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# MINIMUM DISTANCE ESTIMATION OF NONSTATIONARY TIME SERIES MODELS

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This paper analyzes the limit distribution of minimum distance (MD) estimators for nonstationary time series models that involve nonlinear parameter restrictions. A rotation for the restricted parameter space is constructed to separate the components of the MD estimator that converge at different rates. We derive regularity conditions for the restriction function that are easier to verify than the stochastic equicontinuity conditions that arise from direct estimation of the restricted parameters. The sequence of matrices that is used to weigh the discrepancy between the unrestricted estimates and the restriction function is allowed to have a stochastic limit. For MD estimators based on unrestricted estimators with a mixed normal asymptotic distribution the optimal weight matrix is derived and a goodness-of-fit test is proposed. Our estimation theory is illustrated in the context of a permanent-income model and a present-value model.

## 1. INTRODUCTION

This paper considers the limit distribution of minimum distance (MD) estimators for time series models that involve deterministic or stochastic trends. The models are indexed by a $q \times 1$ vector of parameters $a$. At the pseudo-true value $a_0$, the parameter vector satisfies the nonlinear restriction $a_0 = g(b_0)$, where $b_0$ is a lower dimensional $p \times 1$ vector. Such a restriction can, e.g., stem from an

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optimization-based economic model. We assume that an estimator $\hat{a}_T$ for the unrestricted parameter vector is available. Our main interest is in analyzing the limit distribution of MD estimators $\hat{b}_T$ of $b_0$ and testing whether the restriction $a_0 = g(b_0)$ is satisfied.

In principle, the time series model could be reparameterized in terms of the parameter vector $b$ to estimate $b_0$ directly. However, an attractive alternative is to estimate the unrestricted parameter vector $a_0$ first and then to minimize a measure of discrepancy between $\hat{a}_T$ and $g(b)$. This procedure is known as MD estimation. The distance measure used in our paper is $\|W_T(\hat{a}_T - g(b))\|$, where $\{W_T\}$ is a sequence of weight matrices and $\|\cdot\|$ denotes the Euclidean norm. The properly standardized discrepancy between the unrestricted estimate $\hat{a}_T$ and the restriction function evaluated at the MD estimate $g(\hat{b}_T)$ provides a natural goodness-of-fit measure for the restricted specification. We show that the regularity conditions for consistency and weak convergence of the MD estimator can be stated in terms of equicontinuity conditions for the derivatives of the restriction function and provide useful sufficient conditions. Our conditions are easier to verify than the stochastic equicontinuity conditions (e.g., Saikkonen, 1995) that commonly arise in nonstationary time series models if $b$ is directly estimated.

In the context of linear regression models without trends, the asymptotic properties of MD estimators are well known (e.g., Chamberlain, 1984). The unrestricted estimator $\hat{a}_T$ is $\sqrt{T}$-consistent and has a multivariate normal limit distribution. If the restriction function is smooth, a first-order Taylor expansion of $g(b)$ immediately yields the limit distribution of the MD estimator. The optimal weight matrix is the inverse of the covariance matrix of the unrestricted estimator $\hat{a}_T$.

Two complications arise in the presence of time trends. First, some linear combinations of $\hat{a}_T$ will converge at a faster rate than $\sqrt{T}$. The limit theory of $\hat{a}_T$ is usually based on a rotation that separates the fast converging components from the slow components. The analysis of the MD estimator requires an additional rotation of the restricted parameter vector $b$. We describe the appropriate rotation and how to determine the order of consistency of the rotated MD estimator.

Second, if the model includes stochastic trends, the limit distribution of $\hat{a}_T$ is nonstandard. In the examples studied in this paper the limit distribution of $\hat{a}_T$ is mixed normal with a random covariance matrix. Optimality considerations suggest using a sequence of weight matrices $\{W_T\}$ that converges in distribution to the inverse of this random covariance matrix. In this case the objective function of the MD estimator does not converge to a nonstochastic limit, and the standard consistency argument for extremum estimators (e.g., Amemiya, 1985) cannot be employed. Following the methods used in the empirical process literature (e.g., Kim and Pollard, 1990; van der Waart and Wellner, 1996), we present an argument based on an almost sure representation of the objective function.

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The paper is organized as follows. Section 2 reviews the existing literature and presents two examples to motivate the MD estimation problem. Nonlinear restriction functions are derived from a permanent-income model that is based on a linear-quadratic dynamic programming problem and a present-value model. The two examples highlight the complications in MD estimation that are addressed in this paper. A general definition of the MD estimator is provided in Section 3 and some fundamental assumptions are stated. Section 4 establishes the consistency of the MD estimator, and Section 5 characterizes its limit distribution under various assumptions on the rates of convergence of $\hat{a}_T$ and the smoothness of $g(b)$. In Section 6 we consider the case in which the limit distribution of the unrestricted parameter estimates $\hat{a}_T$ is mixed normal. We define an optimality criterion for the MD estimator and derive the optimal weight matrix. Moreover, a $J$-type test for the hypothesis $a_0 = g(b_0)$ is provided. Section 7 concludes, and the Appendix contains mathematical derivations and proofs.

The notation $\equiv$ is used to signify distributional equivalence, $\Rightarrow$ denotes convergence in distribution, $\xrightarrow{p}$ denotes convergence in probability, $\xrightarrow{a.s.}$ is almost sure convergence, and $\otimes$ is the Kronecker product. We will use $\int B$ and $\int BB'$ to abbreviate integrals of vector Brownian motion $B(r)$ sample paths $\int B(r)dr$ and $\int B(r)B(r)'dr$, respectively. The term $\mathcal{I}_m$ is the $m \times m$ identity matrix, $0_{m\times k}$ is an $m \times k$ matrix of zeros, $\text{diag}(a_1,\ldots,a_m)$ denotes the $m \times m$ diagonal matrix with elements $a_1,\ldots,a_m$, and the vec-operator stacks the columns of a matrix.

## 2. EXAMPLES

Economic theory often imposes restrictions across parameters of reduced-form econometric models. For instance, many structural models involve solutions to stochastic dynamic programming problems, which arise from utility, profit, or social welfare maximization. The optimal decision rule generates nonlinear restrictions across the parameters that govern the joint evolution of state and control variables. To motivate the MD estimation we consider a dynamic programming problem with a quadratic objective function and linear state transition equations. In macroeconomics, linear-quadratic programming problems are frequently used to approximate equilibria in smooth nonlinear model economies (see Ljungqvist and Sargent, 2000).

### Example 1

A consumer chooses consumption $\{C_t\}_{t=0}^{\infty}$ to maximize the expected utility

$$-\frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t(C_t-\epsilon_{1,t}-1)^2\right] \tag{1}$$

subject to the constraints

$$\begin{aligned} I_{t+1} &= \mu(1-\phi)+\phi I_t+\epsilon_{2,t+1},\\ W_{t+1} &= (1+r)(W_t-C_t)+I_{t+1}+\epsilon_{3,t+1}, \end{aligned} \tag{2}$$

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where $W_t$ is wealth, $I_t$ is a stochastic income process, and $r$ is the real interest rate. The exogenous shocks are collected in the vector $\epsilon_t = [\epsilon_{1,t}, \epsilon_{2,t}, \epsilon_{3,t}]'$. We will assume that $\epsilon_t \sim iid(0, \Sigma_{\epsilon\epsilon})$. The shock $\epsilon_{1,t}$ can be interpreted as taste shock that shifts the utility of time $t$ consumption, and $\epsilon_{2,t}$ is the innovation of the observed income process $I_t$. It is assumed that there exists a stochastic income component $\epsilon_{3,t}$ that is unobserved by the econometrician. In other words, we do not require measured wealth, consumption, and income to satisfy the wealth accumulation constraint in equation (2) exactly. The goal is to estimate the structural parameters $r$, $\mu$, and $\phi$.

Under the optimal consumption rule the law of motion for consumption, wealth, and income is

$$\begin{bmatrix} C_t \\ \Delta W_t \\ I_t \end{bmatrix} = \begin{bmatrix} \frac{r}{1+r} & \frac{\mu}{1+r} & \frac{\phi r(2-\phi)}{(1+r-\phi)(1+r)} \\ 0 & 0 & \frac{\phi(1-\phi)}{1+r-\phi} \\ 0 & \mu & \phi \end{bmatrix} \begin{bmatrix} W_{t-1} \\ 1 \\ (I_{t-1}-\mu) \end{bmatrix} + \begin{bmatrix} \frac{r}{1+r} & \frac{r}{1+r} & \frac{r}{1+r} \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix} + \begin{bmatrix} -\frac{r^2}{1+r} & 0 & 0 \\ -r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{1,t-1} \\ \epsilon_{2,t-1} \\ \epsilon_{3,t-1} \end{bmatrix}. \tag{3}$$

The first row of (3) corresponds to the decision rule for $C_t$ as a function of the state variables. We will explore the estimation of this system for a stationary ($0 \leq \phi < 1$) and nonstationary income ($\phi = 1$) process. Although our version of the permanent-income model is more stylized than specifications that are estimated in practice, it can be solved analytically and allows us to illustrate the MD estimation problem.[1]

### 2.1. Restricted Cointegration Relationships

Suppose $\phi = 1$ and $\mu = 0$ in Example 1. Define $y_{1,t} = C_t$ and $y_{2,t} = [W_t, I_t]$. According to the permanent-income model all three variables are integrated of order one ($I(1)$). In this case (3) has the form of a cointegration regression model:

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} A' \\ \mathcal{I}_2 \end{bmatrix} y_{2,t-1} + u_t, \tag{4}$$

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where $u_t$ is a moving-average (MA) process of $\epsilon_t$ and $\epsilon_{t-1}$. Let $a = \text{vec}(A)$ and $b = r$. The restriction imposed by the optimal decision rule on the cointegration vector $A$ is

$$g(b) = \left[ \frac{b}{1+b}, \frac{1}{1+b} \right]'. \tag{5}$$

The distribution theory for estimators of the unrestricted cointegration vector $A$ in (4) is well developed. The estimators typically converge at a rate $T^{-1}$. The maximum likelihood estimator (MLE) (Phillips, 1991) and the fully modified least squares estimator (Phillips and Hansen, 1990) have a mixed-normal ($\mathcal{MN}$) limit distribution with a random covariance matrix.[2]

Several results concerning the estimation of the restricted cointegration vector have been published. Phillips (1991) develops a theory of optimal inference for cointegration regressions based on the likelihood function for $[y'_{1,t}, y'_{2,t}]'$. He derives the limit distribution of $\hat{b}_T$ for linear restriction functions. Moreover, Saikkonen (1993) studies a general approach for the estimation of cointegration vectors with linear restrictions.

Saikkonen (1995) extends the analysis of the MLE to the case in which the restriction function is nonlinear and twice differentiable. He provides stochastic equicontinuity conditions to make the conventional Taylor approximation approach valid. Unfortunately, it is in general difficult to verify these conditions (Saikkonen, 1995, p. 893). One advantage of the MD approach is that it leads to conditions that only involve the (deterministic) restriction function $g(b)$. Phillips (1993) also investigates the MLE estimation of a cointegration model in which nonlinear restrictions are imposed on the cointegration parameters.

Nagaraj and Fuller (1991) study a univariate nonstationary autoregressive time series regression model with restrictions across parameters that are estimated at different rates. The restrictions are given by an implicit function. Nagaraj and Fuller show that the constrained nonlinear least squares estimator is consistent and derive its limit distribution under a stochastic equicontinuity condition for the restriction function.

### 2.2. Restrictions between Short-Run and Long-Run Dynamics

Even if the income process is stationary ($0 \leq \phi < 1$, $\mu > 0$) both consumption and wealth are $I(1)$ processes under the optimal consumption choice (see, e.g., Hall, 1978). In this case the optimal decision rule creates restrictions between parameters that are associated with long-run relationships and parameters that control the short-run dynamics. Define $y_{1,t} = C_t$, $y_{2,t} = [\Delta W_t, I_t]'$, $x_{1,t} = W_{t-1}$, $x_{2,t} = [1, I_{t-1}]'$, $y_t = [y_{1,t}, y'_{2,t}]'$, and $x_t = [x_{1,t}, x'_{2,t}]'$. The regressor $x_{1,t}$ is $I(1)$, whereas $x_{2,t}$ is stationary. The consumption model is nested in the following general specification:

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$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} A_{11}' & A_{21}' \\ 0 & A_{22}' \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}, \tag{6}$$

where $u_t$ is an MA(1) process. Define $a_{ij} = \text{vec}(A_{ij})$. The unrestricted parameter vector is $a = [a_{11}', a_{21}', a_{22}']'$ and $b = [r, \mu, \phi]'$ is composed of the structural parameters. The restriction function $g(b)$ is

$$g(b) = \left[ \frac{r}{1+r}, \frac{\mu}{1+r} - \frac{\mu\phi r(2-\phi)}{(1+r)(1+r-\phi)}, \frac{\phi r(2-\phi)}{(1+r-\phi)(1+r)}, \right.$$
$$\left. - \frac{\mu\phi(1-\phi)}{1+r-\phi}, \frac{\phi(1-\phi)}{1+r-\phi}, \mu(1-\phi), \phi \right]'. \tag{7}$$

Because the errors $u_{1,t}$ and $u_{2,t}$ are uncorrelated with the stationary regressors $x_{2,t}$, (6) can be estimated by quasi-maximum likelihood ignoring the MA(1) structure of $u_t$. The limit distribution of the estimator $\hat{a}_T$ is of the form

$$D_T R(\hat{a}_T - a_0) \Rightarrow \eta^{1/2} Z, \tag{8}$$

where

$$D_T = \begin{bmatrix} T & 0 & 0 \\ 0 & T^{-1/2}\mathcal{I}_2 & 0 \\ 0 & 0 & T^{-1/2}\mathcal{I}_4 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R_* & 0 \\ 0 & 0 & \mathcal{I}_2 \otimes R_* \end{bmatrix},$$

$$R_* = \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix},$$

and $Z \equiv \mathcal{N}(0, \mathcal{I}_7)$. The matrix $R$ rotates the unrestricted parameters and its inverse removes the mean $\mu$ from the regressor $I_{t-1}$. The diagonal matrix $D_T$ contains the rates of convergence of the rotated parameters. Assume that the partial sum process of $\Delta W_t$ converges to a vector Brownian motion: $T^{-1/2} \sum_{t=1}^{[Tr]} \Delta W_t \Rightarrow B(r) \equiv BM(\Omega)$. Here $\Omega$ is the long-run variance of $\Delta W_t$ defined as $\lim_{T\to\infty}(1/T)\mathbb{E}[(\sum_{t=1}^{T} \Delta W_t)(\sum_{t=1}^{T} \Delta W_t)']$. The random covariance matrix $\eta$ is given by

$$\eta = \begin{bmatrix} (\Sigma_{11.2} \otimes Q_{11.2}^{-1}) & -(\Sigma_{11.2} \otimes Q_{11.2}^{-1} Q_{12} Q_{22}^{-1}) & 0 \\ -(\Sigma_{11.2} \otimes Q_{22}^{-1} Q_{21} Q_{11.2}^{-1}) & (\Sigma_{11.2} \otimes Q_{22.1}^{-1}) + (\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \otimes Q_{22}^{-1}) & (\Sigma_{12} \otimes Q_{22}^{-1}) \\ 0 & (\Sigma_{21} \otimes Q_{22}^{-1}) & (\Sigma_{22} \otimes Q_{22}^{-1}) \end{bmatrix}. \tag{9}$$

Let $\Sigma$ be the long-run variance of $u_t$ and $\Sigma_{ij}$ be the long-run covariance between $u_{i,t}$ and $u_{j,t}$. The various elements of $\eta$ are defined as follows: $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, $Q_{11} = \int_0^1 B^2$, $Q_{12} = [\int_0^1 B, 0]$, $Q_{21} = Q_{12}'$, $Q_{22} = \mathbb{E}[R_*^{-1} x_{2,t} x_{2,t}' R_*^{-1\prime}]$, $Q_{11.2} = Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}$, and $Q_{22.1} = Q_{22} - Q_{21}Q_{11}^{-1}Q_{12}$.

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Define the rotated unrestricted parameters $\tilde{a} = Ra$ and let $\tilde{a} = [\tilde{a}_1, \tilde{a}_2]'$, where $\tilde{a}_1$ corresponds to the cointegration parameter $a_{11}$ and all remaining parameters are collected in $\tilde{a}_2$. The permanent-income model has the following features. First, the function $\tilde{g}(b) = Rg(b)$ creates restrictions across $\tilde{a}_1$ and $\tilde{a}_2$, which are estimated at the rates $T^{-1}$ and $T^{-1/2}$, respectively. Second, the fast parameter estimate $\hat{\tilde{a}}_{1,T}$ is asymptotically correlated with the slow parameter estimates $\hat{\tilde{a}}_{2,T}$ because of the intercept $\mu/(1 + r)$ in the cointegration relationship of $C_t$ and $W_t$. Third, the rotated restriction function is block-diagonal. The vector $b$ can be expressed as $b = [b_1, b_2']'$ and the restriction function is of the form $\tilde{g}(b) = [\tilde{g}_1(b_1)', \tilde{g}_2(b_1, b_2)']'$.

There are many other economic models that generate restrictions between parameters that control short-run dynamics and long-run relationships. A well-known example is the present-value model, which has been widely studied in the empirical finance literature, e.g., by Campbell and Shiller (1987).

**Example 2**

Let $y_{1,t}$ be a stock price and $y_{2,t}$ a dividend payment. A risk-neutral investor is indifferent between the stock and a bond that guarantees to pay the interest rate $r$ if

$$y_{1,t} = \frac{1}{1+r} I\!E_t[y_{1,t+1} + y_{2,t+1}]. \tag{10}$$

Campbell and Shiller essentially model the joint behavior of $y_{1,t}$ and $y_{2,t}$ as

$$\begin{bmatrix} y_{1,t} - a_1 y_{2,t-1} \\ \Delta y_{2,t} \end{bmatrix} = (\mathcal{I}_2 - A_2' L)^{-1} \epsilon_t, \quad \epsilon_t \sim iid(0, \Sigma_{\epsilon\epsilon}), \tag{11}$$

where $L$ denotes the lag operator. If the cointegration parameter $a_1 = 1/r$ then the linear combination $y_{1,t} - (1/r)y_{2,t}$ is called the spread and reflects the difference between the stock price and the present discounted value of future dividends under the assumption that $y_{2,t+j} = y_{2,t}$ for all $j$.

The model assumes that spread and dividend growth follow a stationary VAR(1) process. In addition to $a_1 = 1/r$ equation (10) imposes restrictions on the matrix of short-run dynamics $A_2$. Let $a_2 = \text{vec}(A_2)$, $a = [a_1, a_2']'$ and define the parameter vector $b = [r, b_2, b_3]'$. The restriction function for this model is

$$g(b) = \left[ \frac{1}{r}, b_2, b_3, 1 + r - b_2, -b_3 - \frac{1+r}{r} \right]. \tag{12}$$

The unrestricted parameter vector $a$ can be estimated, e.g., by maximum likelihood. The MLE has the property that $a_1$ is estimated at rate $T^{-1}$ and $a_2$ at rate $T^{-1/2}$. The restriction function is block-diagonal. However, unlike in Example 1 the estimators $\hat{a}_{1,T}$ and $\hat{a}_{2,T}$ are asymptotically uncorrelated because

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there is no intercept in the cointegration relationship between stock prices and dividends.

After the first draft of this paper was written we learned that Elliott (2000) has analyzed an MD estimator for $b$ in the context of the cointegration regression model (4). He applies the MD estimator to the six-variable cointegration model of King, Plosser, Stock, and Watson (1991) in which the cointegration coefficients have to satisfy linear exclusion restrictions. He does not provide examples of nonlinear restrictions across cointegration parameters. Although his asymptotic theory suffices to analyze the permanent-income model with an $I(1)$ income process, it is not general enough to be applied to the important class of problems in which $g(b)$ imposes restrictions across parameters that are estimated at different rates, such as in the permanent-income model with stationary income process and the present-value model. Thus, our paper provides a much more complete theory of MD estimators for nonstationary time series models, which encompasses Elliot's results as a special case.

## 3. MINIMUM DISTANCE ESTIMATION

This section provides a rigorous definition of the MD estimator and introduces regularity conditions that are used throughout the paper. Let $Y_T(\omega)$ be a data matrix of sample size $T$ defined on the probability space $(\Omega, \mathcal{F}, I\!P)$. Moreover, let $\{\hat{a}_T\}$ be a sequence of estimators of a $q$-dimensional parameter vector $a \in \mathcal{A} \subset I\!R^q$. It is assumed that there exists a unique pseudo-true value $a_0 \in \mathcal{A}$ that is consistently estimated by $\{\hat{a}_T\}$, i.e., $\hat{a}_T \xrightarrow{p} a_0$ under $I\!P$. Let $b \in \mathcal{B} \subset I\!R^p$ be a second parameter vector and $g(\cdot)$ be a mapping from $\mathcal{B}$ to $\mathcal{A}$. We assume that $p \leq q$ and $\{a \in \mathcal{A} : a = g(b), b \in \mathcal{B}\} \subset \mathcal{A}$. Thus, $g(b)$ can be interpreted as a restriction on the parameter space $\mathcal{A}$. We will make the following assumptions with respect to $g(b)$ and $\mathcal{B}$.

Assumption 1 (Parameter Restriction $(I)$).

(i) The parameter space $\mathcal{B}$ is compact.
(ii) The restriction function $g(b)$ is continuous.
(iii) There is a unique $b_0$ in the interior of $\mathcal{B}$ such that $g(b_0) = a_0$.

Suppose $\{\widetilde{W}_T\}$ is a sequence of $q \times q$ weight matrices. Any vector $\hat{b}_T$ that minimizes the criterion function

$$Q_T(b) = \frac{1}{2} \| \widetilde{W}_T(\hat{a}_T - g(b)) \| \tag{13}$$

will be called an MD estimator of $b$. The estimator $\hat{b}_T$ exists because $Q_T(b)$ is continuous in $b$ on the compact set $\mathcal{B}$ by Assumption 1. Moreover, $\hat{b}_T$ is measurable as a result of Lemma 2 of Jennrich (1969). We will now impose condi-

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tions on the joint asymptotic behavior of the unrestricted estimator $\hat{a}_T$ and the weight matrix $\widetilde{W}_T$.[3]

Assumption 2 (Unrestricted Parameter Estimation and Weight Matrix).

(i) $\widetilde{W}_T$ and $\hat{a}_T$ are defined on the same probability space.
(ii) There exist a nonstochastic and invertible matrix $R$ and a nonstochastic diagonal matrix $D_T$ whose elements tend to infinity as $T \to \infty$ such that

$$\begin{bmatrix} D_T R(\hat{a}_T - a_0) \\ \text{vec}(\widetilde{W}_T R^{-1} D_T^{-1}) \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha \\ \text{vec}(W) \end{bmatrix},$$

where $\alpha$ is a $q \times 1$ random vector and $W$ a $q \times q$ random matrix that is nonsingular with probability one.

Assumption 2 implies that the limit distribution of the unrestricted estimator $\hat{a}_T$ is equivalent to the distribution of the random variable $\alpha$. The matrix $R$ rotates the unrestricted parameter space and separates directions in which convergence of $\hat{a}_T$ is fast from directions in which convergence is slow. The diagonal elements of $D_T$ correspond to the convergence rates for these different directions. More weight should be assigned to the rotated elements of $\hat{a}_T$ that provide precise measurements of the corresponding $a_0$ elements. Therefore, it is assumed that the weight matrix $\widetilde{W}_T$ is designed to grow more rapidly along the directions in which the convergence of $\hat{a}_T$ to $a_0$ is fast. For the remainder of the paper we define the standardized weight matrix $W_T = \widetilde{W}_T R^{-1} D_T^{-1}$. Assumption 2 allows the limit of the sequence $\{W_T\}$ to be random.

## 4. CONSISTENCY

The MD estimator is consistent provided that the unrestricted estimator $\hat{a}_T$ is consistent and $b_0$ is uniquely identifiable based on the restriction $b_0 = g(a_0)$. The result is formally stated in the following theorem and proved in the Appendix.

THEOREM 1 (Consistency of MD Estimator). *If Assumptions 1 and 2 are satisfied, then* $\hat{b}_T \xrightarrow{p} b_0$ *as* $T \to \infty$.

The traditional proof of the consistency of extremum estimators is based on the uniform convergence of the random sample objective function to a nonrandom limit function coupled with some identification condition for the "true" parameter values (e.g., Amemiya, 1985). However, this traditional method is not applicable to our MD estimator for two reasons. First, because the weight matrix converges in distribution to a random matrix, the objective function $Q_T(b)$ converges for each value of $b$ to a random variable. Second, in time series models with trends, the convergence rate of $Q_T(b)$ will generally depend on $b$.

The key idea for overcoming the first difficulty is to work with an almost sure representation of the probability distributions of $\hat{a}_T$ and $\widetilde{W}_T$. This idea has

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been used in the empirical process literature to derive limit distributions for extremum estimators. Kim and Pollard (1990), e.g., employ Dudley's almost sure representation. We use the Skorohod representation in this paper (see, e.g., Billingsley, 1986). To cope with the different convergence rates, Lemma 1 of Wu (1981) is employed (see the Appendix).

## 5. LIMIT DISTRIBUTION

Without loss of generality it is assumed that the diagonal elements of the matrix $D_T$ are equal to $T^{\nu_j}$, $j = 1,\ldots,q$, where $\nu_j \geq \nu_{j+1} > 0$. This section will develop the limit distribution of $\hat{b}_T$ under various assumptions on the restriction function $g(b)$. Define $\alpha_T = D_T R(\hat{a}_T - a_0)$ and $W_T = \widehat{W}_T R^{-1} D_T^{-1}$. The objective function of the MD estimator can be rewritten as

$$Q_T(b) = Q_T(b_0) - \alpha_T' W_T' W_T D_T R(g(b) - a_0) + \frac{1}{2}(g(b) - a_0)' R' D_T' W_T' W_T D_T R(g(b) - a_0). \quad (14)$$

We will begin with a linear restriction function and then consider Taylor approximations to nonlinear restriction functions subsequently. Throughout this section we will state additional assumptions on the restriction function $g(b)$ and the domain $\mathcal{B}$ of $b$. We use $g^{(1)}(b_*)$ to denote the $q \times p$ matrix $(\partial g/\partial b')|_{b=b_*}$ of first derivatives.

### 5.1. Linear Restriction Functions

Suppose that $g(b) = Gb$, where $G$ is a $q \times p$ matrix. In this case the objective function is quadratic in $b$. Thus, $Q_T(b) = Q_{q,T}(b)$, where $Q_{q,T}(b)$ is defined as

$$Q_{q,T}(b) = Q_{q,T}(b_0) - \alpha_T' W_T' W_T D_T RG(b - b_0) + \frac{1}{2}(b - b_0)' G' R' D_T' W_T' W_T D_T RG(b - b_0). \quad (15)$$

To analyze the limit distribution of the MD estimator, the restricted parameter space has to be rotated. Define $\tilde{g}(b) = Rg(b)$, $\tilde{G} = RG$, and the $p \times p$ matrix $G^*$ consisting of the first $p$ linearly independent rows of $\tilde{G}$. Moreover, define the function $\iota(j)$ such that the $j$th row of $G^*$ equals the $\iota(j)$th row of $\tilde{G}$. Decompose $G^* = L_* U_*$, where $L_*$ is lower triangular and $U_*$ is upper triangular. Let

$$\Lambda_T = \mathrm{diag}(T^{\nu_{\iota(1)}},\ldots,T^{\nu_{\iota(p)}})U_*, \quad \text{and} \quad \Gamma_T = D_T \tilde{G} \Lambda_T^{-1}, \quad (16)$$

where $T^{\nu_{\iota(j)}}$ is the convergence rate that corresponds to the $j$th row of $G^*$. Define the local parameter vector $s = \Lambda_T(b - b_0)$ with domain $\mathcal{S} = \Lambda_T(\mathcal{B} - b_0)$. The role of the upper triangular matrix $U_*$ in $\Lambda_T$ is to rotate the restricted

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parameter $b$. The matrix $\text{diag}(T^{\nu_{\iota(1)}},\ldots,T^{\nu_{\iota(p)}})$ in $\Lambda_T$ controls the convergence rates of the rotated parameter estimates. The sample objective function of the MD estimator in terms of $s$ is

$$Q_{q,T}(b_0+\Lambda_T^{-1}s)=Q_{q,T}(b_0)-\alpha_T'W_T'W_T\Gamma_T s+\frac{1}{2}s'\Gamma_T'W_T'W_T\Gamma_T s. \tag{17}$$

**Example 3**

Assume that the parameter vectors $a$ and $b$ are $5\times 1$ and $3\times 1$, respectively, and $R=\mathcal{I}_5$. Thus, $\tilde{G}=G$. The term $A_{ij}$ denotes element $ij$ of a matrix $A$. The terms $A_{i.}$ and $A_{.j}$ denote its $i$th row and $j$th column, respectively. Suppose $D_T=\text{diag}(T^{3/2},T,T,T^{1/2},T^{1/2})$ and $G_{3.}=\lambda_1 G_{1.}+\lambda_2 G_{2.}$ for some scalars $\lambda_1,\lambda_2$. In this case $G^*=[G_{1.}',G_{2.}',G_{4.}']'$. The function $\iota(j)$ takes the values $\iota(1)=1$, $\iota(2)=2$, $\iota(3)=4$. Therefore, $\nu_{\iota(1)}=\frac{3}{2}$, $\nu_{\iota(2)}=1$, $\nu_{\iota(3)}=\frac{1}{2}$. Because $L_*$ and $U_*$ are defined through an LU decomposition of $G^*$, it follows that $G_{i.}^*[U_*^{-1}]_{.j}=0$ for $j>i$. Because $G_{3.}$ is a linear combination of $G_{1.}^*$ and $G_{2.}^*$ we can deduce

$$G_{3.}[U_*^{-1}]_{.3}=\lambda_1 G_{1.}^*[U_*^{-1}]_{.3}+\lambda_2 G_{2.}^*[U_*^{-1}]_{.3}=0. \tag{18}$$

Therefore,

$$\Gamma_T=\begin{bmatrix} G_{1.}^*[U_*^{-1}]_{.1}T^{3/2-3/2} & 0 & 0\\ G_{2.}^*[U_*^{-1}]_{.1}T^{1-3/2} & G_{2.}^*[U_*^{-1}]_{.2}T^{1-1} & 0\\ G_{3.}^*[U_*^{-1}]_{.1}T^{1-3/2} & G_{3.}^*[U_*^{-1}]_{.2}T^{1-1} & 0\\ G_{4.}^*[U_*^{-1}]_{.1}T^{1/2-3/2} & G_{4.}^*[U_*^{-1}]_{.2}T^{1/2-1} & G_{4.}^*[U_*^{-1}]_{.3}T^{1/2-1/2}\\ G_{5.}^*[U_*^{-1}]_{.1}T^{1/2-3/2} & G_{5.}^*[U_*^{-1}]_{.2}T^{1/2-1} & G_{5.}^*[U_*^{-1}]_{.3}T^{1/2-1/2} \end{bmatrix}$$

$$\to\begin{bmatrix} G_{1.}^*[U_*^{-1}]_{.1} & 0 & 0\\ 0 & G_{2.}^*[U_*^{-1}]_{.2} & 0\\ 0 & G_{3.}^*[U_*^{-1}]_{.2} & 0\\ 0 & 0 & G_{4.}^*[U_*^{-1}]_{.3}\\ 0 & 0 & G_{5.}^*[U_*^{-1}]_{.3} \end{bmatrix}. \tag{19}$$

The asymptotic behavior of $\Gamma_T$ is summarized in the following lemma.

LEMMA 1. *Suppose that $g(b)=Gb$. Then there exists a matrix $\Gamma$ with full row rank such that $\lim_{T\to\infty}\Gamma_T=\Gamma$.*

Because it is assumed that the "true" parameter is in the interior of $\mathcal{B}$, the parameter space $\mathcal{S}$ of the local parameter $s$ expands to $\mathbb{R}^p$ as $T\to\infty$. Define

$$\hat{s}_{q,T}=(\Gamma_T'W_T'W_T\Gamma_T)^{-1}\Gamma_T'W_T'W_T\alpha_T \tag{20}$$

as the minimizer of the quadratic sample objective function $Q_{q,T}(b_0+\Lambda_T^{-1}s)$ over $\mathbb{R}^p$.

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THEOREM 2. *Suppose Assumptions 1 and 2 are satisfied and the restriction is of the form* $g(b) = Gb$. *Then* $\hat{s}_{q,T} \Rightarrow (\Gamma' W' W \Gamma)^{-1} \Gamma' W' W \alpha$ *and* $\Lambda_T(\hat{b}_T - b_0) = \hat{s}_{q,T} + o_p(1)$.

The proof of Theorem 2 follows from Assumption 2 and Lemma 1. The $o_p(1)$ term arises because for small sample sizes the objective function might attain its minimum on the boundary of the parameter space and not satisfy the first-order conditions.

Remark. If $\Lambda_T$ is nondiagonal and its elements converge at different rates then the (nonrotated) estimate $\hat{b}_T$ may converge at a slow rate and its limit distribution may have a degenerate covariance matrix. For example, suppose $p = 2$ and $\Lambda_T = \mathrm{diag}(T, \sqrt{T}) U_*$. If $U_{*12} \neq 0$, we deduce from Theorem 2 that

$$\begin{bmatrix} \sqrt{T}(\hat{b}_{1,T} - b_{1,0}) \\ \sqrt{T}(\hat{b}_{2,T} - b_{2,0}) \end{bmatrix} = \begin{bmatrix} -\dfrac{U_{*12}}{U_{*11} U_{*22}} \hat{s}_{2,qT} \\ \dfrac{1}{U_{*22}} \hat{s}_{2,qT} \end{bmatrix} + o_p(1). \tag{21}$$

However, if $U_{*12} = 0$, then

$$\begin{bmatrix} T(\hat{b}_{1,T} - b_{1,0}) \\ \sqrt{T}(\hat{b}_{2,T} - b_{2,0}) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{U_{*11}} \hat{s}_{1,qT} \\ \dfrac{1}{U_{*22}} \hat{s}_{2,qT} \end{bmatrix} + o_p(1) \tag{22}$$

and the limit distribution of the original parameter vector is nondegenerate.

### 5.2. Block-Triangular Restriction Matrices

The restriction functions in the examples of Section 2 are block-triangular. In this section we study the linear case. Suppose that $R = \mathcal{I}$ and the unrestricted parameter vector $a$ can be partitioned as follows: $a = [a_1', a_2']'$. Assume that the subvector $a_1$ consists of long-run parameters that can be estimated at a fast rate, say, $\nu_1 = 1$, and that $a_2$ consists of short-run parameters that are estimated at a slower rate $T^{\nu_2}$, e.g., $\nu_2 = \frac{1}{2}$. We refer to the restrictions as block-triangular if the matrix $G$ and the restricted parameter $b$ can be partitioned as follows:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{23}$$

where $G_{11}$ and $G_{22}$ have full row rank. The $b_i$'s are $p_i \times 1$ subvectors of $b$. The subvector $b_2$ does not restrict the long-run parameters $a_1$. The rank condition guarantees that it is possible to solve for $b_1$ based on $a_1$ and for $b_2$ based on $a_2$ conditional on $a_1$. This case is also discussed in Phillips (1991, Remark (m)).

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Before examining the consequences of the block-triangular structure we provide a general definition.

DEFINITION 1 (Block-Triangular Restrictions). *Let $\tilde{a} = Ra$. Partition the rotated unrestricted parameter vector $\tilde{a} = [\tilde{a}_1', \ldots, \tilde{a}_L']'$ such that the estimators of the elements of $\tilde{a}_l$ converge all at the same rate $T^{\nu_l}$, $l = 1, \ldots, L$. Let $\tilde{g}_l(b)$ be the rotated restriction function corresponding to $\tilde{a}_l$ and $G_{lk}$ be the submatrix of $G$ that conforms with the partitions of $\tilde{a}$ and $b$. The restriction function is block-triangular if it is possible to rearrange the elements of $b$ such that the restricted parameter vector can be partitioned into $k = 1, \ldots, K \leq L$ subvectors $b_k$ and (i) $\tilde{g}_l(b) = f_l([b_1', \ldots, b_l']')$, $l = 1, \ldots, K$, and (ii) $G_{ll}$ has full row rank, $l = 1, \ldots, K$.*

As a result of the block-triangular structure, the matrices $G^*$ and $U_*$, defined previously, are of the form

$$G^* = \begin{bmatrix} G_{11}^* & 0 \\ G_{21}^* & G_{22}^* \end{bmatrix}, \qquad U_* = \begin{bmatrix} U_{11}^* & 0 \\ 0 & U_{22}^* \end{bmatrix}, \tag{24}$$

where $G_{11}^*$ consists of the first $p_1$ linearly independent rows of $G_{11}$ and $U_*$ corresponds to the upper triangular matrix of the LU decomposition of $G_*$. The matrix $\Gamma_T$ converges to

$$\Gamma_T = \begin{bmatrix} T^{\nu_1} G_{11} & 0 \\ T^{\nu_2} G_{21} & T^{\nu_2} G_{22} \end{bmatrix} \begin{bmatrix} T^{-\nu_1} U_{11}^{*-1} & 0 \\ 0 & T^{-\nu_2} U_{22}^{*-1} \end{bmatrix} \to \begin{bmatrix} \Gamma_{11} & 0 \\ 0 & \Gamma_{22} \end{bmatrix}, \tag{25}$$

where $\Gamma_{ii} = G_{ii} U_{ii}^{*-1}$. Define the selection matrices $M_1' = [\mathcal{I}_{p_1}, 0_{p_1 \times p_2}]$ and $M_2' = [0_{p_2 \times p_1}, \mathcal{I}_{p_2}]$. The limit distribution of the subvectors $\hat{b}_{i,T}$ is given by

$$T^{\nu_i}(\hat{b}_{i,T} - b_i) \Rightarrow U_{ii}^{*-1} M_i' (\Gamma' W' W \Gamma)^{-1} \Gamma' W' W \alpha, \qquad i = 1,2. \tag{26}$$

Thus, the parameters $b_1$ that enter the long-run parameters $a_1$ can be estimated at the fast rate $T^{\nu_1}$, whereas $b_2$ is estimated at the slower rate $T^{\nu_2}$. In general, the limit distribution for both subvectors depends on the entire vector $\alpha$. Thus, even the restrictions embodied in the short-run parameters $a_2$ are informative with respect to $b_1$.

If the limit weight matrix $W$ is with probability one of the form

$$W = \begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix}, \tag{27}$$

where the partitions of $W$ correspond to the partitions of $a$, then the limit distribution simplifies considerably:

$$T^{\nu_i}(\hat{b}_{i,T} - b_i) \Rightarrow (G_{ii}' W_{ii}' W_{ii} G_{ii})^{-1} G_{ii} W_{ii}' W_{ii} \alpha_i, \qquad i = 1,2. \tag{28}$$

The simplification arises because the system is now block-diagonal and $U_{ii}^* \Gamma_{ii} = G_{ii}$. Thus, if the weight matrix cross-terms between short-run and

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long-run parameters are zero, the distribution of the long-run parameter estimates $\hat{a}_{1,T}$ does not affect the limit of $\hat{b}_{2,T}$ and vice versa, and the limit distribution of $\hat{b}_{1,T}$ is solely determined through the distribution of $\hat{\alpha}_{1,T}$. Now consider the following two-step estimator $\tilde{b}_T$ of the restricted parameter vector $b$.

Two-step Estimation Procedure.

(i) Estimate $b_1$ according to

$$\tilde{b}_{1,T} = \text{argmin}_{b_1 \in \mathcal{B}_1} \frac{1}{2} \| \widetilde{W}_{11,T}(\hat{a}_{1,T} - G_{11} b_1) \|,$$

where $T^{-\nu_1} \widetilde{W}_{11,T} \Rightarrow W_{11}$.

(ii) Estimate $b_2$ according to

$$\tilde{b}_{2,T} = \text{argmin}_{b_2 \in \mathcal{B}_2} \frac{1}{2} \| \widetilde{W}_{22,T}(\hat{a}_{1,T} - G_{21} \tilde{b}_{1,T} - G_{22} b_2) \|,$$

where $T^{-\nu_2} \widetilde{W}_{22,T} \Rightarrow W_{22}$.

In the second step, $b_1$ is replaced by its first-step estimate. Conventional arguments imply that the estimation uncertainty about $b_{1,T}$ does not affect the limit distribution of the estimator for $b_{2,T}$ because $b_{1,T}$ is estimated at a faster rate. Thus, the limit distribution of the two-step estimator is equivalent to the limit of the MD estimator with block-diagonal weight matrix, given (28). The arguments provided in this section easily extend to the case of more than two different convergence rates ($L > 2$).

### 5.3. Nonlinear Restriction Functions

To derive the limit distribution of the MD estimator for nonlinear restriction functions we will impose smoothness conditions and use a Taylor series approximation of the form

$$g(b) = g(b_0) + G(b - b_0) + \Phi[b_+, b_0](b - b_0), \tag{29}$$

where $G = g^{(1)}(b_0)$ and $\Phi[b_+, b_0] = g^{(1)}(b_+) - g^{(1)}(b_0)$ and $b_+$ is located between $b_0$ and $b$. Define $\tilde{G}$, $G^*$, $L_*$, $U_*$, $\Lambda_T$, and $\Gamma_T$ as in Section 5.1. The term $D_T R(g(b) - a_0)$ that appears in the objective function of the MD estimator (equation (14)) is approximated by

$$D_T R(g(b) - a_0) = \Gamma_T s + D_T R \Phi[b_+, b_0] \Lambda_T^{-1} s, \tag{30}$$

where $s$ is the local parameter $s = \Lambda_T(b - b_0)$. To be able to bound the remainder term $\Phi[\cdot, \cdot]$, we impose some smoothness conditions on $g(b)$.

Assumption 3 (Parameter Restriction (II)).

(i) The parameter restriction function $g(b)$ is differentiable in a neighborhood of $b_0$.
(ii) For any sequence $\delta_T \to 0$, $\sup_{\|b_1 - b_0\| \leq \delta_T} \| D_T R \Phi[b_1, b_0] \Lambda_T^{-1} \| = o(1)$.

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Assumption 3(ii) is an equicontinuity condition for the first derivative of the restriction function that allows us to use the conventional quadratic approximation of the objective function. An advantage of the MD approach is that one only has to verify a deterministic condition. Maximum likelihood estimators, such as the one discussed by Saikkonen (1995) for the restricted cointegrated regression model and the constrained least squares estimator proposed by Nagaraj and Fuller (1991, see Assumption 4), require the verification of stochastic equicontinuity conditions.

The following assumption provides a sufficient condition for Assumption 3. A proof can be found in the Appendix.

Assumption 3* (Parameter Restriction [Sufficient Condition]). Suppose that $\partial g(b)/\partial b$ is continuous in a neighborhood of the true parameter $b_0$. Moreover, at least one of the following conditions is satisfied.

(i) For each pair $i, j$ such that $\nu_i > \nu_{\iota(j)}$, there exists an $\epsilon > 0$ such that the $ij$th element of the matrix $R\Phi[b_1, b_0]U_*^{-1}$ is equal to zero for $\|b_1 - b_0\| < \epsilon$.
(ii) For each $i$ such that $\nu_i > \nu_{\iota(p)}$ the $i$th component $\tilde{g}_i(b) = \sum_{j=1}^{q} R_{ij} g_j(b)$ of the rotated restriction function is linear in $b$.
(iii) The rotated restriction function $Rg(b)$ is block-triangular, as in Definition 1.

Remark. Conditions (ii) and (iii) are special cases of condition (i) that are easy to verify and cover many economic models. For instance, the restrictions imposed by both the permanent-income model (stationary income process) and the present-value model have a block-triangular structure and satisfy Assumption 3*(iii). Moreover, from (7) and (12) it is easy to see that the first derivative of $g(b)$ is continuous on $\mathcal{B}$ if $\mathcal{B}$ is compact.

The limit distribution of the MD estimator is obtained by showing that the sample objective function $Q_T(b_0 + \Lambda_T^{-1}s)$ can be approximated by the quadratic function $Q_{q,T}(b_0 + \Lambda_T^{-1}s)$ given in (17).[4] Assumption 3 guarantees that the contributions of the remainder term from the Taylor series approximation are asymptotically negligible. The following theorem characterizes the large sample behavior of $\hat{b}_T$.

THEOREM 3. *Suppose that Assumptions 1–3 are satisfied. Then (i)* $\|\Lambda_T(\hat{b}_T - b_0)\| = O_p(1)$ and (ii) $\Lambda_T(\hat{b}_T - b_0) = \hat{s}_{q,T} + o_p(1)$.

The first part of Theorem 1 characterizes the order of consistency. The second part states that the limit distribution of the nonlinear MD estimator $\hat{b}_T$ is equivalent to the distribution of the local estimator $\hat{s}_{q,T}$, which minimizes the quadratic approximation of the sample objective function $Q_T(b_0 + \Lambda_T^{-1}s)$. The limit distribution of $\hat{s}_{q,T}$ is given in Theorem 2.

## 6. MIXED NORMALITY OF THE UNRESTRICTED ESTIMATOR

The limit distribution of the MD estimator generally depends on the choice of the sequence of weight matrices $\{\widetilde{W}_T\}$. If the asymptotic distribution of the un-

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restricted estimator $\hat{a}_T$ is mixed normal, it is possible to develop an optimality theory for the MD estimation. Moreover, we can construct a test for the null hypothesis that $a_0 = g(b_0)$ for a $b_0 \in \mathcal{B}$. The mixed normal case is of great practical importance. It arises, e.g., in the examples considered in Section 2. Park and Phillips (2000) show that the maximum likelihood estimator of the regression coefficients in a nonstationary binary choice model also has a mixed normal limit distribution.

### 6.1. Optimal Weight Matrix

Consider the following class $\mathcal{M}$ of MD estimators.

DEFINITION 2. *Let $\mathcal{M}$ be the class of MD estimators such that $\hat{b}_T$, $\hat{a}_T$, and $\widetilde{W}_T$ satisfy Assumption 2, $\alpha \equiv \eta^{1/2} Z$, where $\eta$ is a $q \times q$ random matrix that is positive definite with probability one and $Z$ is a $q \times 1$ standard normal random vector that is independent of $\eta$ and $W$.*

A common criterion for asymptotic efficiency of an estimator is the concentration of the limit distribution (e.g., Basawa and Scott, 1983). This criterion does not require the competing estimates to be asymptotically normal and has been widely used in the nonstationary time series literature, such as Saikkonen (1991), Phillips (1991), and Jeganathan (1995).

DEFINITION 3 (Efficiency). *Let $\hat{b}_T$ and $b_T^*$ be two estimators for $b_0$ belonging to class $\mathcal{M}$. The term $b_T^*$ is asymptotically more efficient than $\hat{b}$ if*

$$\lim_{T\to\infty} I\!P\{\Lambda_T(\hat{b}_T - b_0) \in C\} \leq \lim_{T\to\infty} I\!P\{\Lambda_T(b_T^* - b_0) \in C\}$$

*for any convex set $C \subset \mathbb{R}^p$, symmetric about the origin. If the inequality holds for all $\hat{b}_T \in \mathcal{M}$, then $b_T^*$ is asymptotically efficient within the class $\mathcal{M}$.*

We obtain the following result with respect to the optimal choice of weight matrix.

THEOREM 4 (Optimal Weight Matrix). *Suppose that the limit distribution $\alpha$ of the unrestricted estimator $\hat{a}_T$ is mixed normal $\mathcal{MN}(0, \eta)$. Then an optimal sequence of weight matrices $\{\widetilde{W}_T^o\}$ is a sequence of random matrices such that*

$$\begin{bmatrix} D_T^{-1'} R^{-1'} \widetilde{W}_T^{o'} \widetilde{W}_T^o R^{-1} D_T^{-1} \\ D_T R(\hat{a}_T - a_0) \end{bmatrix} \Rightarrow \begin{bmatrix} \eta^{-1} \\ \alpha \end{bmatrix}, \tag{31}$$

*where $\eta$ is the random variance in the mixed normal distribution.*

If $b_0$ is in the interior of the parameter space, Theorems 2 and 3 imply that the limit distribution of the MD estimator is mixed normal of the form

$$\Lambda_T(\hat{b}_T - b) \Rightarrow \mathcal{MN}\big(0, (\Gamma' \eta^{-1} \Gamma)^{-1}\big).$$

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Now reconsider the special case discussed in Section 5.2. We argued that the two-step estimation $\tilde{b}_{1,T}$ and $\tilde{b}_{2,T}$ can be interpreted as a MD estimator with a block-diagonal weight matrix. Let $\eta_{ij}$ denote the partitions of $\eta$ that correspond to the partitions of the unrestricted parameter vector $a$. To implement the two-step estimation procedure it is reasonable to choose the weight matrices such that $T^{-2\nu_i}\widetilde{W}_{ii,T}'\widetilde{W}_{ii,T} \Rightarrow \eta_{ii}^{-1}$. This yields the following limit distribution:

$$T^{\nu_i}(\tilde{b}_{i,T} - b_i) \Rightarrow \mathcal{MN}\big(0, (G_{ii}'\eta_{ii}^{-1}G_{ii})^{-1}\big).$$

However, Theorem 4 implies that the two-step estimator $\tilde{b}_T$ is dominated by MD estimators for which $D_T^{-1'}R^{-1\prime}\widetilde{W}_T^{o\prime}\widetilde{W}_T^{o}R^{-1}D_T^{-1}$ converges to $\eta^{-1}$. Thus, whenever $\eta$ is not block-diagonal, it is inefficient to disregard the information in the short-run parameter estimates $\hat{a}_{2,T}$ about the parameters $b_1$. It is also inefficient to estimate $b_2$ by treating $b_1$ as known.

Remark. The restriction function for the present-value model in Section 2 is block-diagonal, and the asymptotic covariance matrix of the unrestricted parameter estimates turns out to be block-diagonal. Hence, the two-step MD estimator is asymptotically efficient in Example 2. In the permanent-income model (Example 1) the asymptotic covariance matrix of the rotated parameter estimates, however, is not block-diagonal (see equation (9)) and the two-step procedure is inefficient.

### 6.2. Testing the Validity of the Restriction

To test the validity of the imposed restrictions we consider the following $J$-test statistic:

$$J_T = [\hat{a}_T - g(\hat{b}_T)]'\widetilde{W}_T'\widetilde{W}_T[\hat{a}_T - g(\hat{b}_T)].$$

Under the assumption that $b_0$ is in the interior of $\mathcal{B}$ and $\hat{a}_T$ has a mixed normal limit distribution, we obtain the following result.

THEOREM 5. *Suppose that Assumptions 1–3 are satisfied. Assume that the normalized unrestricted estimator* $D_T R(\hat{a}_T - a) \Rightarrow \mathcal{MN}(0, \eta)$.

*(i) Then*

$$J_T \Rightarrow \sum_{i=1}^{q-p} d_i \chi_1^2(i),$$

*where* $\chi_1^2(i)$ *denote independent and identically distributed (i.i.d.)* $\chi^2$ *random variables with one degree of freedom that are independent of* $d_i$. *The* $d_i$*'s are nonzero random variables that correspond to the eigenvalues of*

$$\eta'^{1/2}W'(\mathcal{I}_q - W\Gamma(\Gamma'W'W\Gamma)^{-1}\Gamma'W')W\eta^{1/2}.$$

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*(ii) Under a sequence of optimal weight matrices* $\{\widetilde{W}_T^o\}$ *(defined in Theorem 4)*

$$J_T \Rightarrow \chi^2_{q-p},$$

*where* $\chi^2_{p-q}$ *is a* $\chi^2$ *random variable with* $q - p$ *degrees of freedom.*

## 7. CONCLUSIONS

We have studied the asymptotic properties of the MD estimator in nonstationary time series models that are linear in the variables but involve nonlinear parameter restrictions. We analyze two applications of the proposed MD estimator: a permanent-income model based on a linear-quadratic dynamic programming problem and a present-value model.

To construct optimal MD estimators we allow the criterion function of the estimator to depend on a sequence of weight matrices that converges to a stochastic limit. We show the consistency of the estimator using a Skorohod representation of the weakly converging objective function and derive the limit distribution of the MD estimator for smooth restriction functions. Our analysis relies on an equicontinuity condition for the parameter restriction function that allows a conventional first-order Taylor series approximation. If the equicontinuity condition is violated and some of the remainder terms are being amplified through convergence rate differentials of the unrestricted estimators, higher order expansions of the restriction function may become necessary. However, the results will be very model specific and difficult to generalize.

### NOTES

1. Detailed calculations for Example 1 are available from the authors upon request.
2. See also Park (1992), Saikkonen (1991), and Stock and Watson (1993).
3. Our main interest is the limit distribution of the MD estimator. To establish its consistency, it is not necessary to make such detailed assumptions about the limit distribution of $\hat{a}_T$.
4. An advantage of deriving limit distributions of extremum estimators through quadratic approximations of sample objective functions is that the derivation can be extended to the case in which $b_0$ is on the boundary of $\mathcal{B}$. Based on the very general results in Andrews (1999) this extension is straightforward. However, the boundary case is not pursued in this paper.

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# APPENDIX: PROOFS

LEMMA 2 (Lemma 1 in Wu, 1981; the proof can also be found in Wu). *Suppose that for any* $\delta > 0$

$$\liminf_{T\to\infty} \inf_{\|b-b_0\|\geq\delta} \big(Q_T(b) - Q_T(b_0)\big) > 0 \text{ a.s.} \quad (\textit{or in probability}).$$

*Then* $\hat{b}_T \to b_0$ *a.s. (or in probability) as* $T \to \infty$.

**Proof of Theorem 1.** Define $\alpha_T = D_T R(\hat{a}_T - a_0)$ and $W_T = \widetilde{W}_T R^{-1} D_T^{-1}$. By Assumption 2

$$[\alpha_T', \mathrm{vec}(W_T)']' \Rightarrow [\alpha', \mathrm{vec}(W)']'.$$

Using the Skorohod construction (e.g., Billingsley, 1986) one can find a probability space $(\Omega^*, \mathcal{F}^*, I\!P^*)$ with random variables $\alpha^*, \alpha_T^*, W^*, W_T^*$ that are distributionally equivalent to $\alpha, \alpha_T, W, W_T$, respectively, and $[\alpha_T^{*'}, \mathrm{vec}(W_T^*)'] \xrightarrow{a.s.} [\alpha^{*'}, \mathrm{vec}(W^*)']$ in $[I\!P^*]$.

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Define $q_T(b) = D_T R(g(b) - a_0)$. From the uniqueness assumption that $a_0 = g(b)$ only if $b = b_0$, it follows that $q_T(b) = 0$ if and only if $b = b_0$. Furthermore, under Assumptions 1 and 2, we have

$$\inf_{\|b-b_0\|\geq\delta} \|q_T(b)\|^2 \geq \lambda_{\min}(D_T' D_T)\lambda_{\min}(R'R) \inf_{\|b-b_0\|\geq\delta} \|g(b) - g(b_0)\|^2 \to \infty, \tag{A.1}$$

where $\lambda_{\min}(A)$ denotes the minimum eigenvalue of matrix $A$.

The MD objective function can be rewritten as

$$Q_T(b) = \frac{1}{2}\alpha_T' W_T' W_T \alpha_T - q_T(b)' W_T' W_T \alpha_T + \frac{1}{2} q_T(b)' W_T' W_T q_T(b).$$

Moreover, we define

$$Q_T^*(b) = \frac{1}{2}\alpha_T^{*'} W_T^{*'} W_T^* \alpha_T^* - q_T' W_T^{*'} W_T^* \alpha_T^* + \frac{1}{2} q_T' W_T^{*'} W_T^* q_T(b),$$

which is distributionally equivalent to $Q_T(b)$. Let $\hat{b}_T^*$ be the MD estimator based on the objective function $Q_T^*(b)$. We will use Lemma 2 to show that $\hat{b}_T^* \xrightarrow{a.s.} b_0$. Because $\hat{b}_T^* \equiv \hat{b}_T$ on the original probability space, it can be deduced that $\hat{b}_T \xrightarrow{p} b_0$.

We show that the sufficient condition in Lemma 2 is satisfied. For a given $\delta > 0$

$$\begin{aligned}
&\liminf_{T\to\infty} \inf_{\|b-b_0\|\geq\delta} \left(Q_T^*(b) - Q_T^*(b_0)\right) \\
&\quad = \liminf_{T\to\infty} \inf_{\|b-b_0\|\geq\delta} \left[ q_T(b)' W_T^{*'} W_T^* q_T(b) \left( \frac{1}{2} - \frac{q_T(b)' W_T^{*'} W_T^* \alpha_T^*}{q_T(b)' W_T^{*'} W_T^* q_T(b)} \right) \right] \\
&\quad \geq \liminf_{T\to\infty} \left( \inf_{\|b-b_0\|\geq\delta} \|W_T^* q_T(b)\|^2 \right) \left( \frac{1}{2} - \sup_{\|b-b_0\|\geq\delta} \frac{|q_T(b)' W_T^{*'} W_T^* \alpha_T^*|}{\|W_T^* q_T(b)\|^2} \right).
\end{aligned}$$

The equality and the inequality hold because $W_T^*$ converges almost surely to a nonsingular matrix by Assumption 2 and in consequence together with (A.1) we have

$$\liminf_{T\to\infty} \left( \inf_{\|b-b_0\|\geq\delta} \|W_T^* q_T(b)\|^2 \right) > 0.$$

The Cauchy–Schwarz inequality implies that

$$\sup_{\|b-b_0\|\geq\delta} \frac{|q_T(b)' W_T^{*'} W_T^* \alpha_T^*|}{\|W_T^* q_T(b)\|^2} \leq \frac{\|W_T^* \alpha_T^*\|}{\inf_{\|b-b_0\|\geq\delta} \|W_T^* q_T(b)\|} \to 0$$

almost surely, because $\|W_T^* \alpha_T^*\| \to \|W\alpha\|$ and $\|W_T^* q_T(b)\| \to \infty$ almost surely for $\|b - b_0\| > \delta$ as in (A.1). Thus, it can be deduced that

$$\liminf_{T\to\infty} \inf_{\|b-b_0\|\geq\delta} \left(Q_T^*(b) - Q_T^*(b_0)\right) > 0. \qquad \blacksquare$$

**Proof of Lemma 1.** Here $\Gamma_T = D_T R G \Lambda_T^{-1}$. Let $\Gamma_{T,ij}$ and $\Gamma_{ij}$ denote the elements $ij$ of the matrices $\Gamma_T$ and $\Gamma$, respectively. Suppose the first $p$ rows of $\tilde{G}$ are linearly independent such that $\nu_{(j)} = \nu_j$. Then $\Gamma_{T,ij} = T^{\nu_i - \nu_j}\Gamma_{ij}$. If $i \geq j$ then $\nu_i \leq \nu_j$ and $T^{\nu_i - \nu_j}$ is $O(1)$. If $j > i$, then $\Gamma_{ij} = 0$ because $G^* U_*^{-1}$ is lower triangular. Moreover, $\Gamma_{T,ii} = \Gamma_{ii} \neq 0$

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because the diagonal elements of $L_*$ are nonzero because $G^*$ has full rank. Therefore, $\Gamma$ has full row rank. The argument can be easily extended to the case in which there is linear dependence among the first $p$ rows of $\tilde{G}$, by noting that $\nu_{\iota(j)} \leq \nu_j$ and $G_{i.}[U_*^{-1}]_{.j} = 0$ for $i < \iota(j)$. ■

**Proof of Sufficiency of Assumption 3*.** Part (i). Because $\partial g(b)/\partial b$ is continuous and the parameter space $\mathcal{B}$ is compact, we can deduce that $\partial g(b)/\partial b$ is uniformly continuous around $b_0$. Suppose $\nu_i \leq \nu_{\iota(j)}$. Then

$$\sup_{\|b_1-b_0\|\leq\delta_T} |[D_T R\Phi[b_1,b_0]\Lambda_T^{-1}]_{(ij)}| = T^{\nu_i-\nu_{\iota(j)}} \sup_{\|b_1-b_0\|\leq\delta_T} |[D_T R\Phi[b_1,b_0]U_*^{-1}]_{(ij)}| \to 0$$

because $T^{\nu_i-\nu_{\iota(j)}}$ is $O(1)$ and $\partial g(b)/\partial b'$ is uniformly continuous around $b_0$.

If $\nu_i > \nu_{\iota(j)}$,

$$\sup_{\|b_1-b_0\|\leq\delta_T} |[D_T R\Phi[b_1,b_0]\Lambda_T^{-1}]_{(ij)}| = T^{\nu_i-\nu_{\iota(j)}}[D_T R\Phi[b_1,b_0]U_*^{-1}]_{(ij)} = 0$$

by Assumption 3*(ii).

To prove (ii) and (iii) we will verify that condition (i) is satisfied.

Part (ii). Define $\tilde{\Phi}[b_1,b_0] = R\Phi[b_1,b_0]$. Now consider

$$\left[D_T\tilde{\Phi}[b_1,b_0]\Lambda_T^{-1}\right]_{ij} = T^{\nu_i-\nu_{\iota(j)}}\left(\sum_{l=1}^{p}\tilde{\Phi}_{il}[b_1,b_0][U_*^{-1}]_{lj}\right).$$

For $\nu_i > \nu_{\iota(j)}$, $[D_T\tilde{\Phi}[b_1,b_0]\Lambda_T^{-1}]_{ij} = 0$ because of the linearity assumption of $\tilde{g}_i(b)$ in $b$ so that $\tilde{\Phi}_{ij}[b_1,b_0] = 0$ for $j = 1,\ldots,p$.

Part (iii). Let $\tilde{\Phi}_{lk}[b_1,b_0]$ and $[U_*^{-1}]_{lk}$ denote the submatrices of $\tilde{\Phi}[b_1,b_0]$ and $U_*^{-1}$ that conform with the partitions of $\tilde{a}$ and $b$ in Definition 1. As a result of the block-triangular structure, $\tilde{\Phi}[b_1,b_0] = 0$ for $k > l$. Moreover, $[U_*^{-1}]_{lk} = 0$ for $l \neq k$. Thus,

$$\begin{aligned}\left[D_T\tilde{\Phi}[b_1,b_0]\Lambda_T^{-1}\right]_{lk} &= T^{\nu_l-\nu_k}\sum_{j=1}^{K}\tilde{\Phi}_{lj}[b_1,b_0][U_*^{-1}]_{jk}\\ &= T^{\nu_l-\nu_k}\sum_{j=1}^{l}\tilde{\Phi}_{lj}[b_1,b_0][U_*^{-1}]_{jk},\end{aligned}$$

which is zero whenever $k > l$. Hence condition (i) is satisfied. ■

**Proof of Theorem 3.** Part (i). The proof is similar to the proof of Theorem 1 in Andrews (1999). Write $s = \Lambda_T(b - b_0)$. The objective function is of the form

$$\begin{aligned}Q_T(b) = Q_T(b_0) &- \alpha_T' W_T' W_T \Gamma_T s + \frac{1}{2} s'\Gamma_T' W_T' W_T \Gamma_T s\\ &- \alpha_T' W_T' W_T D_T R\Phi[b_+,b_0]\Lambda_T^{-1}(\Gamma_T' W_T' W_T \Gamma_T)^{-1}\Gamma_T' W_T' W_T \Gamma_T s\\ &+ s'\Gamma_T' W_T' W_T D_T R\Phi[b_+,b_0]\Lambda_T^{-1}(\Gamma_T' W_T' W_T \Gamma_T)^{-1}\Gamma_T' W_T' W_T \Gamma_T s\\ &+ \frac{1}{2} s'\Gamma_T' W_T' W_T \Gamma_T(\Gamma_T' W_T' W_T \Gamma_T)^{-1}\Lambda_T^{-1'}\Phi'[b_+,b_0]R' D_T' W_T'\\ &\times W_T R\Phi[b_+,b_0]\Lambda_T^{-1}(\Gamma_T' W_T' W_T \Gamma_T)^{-1}\Gamma_T' W_T' W_T \Gamma_T s.\end{aligned}$$

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Let $\hat{b}_T$ be the MD estimator and $\hat{s}_T = \Lambda_T(\hat{b}_T - b_0)$. Then

$$\begin{aligned}0 &\leq Q_T(b_0) - Q_T(\hat{b}_T)\\ &= O_p(1)\|W_T\Gamma_T\hat{s}_T\| - \frac{1}{2}\|W_T\Gamma_T\hat{s}_T\|^2 + o_p(1)\|W_T\Gamma_T\hat{s}_T\|\\ &\quad - o_p(1)\|W_T\Gamma_T\hat{s}_T\| - \frac{1}{2}o_p(1)\|W_T\Gamma_T\hat{s}_T\|,\end{aligned}$$

where the last equality holds because $\alpha_T = O_p(1)$ and by Assumption 3. Denote the $O_p(1)$ term by $\xi_T$ and rewrite the inequality as

$$\frac{1}{2}\|W_T\Gamma_T\hat{s}_T\|^2 \leq \xi_T\|W_T\Gamma_T\hat{s}_T\| + o_p(1).$$

Thus,

$$\left(\|W_T\Gamma_T\hat{s}_T\| - \xi_T\right)^2 \leq O_p(1)$$

and therefore

$$\|\hat{s}_T\| \leq \|(\Gamma_T'W_T'W_T\Gamma_T)^{-1}\|\|W_T\Gamma_T\hat{s}_T\| = O_p(1),$$

which implies the desired result.

Part (ii). This follows from Theorem 3(a) in Andrews (1999). ■

**Proof of Theorem 4.** We follow the arguments in Theorem 3.1 in Saikkonen (1991). Recall that $\alpha \equiv \eta^{1/2}Z$, where $Z \equiv \mathcal{N}(0, \mathcal{I}_q)$ and $Z$ is independent of the random covariance matrix $\eta$ and the limit weight matrix $W$. Let

$$\theta = (\Gamma'\eta^{-1}\Gamma)^{-1}\Gamma'\eta^{-1}\alpha, \qquad \psi = (\Gamma'W'W\Gamma)^{-1}(\Gamma'W'W\alpha),$$

and $\phi = \psi - \theta$. It can be easily verified that $\mathbb{E}[\theta\phi'|\eta, W] = 0$, which implies that $\phi$ and $\theta$ are independent conditional on $\eta$ and $W$. Let $C$ be any convex set, symmetric about the origin, and $\hat{b}_T^o$ an MD estimator with an optimal sequence of weight matrices. Then for any MD estimator in $\mathcal{M}$:

$$\begin{aligned}\lim_{T\to\infty}\mathbb{P}\{\Lambda_T(\hat{b}_T - b_0) \in C\} &= \mathbb{E}\left[\mathbb{P}[\{\theta + \phi \in C\}|\eta, W]\right]\\ &\leq \mathbb{E}\left[\mathbb{P}[\{\theta \in C\}|\eta, W]\right]\\ &= \lim_{T\to\infty}\mathbb{P}\{\Lambda_T(\hat{b}_T^o - b_0) \in C\}.\end{aligned}$$

The inequality follows from Lemma 2.3.1 in Basawa and Scott (1983). ■

**Proof of Theorem 5.** Write $\hat{s}_T = \Lambda_T(\hat{b}_T - b_0)$. Under the assumptions of the theorem, we may write

$$J_T = \alpha_T'W_T'W_T\alpha_T - 2\alpha_T'W_T'W_T\Gamma_T\hat{s}_T + \hat{s}_T'\Gamma_T'W_T'W_T\Gamma_T\hat{s}_T + o_p(1). \qquad \textbf{(A.2)}$$

Because $\Gamma_T$ is asymptotically full rank and $b_0$ is in the interior of $\mathcal{B}$, we may write

$$\hat{s}_T = (\Gamma_T'W_T'W_T\Gamma_T)^{-1}\Gamma_T'W_T'W_T\alpha_T + o_p(1).$$

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Replacing $\hat{s}_T$ in (A.2), we have

$$J_T = \alpha_T' W_T'(\mathcal{I}_q - W_T\Gamma_T(\Gamma_T' W_T' W_T \Gamma_T)^{-1}\Gamma_T' W_T')W_T\alpha_T + o_p(1).$$

Under the assumptions, it follows that

$$J_T \Rightarrow Z_q' \eta'^{1/2} W'(\mathcal{I}_q - W\Gamma(\Gamma' W' W\Gamma)^{-1}\Gamma' W')W\eta^{1/2} Z_q$$

as $T \to \infty$. Notice that $\mathcal{I}_q - W\Gamma(\Gamma' W' W\Gamma)^{-1}\Gamma' W'$ is an idempotent (random) matrix of rank $q - p$ with probability one and recall that $W\eta^{1/2}$ is of full rank with probability one. From the spectral decomposition of $\eta'^{1/2} W'(\mathcal{I}_q - W\Gamma(\Gamma' W' W\Gamma)^{-1}\Gamma' W')W\eta^{1/2}$ and $Z_q$ being independent of $\eta'^{1/2} W'(\mathcal{I}_q - W\Gamma(\Gamma' W' W\Gamma)^{-1}\Gamma' W')W\eta^{1/2}$, the result in part (i) follows.

Part (ii) is straightforward because $W\eta^{1/2} = \mathcal{I}_q$ and all the nonzero eigenvalues of $\mathcal{I}_q - W\Gamma(\Gamma' W' W\Gamma)^{-1}\Gamma' W'$ are 1's. ■

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