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Uncertainty Quantification of Derivative Instruments

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Abstract. Model and parameter uncertainties are common whenever some parametric model is selected to value a derivative instrument. Combining the Monte Carlo method with the Smolyak interpolation algorithm, we propose an accurate efficient numerical procedure to quantify the uncertainty embedded in complex derivatives. Except for the value function being sufficiently smooth with respect to the model parameters, there are no requirements on the payoff or candidate models. Numerical tests carried out quantify the uncertainty of Bermudan put options and down-and-out put options under the Heston model, with each model parameter specified in an interval.

AMS subject classifications: 62P05, 65C05, 65C50, 68U20

Key words: Parameter uncertainty, Derivative pricing, Smolyak algorithm, Monte Carlo, Entropy

1. Introduction

In the context of pricing and hedging exotic over-the-counter (OTC) derivatives, the price process of the underlying asset can be described by different types of parametric models, such as stochastic volatility models (e.g. Ref. [26]) and fraction Brownian motion models (e.g. Ref. [33]). The parameters of these models can be estimated by model calibration, but there are two issues: (1) different kinds of models can be perfectly calibrated to the same market data [38]; and (2) different calibration methods may yield different estimations for each parameter in a specific model [21].

These empirical issues, together with limited knowledge of the market dynamics, confront an agent with ambiguity about which model is the best to value a target derivative, especially since the calibrated models can provide quite different values for the target derivative. Indeed, model and parameter ambiguities are ubiquitous whenever a parametric model is employed, in two different categories according to Ref. [30] — viz. risk and uncertainty. Model (parameter) risk relates to those settings where the probabilities

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of the candidate models (parameter realisations) are known, while model (parameter) uncertainty arises from a lack of knowledge of the probabilistic information on these models (parameter realisations). Here we focus on numerically quantifying model (parameter) uncertainty embedded in a derivative instrument

Model uncertainty is a growing concern for regulation and risk management [6,17], and researchers have proposed both model-independent and model-dependent approaches to deal with model uncertainty in applications. A model-independent approach for the value bounds of a derivative is an effective way to eliminate model (parameter) uncertainty, by providing an agent with a conservative reference to quote a target derivative — see e.g. Refs. [5, 9, 12, 27, 40, 41]. However, these model-independent approaches may not work when the target derivative is too complex to be hedged model-independently, and a robust alternative is to employ a model-dependent approach by considering a set of plausible models to value the derivative. At the parameter level, a confidence interval rather than the point estimation of each model parameter can be employed, to account for parameter uncertainty in the setting of option pricing. We believe this idea was first proposed in Refs. [2] and [32], and traders and institutions often attack model (parameter) uncertainty through a worst-case approach — e.g. by stress testing portfolios [37]. Following a worst-case approach, Cont [11] proposed an uncertainty measure — viz. the spread between lower and upper bounds of the target derivative value when a set of plausible models are employed, which accounts for both hedgeable and unhedgeable risk.

However, although a set of meaningful axioms is satisfied, to date little attention has been paid on how to calculate this uncertainty measure accurately for exotic derivatives under complex parametric models — but here we propose an accurate and efficient numerical method to calculate the worst-case value of a derivative instrument under model and parameter uncertainties, combining the Smolyak sparse-grid interpolation approximation with the Monte Carlo-based optimisation method. The only assumption we make is that the derivative value function is smooth enough with respect to the model parameters, which should be a premise feature whenever a parametric model is employed to value an exotic derivative, for otherwise the model user suffers an additional risk if the model parameters are not estimated with high accuracy. Our second objective is to use entropy to complement the uncertainty measure proposed by Cont [11] to assess the uncertainty of the target derivative value with respect to model and parameter uncertainties. In some cases, his uncertainty measure may have the same value spread (not value bounds) for two different derivatives, when cannot be used to distinguish the uncertainty embedded in them. The entropy measure can be used as an alternative, to further quantify their uncertainty from the perspective of the amount of information provided by the ensemble of derivative values.

Our work reflects two strands in the literature. The first and foremost is the calculation of robust values of complex exotic derivatives in a model-dependent framework, where the model parameters are specified in terms of an interval rather than a point in each parameter direction — e.g. the famous volatility uncertainty model [2,32], where the underlying asset price follows

$$dS_t = rS_t dt + \sigma_t S_t dW_t, \quad \sigma_t \in [\sigma_{min}, \sigma_{max}].$$

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In particular, we consider *general* parametric models with constant parameters (e.g. Heston's model [26]), which can be selected from corresponding intervals. The method we propose can be used to calculate the worst-case value of an exotic derivative (that may be but is not necessarily European style [2, 32]) over all realisations of model parameters. The second strand an investigation of parameter risk, where a probability distribution is assigned to the set of candidate models or parameter realisations [3, 14, 22]. The value of a target derivative is the weighted average of the model-based prices over all these candidate models or parameter realisations. However, it is time-consuming to calculate the model prices of an exotic option with thousands of realisations of the model parameters when we implement these methods. In the setting of parameter uncertainty[†], a large ensemble of model parameters can be generated according to the uniform distribution in each direction, because we have no preferred parameter realisations. It is notable that both objective distribution [3, 14, 22] or uniform distribution involve a large ensemble of target derivative values that must be calculated with a large ensemble of model parameters. The interpolation-based approach used in the setting of parameter uncertainty can also be employed to reduce the computational cost in the setting of parameter risk, but we do not outline details in that setting here as we focus on parameter uncertainty. In passing, we note that Gaß *et al.* [18, 19] have proposed approximating the value of a derivative via tensorised polynomial interpolation and magic point interpolation, but tensorised polynomial interpolation suffers from the so-called 'curse of dimensionality' [46]. Magic point interpolation seems to be a promising tool, but its application in the general setting and a comparison with the Smolyak sparse-grid interpolation has not yet been pursued.

Our main objective is formulated in Section 2. Based on the Smolyak multi-dimensional interpolation algorithm, a Monte Carlo-based method to quantify the uncertainty of a derivative instrument with respect to the parameter uncertainty is developed in Section 3. Section 4 numerically illustrates how the uncertainty of Bermudan put options and down-and-out put options under the Heston model is quantified, and our concluding remarks are made in Section 5.

2. Worst-case Valuation and Uncertainty Quantification

As mentioned, the entropy measure is proposed as complementary to the uncertainty measure proposed by Cont [11] for uncertainty. The key aspect of these two uncertainty measures is to efficiently calculate a large ensemble of derivative values for a given set of pricing models. This key issue will be resolved in Section 3, following our discussion of essential concepts associated with uncertainty measures in this section.

[†]Cont [11] proposes to incorporate all candidate model types into a single super-model, such that model uncertainty can be seen as a special case of parameter uncertainty. Hereafter, unless otherwise stated the term 'parameter uncertainty' is used to include both model uncertainty and parameter uncertainty.

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2.1. Pricing models and uncertainty

Generally speaking, when the price process of an underlying asset is modelled by a stochastic system, the parametric model implicitly specifies a probability distribution on the future state of the underlying asset. A parametric model can thus be characterised by a probability measure $\mathbb{Q}_{\boldsymbol{\theta}}$, where $\boldsymbol{\theta}$ represents the vector of model parameters. A pricing model corresponds to a risk-neutral measure $\mathbb{Q}_{\boldsymbol{\theta}}$, under which the discounted asset price $(S_t)_{t\in[0,T]}$ is a martingale with respect to its own history $(\mathscr{F}_t)_{t\in[0,T]}$.

The parameters $\boldsymbol{\theta}$ can be estimated by calibrating the model to the prices of benchmark instruments that are liquidly traded in markets (the model calibration procedure). Since the benchmark instruments are quoted in terms of bid-ask prices [14, 15, 22] and different optimisation methods can be used to calibrate a parametric model [21], there may exist a set of pricing models where the model-based prices of benchmark instruments lie between the quoted bid and ask prices. Let $\mathscr{Q} = \{\mathbb{Q}_{\boldsymbol{\theta}_1}, \cdots, \mathbb{Q}_{\boldsymbol{\theta}_J}\}$ be a family of candidate pricing models, with parameters specified in terms of boxes — i.e. $\boldsymbol{\theta}_j \in D_j$ for $j = 1, \cdots, J$, where the boxes D_j account for parameter uncertainty while $\mathscr{Q}$ accounts for model and parameter uncertainty. Since $\mathscr{Q}$ is characterised by parameters, its role can be to represent parameter uncertainty, although the elements of $\mathscr{Q}$ are not necessarily equivalent measures.

2.2. Worst-case valuation under parameter uncertainty

Worst-case valuation is an approach used by an uncertainty-aversion decision maker, who is concerned about the consequence in the worst and best cases within a set of candidate pricing models. This approach is not only used in derivative pricing and robust portfolio theory (e.g. see Ref. [2, 31, 32]), but also in macroeconomics (e.g. see Ref. [24]). We now explore numerical methods for worst-case values.

The worst case for a seller is of course the best case for a buyer (and vice versa), and in the worst-case approach we calculate the value bounds of a derivative. The upper and lower value bounds of a derivative instrument can be defined as

$$\overline{P}(f) = \sup_{\mathbb{Q}\in\mathscr{Q}} P^{\mathbb{Q}}(f) \quad \text{and} \quad \underline{P}(f) = \inf_{\mathbb{Q}\in\mathscr{Q}} P^{\mathbb{Q}}(f), \tag{2.1}$$

where f is the well-defined payoff function of the target derivative and $P^{\mathbb{Q}}$ is the value function under a pricing measure $\mathbb{Q}$, with $P^{\mathbb{Q}}(f)$ a nonlinear function of the model parameters $\boldsymbol{\theta}$. We assume that $P^{\mathbb{Q}}(f)$ is a smooth function of $\boldsymbol{\theta}$ for each type of candidate model, a natural premise whenever a parametric model is employed to price exotic derivatives, for otherwise there would be an additional risk when the model parameters are not estimated to high accuracy.

In practice, if the worst-case value of a derivative can be calculated efficiently under any type of candidate model $\mathbb{Q}_{\boldsymbol{\theta}_j}$ with parameters given in interval form ($\boldsymbol{\theta}_j \in D_j$ for $j = 1, \cdots, J$), it is trivial to calculate the worst-case value over the super-set $\mathscr{Q}$ of candidate models by comparing the worst-case value under each type of candidate model. Without loss of generality, we simplify the analysis by considering a specific type of candidate model with parameters given in the interval form. The nonlinearity of the optimisation

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problem (2.1) requires a global optimisation method where an ensemble of derivative values is needed to quantify the uncertainty embedded in a target derivative, so the Monte Carlo method can be chosen. The basic idea of the direct Monte Carlo method is to search for the solution to the optimisation problem by calculating the derivative value under each martingale measure $\mathbb{Q}$, to not only provide the global solution of the optimisation problem but also the ensemble of derivative values.

Since we are ambiguous about model parameters and have no preference in their realisations, a large ensemble of model parameters can be generated according to the uniform distribution in each parameter direction. According to the principle of maximum entropy, if nothing is known about a distribution except that it belongs to a certain class, then the distribution with the largest entropy could be chosen as the default. The uniform distribution is the maximum entropy distribution among all continuous distributions that are supported in a finite interval [36], so we may choose the uniform distribution as an *auxiliary* distribution to generate a large ensemble of the model parameters on their support intervals. Thus the upper and lower value bounds in (2.1) can be approximated by

$$\overline{P}(f) \approx \max_{\mathbb{Q}\in\bar{\mathscr{Q}}} P^{\mathbb{Q}}(f) \quad \text{and} \quad \underline{P}(f) \approx \min_{\mathbb{Q}\in\bar{\mathscr{Q}}} P^{\mathbb{Q}}(f)\,, \tag{2.2}$$

where $\bar{\mathscr{Q}}$ is a set of model parameter realisations. The larger the size of $\bar{\mathscr{Q}}$, the better (2.2) can approximate (2.1), because a small number of realisations of the model parameters may yield local optimal estimates for $\overline{P}(f)$ and $\underline{P}(f)$.

The basic idea of the Monte Carlo method for an optimisation problem is straightforward — but it can be time-consuming if the original pricing problem has to be solved with a large ensemble of model parameters, since the exotic derivative value must be calculated under each realisation. To reduce the computational cost, we propose an interpolation-based method to calculate the derivative values for any given realisation of the model parameters.

2.3. Uncertainty measures

Given the worst-case value of a target derivative, Cont [11] proposed that parameter uncertainty on a derivative value be defined by the coherent uncertainty measure

$$\mu_{\mathscr{Q}}(f) = \overline{P}(f) - \underline{P}(f)\,, \tag{2.3}$$

and verified some axioms for applications. This uncertainty measure $\mu_{\mathscr{Q}}$ only uses the maximum and minimum of an ensemble of the target derivative values, and we explore introducing more information from the ensemble of derivative values. Entropy generally represents the lack of information about the state of a system observed by an outsider [7], with information decreasing as the entropy increases — and we use the entropy of the ensemble of target derivative values under the following definition to measure its information extent, complementing the coherent uncertainty measure $\mu_{\mathscr{Q}}$:

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Definition 2.1 (Shannon Entropy — cf. Ref. [39]). The Shannon entropy $H(X)$ of a discrete random variable X with distribution $p(x)$ is defined by

$$H(X) = \sum_i p(x_i) \log \frac{1}{p(x_i)} . \tag{2.4}$$

Thus for a given ensemble of the model parameters, if two derivatives have the same value for the uncertainty measure the one with smaller entropy embeds less uncertainty because its ensemble provides more information, where we adopt the empirical distribution to calculate the entropy of a derivative value. We do not claim that the empirical distribution of the derivative value is the posterior distribution, because the uniform distribution is not the posterior distribution of the model parameters, but it does characterise the uncertainty of the derivative value when each combination of the model parameters is equally treated.

In summary, the main problem of uncertainty quantification of derivative instruments to be solved in the next section is as follows.

Problem 1. *Assuming that the model parameters $\boldsymbol{\theta}$ in the underlying asset price process that account for the parameter uncertainty within the input when pricing derivatives are specified in a box D (i.e. $\boldsymbol{\theta} \in D$), we consider how the two uncertainty measures defined by Eqs.* (2.3) *and* (2.4) *can be calculated efficiently to quantify the parameter uncertainty embedded in the value of a financial derivative.*

3. The Smolyak Algorithm and Monte Carlo-based Method

Our Monte Carlo-based method to solve Problem 1 is based on the Smolyak algorithm, an efficient method to set up an interpolation formula for a multivariate function.

3.1. The Smolyak algorithm

Let $D \subset \mathbb{R}^N$ be an N-dimensional bounded domain. Without loss of generality, hereafter we assume that D is the box

$$D = [-1, 1]^N , \quad N \geq 1 . \tag{3.1}$$

Let $f : D \to \mathbb{R}$ be a smooth function to be interpolated by a set of orthogonal basis functions via a finite number of support interpolation nodes. In the setting of a one-dimensional interpolation ($N = 1$) or in each parameter direction $i \in \{1, \cdots, N\}$, the interpolation formula for the objective function f is given by

$$\mathscr{I}^i(f) = \sum_{m=1}^{m_i} f(y_m^i) a_m^i \tag{3.2}$$

based on nodal sets

$$\Theta^i = \{y_1^i, \cdots, y_{m_i}^i\} \subset [-1, 1], \tag{3.3}$$

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where $\{a_m^i\}_{m=1,\cdots,m_i}$ are interpolation basis functions that satisfy the discrete orthogonality property $a_j^i(y_k^i) = \delta_{jk}$. In the multivariate case ($N > 1$), the Smolyak algorithm is [45]

$$\begin{aligned}\mathscr{I}(f) &= \sum_{k+1\le|\mathbf{i}|\le N+k} (-1)^{N+k-|\mathbf{i}|} \cdot \begin{pmatrix} N-1 \\ N+k-|\mathbf{i}| \end{pmatrix} \cdot \left(\mathscr{I}^{i_1} \otimes \cdots \otimes \mathscr{I}^{i_N}\right)(f) \\ &=: A(N+k,N)(f) \end{aligned} \tag{3.4}$$

where $\mathbf{i} = (i_1, \cdots, i_N) \in \mathbb{N}^N$ is a multi-index with $|\mathbf{i}| = \sum_{l=1}^N i_l$ and the integer k denotes the level of the construction, and the tensor product rule is

$$\left(\mathscr{I}^{i_1} \otimes \cdots \otimes \mathscr{I}^{i_N}\right)(f) = \sum_{k_1=1}^{m_{i_1}} \cdots \sum_{k_N=1}^{m_{i_N}} f(y_{k_1}^{i_1}, \cdots, y_{k_N}^{i_N}) \cdot (a_{k_1}^{i_1} \otimes \cdots \otimes a_{k_N}^{i_N}) \,. \tag{3.5}$$

Thus the Smolyak algorithm is a linear combination of tensor product operators on subsets (sparse grids) of full grids, hence to set up $A(N+k,N)$ we only need to evaluate the objective function on the sparse grids

$$\Theta_N \equiv \mathscr{H}(N+k,N) = \bigcup_{k+1\le|\mathbf{i}|\le N+k} (\Theta^{i_1} \times \cdots \times \Theta^{i_N}) \,. \tag{3.6}$$

In the implementation of the Smolyak algorithm (3.4), different strategies can be used to select the interpolation basis functions and the interpolation nodes in each direction. Although there is usually no explicit formula for the total number of interpolation nodes, when the nodal sets in each direction are nested (i.e. $\Theta^i \subset \Theta^j$, $\forall i < j$ the total number of interpolation nodes can reach a minimum (sparse grid method). Lagrange polynomials and piecewise linear functions are two commonly used bases, and the corresponding nodal sets are nested. If the objective function to be interpolated is known to be very smooth, univariate Lagrange polynomial interpolation at the extrema of Chebyshev polynomials is recommended in constructing the Smolyak formula, with the nodes defined as

$$y_j^i = \begin{cases} 0 \,, & m_i = 1,\ j = 1 \,, \\ -\cos\dfrac{\pi(j-1)}{m_i - 1}, & m_i > 1,\ j = 1, \cdots, m_i \,, \end{cases} \tag{3.7}$$

where $m_1 = 1$ and $m_i = 2^{i-1} + 1$ for $i > 1$. The univariate Lagrange polynomial basis functions associated with Θ^i are

$$a_j^i(y) = \begin{cases} 1 \,, & i = 1 \,, \\ \displaystyle\prod_{\substack{k=1 \\ k\ne j}}^{m_i} \frac{y - y_k^i}{y_j^i - y_k^i} \,, & i > 1 \,,\ j = 1, \cdots, m_i \,. \end{cases} \tag{3.8}$$

Barthelmann *et al.* [4] provide the error bound of high-order polynomial interpolation with the Smolyak algorithm in the following theorem.

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Theorem 3.1 (cf. Ref. [4]). *In the space F_N^l defined by $\{f : [-1,1]^N \to \mathbb{R} \mid \partial^{|\mathbf{m}|} f$ continuous, $m_i \le l,\ \forall i\}$ where $\mathbf{m} \in \mathbb{N}_0^N$, $l \in \mathbb{N}_0$ and $\partial^{|\mathbf{m}|}$ is the usual N-variate partial derivative of order $|\mathbf{m}|$, the interpolation error is*

$$\left\| f - A(N+k,N)(f) \right\|_\infty \le C_{N,l} \cdot M^{-l} \cdot (\log M)^{(l+2)(N-1)+1},$$

where $M = \dim(\mathscr{H}(N+k,N)) \approx (2^k/k!)N^k$ is the number of interpolation points.

As Lagrange polynomials have global support whereas the piecewise linear basis functions have local support, piecewise linear interpolation at equidistant support nodes is recommended if the smoothness of the objective function is not certain. The equidistant support nodes are defined by

$$y_j^i = \begin{cases} 0, & m_i = 1,\ j = 1, \\ 2\dfrac{j-1}{m_i - 1}, & m_i > 1,\ j = 1, \cdots, m_i, \end{cases} \tag{3.9}$$

where $m_1 = 1$ and $m_i = 2^{i-1} + 1$ for $i > 1$. Correspondingly, the piecewise basis functions are defined as $a_1^1(y) = 1$ for $i = 1$, and for $i > 1$

$$a_j^i(y) = \begin{cases} 1 - \dfrac{m_i}{2}|y - y_j^i|, & \text{if } |y - y_j^i| < \dfrac{2}{m_i}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.10}$$

where $j = 1, \cdots, m_i$. An error bound for piecewise linear interpolation with the Smolyak algorithm was provided by Klimke & Wohlmuth [29] as follows.

Theorem 3.2 (cf. Ref. [29]). *In the space F_N^2 defined by $\{f : [-1,1]^N \to \mathbb{R} \mid \partial^{|\mathbf{m}|} f$ continuous, $m_i \le 2,\ \forall i\}$ where $\mathbf{m} \in \mathbb{N}_0^N$ and $\partial^{|\mathbf{m}|}$ is the usual N-variate partial derivative of order $|\mathbf{m}|$, the piecewise linear interpolation error is*

$$\left\| f - A(N+k,N)(f) \right\|_\infty \le C_N \cdot M^{-2} \cdot \left(\log_2 M\right)^{3\cdot(N-1)},$$

where $M = \dim(\mathscr{H}(N+k,N)) \approx (2^k/k!)N^k$ is the number of interpolation points.

Apart from Lagrange or piecewise interpolation, Refs. [34,42,47] and references therein provide other strategies to choose the interpolation basis functions and interpolation nodes. While the sparse grid method is adopted here, many other collocation approximations can be used for uncertainty quantification — cf. the two comprehensive reviews in Refs. [35,43] for more details.

3.2. Monte Carlo-based method

Our Monte Carlo(MC)-based method to quantify the uncertainty of a derivative value can be summarised as follows:

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1. set up the interpolation formula (3.4) for the target value function;
2. get a huge ensemble of the target derivative by substituting a huge ensemble of the model parameters into (3.4); and
3. calculate the uncertainty measures.

Indeed, most attention goes to setting up the interpolation formula (3.4), and as the computation then only involves interpolation basis functions (e.g. polynomials or piecewise linear functions) the cost can be very small – it depends on the number of interpolation nodes and the computational cost of the value function on these nodes. The selection of the interpolation nodes is independent of the value function, so the nodes can be selected beforehand and the value function on these nodes can be calculated in a parallelised way. Setting up the interpolation formula (3.4) or working with the classical Monte Carlo method involves the calculation of the value function on a set of realisations of the model parameters, which can all be done in a parallelised way, where we respectively calculate the derivative values on the fixed nodes and on random nodes. Compared with the classical Monte Carlo method, the construction of the interpolation formula (3.4) requires far fewer nodes.

Essentially, the MC-based method greatly reduces the computational cost, as unlike the standard Monte Carlo method for uncertainty quantification it avoids solving the original pricing problem for each combination of the model parameters. Since a large ensemble of derivative values are calculated, the MC-based method is actually a global optimisation method that provides not only the maximum and minimum of the derivative value but also additional information to calculate the entropy measure. In addition, a trivial further application of the proposed method is to calculate the weighted average of the derivative value in the setting of parameter risk. Multivariate interpolation can also be generalised to quantify the statistics of the solution of differential equations with random inputs — e.g. parameters and boundary/initial conditions [46].

4. Numerical Tests

Although different types of popular models can be collected to account for the model and parameter uncertainty, without loss of generality we considered one type of candidate model and model parameters in specific intervals. These intervals can be viewed as the confidence interval for each model parameter according to the bid-ask prices of benchmarks. In particular, the MC-based method was used to quantify the uncertainty of Bermudan and Barrier options under the Heston model with parameter uncertainty.

4.1. Pricing models and target derivatives

The Heston model [26] is a popular stochastic volatility model, where the asset price process (S) and the variance process (v) under a risk-neutral probability measure $\mathbb{Q}$ evolve according to the following system of stochastic differential equations:

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t dW_t\,, & S_0 > 0\,, \\ dv_t = \kappa(\eta - v_t)dt + \sigma\sqrt{v_t}\, d\widetilde{W}_t\,, & v_0 = \sigma_0^2 > 0\,, \end{cases} \tag{4.1}$$

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where $(W_t)_{0\le t\le T}$ and $(\widetilde{W}_t)_{0\le t\le T}$ are correlated Brownian motions satisfying $dW_t d\widetilde{W}_t = \rho dt$. The parameter η is the long-term average variance, κ is the speed of the mean-reversion of the variance, the parameter σ is the volatility of the variance that scales the diffusion term of the variance process, and r is the risk-free interest rate.

There does not seem to be any previous extensive results on the monotonicity and smoothness of the value functions of American (Bermudan) options or Barrier (down-and-out) options, with respect to each parameter in the Heston model[‡]. On the other hand, the pricing problem for American options or Barrier options has to be solved with numerical methods — e.g. see Refs. [10, 16, 20, 23, 28, 48]. We considered Bermudan put options and the down-and-out put options to illustrate how to use the MC-based method to quantify their uncertainty under the Heston model. Rather than checking the smoothness of the value function, we checked the accuracy of the interpolation formula for each option in our numerical tests. A PDE-based method was employed to calculate the option value on the interpolation grids in the parameter space. When more candidate models were added, the worst-case value could become worse — i.e. the lower bound decreases while the upper bound increases.

Let $P(s,v,t;\mathbf{Z})$ be the fair value of an option under the Heston model with a parameter set $\mathbf{Z}$, if at t time units before the given maturity time T the underlying asset price $s \ge 0$ and the variance $v \ge 0$. The Heston spatial differential operator is

$$\mathscr{A} := \frac{1}{2}s^2 v\frac{\partial^2}{\partial s^2} + \rho\sigma sv\frac{\partial^2}{\partial s\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2}{\partial v^2} + rs\frac{\partial}{\partial s} + \kappa(\eta - v)\frac{\partial}{\partial v} - r\,, \tag{4.2}$$

which when applied to P defines the valuation problem for the American put options and down-and-out options.

1. American and Bermudan options.

   The valuation problem for American put options under the Heston model reduces to solving the following so-called partial differential complementarity problem (PDCP):

$$\begin{cases} \dfrac{\partial P}{\partial t} \ge \mathscr{A}P\,, \quad P \ge i\,, \\ (P-i)\left(\dfrac{\partial P}{\partial t} - AP\right) = 0\,, \end{cases} \tag{4.3}$$

   where $s > 0$, $v > 0$, $0 < t \le T$, $i = \max(K-s,0)$, $\mathscr{A}$ is defined in (4.2), and the abbreviation $P(s,v,t)$ of $P(s,v,t;\mathbf{Z})$ is the option value. The initial condition and boundary conditions adopted on the truncated domain $[0,S_{max}]\times[0,v_{max}]$ for the PDCP (4.3) were

$$P(s,v,0) = \max(K-s,0)\,, \tag{4.4a}$$

[‡]Assing *et al.* [1] prove the monotonicity of an American option value with respect to the initial values of the underlying price and the volatility process under two-dimensional models, including the Heston model. Compared with the out-of-the-money options, the at-the-money and in-the-money options are more sensitive to volatility — e.g. see Ref. [8].

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$$P(0,v,t) = K\,, \tag{4.4b}$$

$$\frac{\partial P}{\partial s}(S_{\max},v,t) = 0\,, \tag{4.4c}$$

$$\frac{\partial P}{\partial v}(s,v_{max},t) = 0\,, \tag{4.4d}$$

where $S_{\max}$ and $v_{\max}$ were set to be $14K$ and 5, respectively. The Heston PDCP problem (4.3) was implicitly assumed to be fulfilled at $v = 0$. Since this problem was solved on discrete temporal grid points, we actually calculated the value of Bermudan options, the discrete counterparts of American options.

2. Down-and-out put options.

   Given the lower bound B of a down-and-out put option, its value on the truncated domain $[B, S_{\max}] \times [0, v_{\max}]$ satisfies the PDE

$$\frac{\partial P}{\partial t} = \mathscr{A} P \tag{4.5}$$

   with the initial and boundary conditions:

$$P(s,v,0) = \max(K-s,0)\,, \tag{4.6a}$$

$$P(B,v,t) = 0\,, \tag{4.6b}$$

$$\frac{\partial P}{\partial s}(S_{\max},v,t) = 0\,, \tag{4.6c}$$

$$\frac{\partial P}{\partial v}(s,v_{max},t) = 0\,, \tag{4.6d}$$

   where $\mathscr{A}$ is defined in (4.2) and the abbreviation $P(s,v,t)$ of $P(s,v,t;\mathbf{Z})$ is the value of the down-and-out put option. We assumed that (4.5) holds at $v = 0$, and $S_{\max}$ and $v_{\max}$ were set to be $14K$ and 5, respectively .

We assumed that r is a constant and let $\mathbf{Z} = (v_0, \lambda, \eta, \kappa, \rho)$ be the other model parameters in the Heston model. Without loss of generality, we assumed that the other model parameters lie in the interval $[-1,1]$ (i.e. $\mathbf{Z} \in D = [-1,1]^5$), for otherwise $[-1,1]$ can be scaled into the confidence interval of each parameter. Then the worst-case values (2.1) are

$$\begin{aligned} \overline{P}(S_0,V_0,T;\mathbf{Z},r) &= \sup_{\mathbf{Z}\in D} P(S_0,v_0,T;\mathbf{Z},r)\,, \\ \underline{P}(S_0,v_0,T;\mathbf{Z},r) &= \inf_{\mathbf{Z}\in D} P(S_0,v_0,T;\mathbf{Z},r)\,, \end{aligned} \tag{4.7}$$

where P is the solution of (4.3) or (4.5) with corresponding conditions for American put options or down-and-out put options, respectively. We presumed $P(S_0,V_0,T;\mathbf{Z},r)$ to be smooth in $\mathbf{Z}$ in both cases, so in Eqs. (4.7) sup and inf are max and min, respectively. Given the option parameters and r, $P(S_0,v_0,T;\mathbf{Z},r)$ as a function of $\mathbf{Z}$ is approximated by the Smolyak interpolation formula and denoted by $\widetilde{P}(\mathbf{Z})$. To set up $\widetilde{P}(\mathbf{Z})$, we used the ADI

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Table 1: Heston's model parameters under uncertainty.

Z	min	max
v_0	0.015	0.025
κ	0.7	2
λ	0.15	0.25
η	0.01	0.02
ρ	-1	-0.5

scheme [23, 28] to solve the original pricing problem with the model parameters given by the interpolation nodes. We could have used another numerical method to calculate the fair values of the American or Barrier options — e.g. see Ref. [44] for a numerical comparison of methods for option pricing problems. The specific steps to quantify the uncertainty of these derivatives in using the proposed MC-based method were given in Section 3.

4.2. Data set

The intervals for the parameters can be their confidence intervals or specified according to an agents' private information. According to the data of the calibrated Heston model to the market data in Ref. [21], κ and ρ vary over relatively big ranges compared with the ranges of the other parameters if different calibration methods are used.

We assumed the intervals corresponding to best estimates of the model parameters listed in Table 1, to price the Bermudan put options and down-and-out put options. The other parameters adopted for the Bermudan put options were

$$r = 0.04\,, \quad T = 0.25 \text{ year}\,, \quad S_0 = 100\,, \quad K = \{80, 100, 110\}\,, \quad \sharp \text{ exercise times: } 60\,, \tag{4.8}$$

which cover the in-the-money, out-of-the-money and at-the-money put options. The parameters of down-and-out put options were

$$r = 0.04\,, \quad T = 0.25 \text{ year}\,, \quad S_0 = 100\,, \quad K = \{80, 100, 110\}, \quad B = 70\,. \tag{4.9}$$

4.3. Efficiency and accuracy

From numerical experience related to the smoothness of derivative value functions [8], we selected Lagrange polynomial interpolation to approximate the out-of-the-money options, but piecewise linear interpolation for the at-the-money and in-the-money options. To select an accurate enough approximation formula, we set up the Smolyak interpolation formula at level $k = 2$ and $k = 3$ for the target options. The number of interpolation nodes we considered is outlined in Table 2, such that we could solve the original pricing problem on the sparse grids in a practical and efficient parallelised way. Of course, the total computational cost of setting up the interpolation formula relies on the computational cost of the pricing problem on each interpolation node.

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Table 2: Number of grid points and the mean value of the error of the approximation formula for Bermudan and down-and-out put options.

		Level2 (♯ grid points: 61)		Level3(♯ grid points: 241)	
Derivatives	Strike	AbsError	RelError	AbsError	RelError
Bermudan put	80	$0.0928\cdot10^{-3}$	$0.5758\cdot10^{-2}$	$0.0047\cdot10^{-3}$	$0.3004\cdot10^{-3}$
	100	$2.7975\cdot10^{-3}$	$0.1194\cdot10^{-2}$	$0.6668\cdot10^{-3}$	$0.2839\cdot10^{-3}$
	110	$1.8079\cdot10^{-3}$	$0.1808\cdot10^{-3}$	$0.9100\cdot10^{-3}$	$0.0910\cdot10^{-3}$
Down-and-out put	80	$0.1322\cdot10^{-3}$	$0.2256\cdot10^{-2}$	$0.0044\cdot10^{-3}$	$0.3365\cdot10^{-3}$
	100	$3.3613\cdot10^{-3}$	$0.1470\cdot10^{-2}$	$0.7861\cdot10^{-3}$	$0.3418\cdot10^{-3}$
	110	$2.0383\cdot10^{-3}$	$0.2247\cdot10^{-3}$	$0.4477\cdot10^{-3}$	$0.0493\cdot10^{-3}$

To check the accuracy of the interpolation formulas, we followed four steps:

1. randomly generate fifty realisations of the model parameters distributed according to independent uniform distributions on the candidate intervals of the model parameters, used to calculate the value of all target options in the following two steps;
2. substitute each combination of the model parameters into the corresponding interpolation formula ($k = 2, 3$) for the target option value, to calculate an ensemble of option values $\widetilde{P}_i$ ($i = 1, \cdots, 50$) in an efficient way;
3. calculate a reference value $\hat{P}_i$ ($i = 1, \cdots, 50$) of a target option with the ADI scheme for each combination of the model parameters; and
4. measure the accuracy by the the absolute error (AbsError, $|\widetilde{P}_i - \hat{P}_i|$) or the relative error (RelError, $|\widetilde{P}_i - \hat{P}_i|/\hat{P}_i$).

Table 2 summarises the mean value of the absolute error and the relative error of the approximation formula for each target option. Compared with the interpolation formulas at level 2 ($k = 2$), the approximation formulas at level 3 ($k = 3$) are accurate enough for the target options, so they were selected to calculate the uncertainty measure as discussed in the next subsection.

4.4. Uncertainty measures

Given the accurate interpolation formulas at level $k = 3$ for the Bermudan put options and the down-and-out put options, it is efficient to get a large ensemble of the target option values by substituting a huge ensemble of the model parameters into the approximation formula. Different ensembles of the model parameters were first generated with the ensemble size of the model parameters starting from 2500, increasing by 50 each time and ending with 7500. Since the computation of the option value with the approximation formula only involves the computation of simple basis functions, one may enlarge the ensemble size to get more accurate results (an advantage of the MC-based method). Each ensemble was substituted into the approximation formula for the target options, and the resulting ensemble of option values used to calculate the uncertainty measure. We first calculated the coherent uncertainty measure by searching for the maximum and minimum values of the

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Table 3: Mean value of uncertainty measures for Bermudan put options and down-and-out put options.

	strike	spread	entropy
Bermudan put	80	0.0460	3.6935
	100	0.7323	4.1843
	110	0.0164	1.8967
Down-and-out put	80	0.0269	3.8468
	100	0.7226	4.1598
	110	0.4559	3.9207

target options in different ensembles. In each ensemble, the interval between the minimum and maximum values was separated into 20 subintervals of the same size. The frequencies of the option value falling into these subintervals work as the probability distribution to calculate the entropy measure (2.4). For each option, we calculated the entropy and spread between the maximum and minimum option values of each ensemble (total 150).

1. Coherent uncertainty measure.

   The left three sub-figures in Fig. 1 show the influence of parameter uncertainty on the Bermudan put options. There appears to be a much bigger effect on the at-the-money Bermudan put option than on the Bermudan put options with $K = 80, 110$, and for the down-and-out put options as shown in the left three sub-figures in Fig. 2. The value bounds on the options appear to not change much when the ensemble size increases, and one of them may be consistent with the global maximum and the other one may accidentally correspond to the global minimum of the option value if only two realisations of the model parameters are used. Adding more realisations cannot change the value bounds, but we do not know the appropriate size of the ensemble beforehand to provide results of high accuracy. It is conservative to use a large ensemble, as the computation mainly involves the simple basis functions after the interpolation formula (3.4) has been set up.
2. Entropy measure.

   The sub-figure for the entropy measure in Fig. 1 shows that among the three Bermudan put options, the entropy value of the at-the-money option is the largest and the option with $K = 80$ is the smallest. In Fig. 2, the sub-figure for the entropy measure shows that the at-the-money down-and-out put option has the largest entropy measure. while the entropy values of the options with $K = 80, 110$ are similar.

Table 3 summarises the average values of the coherent uncertainty measure (2.3) and the entropy measure (2.4) for the target options. These quantitative results are consistent with our qualitative observations in Figs. 1 and 2. For both Bermudan put options and down-and-out put options, the values of the coherent uncertainty measure and of the entropy measure for the at-the-money option are larger than those for the other two options, so parameter uncertainty has more influence on the at-the-money Bermudan (down-and-out) put option.

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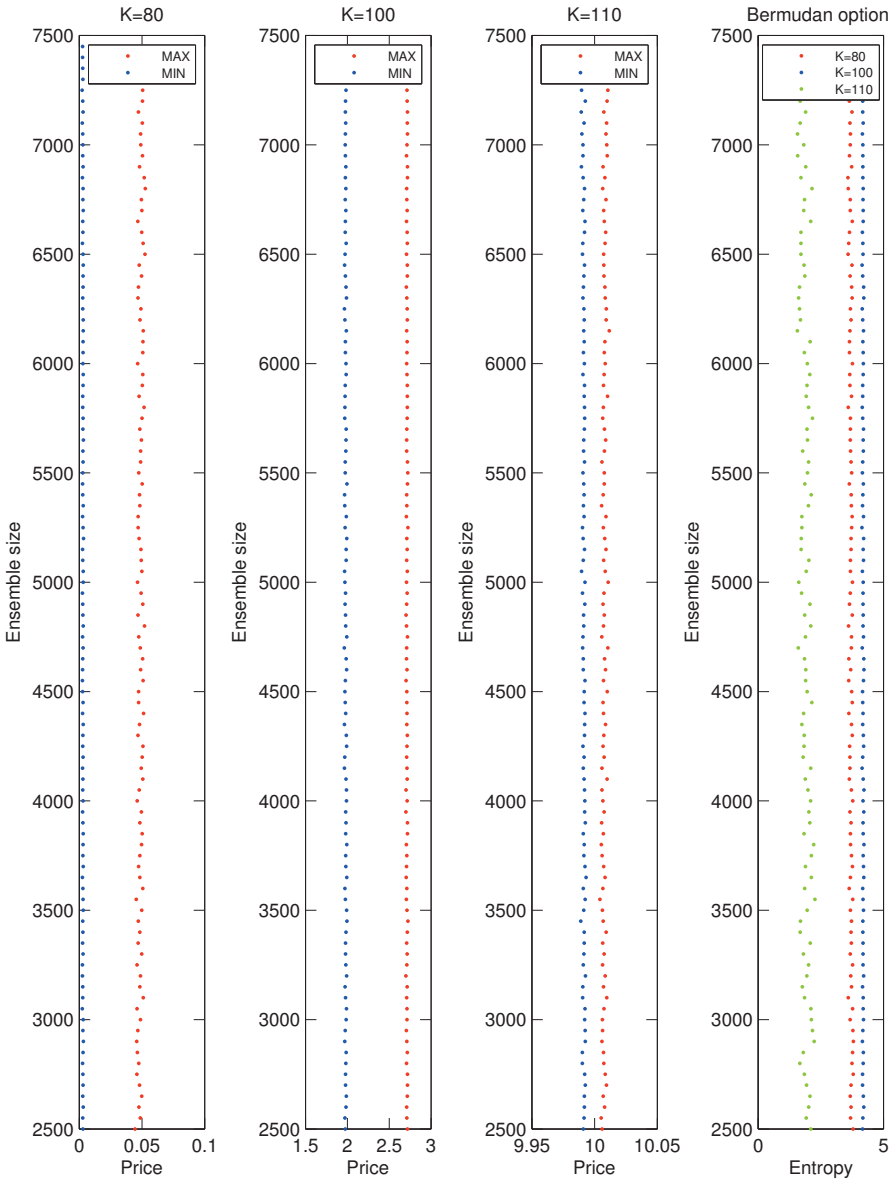


Figure 1: Coherent uncertainty measure and entropy measure of Bermudan put options.

The value of the uncertainty measure for different types of target options may also be compared, such as the Bermudan put option and down-and-out put option. According to the coherent uncertainty measure, parameter uncertainty under the Heston model has more influence on the Bermudan put options with strike $K = 80, 100$ than on the down-and-out put options with the same strike. However, compared with the Bermudan put option with strike $K = 110$, the down-and-out put option with the same strike is much more affected by the parameter uncertainty under the Heston model.

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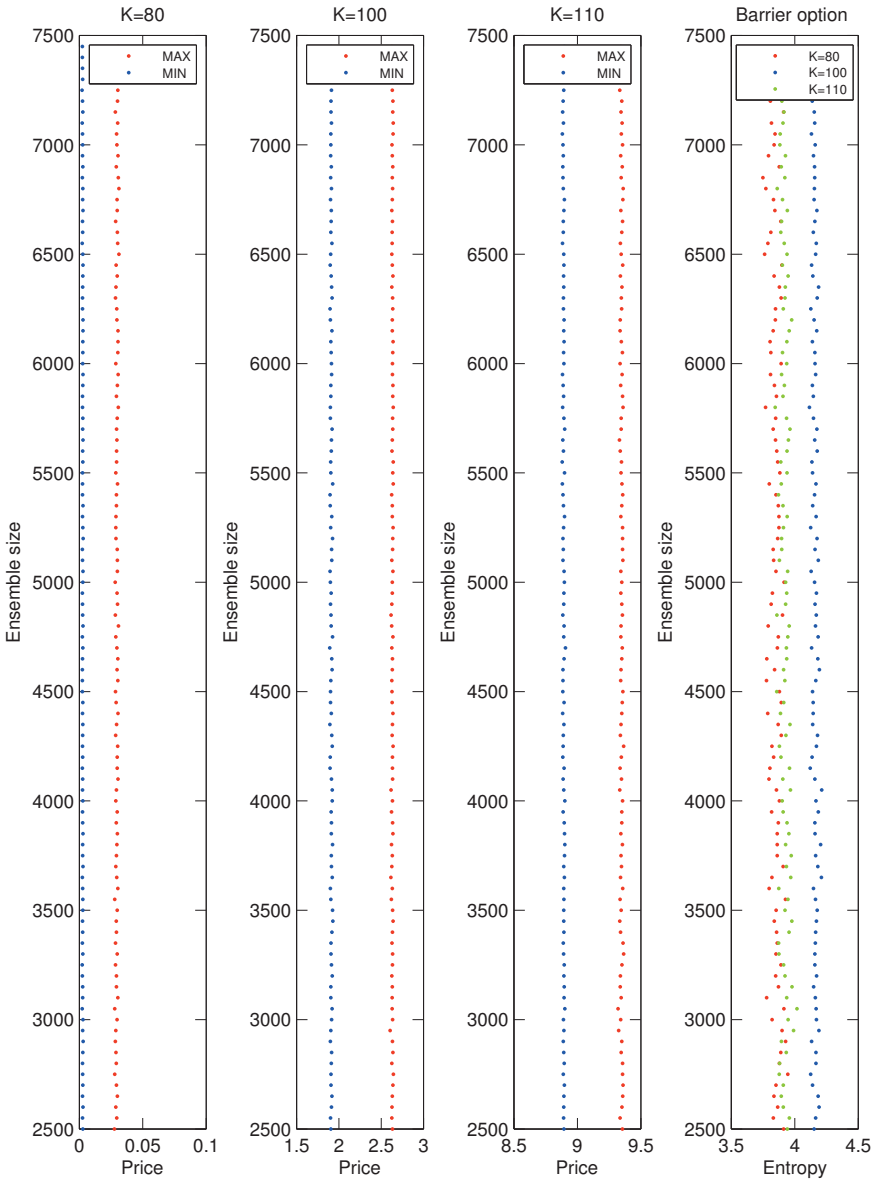


Figure 2: Coherent uncertainty measure and entropy measure of down-and-out put options.

On the other hand, we can use the entropy measure to characterise how much information we get from the ensemble of option values, as a complementary measure of the impact of parameter uncertainty if two options have the same value for the coherent uncertainty measure, In our numerical experiments, we did not find two options with the same value spread — so we assessed the parameter uncertainty for different options according to the coherent uncertainty measure and the extent of the information provided by the ensemble according to entropy. Although parameter uncertainty had more influence on the Bermu-

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dan put option with strike $K = 80$ than on the down-and-out put option with the same strike, according to the coherent uncertainty measure, the ensemble of the first provided more information than the information provided by the ensemble of the second according to the average value of entropy.

Overall, these numerical results provide an agent with the following information. First, if the Heston model is chosen as the candidate model to value Bermudan options and down-and-out put options, the bid-ask spread of the at-the-money option could be higher than those of the options with another strike. Second, compared with the Bermudan put option with strike $K = 110$, the bid-ask spread of the Bermudan put option with strike $K = 80$ could be higher, because more uncertainty is embedded in this option in terms of coherent uncertainty measure and the entropy measure. However, the bid-ask spread of the down-and-out put option with strike $K = 80$ could be smaller than that of the down-and-out put option with strike $K = 110$. Third, the bid-ask spread of Bermudan put options with strike $K = 80, 100$ could be bigger than their counterpart down-and-out put options, whereas the value spread of the Bermudan put option with strike $K = 110$ could be smaller than that the down-and-out put option with the same strike. The final market price is left to a bargaining process in the over-the-counter market.

5. Conclusion

Model and parameter uncertainties are inherent in the setting of pricing and hedging of a derivative instrument in a parametric model. These uncertainties can be measured by the coherent uncertainty measure (2.3) and the entropy measure (2.4). The value of the coherent uncertainty measure can provide financial institutions with quantitative information to assess the need for valuation adjustments, which is of great importance for both regulatory purposes and managing risk with associated trading activities. The entropy measure (2.4) is complementary to the coherent uncertainty measure (2.3), in the sense that the entropy of the ensemble of the option value is associated with lack of information about the state of the option value observed by an agent (outsider).

We have proposed an accurate and efficient numerical method to calculate the coherent uncertainty measure and the entropy measure. Assuming smoothness of the value function of a target derivative with respect to model parameters, one can approximate it by an interpolation formula. The Smolyak algorithm is an efficient way to set up the interpolation formula when there are multiple model parameters. Thus a large ensemble of target derivative values can be calculated by substituting a large ensemble of the model parameters into the interpolation formula, rather than solving the original pricing problem. Subsequently, these ensembles were employed to calculate the uncertainty measures.

In the setting of parameter uncertainty, the ensemble of the model parameters can be generated according to independent uniform distributions because we have no preferred realisations of the model parameters, and the uniform distribution is the maximum entropy distribution supported on a finite interval. When parameter uncertainty is reduced to parameter risk by assigning a probability measure (not necessarily the uniform distribution) to the ensemble of the model parameters according to Bayesian posterior distribution or

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calibration error, the Monte Carlo-based method we have proposed remains an efficient tool to calculate the corresponding ensemble of target derivative values, especially when tens of thousands of parameter realisations are generated in the setting of parameter risk.

The Monte Carlo-based method could also be a reusable simulation infrastructure, in the sense that the interpolation formula can be used in a reproducible manner if all model parameters are taken into consideration, to constitute an invaluable tool in a model validation procedure [25]. Other applications of the Monte Carlo-based method are the subject of ongoing research.

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