for every $w \in A, z \in B$ and every $T \geq T_1$ we have $|\varphi(\Phi^T w) - \varphi(\Phi^T \wedge (w, z))| < \delta\beta$. Thus for $T \geq T_1$ we obtain

$$\begin{aligned}(1 + \delta)^{-1} \int_C (\varphi \circ \Phi^T) \, d\eta - \delta\beta\eta(C) &\leq \eta^{ss}(B) \int_A (\varphi \circ \Phi^T) \, d\eta_A \\ &\leq (1 + \delta) \int_C (\varphi \circ \Phi^T) \, d\eta + \delta\beta\eta(C). \quad (**)\end{aligned}$$

Now η is a Gibbs equilibrium state and hence mixing with respect to the geodesic

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flow. This means that there is a number $T_0 \geq T_1$ such that

$$(1+\delta)^{-1}\beta\eta(C) \leq \int_C (\varphi \circ \Phi^T)\, d\eta \leq (1+\delta)\beta\eta(C) \qquad (***)$$

for all $T \geq T_0$. Equations $(*)$–$(***)$ then imply that

$$((1+\delta)^{-3} - (1+\delta)\delta)\beta\eta^u(A) \leq \int_A (\varphi \circ \Phi^T)\, d\eta^u \leq ((1+\delta)^3 + (1+\delta)\delta)\beta\eta^u(A)$$

for all $T \geq T_0$ which is just the required inequality. □

Again, let $v \in T^1M$ and let $B \subset W^{su}(v)$ be an open, relatively compact neighborhood of v in $W^{su}(v)$. As above, the measures η^{su} on $W^{su}(v)$ induce a Borel probability measure η_B on B.

COROLLARY 3.3. *For every continuous function φ on T^1M the limit $\lim_{T\to\infty} \int_B (\varphi \circ \Phi^T)\, d\eta_B$ exists and equals $\int \varphi\, d\eta$.*

Proof. As in the proof of Lemma 3.2 we may assume that φ is non-negative and that $\int \varphi\, d\eta = \beta > 0$. For $v \in T^1M$ write $f(v) = (d/dt)\eta^{su} \circ \Phi^t\,|_{t=0}\,(v)$; by our assumption on η, the function f is Hölder continuous and hence bounded in absolute value by a constant $\kappa > 0$.

Choose $\delta > 0$ and let $t_0 = (\log(1+\delta))/\kappa$; since φ is continuous and hence uniformly continuous on T^1M we can find a number $\tau \in (0, t_0]$ in such a way that $|\varphi(\Phi^t w) - \varphi(w)| \leq \delta\beta$ for all $w \in T^1M$ and all $t \in (-\tau, \tau)$. Define $A = \cup_{-\tau<t<\tau} \Phi^t B$; then A is an open, relative compact subset of $W^u(v)$. Since $e^{-\kappa t}\eta^{su}(B) \leq \eta^{su}(\Phi^t B) \leq e^{\kappa t}\eta^{su}(B)$ for all $t \in \mathbb{R}$ we have

$$(1+\delta)^{-1} 2\tau \eta^{su}(B) \leq \eta^u(A) \leq (1+\delta) 2\tau \eta^{su}(B). \qquad (*)$$

Moreover, the choice of τ implies that

$$\begin{aligned} &(1+\delta)^{-1}\left[\int_A (\varphi \circ \Phi^T)\, d\eta^u - \delta\beta\eta^u(A)\right] \\ &\quad \leq \quad 2\tau \int_B (\varphi \circ \Phi^T)\, d\eta^{su} \leq (1+\delta)\left[\int_A (\varphi \circ \Phi^T)\, d\eta^u + \delta\beta\eta^u(A)\right]. \end{aligned} \qquad (**)$$

Since $\lim_{T\to\infty} \int_A (\varphi \circ \Phi^T)\, d\eta^u = \beta\eta^u(A)$ by Lemma 3.2, we obtain from this that

$$\begin{aligned} (1+\delta)^{-1}((1+\delta)^{-1} - \delta(1+\delta))\beta \quad &\leq \quad \liminf_{T\to\infty} \int_B (\varphi \circ \Phi^T)\, d\eta_B \\ &\leq \quad \limsup_{T\to\infty} \int_B (\varphi \circ \Phi^T)\, d\eta_B \leq (1+\delta)^3\beta. \end{aligned}$$

Since $\delta > 0$ was arbitrarily chosen this shows the corollary. □

As an application we obtain the following.

PROPOSITION 3.4. *For every $p \in M$, every continuous function ρ on T^1_pM and every continuous function φ on T^1M the limit $\lim_{T\to\infty} \int_{T^1_pM} (\varphi \circ \Phi^T)\rho\, d\eta^p$ exists and equals $(\int \varphi\, d\eta)(\int \rho\, d\eta^p)$.*

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Proof. As before, we may assume that the functions φ on T^1M and ρ on T^1_pM are non-negative and satisfy $\int \varphi\, d\eta = \beta > 0$, and $\int \rho\, d\eta^p = \kappa > 0$. Let $\varepsilon \in (0,1)$; again, we are looking for a number $T_0 > 0$ such that

$$\kappa\beta(1-\varepsilon) \leq \int_{T^1_pM} (\varphi \circ \Phi^T)\rho\, d\eta^p \leq \kappa\beta(1+\varepsilon)$$

for all $T \geq T_0$.

For this let $\delta > 0$ be sufficiently small so that $(1+\delta)^6 < 1+\varepsilon/2$ and $(1+\delta)^{-5}(1-\delta) > 1-\varepsilon/2$.

Choose $c = \max\{1, \varphi(w) \mid w \in T^1M\}$; then there is a number $\chi > 0$ such that with $U = \rho^{-1}(\chi, \infty)$ we have $c\int_{T^1_pM-U} \rho\, d\eta^p < \varepsilon\kappa\beta/2$.

Let d be the distance on T^1M induced by the Sasaki metric; then there is a number $\tau > 0$ such that $\mid \varphi(w) - \varphi(\overline{w}) \mid < \rho\delta(1+\delta)^{-2}$ whenever $d(w, \overline{w}) < \tau$. Choose finitely many points $v_1, \ldots, v_\ell \in U$ and open neighborhoods U_i of v_i in U with the following properties:

(i) $U_i \cap U_j = \emptyset$ for $i \neq j$ and $\eta^p(\cup_{i=1}^{\ell} U_i) = \eta^p(U)$;

(ii) $\rho(w)/\rho(v_i) \in ((1+\delta)^{-1}, 1+\delta)$ for $w \in U_i$, $i \in \{1, \ldots, \ell\}$;

(iii) for every $i \in \{1, \ldots, \ell\}$ there is a homeomorphism Ψ_i of U_i onto a neighborhood V_i of v_i in $W^{su}(v_i)$ such that $\Psi_i(w) \in W^s(w)$ for $w \in U_i$ and $d(w, \Psi_i(w)) < \tau$;

(iv) for every $i \in \{1, \ldots, \ell\}$ the Jacobian of Ψ_i as a map of (U_i, η^p) onto (V_i, η^{su}) has its range in $((1+\delta)^{-1}, 1+\delta)$.

By Corollary 3.3 there is a number $T_0 > 0$ such that $(1+\delta)^{-1}\beta\eta^{su}(V_i) \leq \int_{V_i} (\varphi \circ \Phi^T)\, d\eta^{su} \leq (1+\delta)\beta\eta^{su}(V_i)$ for every $i \in \{1, \ldots, \ell\}$. Since $d(\Phi^T w, \Phi^T \Psi_i(w)) \leq d(w, \Psi_i(w))$ for all sufficiently large $T > 0$, this shows together with (iii) and (iv) that

$$d^{-3}(1-\delta)\eta^p(U_i)\beta \leq \int_{U_i} \varphi \circ \Phi^R\, d\eta^p \leq d(d^2+\delta)\beta\eta^p(U_i).$$

Now (ii) implies that

$$d^{-1}\rho(v_i)\int_{U_i} (\varphi \circ \Phi^T)\, d\eta^p \leq \int_{U_i} (\varphi \circ \Phi^T)\rho\, d\eta^p \leq d\rho(v_i)\int_{U_i} (\varphi \circ \Phi^T)\, d\eta^p$$

and consequently for $T > T_0$ we obtain

$$\begin{aligned}(1-\varepsilon)\beta\kappa &\leq d^{-4}(1-\delta)\beta\int_U \rho\, d\eta^p \leq \int_U (\varphi \circ \Phi^T)\rho\, d\eta^p \leq \int_{T^1_pM} (\varphi \circ \Phi^T)\rho\, d\eta^p \\ &\leq \int_U (\varphi \circ \Phi^T)\rho\, d\eta^p + \kappa\beta\varepsilon/2 \leq (1+\varepsilon)\kappa\beta\end{aligned}$$

as required. □

For $p \in M$ now denote by λ_p the normalized Lebesgue measure on the round sphere T^1_pM. As an immediate corollary of the above consideration we obtain the following.

COROLLARY 3.5. *The measures $\Phi^T(\lambda_p)$ converge as $T \to \infty$ weakly to the Lebesgue–Liouville measure λ on T^1M.*

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REFERENCES

[Bn] F. Bonahon. The geometry of Teichmüller space via geodesic currents. *Inventiones Math*. **92** (1988). 139–162.

[B] M. Bourdon. Une remarque sur le birapport au bord des CAT (-1)-espaces. *Publ. Math. IHES*.

[F] H. Federer. *Geometric Measure Theory*. Springer, New York, 1969.

[GH] E. Ghys and P. de la Harpe. *Sur les groupes hyperboliques d'après Mikhael Gromov*. Birkhäuser, Boston, 1990.

[H1] U. Hamenstädt. Time preserving conjugacies of geodesic flows. *Ergod. Th. & Dynam. Sys.* **12** (1992), 67–74.

[H2] U. Hamenstädt. Harmonic measures, Hausdorff measures and eigenfunctions. *J. Diff. Geom.* **44** (1996), 1–31.

[H3] U. Hamenstädt. A new description of the Bowen Margulis measure. *Ergod. Th. & Dynam. Sys.* **9** (1989), 455–464.

[K] V. Kaimanovich. Invariant measures of the geodesic flow and measures at infinity on negatively curved manifolds. *Ann. I.H.P. (Physique Théorique)* **53** (1990), 361–393.

[Kn] G. Knieper. Spherical means on compact Riemannian manifolds of negative curvature. *Differential Geometry and its Appl.* **4** (1994), 361–390.

[L1] F. Ledrappier. A renewal theorem for the distance in negative curvature. *Proc. Symp. Pure Math.* **57** (1995), 351–360.

[L2] F. Ledrappier. Central limit theorem in negative curvature. *Ann. Prob.* **23** (1995), 1219–1233.

[O1] J.-P. Otal. Le spectre marqué des longeurs des surfaces à courbure négative. *Ann. Math.* **131** (1990), 151–162.

[O2] J.-P. Otal. Sur la geometrie symplectique de l'espace des géodésiques d'une variété à courbure négative. *Rev. Mat. Iberoam.* **8** (1992), 441–456.

[P] W. Parry. Synchronisation of canonical measures for hyperbolic attractors. *Commun. Math. Phys.* **106** (1986), 267–275.

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