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Fitted Mesh Method for a Class of Singularly Perturbed Differential-Difference Equations

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Abstract. This paper deals with a more general class of singularly perturbed boundary value problem for a differential-difference equations with small shifts. In particular, the numerical study for the problems where second order derivative is multiplied by a small parameter ε and the shifts depend on the small parameter ε has been considered. The fitted-mesh technique is employed to generate a piecewise-uniform mesh, condensed in the neighborhood of the boundary layer. The cubic B-spline basis functions with fitted-mesh are considered in the procedure which yield a tridiagonal system which can be solved efficiently by using any well-known algorithm. The stability and parameter-uniform convergence analysis of the proposed method have been discussed. The method has been shown to have almost second-order parameter-uniform convergence. The effect of small parameters on the boundary layer has also been discussed. To demonstrate the performance of the proposed scheme, several numerical experiments have been carried out.

AMS subject classifications: 34K10; 34K28; 65D05

Key words: Singular perturbation, differential-difference equations, fitted-mesh, B-spline collocation method, boundary layer.

1. Introduction

In this paper, we consider the numerical approximation of the more general singularly perturbed differential-difference equation with small delay as well as advance with layer behavior. DDEs of this type arise naturally in the theoretical analysis of neuronal variability. There have been many advanced models of nerve membrane potential in the presence of random synaptic input. Reviews can be found in Fienberg [9], Holden [13], Segundo *et al.* [32]. In 1965, Stein [36] has given a differential-difference equation model incorporating stochastic effects due to neuronal variability and approximate the solution using Monte Carlo techniques. Stein's model contains the following assumptions:

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- Excitatory impulses arrive according to a Poisson process $\pi(f_e, t)$, each event of which leads to an instantaneous increase in the membrane depolarization $V(t)$ by a_e, whereas inhibitory current impulses arrive at event times in a second Poisson process $\pi(f_i, t)$, which is independent of $\pi(f_e, t)$ and causes $V(t)$ to decrease by a_i .
- If depolarization reaches a threshold of r units, the neuron fires an impulse.
- After each neuronal firing there is a refractory period of duration t_0, during which the impulses have no effect and the membrane depolarization $V(t)$, is reset to zero.
- At times $t > t_0$, each impulse produces unit depolarization.
- For sub-threshold levels, the depolarization decays exponentially among impulses with time constant μ.

This model and its modifications have been used as a basis for many studies devoted to the theoretical description of neuronal activities [3, 34, 37, 40]. One of the principle difficulty with the application of this model lies in solving the delay-differential equations that form the mathematical expression of the model [40]. Though there have been extensive studies of the properties of the solutions of many kinds of functional equations [2, 11] a little progress has been made on equations of type (1.1) with both forward (advance) and backward (delay) delays. These applications motivates the approximation of DDEs of Stein's model type.

In 1967, Stein [35] generalized this model to deal with a distribution of postsynaptic potential amplitudes. Johannesma [14] and Tuckwell [37] included the reversal potentials into account. Various other models for neuronal activity have been proposed and many are discussed in Holden's book [13].

We state a model problem for a general boundary value problem for a singularly perturbed differential-difference equation containing both type of shifts (negative as well as positive shifts)

$$\varepsilon^2 y''(x) + p(x)y'(x) + q(x)y(x-\delta) + r(x)y(x) + s(x)y(x+\eta) = f(x), \tag{1.1}$$

$\forall\, x \in \Omega = (0, 1)$ and subject to the interval conditions

$$y(x) = \phi(x) \quad \text{for } -\delta(\varepsilon) \le x \le 0, \quad y(x) = \psi(x) \quad \text{for } 1 \le x \le 1 + \eta(\varepsilon), \tag{1.2}$$

where ε is a small parameter $0 < \varepsilon \ll 1$, $\delta(\varepsilon)$ and $\eta(\varepsilon)$ are of $o(\varepsilon)$. The functions $p(x)$, $q(x)$, $r(x)$, $s(x)$, $f(x)$, $\phi(x)$ and $\psi(x)$ are sufficiently smooth. For a function $y(x)$ to be smooth solution of problem (1.1)–(1.2), it must be continuous on $[0, 1]$ and continuously differentiable on (0, 1) satisfying (1.1). It is well-known [5, 27] for $\delta(\varepsilon) = \eta(\varepsilon) = 0$, the solution of the boundary value problem of the DDEs (1.1)–(1.2) exhibits layer at the left end or right end of the interval depending on the sign of $p(x)$.

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In case of $p(x) = 0$ one may have layer behavior or oscillatory behavior depending on sign of $q(x) + r(x) + s(x)$, *i.e.*, for $q(x) + r(x) + s(x) < 0$, throughout the interval $[0, 1]$, two boundary layers exist at $x = 0$ and $x = 1$ and for $q(x)+r(x)+s(x) > 0$, throughout the interval $[0, 1]$, solution has oscillatory behavior. A solution having classical turning points occur if $q(x)+r(x)+s(x)$ changes the sign at some interior points in the interval $(0, 1)$. However, the layer is maintained for problem (1.1)–(1.2) for $\delta(\varepsilon)$ and $\eta(\varepsilon)$ sufficiently small [21].

It has been observed that often the shifts are very small and affect the solution significantly. Lange and Miura [22] have shown the effect of very small shifts (of the order of ε) on the solution and pointed out that they drastically affect the solution and therefore can not be neglected. Further studies of the effect of δ and η on the layer behavior of the solution have been carried out by Kadalbajoo and Sharma [17–19]. A numerical method comprising of upwind finite difference operator on an adaptive grid, which is formed by equidistributing the arc-length monitor function, has constructed in [24]. The method was proved to be robust, in the sense that the discrete solution obtained converges in the maximum norm to the exact solution uniformly with respect to the perturbation parameter. In [25], a defect-correction method based on finite difference scheme for solving a singularly perturbed delay differential equation has been considered. The analysis of an upwind scheme for singularly perturbed differential-difference equation on a grid which is formed by an equidistributed arc-length monitor function has been given in [26]. For the occurrence and further motivation for solving such problems, the readers may refer to [4, 21–23, 35, 36] and the references therein.

Parameter-uniform numerical methods [6, 28] are methods whose numerical approximations U^N satisfy error bounds of the form

$$\|u_\varepsilon - U^N\| \leq C\vartheta(N), \quad \vartheta(N) \to 0 \ \text{ as } \ N \to \infty,$$

where u_ε is the solution of the continuous problem, $\|.\|$ is the maximum point-wise norm, N is the number of mesh points (independent of ε) used and C is positive constant which is independent of both ε and N. In other words, the numerical approximations U^N converge to u_ε for all values of ε in the range $0 < \varepsilon \ll 1$.

It is well-known that standard discretization methods for solving singular perturbation problems are unstable and fail to give accurate results when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods for these problems, whose accuracy do not depend on the parameter value ε, *i.e.*, methods that are convergent ε-uniformly [5, 8, 30]. There are essentially two strategies to design schemes which have small truncation errors inside the layer region(s). The first approach which forms the class of fitted-mesh methods consists in choosing a fine mesh in the layer region(s). The second approach is in the context of the fitted operator methods in which the mesh remains uniform and the difference schemes reflect the qualitative behavior of the solution(s) inside the layer region(s). A nice discussion using one or both of the above strategies can be found in Miller [5].

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The fitted-mesh methods have probably received less detailed attention in the literature, than the construction of an appropriate fitted operator methods. In 1996, Miller *et al.* [28] established the great importance of fitted-mesh methods for solving singular perturbation problems. There are some problems for which no parameter-uniform method can be constructed using a fitted operator approach on a uniform mesh while for such problem a parameter-uniform fitted-mesh method can be constructed [7]. In this paper our aim is to construct a parameter-uniform numerical method based on Shishkin mesh for a more general singularly perturbed differential-difference equations with small delay as well as advance with layer behavior. The numerical method comprises a B-spline collocation method on a piecewise-uniform mesh which is condensed in the boundary layer. We first approximate the terms containing small shifts by Taylor's series expansion and then apply the fitted-mesh method, provided shifts are of $o(\varepsilon)$.

Throughout this paper, C denotes generic positive constant that is independent of ε and mesh parameter N which may assume different values at different places but remains to be constant. $\|.\|$ and $\|.\|_\infty$ denote the pointwise maximum norm and global maximum norm respectively over the appropriate domain of the independent variable, *i.e.*, $\|u\| = \max\limits_{x\in\bar{\Omega}_N} \mid u(x_i) \mid$ and $\|u\|_\infty = \max\limits_{x\in\bar{\Omega}} \mid u(x) \mid$.

2. The continuous problem

In this section, we consider the numerical treatment for the model problem (1.1)–(1.2). We have by Taylor's series approximation for the terms containing the small shifts in the problem

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) + O(\delta^3),$$
$$y(x+\eta) \approx y(x) + \eta y'(x) + \frac{\eta^2}{2} y''(x) + O(\eta^3).$$

Thus we have from (1.1) the approximating equation

$$c_\varepsilon(x)y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \tag{2.1}$$

$$y(0) = \phi_0 = \phi(0), \quad y(1) = \psi_1 = \psi(1), \tag{2.2}$$

where $c_\varepsilon(x) = \varepsilon^2 + \frac{\delta^2}{2}q(x) + \frac{\eta^2}{2}s(x)$, $a(x) = p(x) - \delta q(x) + \eta s(x)$, $b(x) = q(x) + r(x) + s(x)$. Furthermore we assume that $0 < c_\varepsilon(x) < \varepsilon^2 + \delta^2 M_1 + \eta^2 M_2 = c_\varepsilon$, where $2M_1$ and $2M_2$ are the upper bounds for $q(x)$ and $s(x)$ respectively. It is also assumed that $b(x) \le -\theta < 0$, for some positive constant θ.

Since we have taken the Taylor's series expansion upto second order terms so the BVP (2.1) is an approximate differential equation that differs from the original equation (1.1) by a term that is of $O(\delta^3 y''', \eta^3 y''')$. Here, we assume shifts, δ and η to be

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sufficiently small of $o(\varepsilon)$, so the solution of the problem (2.1)–(2.2) will provide a good approximation to the solution of the problem (1.1)–(1.2). The case where δ and η are of $O(\varepsilon)$ is under consideration by the author. The differential operator L_ε corresponding to the boundary value problem (2.1)–(2.2) is defined by

$$L_\varepsilon y(x) \equiv c_\varepsilon(x)y''(x) + a(x)y'(x) + b(x)y(x).$$

We consider the case when the solution of the model problem (2.1)–(2.2) exhibits a layer behavior on the left side on the interval $[0, 1]$, *i.e.*, it is assumed that $a(x) \geq M > 0,\ \forall\, x \in \bar{\Omega}$, where M is a positive constant. The other case, *i.e.*, when $a(x) \leq -M < 0,\ \forall\, x \in \bar{\Omega}$, the layer occurs at the right end of the interval can be treated similarly.

We now give some basic properties of the solution and its derivatives of the problem (2.1)–(2.2). For their proofs the readers can refer to [5].

Lemma 2.1. *(Minimum principle) Let $\varphi(x)$ be a smooth function satisfying $\varphi(0) \geq 0$, $\varphi(1) \geq 0$ and $L_\varepsilon\varphi(x) \leq 0,\ \forall\, x \in \Omega$. Then $\varphi(x) \geq 0,\ \forall\, x \in \bar{\Omega}$.*

Lemma 2.2. *(Stability) The solution $y(x)$ of* (2.1)–(2.2) *is bounded and satisfies*

$$\|y\|_\infty \leq \theta^{-1}\|f\|_\infty + \max(|\ \phi_0\ |, |\ \psi_1\ |), \quad \forall\, x \in \bar{\Omega}.$$

Lemma 2.3. *Let $y(x)$ be the solution of* (2.1)–(2.2) *and let $a(x)$, $b(x)$ and $f(x)$ be twice continuously differentiable functions, then*

$$|\ y^{(k)}(x)\ | \leq C\left(1 + c_\varepsilon^{-k}e^{-Mx/c_\varepsilon}\right) \quad \text{for } k = 0,\ 1,\ 2,\ 3.$$

These bounds for the derivatives of $y(x)$ were first obtained by Miller *et al.* [28], using the techniques based on Kellogg *et al.* [20].

3. Mesh selection strategy

It is known that on an equidistant mesh no scheme can attain convergence at all mesh points uniformly in ε, unless its coefficients have an exponential property. Therefore, unless we use a specially chosen mesh we will not be able to get ε-uniform convergence at all the mesh points [30]. If *a priori* information is available about the exact solution then we could go for a highly non-uniform grids like given in [1], [10] and [39]. The simplest possible non-uniform mesh, namely a piecewise-uniform mesh proposed by Shishkin [33], is sufficient for the construction of an ε-uniform method. Shishkin mesh is fine near layers but coarser otherwise. In this section, we construct the piecewise-uniform mesh in such a way that more mesh points are generated in the boundary layer regions than outside these regions. In the case of left boundary layer, we divide the given interval $[0, 1]$ into two subintervals $[0, \tau]$ and $[\tau, 1]$, where the transition parameter τ is choosen such that $\tau = \min(0.5, K(c_\varepsilon/M)\ln N)$, where K is some positive constant depends on the scheme being used, N is the number of discretization

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points. It is assumed that $N = 2^m$, where $m \geq 2$ is an integer, which guarantees that there is at least one point in the boundary layer. Then we have

$$x_i = \begin{cases} ih_i, & \text{if} \quad i = 0, 1, \ldots, N/2, \\ \tau + (i - N/2)h_i, & \text{if} \quad i = N/2 + 1, \ldots, N, \end{cases}$$

where

$$h_i = \begin{cases} h_1 = 2\tau/N, & \text{if} \quad i = 1, \ldots, N/2, \\ h_2 = 2(1-\tau)/N, & \text{if} \quad i = N/2 + 1, \ldots, N, \end{cases}$$

i.e., the piecewise-uniform mesh spacing h_i is $2\tau/N$ for the interval $[0, \tau]$ and $2(1-\tau)/N$ for the interval $[\tau, 1]$.

Let $\bar{\Omega}_N = \{x_i\}_{i=0}^N$ be the set of mesh points. If the transition point τ is chosen independently of N, then one cannot obtain a convergence result that is uniform in ε (see ref. [7]).

4. Description of the method

In this section, we describe the B-spline collocation method to obtain the approximate solution of boundary value problem (2.1)–(2.2). Let $P : 0 = x_0 < x_1 < x_2 \ldots < x_{N-1} < x_N = 1$ be the partition of $\bar{\Omega}$, with piecewise uniform spacing h_i defined in previous section.

We include two more points on each side of the partition P as $x_{-2} < x_{-1} < x_0$ and $x_{N+2} > x_{N+1} > x_N$. Then partition P becomes $P : x_{-2} < x_{-1} < x_0 = 0 < x_1 < x_2 \ldots < x_{N-1} < x_N = 1 < x_{N+1} < x_{N+2}$. Let $L_2(\bar{\Omega})$ be the space of all square integrable functions on $\bar{\Omega}$ and X is a linear subspace of $L_2(\bar{\Omega})$. We use the cubic B-spline basis functions $B_i(x)$ for $i = 0(1)N$, (see ref. [29]).

It is easy to see that each $B_i(x)$ is also a piecewise cubic with knots at $\bar{\Omega}_N$ and $B_i(x) \in X$, also $B_i(x)$ is twice continuously differentiable $\forall\, x \in \mathbb{R}$. Also the values of $B_i(x)$, $B_i'(x)$ and $B_i''(x)$ at the nodal points x_i's are given as in table given below:

Table 1: B-Spline basis values

	Nodal values				
	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}
$B_i(x)$	0	1	4	1	0
$B_i'(x)$	0	$3/h$	0	$-3/h$	0
$B_i''(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

Let $\boldsymbol{B} = \{B_{-1}, B_0, B_1, \ldots, B_N, B_{N+1}\}$ and

$$\Phi_3(\bar{\Omega}_N) = \left\{ \Psi_i : \Psi_i = \sum_{i=-1}^{N+1} k_i B_i, \quad k_i \in \mathbb{R} \right\}.$$

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The functions $B_i(x)$ are linearly independent on $\bar{\Omega}$, thus $\Phi_3(\bar{\Omega}_N)$ is $(N+3)$-dimensional subspace of X (see ref. [31]). We seek a function $S(x) \in \Phi_3(\bar{\Omega}_N)$ that approximates the solution of boundary value problem (2.1)–(2.2), represented as

$$S(x) = \sum_{i=-1}^{N+1} \alpha_i B_i(x), \tag{4.1}$$

where α_i are unknown real coefficients. Here we have introduced two extra cubic B-splines, B_{-1} and B_{N+1} to satisfy the boundary conditions. Therefore, we have

$$LS(x_i) = f(x_i), \qquad 0 \le i \le N, \tag{4.2}$$

$$S(x_0) = \phi_0, \qquad S(x_N) = \psi_1. \tag{4.3}$$

Using Table 1 and Eq. (4.1) the system of collocation Eqs. (4.2) gives a system of $(N+1)$ linear equations in $(N+3)$ unknowns

$$r_i^- \alpha_{i-1} + r_i^c \alpha_i + r_i^+ \alpha_{i+1} = h_i^2 f_i, \quad 0 \le i \le N, \tag{4.4}$$

where

$$r_i^- = 6\left(\varepsilon^2 + \frac{\delta^2}{2} q_i + \frac{\eta^2}{2} s_i\right) - 3a_i h_i + b_i h_i^2,$$

$$r_i^c = -12\left(\varepsilon^2 + \frac{\delta^2}{2} q_i + \frac{\eta^2}{2} s_i\right) + 4b_i h_i^2,$$

$$r_i^+ = 6\left(\varepsilon^2 + \frac{\delta^2}{2} q_i + \frac{\eta^2}{2} s_i\right) + 3a_i h_i + b_i h_i^2,$$

where $a(x_i) = a_i$, $b(x_i) = b_i$, $q(x_i) = q_i$, $s(x_i) = s_i$ and $f(x_i) = f_i$. The given boundary conditions become

$$\alpha_{-1} + 4\alpha_0 + \alpha_1 = \phi_0, \tag{4.5a}$$

$$\alpha_{N-1} + 4\alpha_N + \alpha_{N+1} = \psi_1. \tag{4.5b}$$

Thus the Eqs. (4.4) and (4.5) lead to an $(N+3)\times(N+3)$ system with $(N+3)$ unknowns $\alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_{N+1}$. Now eliminating α_{-1} from the first equation of (4.4) and (4.5a), we get

$$(r_0^c - 4r_0^-)\alpha_0 + (r_0^+ - r_0^-)\alpha_1 = h_0^2 f_0 - \phi_0 r_0^-. \tag{4.6}$$

Similarly, eliminating α_{N+1} from the last equation of (4.4) and (4.5b), we obtain

$$(r_N^- - r_N^+)\alpha_{N-1} + (r_N^c - 4r_N^+)\alpha_N = h_N^2 f_N - \psi_1 r_N^+. \tag{4.7}$$

Now taking Eqs. (4.6) and (4.7) with the second through (N-1)st equations of (4.4), we get a system

$$A\alpha = d, \tag{4.8}$$

in $(N+1)$ unknowns $\alpha_0, \alpha_1, \ldots, \alpha_{N-1}, \alpha_N$ with right hand side $d = (d_0, d_1, \ldots, d_{N-1}, d_N)^t$.

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The elements of the tridiagonal matrix $A = (a_{i,j})$ are given by

$$a_{i,j} = \begin{cases} r_0^c - 4r_0^-, & i = j = 0, \\ r_0^+ - r_0^-, & i = 0, j = 1, \\ r_i^-, & i = j+1, 0 \le j \le N-2, \\ r_i^c, & 1 \le i, j \le N-1, \\ r_i^+, & i = j-1, 2 \le j \le N, \\ r_N^- - r_N^+, & i = N, j = N-1, \\ r_N^c - 4r_N^+, & i = j = N, \\ 0, & \forall \mid i-j \mid > 1, \end{cases}$$

and the elements of the column vector d are given by

$$d_j = \begin{cases} h_j^2 f_j - r_j^- \phi_0, & j = 0, \\ h_j^2 f_j, & 1 \le j \le N-1, \\ h_j^2 f_j - \psi_1 r_j^+, & j = N. \end{cases}$$

It can be easily seen that for sufficiently small values of h_i the matrix A is strictly diagonally dominant and hence nonsingular. Since A is nonsingular, we can solve the system $A\alpha = d$ for $\alpha_0, \alpha_1, \ldots, \alpha_N$ and substitute into the boundary conditions (4.5a) and (4.5b) to obtain α_{-1} and α_{N+1}. Hence the method of collocation using a basis of cubic B-splines applied to problem (2.1)–(2.2) has a unique solution $S(x)$ given by (4.1).

5. Parameter-uniform convergence

In this section, we shall show that the B-spline collocation method described in previous section is parameter-uniform convergent of order almost two. Let we define

$$\bar{h} = \max_{1 \le i \le N} \{x_i - x_{i-1}\} = \max\{h_1, h_2\}, \quad \underline{h} = \min_{1 \le i \le N} \{x_i - x_{i-1}\} = \min\{h_1, h_2\},$$

we also define $\beta = \bar{h}/\underline{h}$. For the derivation of uniform convergence, we use the following lemma [29] proved in [15].

Lemma 5.1. *The B-splines $B_{-1}, B_0, \ldots, B_{N+1}$, satisfy the inequality*

$$\sum_{i=-1}^{N+1} \mid B_i(x) \mid \le 10, \qquad 0 \le x \le 1.$$

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Theorem 5.1. *Let $S(x)$ be the collocation approximation from the space of cubic splines $\phi_3(\bar{\Omega}_N)$ to the solution $y(x)$ of the boundary value problem* (2.1)–(2.2). *If $f \in \mathbb{C}^2[0,1]$, then the parameter-uniform error estimate is given by*

$$\sup_{0<\varepsilon\ll 1} \max_{0\le i\le N} \mid y(x_i) - S(x_i) \mid \le CN^{-2}\ln^3 N,$$

where C is a positive constant independent of ε, δ, η and N.

Proof. To estimate the error $\mid y(x)-S(x) \mid$, let $Y(x)$ be the unique spline interpolate from $\Phi_3(\bar{\Omega}_N)$ to the solution $y(x)$ of our boundary value problem (2.1)–(2.2) given by

$$Y(x) = \sum_{i=-1}^{N+1} \tilde{\alpha}_i B_i(x). \tag{5.1}$$

If $f(x) \in \mathbb{C}^2[0,1]$ then $y(x) \in \mathbb{C}^4[0,1]$ and it follows from Hall error estimates [12] that

$$\mid D^j(y(x) - Y(x)) \mid \le \lambda_j \mid y^{(4)}(x) \mid \bar{h}^{4-j}, \quad j = 0,1,2, \tag{5.2}$$

where the λ_j are constants given by $\lambda_0 = 5/384$, $\lambda_1 = (1/216)(9+\sqrt{3})$, $\lambda_2 = (1/12)(3\beta+1)$. It follows immediately from the estimates (5.2) and using Lemma 2.3 that

$$\begin{aligned}
\mid Ly(x_i) - LY(x_i) \mid \le & c_\varepsilon \mid y''(x_i) - Y''(x_i) \mid + \mid a(x_i) \mid\mid y'(x_i) \\
& - Y'(x_i) \mid + \mid b(x_i) \mid\mid y(x_i) - Y(x_i) \mid \\
\le & \Big(c_\varepsilon\lambda_2\bar{h}^2 + \|a\|_\infty\lambda_1\bar{h}^3 + \|b\|_\infty\lambda_0\bar{h}^4\Big) \mid y^{(4)}(x_i) \mid \\
\le & C\Big(c_\varepsilon\lambda_2\bar{h}^2 + \|a\|_\infty\lambda_1\bar{h}^3 + \|b\|_\infty\lambda_0\bar{h}^4\Big)\Big(1 + c_\varepsilon^{-4}e^{-Mx_i/c_\varepsilon}\Big).
\end{aligned} \tag{5.3}$$

Now there arise two cases:
Case I. $\tau = 1/2$. In this case, mesh is uniform with spacing $1/N$ and $K(c_\varepsilon/M)\ln N \ge 1/2$ this gives $c_\varepsilon^{-1} \le C\ln N$ also $\bar{h} = 1/N$, therefore from (5.3) we get

$$\mid Ly(x_i) - LY(x_i) \mid \le CN^{-2}\ln^3 N. \tag{5.4}$$

Case II. $\tau = K(c_\varepsilon/M)\ln N$. In this case, mesh is piecewise-uniform with spacing $2\tau/N$ in $[0,\tau]$ and $2(1-\tau)/N$ in $[\tau,1]$. First suppose i satisfies $N/2 < i \le N$. Then it is easy to know from Lemma 2.3 that in the regular region *i.e.*, $(\tau,1)$

$$\mid y^{(k)}(x) \mid \le C,$$

therefore we have immediately from (5.3)

$$\mid Ly(x_i) - LY(x_i) \mid \le CN^{-2}. \tag{5.5}$$

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On the other hand if i satisfies $1 \leq i \leq N/2$, in this case we have $h_1 = 2\tau/N = 2K(c_\varepsilon/M)N^{-1}\ln N$, this gives $h_1/c_\varepsilon = CN^{-1}\ln N$, using in (5.3) we get

$$| Ly(x_i) - LY(x_i) | \leq CN^{-2}\ln^2 N. \tag{5.6}$$

Combining (5.5) and (5.6), we get

$$| Ly(x_i) - LY(x_i) | \leq CN^{-2}\ln^2 N. \tag{5.7}$$

Finally, on combining (5.4) and (5.7), we obtain

$$| Ly(x_i) - LY(x_i) | \leq CN^{-2}\ln^3 N. \tag{5.8}$$

Thus we have

$$\begin{aligned} | LS(x_i) - LY(x_i) | &= | f(x_i) - LY(x_i) | \\ &= | Ly(x_i) - LY(x_i) | \leq CN^{-2}\ln^3 N. \end{aligned} \tag{5.9}$$

Now suppose that $LY(x_i) = \tilde{f}(x_i)\ \forall\, 0 \leq i \leq N$ with the boundary conditions

$$Y(x_0) = \phi_0, \quad Y(x_N) = \psi_1,$$

leads to the linear system $A\tilde{\alpha} = \tilde{d}$, then it follows that

$$A(\alpha - \tilde{\alpha}) = (d - \tilde{d}), \tag{5.10}$$

where

$$\begin{aligned} &\alpha - \tilde{\alpha} = (\alpha_0 - \tilde{\alpha}_0, \alpha_1 - \tilde{\alpha}_1, \ldots, \alpha_N - \tilde{\alpha}_N)^t, \\ &d - \tilde{d} = (h^2(f(x_0) - \tilde{f}(x_0)), h^2(f(x_1) - \tilde{f}(x_1)), \ldots, h^2(f(x_N) - \tilde{f}(x_N)))^t. \end{aligned}$$

Using (5.9) we obtain

$$\|d - \tilde{d}\|_\infty \leq CN^{-4}\ln^3 N. \tag{5.11}$$

We have seen that, for sufficiently small values of h, the coefficient matrix A is strictly diagonally dominant. Also note that, except the first row and the last row of A, the off diagonal elements $a_{i,i-1},\ a_{i,i+1},\ 1 \leq i \leq N-1$, are positive and the main diagonal elements $a_{i,i}$ are negative, therefore

$$| a_{i,i} | -(| a_{i,i-1} | + | a_{i,i+1} |) = -6b_ih_i^2 \geq 6\theta h_i^2 > 0, \quad \text{as} \quad b(x) \leq -\theta < 0.$$

Also from the first row of A, we have

$$| a_{0,0} | - | a_{0,1} | = 36\left(\varepsilon^2 + \frac{\delta^2}{2}q_0 + \frac{\eta^2}{2}s_0\right) - 18a_0h_i.$$

Again from the last row of A, we have

$$| a_{N,N} | - | a_{N,N-1} | = 36\left(\varepsilon^2 + \frac{\delta^2}{2}q_N + \frac{\eta^2}{2}s_N\right) + 6a_Nh_i.$$

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Therefore, by the estimate given in [38], we get

$$\|A^{-1}\|_\infty \leq CN^2. \tag{5.12}$$

From (5.10), (5.11) and (5.12), we get

$$| \alpha_i - \tilde{\alpha}_i | \leq CN^{-2} \ln^3 N, \quad 0 \leq i \leq N.$$

Now to estimate $| \alpha_{-1} - \tilde{\alpha}_{-1} |$ and $| \alpha_{N+1} - \tilde{\alpha}_{N+1} |$, using the boundary condition (4.5a), we obtain

$$| \alpha_{-1} - \tilde{\alpha}_{-1} | \leq CN^{-2} \ln^3 N.$$

Again using the boundary condition (4.5b), we obtain

$$| \alpha_{N+1} - \tilde{\alpha}_{N+1} | \leq CN^{-2} \ln^3 N.$$

Therefore

$$\max_{-1 \leq i \leq N+1} | \alpha_i - \tilde{\alpha}_i | \leq CN^{-2} \ln^3 N. \tag{5.13}$$

The above inequality enables us to estimate $| S(x) - Y(x) |$. We have

$$S(x) - Y(x) = \sum_{i=-1}^{N+1} (\alpha_i - \tilde{\alpha}_i) B_i(x).$$

Thus, using (5.13) and Lemma 5.1, we obtain

$$\begin{aligned} | S(x) - Y(x) | &\leq \max_{-1 \leq i \leq N+1} | \alpha_i - \tilde{\alpha}_i | \sum_{i=-1}^{N+1} | B_i(x) | \\ &\leq CN^{-2} \ln^3 N. \end{aligned}$$

This gives

$$\max_{0 \leq i \leq N} | S(x_i) - Y(x_i) | \leq CN^{-2} \ln^3 N.$$

Therefore using triangle inequality, we obtain

$$\sup_{0 < \varepsilon \ll 1} \max_{0 \leq i \leq N} | y(x_i) - S(x_i) | \leq CN^{-2} \ln^3 N. \qquad \square$$

6. Stability analysis

In this section, we consider the stability of the B-spline collocation method when applied to (2.1)–(2.2). The B-spline collocation method results in a system of linear equations (4.8). Spurious oscillations can occur unless the numerical method is designed to inherit the inverse-monotonicity property of the associated differential operator. Being the system matrix A, as the negative of an M matrix would guarantees

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that the discrete solution will not exhibits spurious oscillations. We want to show that the difference between actual solution and computed solution depends continuously on the perturbations in the prescribed coefficients of the linear system (4.8); that is, the method is stable. Let γ be the solution of the perturbed system *i.e.*,

$$(A+\sigma)\gamma = d + \partial, \tag{6.1}$$

where σ and ∂ are small perturbations in A and d respectively. Subtracting (6.1) from (4.8) we get

$$\alpha - \gamma = (A+\sigma)^{-1}(\sigma\alpha - \partial). \tag{6.2}$$

Now we claim that

$$\|(A+\sigma)^{-1}\|_\infty \le C\|A^{-1}\|_\infty. \tag{6.3}$$

We have

$$\begin{aligned}
\|(A+\sigma)^{-1}\|_\infty &= \|A^{-1}(I+\sigma A^{-1})^{-1}\|_\infty \\
&\le \|A^{-1}\|_\infty \|(I+\sigma A^{-1})^{-1}\|_\infty \\
&\le \frac{\|A^{-1}\|_\infty}{1-\|\sigma A^{-1}\|_\infty} \quad \Big(\text{provided } \|\sigma A^{-1}\|_\infty < 1\Big) \\
&\le \frac{\|A^{-1}\|_\infty}{1-\|\sigma\|_\infty\|A^{-1}\|_\infty} \\
&\le \|A^{-1}\|_\infty \left\{1 + \frac{\|\sigma\|_\infty\|A^{-1}\|_\infty}{1-\|\sigma\|_\infty\|A^{-1}\|_\infty}\right\} \\
&\le C\|A^{-1}\|_\infty,
\end{aligned}$$

also we have

$$\|A^{-1}\|_\infty \le \left[\min_{0\le i\le N}\left(\mid a_{i,i}\mid - \sum_{j\ne i}\mid a_{i,j}\mid\right)\right]^{-1} = C < \infty. \tag{6.4}$$

On combining (6.2), (6.3) and (6.4), we conclude that

$$\|\alpha-\gamma\|_\infty \le C\Big(\|\sigma\|_\infty\|\alpha\|_\infty + \|\partial\|_\infty\Big),$$

which shows the stability of the collocation system.

7. Numerical results and discussions

To validate the theoretical results, we apply the proposed numerical scheme to several test problems with constant and variable coefficients having left and right boundary layers. Since the exact solutions of the considered problems are not known so to calculate the maximum point-wise error and rate of convergence, we use the double mesh principle [5].

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For this, we shall compute y_i^N and y_{2i}^{2N} which are described as follows: Let $\bar{\Omega}_\varepsilon^N$ be a Shishkin mesh with the parameter τ as defined in the paper altered slightly to

$$\bar{\tau} = \min(0.5, K(c_\varepsilon/M)\ln(N/2)).$$

Then, for $i = 0, 1, 2, \ldots N$, the ith point of the mesh $\bar{\Omega}_\varepsilon^N$ coincides with the (2i)th point of the mesh $\bar{\Omega}_\varepsilon^{2N}$. This idea is mainly to obtain better comparison between the solutions from the N, $2N$ mesh intervals. More precisely, in the case of $2N$ mesh intervals, if we use the transition parameter τ as given in this manuscript, then the value of τ will be different in the cases of N and $2N$ mesh intervals, and the comparison of the solutions will not be accurate. Therefore, to fix τ, we made this alternate new definition.

The ε-uniform maximum nodal error is defined by

$$E_N = \max_{0<\varepsilon\ll 1} E_{N,\varepsilon}.$$

The numerical rates of convergence are computed using the formula:

$$r_k \equiv r_{k,\varepsilon} = \log_2(E_{N_k,\varepsilon}/E_{2N_k,\varepsilon}), \quad k = 1, 2, \ldots$$

and the numerical rate of ε-uniform convergence is computed using

$$R_N = \log_2(E_N/E_{2N}).$$

Example 7.1. First, we consider the example

$$\varepsilon^2 y''(x) + y'(x) - 2y(x-\delta) - 5y(x) + y(x+\eta) = 0,$$

under the interval conditions

$$y(x) = 1, \text{ on } -\delta \le x \le 0, \quad y(x) = 0, \text{ on } 1 \le x \le 1+\eta.$$

Example 7.2. Next, we consider the example

$$\begin{aligned}\varepsilon^2 y''(x) - (1+\exp(x^2))y'(x) - x\exp(x)y(x-\delta) - x^2 y(x)\\ - (1+\exp(-x))y(x+\eta) = 1,\end{aligned}$$

under the interval conditions

$$y(x) = 0, \text{ on } -\delta \le x \le 0, \quad y(x) = 1, \text{ on } 1 \le x \le 1+\eta.$$

We also compute the constant in the error estimate, *i.e.*, since we have the theoretical error bound $E_N \le CN^{-2}\ln^3 N$ from Theorem 5.1 we approximate the constant in the error estimate by $C_N = E_N N^2/\ln^3 N$. Table 4 display the values of C_N for the examples taken.

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Table 2: Maximum absolute error for Example 7.1 for $\delta = 10^{-5} \times \varepsilon = \eta$.

ε^2	Number of mesh points N							
	16	32	64	128	256	512	1024	2048
2^{-4}	1.66E-02	6.05E-03	1.98E-03	4.90E-04	1.22E-04	3.05E-05	7.63E-06	1.91E-06
	1.46	1.61	2.01	2.01	2.00	2.00	2.00	
2^{-8}	1.15E-02	4.52E-03	1.60E-03	5.42E-04	1.77E-04	5.59E-05	1.73E-05	5.22E-06
	1.35	1.50	1.56	1.61	1.66	1.69	1.73	
2^{-12}	1.12E-02	4.41E-03	1.57E-03	5.31E-04	1.73E-04	5.47E-05	1.69E-05	5.11E-06
	1.35	1.49	1.56	1.62	1.66	1.69	1.73	
2^{-16}	1.12E-02	4.40E-03	1.57E-03	5.30E-04	1.73E-04	5.46E-05	1.69E-05	5.10E-06
	1.35	1.49	1.56	1.62	1.66	1.69	1.73	
2^{-20}	1.12E-02	4.40E-03	1.57E-03	5.30E-04	1.73E-04	5.46E-05	1.69E-05	5.10E-06
	1.35	1.49	1.56	1.62	1.66	1.69	1.73	
2^{-24}	1.12E-02	4.40E-03	1.57E-03	5.30E-04	1.73E-04	5.46E-05	1.69E-05	5.10E-06
	1.35	1.49	1.56	1.62	1.66	1.69	1.73	
2^{-28}	1.12E-02	4.40E-03	1.57E-03	5.30E-04	1.73E-04	5.46E-05	1.69E-05	5.10E-06
	1.35	1.49	1.56	1.62	1.66	1.69	1.73	
2^{-32}	1.12E-02	4.40E-03	1.57E-03	5.30E-04	1.73E-04	5.46E-05	1.69E-05	5.10E-06
	1.35	1.49	1.56	1.62	1.66	1.69	1.73	
E_N	1.66E-02	6.05E-03	1.98E-03	5.42E-04	1.77E-04	5.59E-05	1.73E-05	5.22E-06
R_N	1.46	1.61	1.87	1.61	1.66	1.69	1.73	

Table 3: Maximum absolute error for Example 7.2 for $\delta = 10^{-5} \times \varepsilon = \eta$.

ε^2	Number of mesh points N							
	16	32	64	128	256	512	1024	2048
2^{-4}	6.14E-02	2.10E-02	7.10E-03	2.30E-03	7.36E-04	2.27E-04	6.96E-05	2.18E-05
	1.55	1.56	1.63	1.64	1.70	1.71	1.67	
2^{-8}	6.01E-02	2.09E-02	7.20E-03	2.40E-03	8.05E-04	2.62E-04	8.46E-05	2.75E-05
	1.52	1.54	1.59	1.58	1.62	1.63	1.62	
2^{-12}	6.01E-02	2.09E-02	7.20E-03	2.40E-03	8.18E-04	2.70E-04	8.93E-05	3.02E-05
	1.52	1.54	1.59	1.55	1.60	1.60	1.56	
2^{-16}	6.01E-02	2.09E-02	7.20E-03	2.40E-03	8.10E-04	2.69E-04	8.96E-05	3.03E-05
	1.52	1.54	1.59	1.57	1.59	1.59	1.56	
2^{-20}	6.01E-02	2.09E-02	7.20E-03	2.40E-03	8.11E-04	2.66E-04	8.71E-05	2.99E-05
	1.52	1.54	1.59	1.57	1.61	1.61	1.54	
2^{-24}	6.01E-02	2.09E-02	7.20E-03	2.40E-03	8.11E-04	2.66E-04	8.73E-05	2.90E-05
	1.52	1.54	1.59	1.57	1.61	1.61	1.59	
2^{-28}	6.01E-02	2.09E-02	7.20E-03	2.40E-03	8.11E-04	2.66E-04	8.74E-05	2.91E-05
	1.52	1.54	1.59	1.57	1.61	1.61	1.59	
2^{-32}	6.01E-02	2.09E-02	7.20E-03	2.40E-03	8.11E-04	2.66E-04	8.74E-05	2.92E-05
	1.52	1.54	1.59	1.57	1.61	1.61	1.58	
E_N	6.14E-02	2.10E-02	7.20E-03	2.40E-03	8.18E-04	2.70E-04	8.96E-05	3.03E-05
R_N	1.55	1.54	1.59	1.55	1.60	1.60	1.56	

8. Conclusion

B-spline collocation method has been taken into account to approximate the solution of a more general class of singularly perturbed differential-difference equations

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Table 4: E_N, R_N and C_N for Examples 7.1 and 7.2.

	Example 1			Example 2		
N	E_N	R_N	C_N	E_N	R_N	C_N
16	1.66E-02	1.46	0.20	6.14E-02	1.52	0.74
32	6.05E-03	1.61	0.15	2.10E-02	1.54	0.05
64	1.98E-03	1.87	0.11	7.20E-03	1.59	0.41
128	5.42E-04	1.61	0.08	2.40E-03	1.57	0.34
256	1.77E-04	1.66	0.07	8.18E-04	1.61	0.31
512	5.59E-05	1.69	0.06	2.70E-04	1.61	0.29
1024	1.73E-05	1.73	0.05	8.96E-05	1.58	0.29
2048	5.22E-06	1.73	0.05	3.03E-05	1.61	0.29

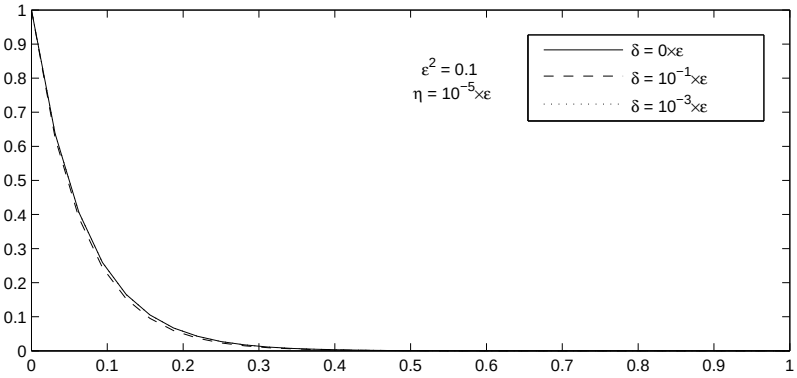


Figure 1: Numerical solution of Example 7.1 for different values of δ using $N = 32$.

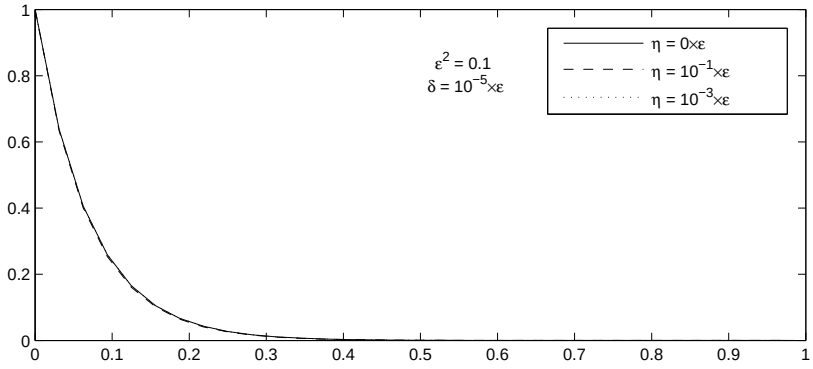


Figure 2: Numerical solution of Example 7.1 for different values of η using $N = 32$.

which arises in the mathematical modeling of a model of neuronal variability. A parameter-uniform error estimate is obtained for the presented scheme. The piecewise-uniform mesh is chosen to grasp the better approximation to the exact solution in the boundary layer region. Instead, if we use the uniform mesh for solving the singular perturbation problem, not only it requires a larger number of mesh points in the given interval, but also the approximate solution oscillates about the exact solution in the

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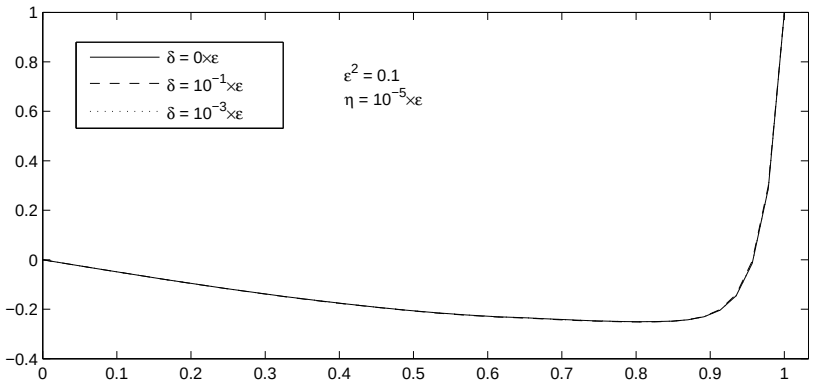


Figure 3: Numerical solution of Example 7.2 for different values of δ using $N = 32$.

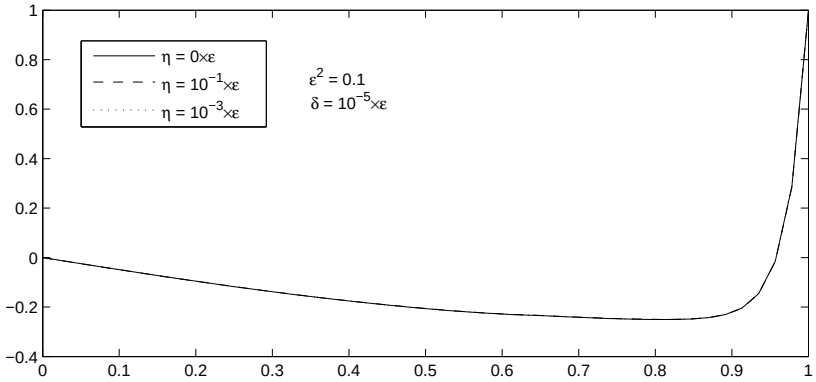


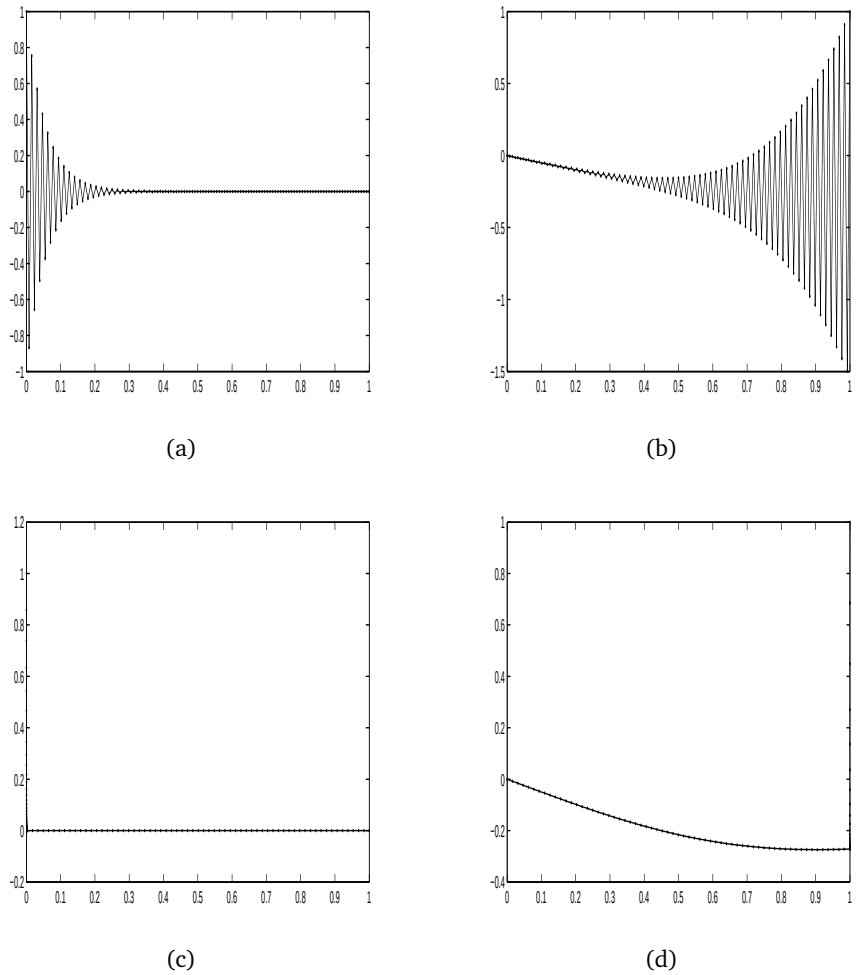
Figure 4: Numerical solution of Example 7.2 for different values of η using $N = 32$.

boundary layer region when the value of ε decreases. So, we considered piecewise-uniform mesh with more mesh points in the boundary layer region. Two numerical experiments are carried out in support of the predicted theory via tabulating the maximum absolute errors in Tables 2 and 3.

To show the effect of the small shifts on the solutions of the problems the graphs of the solutions for different values of negative shift and positive shift have been plotted. Figs. 1 and 2 illustrate that in the case when the solution of the boundary value problem exhibits layer behavior on the left side, the change in the advance affects the solution in similar fashion as the change in delay affects, but reversely while Figs. 3 and 4 illustrate that in the case when the boundary value problem exhibits layer behavior on the right side, the change in the advance affects the solution in similar fashion as the change in delay affects.

In order to observe the oscillations and use of fitted mesh over oscillations, Fig. 5 has been inserted. One can see that the oscillations are present near the boundary layer using uniform mesh however there are no oscillations near the boundary layer using fitted mesh.

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(a) (b)

(c) (d)

Figure 5: Numerical solution profiles of Example 7.1 and 7.2 for $\varepsilon = 2^{-6}$, $\delta = \eta = 10^{-5} \times \varepsilon$ using $N = 128$ (a) Example 7.1 taking uniform mesh (b) Example 7.2 taking uniform mesh (c) Example 7.1 taking piecewise-uniform mesh (d) Example 7.2 taking piecewise-uniform mesh.

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