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ORACLE EFFICIENT VARIABLE SELECTION IN RANDOM AND FIXED EFFECTS PANEL DATA MODELS

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This paper generalizes the results for the Bridge estimator of Huang, Horowitz, and Ma (2008) to linear random and fixed effects panel data models which are allowed to grow in both dimensions. In particular, we show that the Bridge estimator is oracle efficient. It can correctly distinguish between relevant and irrelevant variables and the asymptotic distribution of the estimators of the coefficients of the relevant variables is the same as if only these had been included in the model, i.e. as if an oracle had revealed the true model prior to estimation.

In the case of more explanatory variables than observations we prove that the Marginal Bridge estimator can asymptotically correctly distinguish between relevant and irrelevant explanatory variables if the error terms are Gaussian. Furthermore, a partial orthogonality condition of the same type as in Huang et al. (2008) is needed to restrict the dependence between relevant and irrelevant variables.

1. INTRODUCTION

When building a model one of the first steps is to decide which variables to include. Sometimes theory can guide the researcher toward a set of potential explanatory variables, but which variables in this set are relevant and which are to be left out? Huang, Horowitz, and Ma (2008) showed that the Bridge estimator is able to discriminate between relevant and irrelevant explanatory variables in a cross-section setting with fixed covariates whose number is allowed to increase with the sample size. In fact, oracle efficient estimation has received quite a bit of attention in the statistics literature in recent years; see (among others) Zou (2006), Candes and Tao (2007), Fan and Lv (2008), and Meinshausen and Yu (2009). However, we are not aware of any similar results for panel data models. For the case of fewer explanatory variables than observations we show that the oracle efficiency of the Bridge estimator carries over to linear panel data models

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with random regressors in the random and fixed effects settings. More precisely, it suffices that either the number of cross-sectional units (N) or the number of observations within each cross-sectional unit (T_N) goes to infinity to establish consistency and correct elimination of irrelevant variables. To obtain the oracle efficient asymptotic distribution (the distribution obtained by only including the relevant covariates) of the estimators of the nonzero coefficients, further restrictions are needed. In the classical setting of fixed T_N and large N these restrictions are satisfied. Further sufficient conditions for oracle efficiency are given. By fixing T_N and the number of covariates we obtain as a corollary that the asymptotic distribution of the estimators of the nonzero coefficients is exactly the classical fixed effects or random effects limit law.

If the set of potential explanatory variables is larger than the number of observations we show that the Marginal Bridge estimator of Huang et al. (2008) can be used to distinguish between relevant and irrelevant variables in random and fixed effects panel data models. A partial orthogonality condition restricting the dependence between the relevant and the irrelevant variables of the same type as in Huang et al. (2008) is imposed. Furthermore, the error terms must be Gaussian—a price paid for letting the covariates be random. The random covariates assumption implies that the number of irrelevant covariates cannot grow as fast as in Huang et al. (2008). However, we shall see that they can still grow much faster than the sample size, and so the practical implications are minimal. The number of relevant variables may still be $o(N^{1/2})$ (T_N is considered fixed for comparability to the cross-sectional results).

Furthermore, the Marginal Bridge estimator is very fast to implement, which also makes it useful as an initial screening device to weed out the most irrelevant variables before initiating the actual modeling stage.

Because cross-section data can be viewed as panel data with only one observation per individual, all our results are also valid for cross-section data and hence generalize the results for these.

The plan of the paper is as follows. Section 2 puts forward the general framework. Section 3 introduces the Bridge estimator and its properties, and Section 4 discusses the Marginal Bridge estimator. Section 5 illustrates the results by simulation, and Section 6 concludes. Section 7 contains proofs of the propositions.

2. SETUP AND ASSUMPTIONS

Consider the following linear panel data model:

$$\tilde{y}_{it} = \tilde{\mathbf{x}}_{it}'\beta_0 + c_i + \tilde{\epsilon}_{it}, \qquad i = 1, \ldots, N, \qquad t = 1, \ldots, T_N. \tag{2.1}$$

The term $\tilde{\mathbf{x}}_{it}$ is a $p_N \times 1$ vector of covariates indicating that the number of covariates is allowed to increase with the sample size. The interpretation of (2.1) is that N individuals are observed in T_N time periods, totaling NT_N observations. The c_i indicate the unobserved heterogeneity, i.e., unobserved time invariant variables such as intelligence of an individual or start-up capital of a firm. The $\tilde{\epsilon}_{it}$ are

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the idiosyncratic error terms. Some of the elements of β_0 may be zero. It is our objective to locate these while still estimating the nonzero coefficients consistently.

Both N and T_N are allowed to tend to infinity. However, all results are valid as long as N tends to infinity. Hence, the traditional large N, fixed T_N setting is covered. Note that T_N is indexed by N. Some of our results put no restrictions on how T_N depends on N.

Equation (2.1) can equivalently be written as

$$\tilde{\mathbf{Y}}_{iN} = \tilde{\mathbf{X}}_{iN}\beta_0 + \mathbf{c}_{iN} + \tilde{\epsilon}_{iN}, \qquad i = 1, \ldots, N, \tag{2.2}$$

where $\tilde{\mathbf{Y}}_{iN} = (\tilde{y}_{i1}, \ldots, \tilde{y}_{iT_N})'$, $\tilde{\mathbf{X}}_{iN} = (\tilde{\mathbf{x}}_{i1}, \ldots, \tilde{\mathbf{x}}_{iT_N})'$, $\tilde{\epsilon}_{iN} = (\tilde{\epsilon}_{i1}, \ldots, \tilde{\epsilon}_{iT_N})'$, $\mathbf{c}_{iN} = c_{iN}\iota_{T_N}$, $\iota'_{T_N} = (1, \ldots, 1)$, $i = 1, \ldots, N$.

2.1. Fixed Effects

In the fixed effects setting one assumes

(FE1) Random sampling: $(\tilde{\mathbf{X}}_{iN}, \mathbf{c}_{iN}, \tilde{\epsilon}_{iN})_{i=1}^{N}$ is independent and identically distributed (i.i.d.)

(FE2) $\mathbb{E}\left(\tilde{x}_{itl}^4\right), \mathbb{E}\left(\tilde{\epsilon}_{it}^4\right) < \infty$, $i = 1, \ldots, N$, $t = 1, \ldots, T_N$, $l = 1, \ldots, p_N$

(FE3) (a) $\mathbb{E}(\tilde{\epsilon}_{iN}|\tilde{\mathbf{X}}_{iN}, \mathbf{c}_{iN}) = 0$ and
(b) $\mathbb{E}(\tilde{\epsilon}_{iN}\tilde{\epsilon}'_{iN}|\tilde{\mathbf{X}}_{iN}, \mathbf{c}_{iN}) = \sigma^2\mathbf{I}_{T_N}$, $i = 1, \ldots, N$,

where $\tilde{x}_{itl}$ is the lth covariate of individual i in period t. For our proofs we may replace (FE3) by $\mathbb{E}(\tilde{\epsilon}_{iN}|\tilde{\mathbf{X}}_{iN}) = 0$ and $\mathbb{E}(\tilde{\epsilon}_{iN}\tilde{\epsilon}'_{iN}|\tilde{\mathbf{X}}_{iN}) = \sigma^2\mathbf{I}_{T_N}$, which is less restrictive, but because (FE3) is standard in the literature we stick to this. Next, we carry out the forward orthogonal deviations transform of Arellano (2003, p. 17). This transformation removes the unobserved heterogeneity while keeping the error terms uncorrelated. In particular, define the $(T_N - 1) \times T_N$ matrix

$$\mathbf{D} = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

and multiply (2.2) through by $\left(\mathbf{D}\mathbf{D}'\right)^{-1/2}\mathbf{D}$ to get

$$\mathbf{Y}_{iN} = \mathbf{X}_{iN}\beta_0 + \epsilon_{iN}, \qquad i = 1, \ldots, N, \tag{2.3}$$

where $\mathbf{Y}_{iN} = \left(\mathbf{D}\mathbf{D}'\right)^{-1/2}\mathbf{D}\tilde{\mathbf{Y}}_{iN}$, $\mathbf{X}_{iN} = \left(\mathbf{D}\mathbf{D}'\right)^{-1/2}\mathbf{D}\tilde{\mathbf{X}}_{iN}$ and $\epsilon_{iN} = \left(\mathbf{D}\mathbf{D}'\right)^{-1/2}\mathbf{D}\tilde{\epsilon}_{iN}$. Clearly, (FE3) implies

(FE3′) (a) $\mathbb{E}(\epsilon_{iN}|\mathbf{X}_{iN}) = 0$ and
(b) $\mathbb{E}(\epsilon_{iN}\epsilon'_{iN}|\mathbf{X}_{iN}) = \sigma^2\mathbf{I}_{T_N-1}$,

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which is what will be used in the proofs. Arellano (2003) gives the specific form of the entries of $\left(\mathbf{DD}'\right)^{-1/2}\mathbf{D}$. The number of time series observations for each individual is reduced from T_N to $T_N - 1$ by the forward orthogonal deviations transform. However, for notational convenience, we will keep using T_N for the number of time series observations in the transformed model. In a cross-section setting this transform does not need to be carried out.

Assumption (FE3) (b) $\mathbb{E}(\tilde{\epsilon}_{iN}\tilde{\epsilon}'_{iN}|\tilde{\mathbf{X}}_{iN}, \mathbf{c}_{iN}) = \sigma^2\mathbf{I}_{T_N}$ restricts the $\tilde{\epsilon}_{it}$ to be uncorrelated. This may be relaxed to $\mathbb{E}(\tilde{\epsilon}_{iN}\tilde{\epsilon}'_{iN}|\tilde{\mathbf{X}}_{iN}, \mathbf{c}_{iN}) = \mathbf{S}$ where $\mathbf{S}$ is a known covariance matrix. In this case the forward orthogonal deviations transform is replaced by $\left(\mathbf{DD}'\right)^{-1/2}\mathbf{DS}^{-1/2}$. So heteroskedasticity can be handled as long as the structure of it is known—the applied transformation simply changes accordingly. If the heteroskedasticity is ignored the situation is more subtle, and we will discuss this in more detail in Section 3.

2.2. Random Effects

In the random effects setting (FE1)–(FE3) are maintained and

(RE4) (a) $\mathbb{E}\left(\mathbf{c}_{iN}|\tilde{\mathbf{X}}_{iN}\right) = 0$,
(b) $\mathbb{E}\left(\mathbf{c}_{iN}\mathbf{c}'_{iN}|\tilde{\mathbf{X}}_{iN}\right) = \sigma_c^2\iota_{T_N}\iota'_{T_N}$, and
(c) σ and σ_c are known and finite[1]

are added. Part (a) of this extra assumption restricts the dependence between $\mathbf{X}_{iN}$ and $\mathbf{c}_{iN}$ sufficiently to allow merging the latter with the error term while still being able to prove the desired results. Part (b) specifies the conditional covariance structure of $\mathbf{c}_i$. RE4 (c) is needed to enable us to carry out the generalized least squares (GLS) transform that follows. The gain from these stronger assumptions is that they (as opposed to fixed effects) allow for the inclusion of a covariate that is constant over time and only varies over individuals. Defining $\mathbf{v}_{iN} = \mathbf{c}_{iN} + \tilde{\epsilon}_{iN}$, (FE3) and (RE4) imply $\mathbb{E}\left(\mathbf{v}_{iN}|\tilde{\mathbf{X}}_{iN}\right) = 0$ and

$$\mathbb{E}(\mathbf{v}_{iN}\mathbf{v}'_{iN}|\tilde{\mathbf{X}}_{iN}) = \mathbb{E}([\mathbf{c}_{iN} + \tilde{\epsilon}_{iN}][\mathbf{c}_{iN} + \tilde{\epsilon}_{iN}]'|\tilde{\mathbf{X}}_{iN}) = \begin{pmatrix} \sigma_c^2 + \sigma^2 & \sigma_c^2 & \cdots & \sigma_c^2 \\ \sigma_c^2 & \sigma_c^2 + \sigma^2 & \cdots & \vdots \\ \vdots & & \ddots & \sigma_c^2 \\ \sigma_c^2 & & & \sigma_c^2 + \sigma^2 \end{pmatrix} = \Omega.$$

The presence of the unobserved heterogeneity renders the error terms correlated. Because the structure of the correlation is known, the correlation is easily removed by premultiplying (2.2) by[2] $\sigma\Omega^{-1/2}$ (GLS transform). This yields

$$\mathbf{Y}_{iN} = \mathbf{X}_{iN}\beta_0 + \epsilon_{iN}, \qquad i = 1, \ldots, N, \tag{2.4}$$

where $\mathbf{Y}_{iN} = \sigma\Omega^{-1/2}\tilde{\mathbf{Y}}_{iN}$, $\mathbf{X}_{iN} = \sigma\Omega^{-1/2}\tilde{\mathbf{X}}_{iN}$, and $\epsilon_{iN} = \sigma\Omega^{-1/2}\mathbf{v}_{iN}$. Hence,

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(RE3′) (a) $\mathbb{E}(\epsilon_{iN}|\mathbf{X}_{iN}) = 0$ and
(b) $\mathbb{E}(\epsilon_{iN}\epsilon_{iN}'|\mathbf{X}_{iN}) = \sigma^2\mathbf{I}_{T_N}$,

which is what will be used in the proofs. In a cross-section setting the random effects transform does not need to be carried out. As was the case in the fixed effects setting, any known heteroskedasticity structure can be handled in the random effects setting also, as long as Ω is known.

3. THE BRIDGE ESTIMATOR

The Bridge estimator estimates β_0 by minimizing

$$L_N(\beta) = \sum_{i=1}^{N}\sum_{t=1}^{T_N}\left(y_{it} - \mathbf{x}_{it}'\beta\right)^2 + \lambda_N \sum_{k=1}^{p_N}|\beta_k|^\gamma \tag{3.1}$$

$$= \sum_{j=1}^{NT_N}\left(y_j - \mathbf{x}_j'\beta\right)^2 + \lambda_N \sum_{k=1}^{p_N}|\beta_k|^\gamma, \qquad \gamma > 0, \tag{3.2}$$

where summation from 1 to NT_N indicates summation over all time periods for each individual. (So the first T_N terms in the sum correspond to all T_N observations on individual 1, the next T_N terms to all observations on individual 2, and so on. This convention is adopted in what follows.) The Bridge estimator, denoted $\hat{\beta}_N$, may hence be seen as a sort of penalized/regularized least squares. The objective function consists of two parts; the first part being the least squares objective function and the second part penalizing parameters different from 0. The larger λ_N, the larger the penalty. For $\gamma = 1$ the minimizer of (3.2) could be called the LASSO panel estimator (Tibshirani, 1996). For $\gamma = 2$ it could be called the ridge regression estimator (Tikhonov regularization) for panel data models. In a cross-sectional setting the ridge regression is frequently used to deal with multicollinearity. The Tikhonov regularization is more generally used to solve ill-conditioned (singular) overdetermined systems of linear equations.

Let β_0 denote the true value of β where the dependence on N is suppressed as in Huang et al. (2008). Partition β_0 as $\beta_0 = (\beta_{10}', \beta_{20}')'$ where $\beta_{10} \neq 0$ is $k_N \times 1$ and $\beta_{20} = 0$ is $m_N \times 1$. Hence, the β_{10} are the coefficients corresponding to the relevant variables denoted $\mathbf{w}_{it}$. The β_{20} are the coefficients of the irrelevant variables denoted $\mathbf{z}_{it}$. So $\mathbf{x}_{it}$ is partitioned as $\mathbf{x}_{it} = (\mathbf{w}_{it}', \mathbf{z}_{it}')'$. Accordingly, we define $\mathbf{X}_N = \left(\mathbf{x}_{11}, \ldots, \mathbf{x}_{NT_N}\right)'$, $\mathbf{W}_N = \left(\mathbf{w}_{11}, \ldots, \mathbf{w}_{NT_N}\right)'$, and $\mathbf{Z}_N = \left(\mathbf{z}_{11}, \ldots, \mathbf{z}_{NT_N}\right)'$. The terms $\Sigma_N = (NT_N)^{-1}\mathbf{X}_N'\mathbf{X}_N$ and $\Sigma_{1N} = (NT_N)^{-1}\mathbf{W}_N'\mathbf{W}_N$ are the scaled Gram matrices of $\mathbf{X}_N$ and $\mathbf{W}_N$, respectively. Let ρ_{1N} and ρ_{2N} be the smallest and the largest eigenvalue of Σ_N. Similarly, define τ_{1N} and τ_{2N} as the smallest and the largest eigenvalue of Σ_{1N}. Set $\mathbf{W}_{iN} = \left(\mathbf{w}_{i1}, \ldots, \mathbf{w}_{iT_N}\right)'$ and for $\mathbf{x} \in \mathbf{R}^p$ $\|\mathbf{x}\| = \sqrt{\sum_{k=1}^{p} x_k^2}$ denotes the euclidean norm on $\mathbf{R}^p$ stemming from the dot product. Finally, $\mathbf{x}_k = (x_{1,k}, \ldots, x_{NT_N,k})'$ is the vector containing all observations of the kth explanatory variable.

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Next, we state and discuss the assumptions needed to establish consistency and oracle efficiency of Bridge estimators in random and fixed effects panel data models. Notice how N and T_N enter symmetrically, indicating that what matters is their product, i.e., the total number of observations, and not whether it is N or T_N that gets large (however, some theorems require further assumptions restricting the rate at which T_N increases relative to N).

(A1) $1/(NT_N p_N)\sum_{i=1}^{N}\sum_{t=1}^{T_N}\sum_{k=1}^{p_N} x_{itk}^2$ is bounded in L^1; i.e.,

$$\sup_{1\leq N<\infty}\mathbb{E}\left(\frac{1}{NT_N p_N}\sum_{i=1}^{N}\sum_{t=1}^{T_N}\sum_{k=1}^{p_N}x_{itk}^2\right)=\sup_{1\leq N<\infty}\frac{1}{T_N p_N}\sum_{t=1}^{T_N}\sum_{k=1}^{p_N}\mathbb{E}\left(x_{1tk}^2\right)=K<\infty.$$

(A2) There exist constants $0<\tau_1<\tau_2<\infty$ such that $\tau_1\leq\tau_{1N}\leq\tau_{2N}\leq\tau_2$ almost surely.

(A3) $\lambda_N(k_N/(NT_N))^{1/2}\to 0$.

(A4) $\lambda_N\rho_{1N}^{2-\gamma}(NT_N)^{-\gamma/2}p_N^{\gamma/2-1}\to\infty$ almost surely.

(A5) There exist constants $0<b_0<b_1<\infty$ such that $b_0\leq\min\left\{|\beta_{10j}|\,|1\leq j\leq k_N\right\}\leq\max\left\{|\beta_{10j}|\,|1\leq j\leq k_N\right\}\leq b_1$.

(A6) $(p_N+\lambda_N k_N)/(NT_N\rho_{1N})\to 0$ almost surely.

(A7) $\left(\rho_{1N}\rho_{2N}^{1/2}\right)\Big/\left(p_N^{1/2}\right)\in O_p(1)$.

Assumption (A1) may be dropped altogether if the covariates are normalized as $1/(NT_N)\sum_{j=1}^{NT_N}x_{jk}^2=1/(NT_N)\sum_{i=1}^{N}\sum_{t=1}^{T_N}x_{itk}^2=1$ for all $1\leq k\leq p_N$. Alternatively, (A1) is satisfied if $\{x_{itk}\}$ is bounded in L^2—this is in turn satisfied if, e.g., the covariates are uniformly bounded or if T_N and p_N are fixed constants. If the covariates are identically distributed over time, then the assumption reduces to boundedness of the Cesàro sum $\frac{1}{p_N}\sum_{k=1}^{p_N}\mathbb{E}\left(x_{11k}^2\right)$. Finiteness of p_N or convergence of $\left\{\mathbb{E}\left(x_{11k}^2\right)\right\}_{k=1}^{\infty}$ is sufficient for this. Finally, it may be noted that convergence of $1/(T_N p_N)\sum_{t=1}^{T_N}\sum_{k=1}^{p_N}\mathbb{E}\left(x_{1tk}^2\right)$ is also sufficient for the desired boundedness in L^1.

Huang et al. (2008) mention that Assumption (A2) is likely to be satisfied in sparse systems, where k_N is relatively small.

Regarding condition (A3) one notices that if the number of relevant covariates k_N stays fixed $\lambda_N/(NT_N)^{1/2}\to 0$. Hence, $\lambda_N\in o((NT_N)^{1/2})$.

Assumption (A4): Assume $0<a_1<\rho_{1N}\leq\rho_{2N}<a_2<\infty$ for some constants a_1 and a_2 and that the number of covariates stays constant. Then it must be the case that $\lambda_N(NT_N)^{-\gamma/2}\to\infty$. This excludes $\gamma\geq 1$ by (A3). Hence, $0<\gamma<1$, and $\lambda_N\in o((NT_N)^{1/2})\cap\omega((NT_N)^{\gamma/2})$ where $\omega(g(N))$ is the set of functions that diverge to infinity when divided by $g(N)$ as $N\to\infty$.

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Assumption (A5) requires that the nonzero coefficients are uniformly bounded away from 0 and infinity. This is trivially satisfied if the number of covariates is finite. Also note that all results remain valid (with slight modifications) if b_1 is replaced by a sequence b_{1N} that is allowed to tend to infinity.

By Assumption (A3), Assumption (A6) is satisfied if $0 < a_1 < \rho_{1N} < \rho_{2N} < a_2 < \infty$ for some constants a_1 and a_2 and the number of covariates is finite. Because the Gramian Σ_N is positive semidefinite (A6) also implies that $\rho_{1N} > 0$ for the condition to be well defined. This excludes $p_N > NT_N$ because the rank of Σ_N can be no larger than NT_N.

Assumption (A7) is satisfied if $0 < a_1 < \rho_{1N} < \rho_{2N} < a_2 < \infty$ for some constants a_1 and a_2.

Assumptions (A2)–(A6) are identical[3] to assumptions made in Huang et al. (2008). (A1) and (A7) are not made by Huang et al. (2008), but both these assumptions are redundant if the covariates are normalized as $1/(NT_N)\sum_{j=1}^{NT_N} x_{jk}^2 = 1$ for all $1 \leq k \leq p_N$ as done by these authors.

Our first theorem states that the Bridge estimator is consistent in the random and the fixed effects setting. Throughout we will assume that (FE1)–(FE3) (fixed effects setting) or (FE1)–(FE3) and (RE4) (random effects setting) are satisfied.

THEOREM 1. *Let $\hat{\beta}_N$ denote the minimizer of (3.2). Suppose that $\gamma > 0$ and that conditions (A1), (A3), (A5), and (A6) hold. Then $||\hat{\beta}_N - \beta_N|| \in O_p(\min(h_N, h'_N))$ where $h_N = \rho_{1N}^{-1}(p_N/(NT_N))^{1/2}$ and $h'_N = [(p_N + \lambda_N k_N)/(NT_N\rho_{1N})]^{1/2}$.*

Theorem 1 shows the consistency of the Bridge estimator by Assumption (A6). By considering h_N it follows that if there exists a constant a_1 such that $0 < a_1 < \rho_{1N}$ and p_N is constant the Bridge estimator converges at the same rate as the least squares estimator. The faster the arrival rate of new explanatory variables (p_N increases) the slower the rate of convergence of the Bridge estimator because h_N and h'_N are increasing in p_N. If ρ_{1N} tends to 0 (approaching a singular design) the convergence rate is also slowed down. It is also seen that N and T_N enter symmetrically. This is not immediate on the outset because only independence of $\{\mathbf{X}_{iN}\}_{i=1}^{\infty}$ has been assumed, whereas the T_N rows of each $\mathbf{X}_{iN}$ may have any dependence structure between them. What provides the result is that $\mathbb{E}(\epsilon_{iN}\epsilon'_{iN}|\mathbf{X}_{iN}) = \sigma^2\mathbf{I}_{T_N}$, i.e., the conditional uncorrelatedness of the error terms of each individual. This underscores the importance of orthogonalizing (in L^2) the error terms prior to applying the Bridge estimator. In the presence of unknown arbitrary heteroskedasticity, $\mathbb{E}(\epsilon_{iN}\epsilon'_{iN}|\mathbf{X}_{iN}) = \mathbf{S}$ (which cannot be orthogonalized), we were only able to prove that $||\hat{\beta}_N - \beta_N|| \in O_p(h'_N + (T_N - 1)/\rho_{1N})$. So even in the situation of fixed T_N, consistency requires $\rho_{1N} \to \infty$, which is impossible.

The next theorem reveals that the Bridge estimator performs variable selection and gives the limiting law of the estimator of the nonzero coefficients.

Let $U_{1N} = \alpha' \left(\mathbb{E}\left[(1/T_N)\mathbf{W}'_{1N}\mathbf{W}_{1N}\right]\right)^{-1} T_N^{-1/2}\mathbf{W}'_{1N}\epsilon_{1N}$.

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THEOREM 2. *Assume* $0 < \gamma < 1$. *Then, under (A1)–(A7),*

(i) $\hat{\beta}_{2N} = 0$ *with probability converging to 1.*

(ii) Let k_N *be a fixed number* k, α *be a* $k \times 1$ *vector, and* $s_N = \sqrt{\sigma^2 \alpha' \Sigma_{1N}^{-1} \alpha}$. *If* $\{U_{1N}^2\}_{N=1}^{\infty}$ *is uniformly integrable,*

$$\frac{\max_{1 \leq t \leq T_N} \mathrm{Var}\,(w_{1tl} w_{1tm})}{N} \to 0 \quad \textit{for all } 1 \leq l, m \leq k,$$

and

$$\lim_{N \to \infty} \frac{1}{NT_N} \sum_{j=1}^{NT_N} \mathbb{E}\left(\mathbf{w}_j \mathbf{w}_j'\right) = \lim_{N \to \infty} \mathbb{E}\left(\frac{1}{T_N} \mathbf{W}_{1N}' \mathbf{W}_{1N}\right)$$

exists, then

$$(NT_N)^{1/2} s_N^{-1} \alpha' \left(\hat{\beta}_{1N} - \beta_{10}\right) \xrightarrow{d} N(0, 1).$$

Part (i) states that not only does $\hat{\beta}_{2N} \to 0$ in probability (Theorem 1), the Bridge estimator actually sets $\hat{\beta}_{2N} = 0$ with probability converging to 1. The latter of course implies the former whereas the converse is not true. The fact that $\hat{\beta}_{2N}$ is set exactly equal to 0 with probability converging to 1 means that the Bridge estimator performs variable selection.

Part (ii) of the theorem states that the asymptotic distribution of the estimators of the nonzero coefficients is the same as if the true model had been known in advance—i.e., as if an oracle had revealed which variables to include and which to exclude. This is a very useful result in practice. One simply includes the whole set of potential explanatory variables. The irrelevant ones will be kicked out ($\hat{\beta}_{2N} = 0$ with probability converging to 1), and the relevant ones are estimated with the same asymptotic efficiency as if the irrelevant ones had been left out from the outset. However, notice that the price paid for letting the covariates be random is that k_N must be fixed. Alternatively, one may continue to let k_N increase in N while conditioning on the covariates and establish the limiting law along the lines of Huang et al. (2008).

Next we discuss conditions under which the requirements of part (ii) of Theorem 2 hold. The following theorem gives sufficient conditions under which $\{U_{1N}^2\}_{N=1}^{\infty}$ is uniformly integrable.

THEOREM 3. $\{U_{1N}^2\}_{N=1}^{\infty}$ *is uniformly integrable if either of the following conditions is satisfied:*

(i) $T_N = T$ *for a fixed* T.

(ii) The rows in $\mathbf{W}_{1N}$ *are identically distributed, and* $\mathbf{W}_{iN} \perp\!\!\!\perp \epsilon_{iN}$, $i = 1, \ldots, N$.

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The assumption $\max_{1\leq t\leq T_N} \mathrm{Var}\left(w_{1tl}w_{1tm}\right)/N \to 0$ for all $1 \leq l,m \leq k$ in part (ii) of Theorem 2 is not restrictive. It is clearly satisfied if T_N is fixed. It is also satisfied if $\max_{1\leq t\leq T_N} \mathbb{E}\left(\left[w_{1tl}w_{1tm}\right]^2\right) \leq M < \infty$ for all T_N and $1 \leq l,m \leq k$ (second moments uniformly bounded in t), which in turn is satisfied if the variables themselves are uniformly bounded in t. The assumption is also satisfied if $\mathbf{w}_{1t}$ are identically distributed across t. If the variances are linearly increasing, i.e., $\mathrm{Var}\left(w_{1tl}w_{1tm}\right) = a_{lm}t$ for some $a_{lm} > 0$, it suffices that $T_N/N \to 0$.[4]

If T_N is fixed, $\lim_{N\to\infty} 1/(NT_N)\sum_{j=1}^{NT_N} \mathbb{E}\left(\mathbf{w}_j\mathbf{w}_j'\right) = \lim_{N\to\infty} \mathbb{E}\left((1/T_N)\mathbf{W}_{1N}'\mathbf{W}_{1N}\right)$ exists. The same is true if $\mathbf{w}_{1t}$ is identically distributed across t.

Part (ii) of Theorem 2 is made more precise in the following corollary, which considers the classical situation of fixed T_N. Let $\tilde{\mathbf{W}}_1$ denote the matrix containing the k untransformed relevant variables of individual 1 in all time periods and $\ddot{\tilde{\mathbf{W}}}_1$ its column demeaned version.

COROLLARY 1. *Under the assumptions of Theorem 2, T_N fixed*

(i) and (FE1)–(FE3) and the forward orthogonal deviations transform

$$N^{1/2}\left(\hat{\beta}_{1N} - \beta_{10}\right) \xrightarrow{d} N\left(0, \sigma^2\left[\mathbb{E}\left(\ddot{\tilde{\mathbf{W}}}_1'\ddot{\tilde{\mathbf{W}}}_1\right)\right]^{-1}\right) \tag{3.3}$$

(ii) and (FE1)–(FE3), (RE4), and the GLS transform

$$N^{1/2}\left(\hat{\beta}_{1N} - \beta_{10}\right) \xrightarrow{d} N\left(0, \sigma^2\left[\mathbb{E}\left(\tilde{\mathbf{W}}_1'\Omega^{-1}\tilde{\mathbf{W}}_1\right)\right]^{-1}\right). \tag{3.4}$$

Note that the asymptotic distribution in (3.3) is the same as for a fixed effects estimator with *known* sparsity pattern of β_0. This underscores the oracle property of the panel Bridge estimator. Similarly, (3.4) is the asymptotic distribution of the random effects estimator with *known* sparsity pattern of β_0.

4. THE MARGINAL BRIDGE ESTIMATOR

Because the Bridge estimator is not applicable when $p_N > NT_N$ (though it does allow $p_N \to \infty$) a different approach is needed for this situation. As in Huang et al. (2008) we will employ the Marginal Bridge estimator, which estimates β_0 by minimizing

$$U_N(\beta) = \sum_{k=1}^{p_N}\sum_{j=1}^{NT_N}\left(y_j - x_{jk}\beta_k\right)^2 + \lambda_N\sum_{k=1}^{p_N}|\beta_k|^\gamma \tag{4.1}$$

$$= \sum_{k=1}^{p_N}\left(\sum_{j=1}^{NT_N}\left(y_j - x_{jk}\beta_k\right)^2 + \lambda_N|\beta_k|^\gamma\right). \tag{4.2}$$

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From (4.2) it is clear that the objective function is nothing else than the sum of the marginal objective functions for each variable—hence the name Marginal Bridge estimator. Let $\tilde{\beta}_N$ denote the minimizer of (4.2). We show that the Marginal Bridge estimator is able to correctly distinguish between relevant and irrelevant variables even when there are more explanatory variables than observations ($p_N > NT_N$). Maintain (FE1), assume $\tilde{\epsilon}_{iN}$ is normally distributed for all $1 \leq i \leq N$, and replace (FE2) and (FE3) by

(FE2MB) $\left(\tilde{\mathbf{X}}_{iN}, \mathbf{c}_{iN}\right) \perp\!\!\!\perp \tilde{\epsilon}_{iN}$, with $\tilde{\epsilon}_{iN} \sim N(0, \sigma^2 \mathbf{I}_{T_N})$.[5]

(FE2MB) clearly implies (FE3), whereas the reverse need not be the case (for an example, see Stoyanov, 1997). However, this strengthening is not likely to be of any practical importance because it is hard to imagine *practical* examples where (FE3) is satisfied while (FE2MB) is not. After carrying out either the fixed effects or the random effects transform (FE2MB) implies that $\mathbf{X}_{iN} \perp\!\!\!\perp \epsilon_{iN}$, $\epsilon_{i1}, \ldots, \epsilon_{iT_N}$ is i.i.d. Gaussian for all $1 \leq i \leq N$, $\mathbb{E}(\epsilon_{it}) = 0$, and $\mathbb{E}\left(\epsilon_{it}^2\right) = \sigma^2$. The Gaussianity assumption on the error terms is a price we must pay for working with L^q norms instead of the exponential Orlicz norms in Huang et al. (2008) in the proofs.[6] Working with exponential Orlicz norms did not turn out to be fruitful (even after conditioning arguments) because of the presence of random covariates that rendered some otherwise useful maximal inequalities inapplicable.

Let $K_N = (1, \ldots, k_N)$ denote the active set, i.e., the set of indexes of the relevant variables, and $J_N = (k_N + 1, \ldots, p_N)$ the inactive set, i.e., the set of indexes of the irrelevant variables. Standardize the covariates such that $1/(NT_N)\sum_{j=1}^{NT_N} x_{jk}^2 = 1$ for all $k = 1, \ldots, p_N$. This implies that the covariates have moments of any order because $|x_{jk}| \leq \sqrt{\sum_{j=1}^{NT_N} x_{jk}^2} = (NT_N)^{1/2}$. Because the Gaussian error terms also have moments of any order, this is the real reason enabling us to refrain from any moment assumptions like (FE2). Finally, define $\xi_{Nk} = 1/(NT_N)\sum_{j=1}^{NT_N} \mathbf{w}_j' \beta_{10} x_{jk}$. Make the following assumptions.

(B1) There exists a constant $\xi_0 > 0$ such that $\min_{k \in K_N} |\xi_{Nk}| > \xi_0$ with probability approaching 1.

(B2) $\lambda_N/(NT_N) \to 0$.

(B3) $\frac{k_N}{\left(\lambda_N (NT_N)^{-\gamma/2}\right)^{1/(2-\gamma)}} \to 0$.

(B4) $\frac{m_N}{\left(\lambda_N (NT_N)^{-\gamma/2}\right)^{q/(2-\gamma)}} \to 0$ for some $q \geq 1$.

(B5) For all $\delta > 0$ there exist a $c_0 > 0$ and a $N_0 \in \mathbf{N}$ such that

$$P\left(\frac{\sum_{j=1}^{NT_N} x_{jk} x_{jl}}{(NT_N)^{1/2}} \leq c_0,\ k \in K_N,\ l \in J_N\right) \geq 1 - \delta \text{ for } N \geq N_0.$$

(B6) There exists a constant $0 < b_1 < \infty$ such that $\max_{k \in K_N} |\beta_{10k}| \leq b_1$.

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Assumption (B1) is a technical assumption needed to prove that no variables from the active set will be discarded by the Marginal Bridge. In a fixed regressor setting it is similar to assuming that the covariance between the left-hand side variable and the relevant covariates is bounded away from 0.

Assumption (B2) requires that $\lambda_N \in o(NT_N)$.

Assumption (B3) combined with assumption (B2) implies that $k_N \in o((NT_N)^{1/2})$. In the classical case of fixed T_N this amounts to $k_N \in o(N^{1/2})$. This is in line with the results of Huang et al. (2008).

For $0 < \gamma < 2$, (B3) also implies that $\lambda_N (NT_N)^{-\gamma/2} \to \infty$. Together with (B2) this yields that $\lambda_N \in o(NT_N) \cap \omega\left((NT_N)^{\gamma/2}\right)$.

Using (B2) in (B4) implies $m_N \in o\left((NT_N)^{q/2}\right)$. Because q is arbitrary this indicates that the number of irrelevant variables must be asymptotically dominated by some polynomial in the sample size. Note that the number of irrelevant variables cannot tend to infinity as fast as in Huang et al. (2008) where $m_N \in \exp(o(N))$. As indicated previously the reason is that the exponential Orlicz norms did not carry over straightforwardly to the random covariate setting and so we had to settle with maximal inequalities based on L^q norms, which don't give us as sharp results as exponential Orlicz norms. However, $m_N \in o(N^{q/2})$ (T_N fixed) is not very restrictive in practice because it still allows the number of irrelevant variables to increase at a much higher rate than the sample size as long as this rate is polynomial.

Assumption (B5) is a partial orthogonality assumption limiting the dependence between the variables in the active and the inactive sets. It rules out correlations of -1 or 1.[7] However, it is not too restrictive and as will be seen from the Monte Carlo simulations in Section 5 the Marginal Bridge also works quite well even when the covariates in the active and inactive sets are highly correlated.

Assumption (B6) requires a uniform bound on the size of the coefficients belonging to the relevant variables. This assumption may be relaxed in the same way as Assumption (A5) for the Bridge estimator at the price of a lower growth rate of the number of relevant variables.

Assumptions (B1)–(B6) are similar to the assumptions made in Huang et al. (2008). However, we must assume Gaussianity of the error terms instead of sub-Gaussianity as in Huang et al. (2008).[8] As indicated before, this is a price we pay for letting the covariates be random. The properties of the Marginal Bridge are given in the following theorem.

THEOREM 4. *Under Assumptions (B1)–(B6) and if* $0 < \gamma < 1$,

$$P\left(\tilde{\beta}_{2N} = 0\right) \to 1 \quad \textit{and} \quad P\left(\tilde{\beta}_{1Nk} = 0,\ k \in K_N\right) \to 0. \tag{4.3}$$

Hence, the Marginal Bridge estimator is able to screen out the irrelevant variables while retaining the relevant ones.

The nonzero coefficients are not estimated consistently. To obtain consistent estimates the same two-step procedure as in Huang et al. (2008) can be applied.

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In the first step the Marginal Bridge estimator is applied to distinguish between the relevant and irrelevant variables. In the second step, where only the relevant variables are left, these may be estimated by any consistent estimator (e.g., least squares or the Bridge estimator).

5. SIMULATIONS

In this section the finite-sample properties of the proposed estimators will be investigated. The Bridge estimator will be implemented by means of the minorize-maximize algorithm of Hunter and Li (2005), which in the present case reduces to a series of ridge regressions.

Implementing the Marginal Bridge is very fast. Because $\sum_{j=1}^{NT_N} x_{jk}^2 = NT_N$ for $k = 1, \ldots, p_N$ it follows from Lemma A in Knight and Fu (2000) that $\beta_k = 0$ iff

$$\frac{\lambda_N}{NT_N} > c_\gamma \left| \frac{\sum_{j=1}^{NT_N} y_j x_{jk}}{NT_N} \right|^{2-\gamma}, \tag{5.1}$$

where $c_\gamma = (2/(2-\gamma))\,(2(1-\gamma)/(2-\gamma))^{1-\gamma}$. Hence, variable selection is extremely fast[9] even in vast-dimensional models because the inclusion of a variable is solely based on the criterion (5.1), which roughly amounts to checking whether the correlation between the left-hand-side variable and the covariate is sufficiently high to deem the latter relevant. Notice how only marginal information is used to decide whether a variable is to be included or not. Having decided on the sparsity pattern the second-step estimates of β_{10} are found by means of least squares.[10]

The following issues will be investigated

1. How often the Bridge and the Marginal Bridge estimators select the correct sparsity pattern, i.e., how good they are at distinguishing the active from the inactive set. This is highly relevant in applied work investigating which variables help explain the left-hand-side variable.
2. The median number of variables included, i.e., how well the Bridge and the Marginal Bridge reduce the dimension of the problem. This median is ideally equal to the cardinality of the active set.
3. The explanatory power of the Bridge and the Marginal Bridge. To investigate this, the estimated parameters are used to fit values on a validation data set drawn from the same distribution as the training set.
4. In connection to the explanatory power, how often the procedures retain all relevant explanatory variables. As can be expected, retention of all relevant explanatory variables is important for achieving a good fit. It is also highly desirable if the procedures are to be used as initial screening devices in vast-dimensional data sets.
5. The precision of the parameter estimates using the mean square error of $\hat{\beta}$.
6. The asymptotic distribution of the estimator of the nonzero β_0's. This is done by comparing the standard deviation of $\hat{\beta}_1$ to the corresponding quantities for

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the least squares estimator with only the active set included. The latter (in practice infeasible) estimator will be called the ordinary least squares (OLS) Oracle henceforth.

7. In the $p_N < NT_N$ setting the coverage probabilities of 95% confidence intervals are reported for the Bridge estimator and the OLS Oracle to assess Theorem 2.[11]

The Bridge and the Marginal Bridge estimators will be compared to the LASSO of Tibshirani (1996) estimated by pathwise coordinate descent, the Schwarz information criterion (Bayesian information criterion [BIC]), the OLS Oracle, and OLS on the system including all covariates. Only the Marginal Bridge, the LASSO, and the OLS Oracle are applied when $p_N > NT_N$. To limit the computational burden, BIC is only applied for the designs with 15 or fewer covariates, which implies a maximum of $2^{15} - 1 = 32{,}767$ regressions per Monte Carlo replication. All experiments are carried out with 1,000 replications.

The data are generated from equation (2.2). In all experiments $T_N = 10$. Initial experiments indicated that $\gamma = 0.5$ works quite well for both the Bridge and the Marginal Bridge estimators, and this value will be used throughout. Larger values of γ resulted in larger models. In light of the fact that the LASSO turns out to select larger models than the Bridge estimator, it is no surprise that as γ approaches 1 the median number of variables included by the Bridge estimator increases too. In all experiments, $\tilde{\epsilon}_{it}$ and c_i are $N(0, 1)$.

The regularization parameter λ_N is usually chosen by 10-fold cross validation. Here we try this and also the significantly faster BIC to determine λ_N for the Bridge, the Marginal Bridge, and the LASSO.

5.1. The Experiments

(A) $N = 10$, $\beta_0 = (1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)'$, and the covariates are independent $N(0,1)$ variables.
(B) β_0 is as in (A). The correlation between the kth and lth covariates is $\rho^{|k-l|}$ with $\rho = 0.50$.
(C) As (B) but with $\rho = 0.95$.
(D) As (A) but with $N = 100$.
(E) As (B) but with $N = 100$.
(F) As (C) but with $N = 100$.
(G) $N = 10$ and five relevant explanatory variables with a coefficient of 1; 245 irrelevant variables. All covariates are independent.
(H) $N = 10$ and five relevant explanatory variables with a coefficient of 1; 495 irrelevant variables. All covariates are independent.
(I) $N = 10$ and five relevant explanatory variables with a coefficient of 1; 2495 irrelevant variables. All covariates are independent.

Note that even though the covariates are independent in experiments G–I the maximum spurious sample correlation, i.e., the maximum observed sample correlation

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between covariates, may still be very high (for examples, see Fan and Lv, 2008). In particular, if a relevant and irrelevant covariate are highly correlated it will be difficult to distinguish between these.

5.2. Results

Table 1 contains the results for experiments A–F, where $p_N < NT_N$.

Experiment A reveals that the Bridge, Marginal Bridge, and Schwarz information criterion all perform quite well in the independent covariates setting. They all detect the correct sparsity pattern in more than half of the cases irrespective of whether cross validation or BIC is used to determine λ_N. In all respects their performance is comparable to the OLS Oracle.

As seen from experiment B making the covariates moderately correlated does not cause the performance of the procedures to deteriorate with respect to the fraction of times the right sparsity pattern is chosen or the fraction of times all relevant covariates are retained. However, all procedures get more imprecise. Because this is also the case for the OLS Oracle this is not a particular artifact of the Bridge class of estimators.

Experiment C reveals that as the correlation gets very high the performance of the Bridge and BIC deteriorates. On the other hand the Marginal Bridge continues to detect the right sparsity pattern in more than half of the cases. However, even the latter fails to retain all relevant variables in all cases. The coverage probabilities of the Bridge confidence intervals are significantly below 95%—this is the case in particular when BIC is used to select λ_N. Taking into account that it only retains all relevant variables in 3% of the simulations, this result is not too surprising.

Experiments D–F illuminate the asymptotic properties of the Bridge and the Marginal Bridge. In particular the Marginal Bridge with BIC used to determine λ_N detects the correct sparsity pattern in almost all cases irrespective of the correlation structure imposed on the covariates. The performance of the Bridge also gets significantly better as the sample size is increased, whereas the LASSO only improves moderately. The loss of all procedures is reduced, and the parameters are estimated more precisely. Finally, the coverage probabilities of the Bridge are now close to the nominal rate of 95%.

Note that the Marginal Bridge performs quite well even in the high correlation experiments C and F, indicating that the partial orthogonality assumption (B5) is not overly restrictive.

It is seen that in general the BIC is a better way of determining λ_N than cross validation. BIC detects the correct sparsity pattern more often, and only in experiment C does one find that cross validation is superior with respect to the number of relevant variables retained and coverage probabilities.

Table 2 contains the results for experiments G–I, which investigate the performance of the Marginal Bridge in the $p_N > NT_N$ case.[12] As can be expected the correct sparsity pattern is detected less frequently. However, all relevant

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TABLE 1. Results for experiments A–F where $p_N < NT_N$

	Cross Validation			BIC					
Experiment	Bridge	Marginal Bridge	LASSO	Bridge	Marginal Bridge	LASSO	BIC	OLS Oracle	OLS All
A ($N = 10$)									
Sparsity pattern	0.60	0.52	0.01	0.67	0.78	0.10	0.66	1.00	0.00
Median #Var	5.00	5.00	10.00	5.00	5.00	8.00	5.00	5.00	15.00
Loss	2.11	2.11	2.15	2.09	2.08	2.16	2.10	2.07	2.21
Relevant retained	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Median Beta MSE	0.08	0.07	0.10	0.07	0.06	0.10	0.07	0.06	0.11
Stdv	0.12	0.11	0.12	0.12	0.11	0.12	0.11	0.11	0.12
Cov. Prob.	0.91			0.92				0.94	
B ($N = 10$)									
Sparsity pattern	0.66	0.65	0.05	0.72	0.92	0.29	0.69	1.00	0.00
Median #Var	5.00	5.00	8.00	5.00	5.00	6.00	5.00	5.00	15.00
Loss	2.10	2.10	2.13	2.09	2.07	2.14	2.10	2.07	2.21
Relevant retained	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Median Beta MSE	0.09	0.08	0.10	0.08	0.07	0.09	0.08	0.07	0.14
Stdv	0.14	0.13	0.14	0.14	0.13	0.14	0.13	0.13	0.14
Cov. Prob.	0.93			0.94				0.95	
C ($N = 10$)									
Sparsity pattern	0.08	0.56	0.14	0.02	0.64	0.28	0.01	1.00	0.00
Median #Var	5.00	5.00	7.00	4.00	5.00	6.00	4.00	5.00	15.00
Loss	2.16	2.11	2.10	2.17	2.10	2.10	2.18	2.07	2.21
Relevant retained	0.32	0.89	0.90	0.03	0.69	0.88	0.01	1.00	1.00
Median Beta MSE	0.40	0.29	0.26	0.48	0.28	0.25	0.49	0.22	0.48
Stdv	0.50	0.41	0.35	0.60	0.51	0.36	0.61	0.35	0.37
Cov. Prob.	0.74			0.57				0.95	

Table continues on overleaf

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TABLE 1. *continued*

	Cross Validation			BIC					
Experiment	Bridge	Marginal Bridge	LASSO	Bridge	Marginal Bridge	LASSO	BIC	OLS Oracle	OLS All
D ($N = 100$)									
Sparsity pattern	0.76	0.66	0.02	0.94	0.99	0.46	0.91	1.00	0.00
Median #Var	5.00	5.00	10.00	5.00	5.00	6.00	5.00	5.00	15.00
Loss	2.01	2.01	2.01	2.01	2.01	2.02	2.01	2.01	2.02
Relevant retained	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Median Beta MSE	0.02	0.02	0.03	0.02	0.02	0.03	0.02	0.02	0.03
Stdv	0.03	0.03	0.03	0.03	0.03	0.04	0.03	0.03	0.03
Cov. Prob.	0.95			0.95				0.95	
E ($N = 100$)									
Sparsity pattern	0.77	0.71	0.12	0.93	0.99	0.62	0.91	1.00	0.00
Median #Var	5.00	5.00	8.00	5.00	5.00	5.00	5.00	5.00	15.00
Loss	2.01	2.01	2.01	2.01	2.01	2.01	2.01	2.01	2.02
Relevant retained	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Median Beta MSE	0.02	0.02	0.03	0.02	0.02	0.03	0.02	0.02	0.04
Stdv	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
Cov. Prob.	0.94			0.94				0.95	

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F ($N = 100$)									
Sparsity pattern	0.70	0.70	0.16	0.93	0.99	0.47	0.92	1.00	0.00
Median #Var	5.00	5.00	7.00	5.00	5.00	6.00	5.00	5.00	15.00
Loss	2.01	2.01	2.01	2.01	2.01	2.01	2.01	2.01	2.02
Relevant retained	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Median Beta MSE	0.08	0.08	0.08	0.07	0.07	0.07	0.07	0.07	0.14
Stdv	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11
Cov. Prob.	0.93			0.94				0.95	

Notes: Cross validation and BIC indicate which procedure was used to determine λ_N. Sparsity pattern: The fraction of times the correct sparsity pattern is detected. Median #Var: The median number of variables included. Loss: The mean square error (MSE) when using the estimated parameters on a validation data set drawn from the same distribution as the training set. Relevant retained: The fraction of relevant variables retained in the model. Median Beta MSE: Mean square error of $\hat{\beta}$. Stdv: Standard deviation of the estimated coefficient of the first variable (which is always in the active set). Cov. Prob.: Coverage probability of 95% confidence interval of the estimated coefficient of the first variable.

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TABLE 2. Results for experiments G–I where $p_N > NT_N$

	Cross Validation			BIC					
	Bridge	Marginal Bridge	LASSO	Bridge	Marginal Bridge	LASSO	OLS BIC	OLS Oracle	All
G									
Sparsity pattern		0.14	0.00		0.34	0.04		1.00	
Median #Var		9.00	20.00		6.00	9.00		5.00	
Loss		2.33	2.47		2.23	2.62		2.08	
Relevant retained		0.98	1.00		0.94	1.00		1.00	
Median Beta MSE		0.03	0.04		0.02	0.04		0.01	
Stdv		0.15	0.14		0.16	0.14		0.11	
H									
Sparsity pattern		0.07	0.00		0.23	0.03		1.00	
Median #Var		12.00	23.50		7.00	10.00		5.00	
Loss		2.52	2.58		2.36	2.76		2.09	
Relevant retained		0.94	1.00		0.88	1.00		1.00	
Median Beta MSE		0.03	0.03		0.02	0.04		0.01	
Stdv		0.17	0.14		0.20	0.16		0.11	
I									
Sparsity pattern		0.01	0.00		0.05	0.01		1.00	
Median #Var		19.00	35.00		9.00	11.00		5.00	
Loss		3.14	2.87		2.85	3.23		2.03	
Relevant retained		0.75	1.00		0.67	0.95		1.00	
Median Beta MSE		0.02	0.02		0.01	0.02		0.00	
Stdv		0.24	0.15		0.27	0.17		0.11	

Notes: Cross validation and BIC indicate which procedure was used to determine λ_N. Sparsity pattern: The fraction of times the correct sparsity pattern is detected. Median #Var: The median number of variables included. Loss: The MSE when using the estimated parameters on a validation data set drawn from the same distribution as the training set. Relevant retained: The fraction of relevant variables retained in the model. Median Beta MSE: Calculated as explained in the main text. Stdv: Standard deviation of the estimated coefficient of the first variable (which is always in the active set).

variables are retained very often, and only few irrelevant variables are kept in the model. Hence, the Marginal Bridge is still a very effective tool for dimension reduction.

The LASSO and the Marginal Bridge perform equally well in experiments G and H (slight advantage for the LASSO), whereas the LASSO is superior in experiment I. However, the LASSO also takes much longer to compute, and the models it chooses are bigger. The following idea that builds on the thoughts of Fan and Lv (2008) could potentially improve the performance of the Marginal Bridge: estimate the Marginal Bridge one or several times more using the residuals from the first (previous) step as left-hand-side variables. This will lower the priority of those irrelevant variables that seemed relevant only through their high correlation with some of the relevant variables already included.

6. CONCLUSIONS

This paper introduces the Bridge and Marginal Bridge estimators in a linear panel data setting allowing for both random and fixed effects. When $p < NT_N$ it is shown that the Bridge estimator (and Marginal Bridge) has the oracle property. It sets all coefficients that are truly 0 to 0, and the asymptotic distribution of the estimator of the nonzero coefficients is the same as if the sparsity pattern had been known. Monte Carlo experiments underscore this conclusion and are used to investigate the finite-sample properties of the procedures. They also reveal that the Schwarz information criterion is more useful than 10-fold cross validation for selecting λ_N. This is encouraging because BIC is also faster than cross validation.

When $p > NT_N$ it is shown that the Marginal Bridge estimator still detects the correct sparsity pattern with probability converging to 1. This is true in the random and the fixed effects settings under a partial orthogonality assumption on the covariates. However, the Marginal Bridge works well even when the relevant and irrelevant covariates are highly correlated. Furthermore, the Marginal Bridge estimates are extremely fast to compute because only marginal information is used to decide whether a variable is relevant or not. The Marginal Bridge is also shown to perform well in the $p_N < NT_N$ setting. In the $p_N > NT_N$ setting the Marginal Bridge does not always retain all relevant variables. An iterative procedure was proposed to solve this problem. Working out the properties of this procedure is left for future research.

7. PROOFS

LEMMA 1. *Let* $\mathbf{u}$ *be a* $p_N \times 1$ *vector. Then*

$$\mathbb{E}\left(\sup_{\|\mathbf{u}\|\le\delta}\left|\sum_{j=1}^{NT_N}\epsilon_j\mathbf{x}_j'\mathbf{u}\right|\,\middle|\,\mathbf{X}_N\right) \le \delta\sigma\,(NT_N p_N)^{1/2}\left(\frac{1}{NT_N p_N}\sum_{i=1}^{N}\sum_{t=1}^{T_N}\sum_{k=1}^{p_N}x_{itk}^2\right)^{1/2}.$$

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Proof.

$$\mathbb{E}\left(\sup_{\|\mathbf{u}\|\leq\delta}\left|\sum_{j=1}^{NT_N}\epsilon_j\mathbf{x}'_j\mathbf{u}\right|^2\middle|\mathbf{X}_N\right)\leq\mathbb{E}\left[\sup_{\|\mathbf{u}\|\leq\delta}\|\mathbf{u}\|^2\left\|\left(\sum_{j=1}^{NT_N}\epsilon_j\mathbf{x}_j\right)\right\|^2\middle|\mathbf{X}_N\right]$$
$$\leq\delta^2\mathbb{E}\left[\left(\sum_{j=1}^{NT_N}\epsilon_j\mathbf{x}'_j\right)\left(\sum_{j=1}^{NT_N}\epsilon_j\mathbf{x}_j\right)\middle|\mathbf{X}_N\right],$$

where the first inequality follows from the Cauchy–Schwarz inequality. Because

$$\mathbb{E}\left(\epsilon_{it}\mathbf{x}'_{it}\epsilon_{is}\mathbf{x}_{is}|\mathbf{X}_N\right)=\mathbb{E}\left(\mathbf{x}'_{it}\mathbf{x}_{is}\epsilon_{it}\epsilon_{is}|\mathbf{X}_{iN}\right)=\mathbf{x}'_{it}\mathbf{x}_{is}\mathbb{E}\left(\epsilon_{it}\epsilon_{is}|\mathbf{X}_{iN}\right)=0,\qquad t\neq s,$$
$$\begin{aligned}\mathbb{E}\left(\epsilon_{it}\mathbf{x}'_{it}\epsilon_{js}\mathbf{x}_{js}|\mathbf{X}_N\right)&=\mathbb{E}\left(\mathbb{E}\left[\epsilon_{it}\mathbf{x}'_{it}\epsilon_{js}\mathbf{x}_{js}|\mathbf{X}_N,\epsilon_{it}\right]|\mathbf{X}_N\right)\\&=\mathbb{E}\left(\epsilon_{it}\mathbf{x}'_{it}\mathbf{x}_{js}\mathbb{E}\left[\epsilon_{js}|\mathbf{X}_N,\epsilon_{it}\right]|\mathbf{X}_N\right)\\&=\mathbb{E}\left(\epsilon_{it}\mathbf{x}'_{it}\mathbf{x}_{js}\mathbb{E}\left[\epsilon_{js}|\mathbf{X}_{jN}\right]|\mathbf{X}_N\right)=0,\qquad i\neq j,\end{aligned}$$
$$\mathbb{E}\left(\epsilon_{it}\mathbf{x}'_{it}\epsilon_{it}\mathbf{x}_{it}|\mathbf{X}_N\right)=\mathbb{E}\left[\mathbf{x}'_{it}\mathbf{x}_{it}\epsilon_{it}\epsilon_{it}|\mathbf{X}_{iN}\right]=\mathbf{x}'_{it}\mathbf{x}_{it}\mathbb{E}[\epsilon_{it}\epsilon_{it}|\mathbf{X}_{iN}]=\sigma^2\mathbf{x}'_{it}\mathbf{x}_{it}.$$

Hence,

$$\mathbb{E}\left(\sup_{\|\mathbf{u}\|\leq\delta}\left|\sum_{j=1}^{NT_N}\epsilon_j\mathbf{x}_j\mathbf{u}\right|^2\middle|\mathbf{X}_N\right)\leq\delta^2\sigma^2\sum_{j=1}^{NT_N}\mathbf{x}'_j\mathbf{x}_j=\delta^2\sigma^2\sum_{i=1}^{N}\sum_{t=1}^{T_N}\mathbf{x}'_{it}\mathbf{x}_{it}$$
$$=\delta^2\sigma^2NT_Np_N\frac{1}{NT_Np_N}\sum_{i=1}^{N}\sum_{t=1}^{T_N}\sum_{k=1}^{p_N}x_{itk}^2,$$

and the result follows from the conditional Jensen inequality. ■

LEMMA 2. *Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be sequences of nonnegative random variables. If there exist an integer N_0 and a constant C such that for $n\geq N_0$*

$$\mathbb{E}\left(\frac{X_n}{Y_n}\right)\leq C,$$

then

$$X_n\in O_p(Y_n).$$

Proof. It suffices to show that for any $\epsilon>0$ $P\left(\{X_n/Y_n>C/\epsilon\}\right)\leq\epsilon$ for $n\geq N_0$. Assume the opposite is true for some $\epsilon>0$ to reach a contradiction. Then

$$\mathbb{E}\left(\frac{X_n}{Y_n}\right)=\int\frac{X_n}{Y_n}dP\geq\int_{\{X_n/Y_n>C/\epsilon\}}\frac{X_n}{Y_n}dP\geq\int_{\{X_n/Y_n>C/\epsilon\}}\frac{C}{\epsilon}dP$$
$$\geq\frac{C}{\epsilon}P\left(\left\{\frac{X_n}{Y_n}>\frac{C}{\epsilon}\right\}\right)>C,$$

which is the desired contradiction. ■

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Proof of Theorem 1. We first show that $\|\hat{\beta}_N - \beta_0\| \in O_p\left(\left[\frac{p_N+\lambda_N k_N}{NT_N\rho_{1N}}\right]^{1/2}\right)$. Because $\hat{\beta}_N$ minimizes (3.2)

$$\sum_{j=1}^{NT_N}\left(y_j - \mathbf{x}_j'\hat{\beta}_N\right)^2 + \lambda_N\sum_{k=1}^{p_N}|\hat{\beta}_{Nk}|^\gamma \le \sum_{j=1}^{NT_N}\left(y_j - \mathbf{x}_j'\beta_0\right)^2 + \lambda_N\sum_{k=1}^{p_N}|\beta_{0k}|^\gamma.$$

Defining $\eta_N = \lambda_N\sum_{k=1}^{p_N}|\beta_{0k}|^\gamma$ this implies

$$\begin{aligned}\eta_N &\ge \sum_{j=1}^{NT_N}\left(y_j - \mathbf{x}_j'\hat{\beta}_N\right)^2 - \sum_{j=1}^{NT_N}\left(y_j - \mathbf{x}_j'\beta_0\right)^2\\ &= \sum_{j=1}^{NT_N}\left(\left[y_j - \mathbf{x}_j'\hat{\beta}_N\right] - \left[y_j - \mathbf{x}_j'\beta_0\right]\right)\left(\left[y_j - \mathbf{x}_j'\hat{\beta}_N\right] + \left[y_j - \mathbf{x}_j'\beta_0\right]\right)\\ &= \sum_{j=1}^{NT_N}\left[\mathbf{x}_j'(\beta_0 - \hat{\beta}_N)\right]^2 + 2\sum_{j=1}^{NT_N}\epsilon_j\mathbf{x}_j'\left(\beta_0 - \hat{\beta}_N\right).\end{aligned}$$

Now define $\delta_N = (NT_N)^{1/2}\Sigma_N^{1/2}(\hat{\beta}_N - \beta_0)$, $\mathbf{D}_N = (NT_N)^{-1/2}\Sigma_N^{-1/2}\mathbf{X}_N'$, and $\epsilon_N = (\epsilon_1, \ldots, \epsilon_{NT_N})'$. With these definitions

$$\begin{aligned}\sum_{j=1}^{NT_N}\left[\mathbf{x}_j'(\beta_0 - \hat{\beta}_N)\right]^2 + 2\sum_{j=1}^{NT_N}\epsilon_j\mathbf{x}_j'\left(\beta_0 - \hat{\beta}_N\right) &= \delta_N'\delta_N - 2(\mathbf{D}_N\epsilon_N)'\delta_N\\ &= ||\delta_N - \mathbf{D}_N\epsilon_N||^2 - ||\mathbf{D}_N\epsilon_N||^2,\end{aligned}$$

which implies

$$||\delta_N - \mathbf{D}_N\epsilon_N||^2 - ||\mathbf{D}_N\epsilon_N||^2 - \eta_N \le 0$$

and by the subadditivity of $x \mapsto x^{1/2}$ yields $||\delta_N - \mathbf{D}_N\epsilon_N|| \le ||\mathbf{D}_N\epsilon_N|| + \eta_N^{1/2}$. By subadditivity of the norm $\|\cdot\|$ this implies $||\delta_N|| \le ||\delta_N - \mathbf{D}_N\epsilon_N|| + ||\mathbf{D}_N\epsilon_N|| \le 2||\mathbf{D}_N\epsilon_N|| + \eta_N^{1/2}$. Because $(x+y)^2 \le 2x^2 + 2y^2$ for $x, y \in \mathbf{R}$ by the convexity of $x \mapsto x^2$ one has

$$||\delta_N||^2 \le 4||\mathbf{D}_N\epsilon_N||^2 + 2\eta_N.$$

Letting $\mathbf{d}_j$ denote the jth column of $\mathbf{D}_N$ we may write $\mathbf{D}_N\epsilon_N = \sum_{j=1}^{NT_N}\mathbf{d}_j\epsilon_j$. Using that $\mathbf{D}_N$ is measurable with respect to $\mathbf{X}_N$, conclude

$$\begin{aligned}\mathbb{E}\left(\mathbf{d}_{it}'\epsilon_{it}\mathbf{d}_{is}\epsilon_{is}\right) &= \mathbb{E}\left(\mathbf{d}_{it}'\mathbf{d}_{is}\mathbb{E}[\epsilon_{it}\epsilon_{is}|\mathbf{X}_N]\right)\\ &= \mathbb{E}\left(\mathbf{d}_{it}'\mathbf{d}_{is}\mathbb{E}[\epsilon_{it}\epsilon_{is}|\mathbf{X}_{iN}]\right) = 0, \qquad s \ne t,\\ \mathbb{E}\left(\mathbf{d}_{it}'\epsilon_{it}\mathbf{d}_{js}\epsilon_{js}\right) &= \mathbb{E}\left(\mathbf{d}_{it}'\epsilon_{it}\mathbf{d}_{js}\mathbb{E}\left[\epsilon_{js}|\mathbf{X}_N, \epsilon_{it}\right]\right)\\ &= \mathbb{E}\left(\mathbf{d}_{it}'\epsilon_{it}\mathbf{d}_{js}\left[\epsilon_{js}|\mathbf{X}_{jN}\right]\right) = 0, \qquad i \ne j,\\ \mathbb{E}\left(\mathbf{d}_{it}'\epsilon_{it}\mathbf{d}_{it}\epsilon_{it}\right) &= \mathbb{E}\left(\mathbf{d}_{it}'\mathbf{d}_{it}\mathbb{E}[\epsilon_{it}\epsilon_{it}|\mathbf{X}_{iN}]\right) = \mathbb{E}\left(\mathbf{d}_{it}'\mathbf{d}_{it}\right) = \sigma^2\mathbb{E}\left(||\mathbf{d}_{it}||^2\right).\end{aligned}$$

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Hence,

$$\mathbb{E}\left(\|\mathbf{D}_N\epsilon_N\|^2\right) = \mathbb{E}\left(\left\|\sum_{j=1}^{NT_N}\mathbf{d}_j\epsilon_j\right\|^2\right) = \mathbb{E}\left[\left(\sum_{j=1}^{NT_N}\mathbf{d}_j\epsilon_j\right)'\left(\sum_{j=1}^{NT_N}\mathbf{d}_j\epsilon_j\right)\right]$$
$$= \sigma^2\mathbb{E}\left(\sum_{j=1}^{NT_N}\|\mathbf{d}_j\|^2\right) = \sigma^2\mathbb{E}\left(\mathrm{tr}\left(\mathbf{D}_N'\mathbf{D}_N\right)\right) = \sigma^2\mathbb{E}\left(\mathrm{tr}\left(\mathbf{D}_N\mathbf{D}_N'\right)\right)$$
$$= \sigma^2\mathrm{tr}\left(\mathbf{I}_{p_N}\right) = \sigma^2 p_N.$$

And so, $\mathbb{E}\left(||\delta_N||^2\right) \leq 4\sigma^2 p_N + 2\eta_N$. Hence,

$$(NT_N)\mathbb{E}\left(\left(\hat{\beta}_N - \beta_0\right)'\Sigma_N\left(\hat{\beta}_N - \beta_0\right)\right) = \mathbb{E}(\delta_N'\delta_N) = \mathbb{E}\left(||\delta_N||^2\right) \leq 4\sigma^2 p_N + 2\eta_N.$$

Because the number of nonzero coefficients is k_N,

$$\eta_N = \lambda_N\sum_{k=1}^{p_N}|\beta_{0j}|^\gamma = \lambda_N\sum_{k=1}^{k_N}|\beta_{0j}|^\gamma \leq \lambda_N k_N b_1^\gamma,$$

where the inequality is a consequence of Assumption (A5). Because ρ_{1N} is the smallest eigenvalue of Σ_N,

$$\rho_{1N}||\hat{\beta}_N - \beta_0||^2 = \rho_{1N}\left(\hat{\beta}_N - \beta_0\right)'\left(\hat{\beta}_N - \beta_0\right) \leq \left(\hat{\beta}_N - \beta_0\right)'\Sigma_N\left(\hat{\beta}_N - \beta_0\right).$$

Hence,

$$\mathbb{E}\left(\rho_{1N}||\hat{\beta}_N - \beta_0||^2\right) \leq \frac{NT_N}{NT_N}\mathbb{E}\left(\left(\hat{\beta}_N - \beta_0\right)'\Sigma_N\left(\hat{\beta}_N - \beta_0\right)\right)$$
$$\leq \frac{4\sigma^2 p_N + 2\eta_N}{NT_N} \leq \frac{4\sigma^2 p_N + 2\lambda_N k_N b_1^\gamma}{NT_N}$$
$$\leq C\frac{p_N + \lambda_N k_N b_1^\gamma}{NT_N}$$

for $C = \max\left(4\sigma^2, 2b_1^\gamma\right)$. This implies

$$\mathbb{E}\left(\frac{||\hat{\beta}_N - \beta_0||^2}{\left(\frac{p_N + \lambda_N k_N}{NT_N\rho_{1N}}\right)}\right) \leq C.$$

By Lemma 2 this establishes $||\hat{\beta}_N - \beta_0|| \in O_p\left([(p_N + \lambda_N k_N)/(NT_N\rho_{1N})]^{1/2}\right)$. Next we show that $||\hat{\beta}_N - \beta_0|| \in O_p\left(\rho_{1N}^{-1}(p_N/(NT_N))^{1/2}\right)$. Like Huang et al. (2008) we use the idea from the proof of Theorem 3.2.5 in Van der Vaart and Wellner (1996). Let $r_N = \rho_{1N}^{-1}(p_N/(NT_N))^{1/2}$. For every

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N partition the parameter space (excluding β_0) into the disjoint shells $S_{l,N} = \{\beta : 2^{l-1} < \|\beta - \beta_0\|/r_N \leq 2^l\}$ where $l \in \mathbb{Z}$. If $2^M < \left\|\hat{\beta}_N - \beta_0\right\|/r_N$ for a given integer M then $\hat{\beta}_N \in \bigcup_{l>M} S_{l,N}$. For the shell that $\hat{\beta}_N$ belongs to, the infimum of the map $\beta \mapsto L_N(\beta) - L_N(\beta_0)$ is nonpositive. Hence, for any $\delta > 0$[13]

$$P\left(\left\|\hat{\beta}_N - \beta_0\right\|/r_N > 2^M\right) \leq \sum_{\substack{l>M \\ 2^{l-1}<\delta/r_N}} P\left(\inf_{\beta\in S_{l,N}} (L_N(\beta) - L_N(\beta_0)) \leq 0\right) + P\left(\left\|\hat{\beta}_N - \beta_0\right\| > \delta\right). \quad \textbf{(7.1)}$$

The last term in (7.1) converges to 0 by the consistency of $\hat{\beta}_N$ shown in the first part of the theorem. The theorem is established by proving that the first term on the right-hand side can be made arbitrarily small by choosing M sufficiently large. To this end let $\beta \in S_{l,N}$ for an arbitrary l summed over and note that

$$\begin{aligned} L_N(\beta) - L_N(\beta_0) &= \sum_{j=1}^{NT_N} \left(y_j - \mathbf{x}_j'\beta\right)^2 + \lambda_N \sum_{k=1}^{k_N} |\beta_{1k}|^\gamma + \lambda_N \sum_{k=1}^{m_N} |\beta_{2k}|^\gamma \\ &\quad - \sum_{j=1}^{NT_N} \left(y_j - \mathbf{x}_j'\beta_0\right)^2 - \lambda_N \sum_{k=1}^{k_N} |\beta_{01k}|^\gamma \\ &\geq \sum_{j=1}^{NT_N} \left(y_j - \mathbf{x}_j'\beta\right)^2 + \lambda_N \sum_{k=1}^{k_N} |\beta_{1k}|^\gamma - \sum_{j=1}^{NT_N} \left(y_j - \mathbf{x}_j'\beta_0\right)^2 \\ &\quad - \lambda_N \sum_{k=1}^{k_N} |\beta_{01k}|^\gamma \\ &= \sum_{j=1}^{NT_N} \left(\mathbf{x}_j'[\beta - \beta_0]\right)^2 - 2\sum_{j=1}^{NT_N} \epsilon_j \mathbf{x}_j'(\beta - \beta_0) \\ &\quad + \lambda_N \sum_{k=1}^{k_N} \left(|\beta_{1k}|^\gamma - |\beta_{01k}|^\gamma\right). \end{aligned} \quad \textbf{(7.2)}$$

Regarding the first term in (7.2),

$$\begin{aligned} \sum_{j=1}^{NT_N} \left(\mathbf{x}_j'[\beta - \beta_0]\right)^2 &= [\beta - \beta_0]' \sum_{j=1}^{NT_N} \mathbf{x}_j\mathbf{x}_j'[\beta - \beta_0] = NT_N[\beta - \beta_0]'\Sigma_N[\beta - \beta_0] \\ &\geq NT_N\|\beta - \beta_0\|^2\rho_{1N} > NT_N 2^{2(l-1)} r_N^2 \rho_{1N}. \end{aligned}$$

Regarding the third term in (7.2) we note that $\beta \in S_{l,N}$ and $2^{l-1} < \delta/r_N$ implies that $\|\beta - \beta_0\|/r_N \leq 2^l < 2\delta/r_N$. Hence, it suffices to consider β's satisfying

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$\|\beta - \beta_0\| < 2\delta$. Because $\delta > 0$ is arbitrary and the entries of β_{01} are bounded uniformly away from the 0 by b_0 the mean value theorem may be applied to conclude that for some ζ_k between β_{1k} and β_{01k}

$$\left|\lambda_N \sum_{k=1}^{k_N} \left(|\beta_{1k}|^\gamma - |\beta_{01k}|^\gamma\right)\right| = \left|\lambda_N \gamma \sum_{k=1}^{k_N} |\zeta_k|^{\gamma-1} \text{sign}(\zeta_k)(\beta_{1k} - \beta_{01k})\right|$$

$$\leq c\lambda_N \sum_{k=1}^{k_N} |\beta_{1k} - \beta_{01k}| \leq c\lambda_N k_N^{1/2} \|\beta - \beta_0\| \leq c\lambda_N k_N^{1/2} 2^l r_N,$$

where $c = \gamma (b_0 - 2\delta)^{\gamma-1}$ and the second to last estimate follows from Jensen's inequality. Hence, on $S_{l,N}$, $\lambda_N \sum_{k=1}^{k_N} (|\beta_{1k}|^\gamma - |\beta_{01k}|^\gamma) \geq -c\lambda_N k_N^{1/2} 2^l r_N$. Therefore, on $S_{l,N}$,

$$L_N(\beta) - L_N(\beta_0) \geq -\left|2 \sum_{j=1}^{NT_N} \epsilon_j \mathbf{x}_j' (\beta - \beta_0)\right| + \rho_{1N} N T_N 2^{2(l-1)} r_N^2 - c\lambda_N k_N^{1/2} 2^l r_N.$$

Hence, by the conditional Markov inequality and Lemma 1,

$$P\left(\inf_{\beta \in S_{l,N}} (L_N(\beta) - L_N(\beta_0)) \leq 0 \middle| \mathbf{X}_N\right)$$

$$\leq P\left(\sup_{\beta \in S_{l,N}} \left|2 \sum_{j=1}^{NT_N} \epsilon_j \mathbf{x}_j' (\beta - \beta_0)\right| \geq \rho_{1N} N T_N 2^{2(l-1)} r_N^2 - c\lambda_N k_N^{1/2} 2^l r_N \middle| \mathbf{X}_N\right)$$

$$\leq \frac{\mathbb{E}\left(\sup_{\beta \in S_{l,N}} \left|2\sum_{j=1}^{NT_N} \epsilon_j \mathbf{x}_j' (\beta - \beta_0)\right| \middle| \mathbf{X}_N\right)}{\rho_{1N} N T_N 2^{2(l-1)} r_N^2 - c\lambda_N k_N^{1/2} 2^l r_N}$$

$$\leq \frac{\sigma (NT_N p_N)^{1/2} 2^l r_N \left(\frac{1}{NT_N p_N} \sum_{i=1}^{N} \sum_{t=1}^{T_N} \sum_{k=1}^{p_N} x_{itk}^2\right)^{1/2}}{\rho_{1N} N T_N 2^{2(l-1)} r_N^2 - c\lambda_N k_N^{1/2} 2^l r_N}$$

$$= \frac{2\sigma \left(\frac{1}{NT_N p_N} \sum_{i=1}^{N} \sum_{t=1}^{T_N} \sum_{k=1}^{p_N} x_{itk}^2\right)^{1/2}}{2^{l-2} - c\lambda_N (k_N/(p_N NT_N))^{1/2}}.$$

By Assumption (A3) $\lambda_N (k_N/(p_N NT_N))^{1/2} \to 0$, and so $2^{l-2} - c\lambda_N (k_N/(NT_N))^{1/2} \geq 2^{l-3}$ for N sufficiently large. Hence, by iterated expectations and Assumption (A1)

$$P\left(\inf_{\beta \in S_{l,N}} (L_N(\beta) - L_N(\beta_0)) \leq 0\right) \leq \frac{\sigma\sqrt{K}}{2^{l-4}}.$$

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Finally, this implies that

$$\sum_{\substack{l>M \\ 2^{l-1}<\delta/r_N}} P\left(\inf_{\beta\in S_{l,N}}(L_N(\beta)-L_N(\beta_0))\le 0\right)\le \sum_{l>M}\frac{\sigma\sqrt{K}}{2^{l-4}},$$

which is convergent, and so the tail can be made arbitrarily small by choosing M sufficiently large. ■

LEMMA 3. *Suppose* $0<\gamma<1$. *Let* $\hat{\beta}_N=\left(\hat{\beta}_{1N},\ \hat{\beta}_{2N}\right)$. *Then* $\hat{\beta}_{2N}=0$ *with probability converging to* 1 *under Assumptions (A1)–(A7).*

Proof. By Theorem 1 $||\hat{\beta}_N-\beta_0||\in O_p(h_N)$ with $h_N=\rho_{1N}^{-1}(p_N/(NT_N))^{1/2}$, and so for all $\epsilon>0$ there exists a constant C such that for N sufficiently large

$$P\left(||\hat{\beta}_N-\beta_0||/h_N>C\right)<\epsilon\Leftrightarrow P\left(||\hat{\beta}_N-\beta_0||\le Ch_N\right)\ge 1-\epsilon.$$

Put differently, $\hat{\beta}_N\in\{\beta:||\beta-\beta_0||\le Ch_N\}$ with probability converging to 1. Let $\hat{\beta}_{1N}=\beta_{10}+h_N\mathbf{u}_1$ and $\hat{\beta}_{2N}=\beta_{20}+h_N\mathbf{u}_2=h_N\mathbf{u}_2$. Choosing $\hat{\beta}_N$ is then equivalent to choosing $\mathbf{u}_1$ and $\mathbf{u}_2$. For $\mathbf{u}=(\mathbf{u}_1',\mathbf{u}_2')'$ one has $||\mathbf{u}||=||\hat{\beta}_N-\beta_0||/h_N$, which is bounded by C with probability approaching 1. Hence, we may assume $||\mathbf{u}||^2=||\mathbf{u}_1||^2+||\mathbf{u}_2||^2\le C^2$ and define

$$V_N(\mathbf{u}_1,\mathbf{u}_2)=L_N(\hat{\beta}_{1N},\hat{\beta}_{2N})=L_N(\beta_{10}+h_N\mathbf{u}_1,h_N\mathbf{u}_2).$$

To establish the lemma it now suffices to show that for any $\mathbf{u}$ with $\|\mathbf{u}\|\le C$, $V_N(\mathbf{u}_1,\mathbf{u}_2)-V_N(\mathbf{u}_1,\mathbf{0})>0$ with probability converging to 1 if $\mathbf{u}_2\ne 0$. Now,

$$\begin{aligned}
V_N(\mathbf{u}_1,\mathbf{u}_2)-V_N(\mathbf{u}_1,\mathbf{0})&=\sum_{j=1}^{NT_N}\left(y_j-\beta_{01}'\mathbf{w}_j-h_N\mathbf{u}_1'\mathbf{w}_j-h_N\mathbf{u}_2'\mathbf{z}_j\right)^2\\
&\quad+\lambda_N\sum_{k=1}^{k_N}|\beta_{01k}+h_Nu_{1k}|^\gamma+\lambda_N\sum_{k=1}^{m_N}|h_Nu_{2k}|^\gamma\\
&\quad-\sum_{j=1}^{NT_N}\left(y_j-\beta_{01}'\mathbf{w}_j-h_N\mathbf{u}_1'\mathbf{w}_j\right)^2-\lambda_N\sum_{k=1}^{k_N}|\beta_{01k}+h_Nu_{1k}|^\gamma\\
&=\sum_{j=1}^{NT_N}-h_N(\mathbf{u}_2'\mathbf{z}_j)\left[2(y_j-\beta_{01}'\mathbf{w}_j-h_N\mathbf{u}_1'\mathbf{w}_j)-h_N\mathbf{u}_2'\mathbf{z}_j\right]\\
&\quad+\lambda_N\sum_{k=1}^{m_N}|h_Nu_{2k}|^\gamma\\
&=h_N^2\sum_{j=1}^{NT_N}(\mathbf{z}_j'\mathbf{u}_2)^2+2h_N^2\sum_{j=1}^{NT_N}(\mathbf{w}_j'\mathbf{u}_1)(\mathbf{z}_j'\mathbf{u}_2)\\
&\quad-2h_N\sum_{j=1}^{NT_N}(\mathbf{z}_j'\mathbf{u}_2)\epsilon_j+\lambda_Nh_N^\gamma\sum_{k=1}^{m_N}|u_{2j}|^\gamma.
\end{aligned}$$

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Regarding the sum of the first two terms, because $2xy \geq -(x^2+y^2)$,

$$h_N^2 \sum_{j=1}^{NT_N} (\mathbf{z}_j'\mathbf{u}_2)^2 + 2h_N^2 \sum_{j=1}^{NT_N} (\mathbf{w}_j'\mathbf{u}_1)(\mathbf{z}_j'\mathbf{u}_2) \geq h_N^2 \sum_{j=1}^{NT_N} (\mathbf{z}_j'\mathbf{u}_2)^2 - h_N^2 \sum_{j=1}^{NT_N} \left[(\mathbf{w}_j'\mathbf{u}_1)^2 + (\mathbf{z}_j'\mathbf{u}_2)^2\right]$$
$$= -h_N^2 N T_N \mathbf{u}_1' \Sigma_{1N} \mathbf{u}_1 \geq -\rho_{1N}^{-2} p_N \tau_2 C^2,$$

where the last inequality follows from Assumption (A2) and the fact that $||\mathbf{u}_1|| \leq C$. Hence,

$$\frac{h_N^2 \sum_{j=1}^{NT_N} (\mathbf{z}_i'\mathbf{u}_2)^2 + 2h_N^2 \sum_{j=1}^{NT_N} (\mathbf{w}_i'\mathbf{u}_1)(\mathbf{z}_i'\mathbf{u}_2)}{p_N \rho_N^{-2}} \geq -\tau_2 C^2.$$

Regarding the third term, it follows from Jensen's inequality (conditional version) that

$$\mathbb{E}\left(\left|\sum_{j=1}^{NT_N} (\mathbf{z}_j'\mathbf{u}_2)\epsilon_j\right| \middle| \mathbf{X}_N\right) \leq \left[\mathbb{E}\left(\sum_{j=1}^{NT_N} (\mathbf{z}_j'\mathbf{u}_2)\epsilon_j\right)^2 \middle| \mathbf{X}_N\right]^{1/2} = \left[\sigma^2 \sum_{j=1}^{NT_N} \left(\mathbf{z}_j'\mathbf{u}_2\right)^2\right]^{1/2}$$
$$= \sigma \left[\sum_{j=1}^{NT_N} \mathbf{u}_2'\mathbf{z}_j\mathbf{z}_j'\mathbf{u}_2\right]^{1/2} = \sigma (NT_N)^{1/2} \left(\mathbf{u}_2' \Sigma_{2N} \mathbf{u}_2\right)^{1/2}$$
$$\leq \sigma (NT_N)^{1/2} \rho_{2N}^{1/2} C,$$

where the last inequality used that

$$\rho_{2N} = \max_{\mathbf{u} \in \mathbf{R}^{p_N}} \frac{\mathbf{u}'\Sigma_N \mathbf{u}}{\mathbf{u}'\mathbf{u}} \geq \max_{\mathbf{u}_2 \in \mathbf{R}^{m_N}} \frac{\left(\mathbf{0}', \mathbf{u}_2'\right) \Sigma_N \left(\mathbf{0}', \mathbf{u}_2'\right)'}{\left(\mathbf{0}', \mathbf{u}_2'\right)\left(\mathbf{0}', \mathbf{u}_2'\right)'} = \max_{\mathbf{u}_2 \in \mathbf{R}^{m_N}} \frac{\mathbf{u}'\Sigma_{2N}\mathbf{u_2}}{\mathbf{u}_2'\mathbf{u}_2}.$$

Hence, because h_N is measurable with respect to $\sigma(\mathbf{X}_N)$,

$$\mathbb{E}\left(\left|-2h_N \sum_{j=1}^{NT_N} (\mathbf{z}_j'\mathbf{u}_2)\epsilon_j\right| \middle| \mathbf{X}_N\right) \leq 2\sigma h_N (NT_N)^{1/2} \rho_{2N}^{1/2} C,$$

and so

$$\mathbb{E}\left(\frac{\left|-2h_N \sum_{j=1}^{NT_N} (\mathbf{z}_j'\mathbf{u}_2)\epsilon_j\right|}{h_N (NT_N)^{1/2} \rho_{2N}^{1/2}}\right) \leq 2\sigma C,$$

which by Lemma 2 shows that $-2h_N \sum_{j=1}^{NT_N} \left(\mathbf{z}_j'\mathbf{u}_2\right)\epsilon_j$ is $O_p\left(h_N (NT_N)^{1/2} \rho_{2N}^{1/2}\right) = O_p\left(\rho_{1N}^{-1} \rho_{2N}^{1/2} p_N^{1/2}\right)$. Hence,

$$\frac{\left|-2h_N \sum_{j=1}^{NT_N} (\mathbf{z}_j'\mathbf{u}_2)\epsilon_j\right|}{p_N \rho_{1N}^{-2}} \in O_p\left(\frac{\rho_{1N}^{-1} \rho_{2N}^{1/2} p_N^{1/2}}{p_N \rho_{1N}^{-2}}\right) = O_p\left(\frac{\rho_{1N} \rho_{2N}^{1/2}}{p_N^{1/2}}\right) \subseteq O_p(1)$$

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by Assumption (A7). Regarding the fourth term, because $\mathbf{u}_2 \neq 0$ we have

$$\lambda_N h_N^{\gamma}/(p_N \rho_{1N}^{-2}) = \lambda_N \left[\rho_{1N}^{-1}(p_N/NT_N)^{1/2}\right]^{\gamma}/(p_N \rho_{1N}^{-2}) = \lambda_N \rho_{1N}^{2-\gamma}(NT_N)^{-\gamma/2} p_N^{\gamma/2-1} \to \infty$$

by Assumption (A4), and so the fourth term diverges to infinity. Because $p_N \rho_{1N}^{-2} \in \Omega_p(1)$ this completes the proof. ■

Proof of Theorem 2. The first part has been established in Lemma 3. Because $\hat{\beta}_N$ is consistent it follows from Assumption (A5) that for an arbitrary $\epsilon > 0$

$$\begin{aligned}
&P\left(\left\{\min\left\{|\hat{\beta}_{1Nj}| \,|1 \leq j \leq k\right\} + \epsilon < b_0\right\}\right) \\
&\quad = P\left(\bigcup_{j=1}^{k}\left\{|\hat{\beta}_{1Nj}| + \epsilon < b_0\right\}\right) \\
&\quad = P\left(\bigcup_{j=1}^{k}\left\{b_0 - |\hat{\beta}_{1Nj}| > \epsilon\right\}\right) \leq P\left(\bigcup_{j=1}^{k}\left\{|\beta_{10j}| - |\hat{\beta}_{1Nj}| > \epsilon\right\}\right) \\
&\quad \leq P\left(\bigcup_{j=1}^{k}\left\{|\beta_{10j} - \hat{\beta}_{1Nj}| > \epsilon\right\}\right) \leq P\left(||\beta_{10} - \hat{\beta}_{1Nj}|| > \epsilon\right) \to 0.
\end{aligned}$$

Choosing $\epsilon = b_0/2$ shows that with probability converging to 1 $\min\left\{|\hat{\beta}_{1Nj}| \,|\, 1 \leq j \leq k\right\} \geq b_0/2$, and so $\hat{\beta}_{1N}$ is bounded away from 0. Hence L_N is differentiable at $\hat{\beta}_{1N}$ with probability converging to 1. And so $\hat{\beta}_{1N}$ satisfies

$$\frac{\partial}{\partial \beta_1} L_N(\hat{\beta}_{1N}, \hat{\beta}_{2N}) = 0.$$

That is,

$$-2\sum_{j=1}^{NT_N}\left(y_j - \mathbf{w}_j'\hat{\beta}_{1N} - \mathbf{z}_j'\hat{\beta}_{2N}\right)\mathbf{w}_j + \lambda_N \gamma\, \psi_N = 0$$

with probability converging to 1 where ψ_N is a $k \times 1$ vector with lth entry given by $\psi_{Nl} = |\hat{\beta}_{1Nl}|^{\gamma-1}\text{sign}(\hat{\beta}_{1Nl})$. This can be rewritten as

$$\begin{aligned}
&-2\sum_{j=1}^{NT_N}\left(\epsilon_j - \mathbf{w}_j'(\hat{\beta}_{1N} - \beta_{10}) - \mathbf{z}_j'\hat{\beta}_{2N}\right)\mathbf{w}_j + \lambda_N \gamma\, \psi_N = 0 \\
&\quad \Leftrightarrow -2\sum_{j=1}^{NT_N}\epsilon_j \mathbf{w}_j + 2\sum_{j=1}^{NT_N}\mathbf{w}_j\mathbf{w}_j'\left(\hat{\beta}_{1N} - \beta_{10}\right) + 2\sum_{j=1}^{NT_N}\mathbf{z}_j'\hat{\beta}_{2N}\mathbf{w}_j + \lambda_N \gamma\, \psi_N = 0 \\
&\quad \Leftrightarrow \Sigma_{1N}\left(\hat{\beta}_{1N} - \beta_{10}\right) = \frac{1}{NT_N}\sum_{j=1}^{NT_N}\epsilon_j \mathbf{w}_j - \frac{\lambda_N \gamma\, \psi_N}{2NT_N} - \frac{1}{NT_N}\sum_{j=1}^{NT_N}\mathbf{z}_j'\hat{\beta}_{2N}\mathbf{w}_j.
\end{aligned}$$

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Hence, for any $k \times 1$ vector α,

$$(NT_N)^{1/2}\alpha'\left(\hat{\beta}_{1N} - \beta_{10}\right) = (NT_N)^{-1/2}\sum_{j=1}^{NT_N}\alpha'\Sigma_{1N}^{-1}\epsilon_j\mathbf{w}_j - (1/2)\gamma\,(NT_N)^{-1/2}\lambda_N\alpha'\Sigma_{1N}^{-1}\psi_N - (NT_N)^{-1/2}\sum_{j=1}^{NT_N}\alpha'\Sigma_{1N}^{-1}\mathbf{z}_j'\hat{\beta}_{2N}\mathbf{w}_j.$$

Because $P(\hat{\beta}_{2N} = 0) \to 1$ the last term equals 0 with probability converging to 1. From the Cauchy–Schwarz inequality in $\mathbf{R}^k$ it follows that

$$|\alpha'\Sigma_{1N}^{-1}\psi_N| \leq ||\alpha'\Sigma_{1N}^{-1}||\,||\psi_N||.$$

Because

$$||\alpha'\Sigma_{1N}^{-1}||^2 = \alpha'\Sigma_{1N}^{-1}\Sigma_{1N}^{-1}\alpha = \alpha'\Sigma_{1N}^{-2}\alpha \leq \tau_{1N}^{-2}||\alpha||^2 \leq \tau_1^{-2}||\alpha||^2$$

by Assumption (A2), we get with probability converging to 1

$$\begin{aligned}|(1/2)(NT_N)^{-1/2}\gamma\,\lambda_N\alpha'\Sigma_{1N}^{-1}\psi_N| &\leq (1/2)(NT_N)^{-1/2}\gamma\,\lambda_N\tau_1^{-1}||\alpha||\,||\psi_N|| \\ &\leq (1/2)(NT_N)^{-1/2}\gamma\,\lambda_N\tau_1^{-1}||\alpha||k^{1/2}(b_0/2)^{(\gamma-1)} \\ &= (1/2)\gamma\,\tau_1^{-1}||\alpha||(b_0/2)^{(\gamma-1)}T_N^{-1/2}N^{-1/2}\lambda_N k^{1/2} \to 0\end{aligned}$$

by Assumption (A3). Hence,

$$(NT_N)^{1/2}\alpha'\left(\hat{\beta}_{1N} - \beta_{10}\right) \in (NT_N)^{-1/2}\sum_{j=1}^{NT_N}\alpha'\Sigma_{1N}^{-1}\epsilon_j\mathbf{w}_j + o_p(1).$$

Because $s_N^{-1} = 1/\sqrt{\sigma^2\alpha'\Sigma_{1N}^{-1}\alpha} \leq \tau_2^{1/2}/(\sigma||\alpha||)$ by Assumption (A2) it is also true that

$$(NT_N)^{1/2}s_N^{-1}\alpha'\left(\hat{\beta}_{1N} - \beta_{10}\right) \in (NT_N)^{-1/2}s_N^{-1}\sum_{j=1}^{NT_N}\alpha'\Sigma_{1N}^{-1}\epsilon_j\mathbf{w}_j + o_p(1). \tag{7.3}$$

Now, defining $\mathbf{W}_{i,N} = (\mathbf{w}_{i1}, \ldots, \mathbf{w}_{iT_N})'$ and $\epsilon_{i,N} = (\epsilon_{i1}, \ldots, \epsilon_{iNT_N})'$ one first notices that[14]

$$\frac{1}{NT_N}\sum_{j=1}^{NT_N}\mathbf{w}_j\mathbf{w}_j' \to \lim_{N\to\infty}\frac{1}{NT_N}\sum_{j=1}^{NT_N}\mathbb{E}\left(\mathbf{w}_j\mathbf{w}_j'\right) = \lim_{N\to\infty}\mathbb{E}\left(\frac{1}{T_N}\mathbf{W}_{1N}'\mathbf{W}_{1N}\right), \tag{7.4}$$

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where the first limit is in probability. To see why (7.4) is true let Z_j be a fixed entry in $\mathbf{w}_j\mathbf{w}_j'$, $j=1,\ldots,NT_N$. Letting $\eta>0$ be arbitrary and using the Markov inequality,

$$
\begin{aligned}
&P\left(\left|\frac{1}{NT_N}\sum_{j=1}^{NT_N}Z_j-\frac{1}{NT_N}\sum_{j=1}^{NT_N}\mathbb{E}\left(Z_j\right)\right|>\eta\right)\\
&\quad\le\frac{\mathbb{E}\left(\frac{1}{NT_N}\sum_{j=1}^{NT_N}\left[Z_j-\mathbb{E}\left(Z_j\right)\right]\right)^2}{\eta^2}\\
&\quad=\frac{\mathbb{E}\left(\sum_{i=1}^{N}\sum_{t=1}^{T_N}\left[Z_{it}-\mathbb{E}\left(Z_{it}\right)\right]\right)^2}{(NT_N\eta)^2}=\frac{\sum_{i=1}^{N}\mathbb{E}\left(\sum_{t=1}^{T_N}\left[Z_{it}-\mathbb{E}\left(Z_{it}\right)\right]\right)^2}{(NT_N\eta)^2}\\
&\quad=\frac{N\left(\sum_{t=1}^{T_N}\mathrm{Var}(Z_{1t})+2\sum_{t=1}^{T_N}\sum_{s>t}^{T_N}\mathrm{Cov}\left(Z_{1t},\ Z_{1s}\right)\right)}{(NT_N\eta)^2}\\
&\quad\le\frac{N\left(T_N\max_{1\le t\le T_N}\mathrm{Var}\left(Z_{1t}\right)+2T_N(T_N-1)/2\max_{1\le t\le T_N}\mathrm{Var}\left(Z_{1t}\right)\right)}{(NT_N\eta)^2}\\
&\quad=\frac{\max_{1\le t\le T_N}\mathrm{Var}\left(Z_{1t}\right)}{N\eta^2}\to 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(NT_N)^{-1/2}s_N^{-1}\sum_{j=1}^{NT_N}\alpha'\Sigma_{1N}^{-1}\epsilon_j\mathbf{w}_j\\
&\quad=\frac{(NT_N)^{-1/2}\sum_{j=1}^{NT_N}\alpha'\left(\frac{1}{NT_N}\sum_{j=1}^{NT_N}\mathbf{w}_j\mathbf{w}_j'\right)^{-1}\epsilon_j\mathbf{w}_j}{\sqrt{\sigma^2\alpha'\left(\frac{1}{NT_N}\sum_{j=1}^{NT_N}\mathbf{w}_j\mathbf{w}_j'\right)^{-1}\alpha}}\\
&\quad\in\frac{N^{-1/2}}{\sqrt{\sigma^2\alpha'\left(\mathbb{E}\left[\frac{1}{T_N}\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]\right)^{-1}\alpha}}\\
&\qquad\times\sum_{i=1}^{N}\alpha'\left(\mathbb{E}\left[\frac{1}{T_N}\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]\right)^{-1}T_N^{-1/2}\mathbf{W}_{iN}'\epsilon_{iN}+o_p(1).
\end{aligned}
$$

Now,

$$
\mathbb{E}\left(\sum_{i=1}^{N}\alpha'\left(\mathbb{E}\left[\frac{1}{T_N}\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]\right)^{-1}T_N^{-1/2}\mathbf{W}_{iN}'\epsilon_{iN}\right)=0
$$

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by iterated expectations and

$$
\begin{aligned}
r_N^2 :&= \mathbb{E}\left[\left(\sum_{i=1}^{N}\alpha'\left(\mathbb{E}\left[\frac{1}{T_N}\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]\right)^{-1}T_N^{-1/2}\mathbf{W}_{iN}'\epsilon_{iN}\right)^2\right]\\
&= \sum_{i=1}^{N}\mathbb{E}\left(\alpha'\left(\mathbb{E}\left[\frac{1}{T_N}\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]\right)^{-1}T_N^{-1/2}\mathbf{W}_{iN}'\epsilon_{iN}\epsilon_{iN}'\right.\\
&\quad\left.\times\,\mathbf{W}_{iN}T_N^{-1/2}\left(\mathbb{E}\left[\frac{1}{T_N}\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]\right)^{-1}\alpha\right)\\
&= \sigma^2 N\alpha'\left(\mathbb{E}\left[\frac{1}{T_N}\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]\right)^{-1}\alpha.
\end{aligned}
$$

Finally, let $U_{iN} = \alpha'\left(\mathbb{E}\left[(1/T_N)\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]\right)^{-1}T_N^{-1/2}\mathbf{W}_{iN}'\epsilon_{iN}$. Because $\mathbb{E}(U_{iN}^2) = \sigma^2\alpha'\left(\mathbb{E}\left[(1/T_N)\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]\right)^{-1}\alpha$ is convergent it is bounded. Furthermore,

$$
\alpha'\mathbb{E}\left(\frac{1}{T_N}\mathbf{W}_{1N}'\mathbf{W}_{1N}\right)\alpha = \mathbb{E}\left(\alpha'\frac{1}{NT_N}\mathbf{W}_N'\mathbf{W}_N\alpha\right) \in \left[\alpha'\alpha\tau_1, \alpha'\alpha\tau_2\right],
$$

and so the eigenvalues of $\mathbb{E}\left[(1/T_N)\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]$ are also contained in $[\tau_1, \tau_2]$. This implies $r_N^2 = \sigma^2 N\alpha'\left(\mathbb{E}\left[(1/T_N)\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]\right)^{-1}\alpha \geq \sigma^2 N\alpha'\alpha/\tau_2$. Hence, the Lindeberg condition is satisfied because, for all $\delta > 0$,

$$
\begin{aligned}
&\lim_{N\to\infty}\left\{r_N^{-2}\sum_{i=1}^{N}\int_{\{|U_{i,N}|>\delta r_N\}}U_{iN}^2\,dP\right\}\\
&\quad\leq \lim_{N\to\infty}\frac{\tau_2}{\alpha'\alpha\sigma^2}\int_{\left\{|U_{1N}|>\delta\sqrt{(\alpha'\alpha\sigma^2)/\tau_2}\sqrt{N}\right\}}U_{1N}^2\,dP = 0
\end{aligned}
$$

because for any $\rho > 0$ (let $\delta\sqrt{(\alpha'\alpha\sigma^2)/\tau_2} = K$)

$$
\begin{aligned}
\lim_{N\to\infty}P\left(U_{1N}^2\mathbf{1}_{\left\{|U_{1N}|>K\sqrt{N}\right\}} > \rho\right) &\leq \lim_{N\to\infty}P\left(\left\{|U_{1N}| > K\sqrt{N}\right\}\right)\\
&\leq \lim_{N\to\infty}\frac{\mathbb{E}\left(U_{1N}^2\right)}{K^2N} = 0
\end{aligned}
$$

and $\left\{U_{1N}^2\right\}_{N=1}^{\infty}$ is uniformly integrable.[15] Hence,

$$
\begin{aligned}
&\frac{N^{-1/2}}{\sqrt{\sigma^2\alpha'\left(\mathbb{E}\left[\frac{1}{T_N}\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]\right)^{-1}\alpha}}\sum_{i=1}^{N}\alpha'\left(\mathbb{E}\left[\frac{1}{T_N}\mathbf{W}_{1N}'\mathbf{W}_{1N}\right]\right)^{-1}T_N^{-1/2}\mathbf{W}_{iN}'\epsilon_{iN}\\
&\quad\xrightarrow{d} N(0,1).
\end{aligned}
$$

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And so, by (7.3),

$$(NT_N)^{1/2}s_N^{-1}\alpha'(\hat{\beta}_{1N}-\beta_{10}) \xrightarrow{d} N(0,1),$$

or, equivalently,

$$(NT_N)^{1/2}(\hat{\beta}_{1N}-\beta_{10}) \xrightarrow{d} N\left(0,\sigma^2\left(\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{T_N}\mathbf{W}'_{1N}\mathbf{W}_{1N}\right]\right)^{-1}\right). \tag{7.5}$$

■

Proof of Theorem 3.

(i) If $T_N = T$ for a fixed T, $U_{1N} = U_1$ for all N where U_1 is defined in the obvious way does not depend on N and belongs to L^2. Hence,

$$\lim_{K\to\infty}\sup_{1\le N<\infty}\int_{\{|U_{1N}|>K\}}U_{1N}^2 dP = \lim_{K\to\infty}\int_{\{|U_1|>K\}}U_1^2 dP = 0$$

by Lebesgue's dominated convergence theorem.

(ii) By the Cauchy–Schwarz inequality

$$|U_{1N}| \le \left\|\alpha'\left(\mathbb{E}\left[\frac{1}{T_N}\mathbf{W}'_{1N}\mathbf{W}_{1N}\right]\right)^{-1}\right\|\left\|T_N^{-1/2}\mathbf{W}'_{1N}\epsilon_{1N}\right\|.$$

Because $\left\|\alpha'\left(\mathbb{E}\left[(1/T_N)\mathbf{W}'_{1N}\mathbf{W}_{1N}\right]\right)^{-1}\right\|$ is convergent it is bounded by a constant C. Hence,

$$U_{1N}^2 \le C^2\left\|T_N^{-1/2}\mathbf{W}'_{1N}\epsilon_{1N}\right\|^2 = C^2\sum_{i=1}^{k}\frac{1}{T_N}\sum_{j=1}^{T_N}\left(\mathbf{W}_{1N}^{ji}\epsilon_{1N}^{j}\right)^2,$$

where $\mathbf{W}_{1N}^{ji}$ is the jth row in the ith column of $\mathbf{W}_{1N}$. Because the rows of $\mathbf{W}_{1N}$ are identically distributed and $\mathbf{W}_{1N}^{ji} \perp\!\!\!\perp \epsilon_{1N}^{j}$ one has (calculate the characteristic functions and notice they are identical) $\mathbf{W}_{1N}^{ji}\epsilon_{1N}^{j} \sim \mathbf{W}_{1N}^{1i}\epsilon_{1N}^{1}$ where $\mathbf{W}_{1N}^{1i}\epsilon_{1N}^{1} \sim Z_i$ for some $Z_i \in L^2$. Hence, $\left\{\left\{\left(\mathbf{W}_{1N}^{ji}\epsilon_{1N}^{j}\right)^2\right\}_{j=1}^{T_N}\right\}_{N=1}^{\infty}$ is uniformly integrable for all $1 \le i \le k$ because

$$\begin{aligned}
&\lim_{K\to\infty}\sup_{1\le N<\infty}\sup_{1\le j\le T_N}\int_{\left\{\left(\mathbf{W}_{1N}^{ji}\epsilon_{1N}^{j}\right)^2>K\right\}}\left(\mathbf{W}_{1N}^{ji}\epsilon_{1N}^{j}\right)^2 dP \\
&\quad= \lim_{K\to\infty}\sup_{1\le N<\infty}\sup_{1\le j\le T_N}\int_{\{x^2>K\}}x^2 dP_{\mathbf{W}_{1N}^{ji}\epsilon_{1N}^{j}}(x) \\
&\quad= \lim_{K\to\infty}\sup_{1\le N<\infty}\sup_{1\le j\le T_N}\int_{\{x^2>K\}}x^2 dP_{Z_i}(x) \\
&\quad= \lim_{K\to\infty}\int_{\{x^2>K\}}x^2 dP_{Z_i}(x) = 0
\end{aligned}$$

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by Lebesgue's dominated convergence theorem. By Hoffmann-Jørgensen (1994, p. 338)[16] this implies that $\left\{(1/T_N)\sum_{j=1}^{T_N}\left(\mathbf{W}_{1N}^{ji}\epsilon_{1N}^{j}\right)^2\right\}_{N=1}^{\infty}$ is uniformly integrable, which in turn implies that $\left\{\sum_{i=1}^{k}(1/T_N)\sum_{j=1}^{T_N}(\mathbf{W}_{1N}^{ji}\epsilon_{1N}^{j})^2\right\}_{N=1}^{\infty}$ is uniformly integrable by Hoffmann-Jørgensen (1994, p. 337).[17] Because $\left\{C^2\sum_{i=1}^{k}(1/T_N)\sum_{j=1}^{T_N}\left(\mathbf{W}_{1N}^{ji}\epsilon_{1N}^{j}\right)^2\right\}_{N=1}^{\infty}$ dominates $\left\{U_{1N}^2\right\}$ this yields the desired result. ■

Proof of Corollary 1. For fixed T_N (7.5) reads

$$N^{1/2}(\hat{\beta}_{1N}-\beta_{10})\xrightarrow{d}N\left(0,\sigma^2\left(\mathbb{E}\left[\mathbf{W}_1'\mathbf{W}_1\right]\right)^{-1}\right),$$

where absence of subscript N indicates that the matrices no longer depend on T_N. In the fixed effects setting $\mathbf{W}_1=\left(\mathbf{D}\mathbf{D}'\right)^{-1/2}\mathbf{D}\tilde{\mathbf{W}}_1$, and so

$$\begin{aligned}\mathbb{E}\left(\mathbf{W}_1'\mathbf{W}_1\right)&=\mathbb{E}\left(\tilde{\mathbf{W}}_1'\mathbf{D}'\left(\mathbf{D}\mathbf{D}'\right)^{-1/2}\left(\mathbf{D}\mathbf{D}'\right)^{-1/2}\mathbf{D}\tilde{\mathbf{W}}_1\right)\\&=\mathbb{E}\left(\tilde{\mathbf{W}}_1'\mathbf{D}'\left(\mathbf{D}\mathbf{D}'\right)^{-1}\mathbf{D}\tilde{\mathbf{W}}_1\right)=\mathbb{E}\left(\ddot{\tilde{\mathbf{W}}}_1'\ddot{\tilde{\mathbf{W}}}_1\right),\end{aligned}$$

where the last equality used that $\mathbf{D}'\left(\mathbf{D}\mathbf{D}'\right)^{-1}\mathbf{D}$ is symmetric and idempotent and that premultiplication of it corresponds to columnwise demeaning.

The proof of part (ii) is similar using $\mathbf{W}_1=\Omega^{-1/2}\tilde{\mathbf{W}}_1$. ■

Next we turn to the properties of the Marginal Bridge estimator.

LEMMA 4. *For q even and any $w_N>0$,*

$$P\left(w_N>\max_{1\leq k\leq m}\left|\sum_{j=1}^{NT}x_{jk}\epsilon_j\right|\right)\geq 1-\frac{Km(NT_N)^{q/2}}{w_N^q}.$$

Proof. By the Markov inequality,

$$\begin{aligned}P\left(w_N>\max_{1\leq k\leq m}\left|\sum_{j=1}^{NT}x_{jk}\epsilon_j\right|\right)&=1-P\left(\max_{1\leq k\leq m}\left|\sum_{j=1}^{NT_N}x_{jk}\epsilon_j\right|\geq w_N\right)\\&\geq 1-\frac{\mathbb{E}\left(\max_{1\leq k\leq m}\left|\sum_{j=1}^{NT_N}x_{jk}\epsilon_j\right|^q\right)}{w_N^q}.\end{aligned}$$

Because for any sequence of random variables $\{Z_k\}_{k=1}^m\subseteq L^q$

$$\mathbb{E}\left(\max_{1\leq k\leq m}Z_k^q\right)\leq\mathbb{E}\left(\sum_{k=1}^{m}Z_k^q\right)\leq m\max_{1\leq k\leq m}\mathbb{E}\left(Z_k^q\right)$$

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it suffices to show that $\mathbb{E}\left(\sum_{j=1}^{NT_N} x_{jk}\epsilon_j\right)^q \leq K(NT_N)^{q/2}$ for all $k \in \{1,\dots,m\}$. By the multinomial theorem

$$\mathbb{E}\left(\sum_{j=1}^{NT_N} x_{jk}\epsilon_j\right)^q = \mathbb{E}\left(\sum_{|\alpha|=q}\binom{q}{\alpha}\prod_{j=1}^{NT_N}(x_{jk}\epsilon_j)^{\alpha_j}\right),$$

where we have used the multi-index convention $\alpha = (\alpha_1,\dots,\alpha_{NT})$ and $|\alpha| = q$ means $\sum_{j=1}^{NT_N}\alpha_j = q$. The sum is taken over all such vectors α whose entries add up to q. Because $(\mathbf{X}_{iN},\epsilon_{iN})$ is i.i.d. and $\mathbf{X}_{iN} \perp\!\!\!\perp \epsilon_{iN}$ it follows that $(\mathbf{X}_{1N},\epsilon_{1N},\dots,\mathbf{X}_{NN},\epsilon_{NN})$ are independent (calculate the characteristic function and observe that it factorizes). This implies $\mathbf{X}_N \perp\!\!\!\perp \epsilon_N$. Finally, it is seen that $\epsilon_{11},\dots,\epsilon_{NT_N}$ are independent (again calculate the characteristic function and observe that it factorizes by using $(\mathbf{X}_{iN},\epsilon_{iN})$ is i.i.d. since uncorrelated Gaussian variables are independent). From these observations it follows that all summands including an $\alpha_j = 1$ for some $j = 1,\dots,NT_N$ equal 0 because $\mathbb{E}(\epsilon_j) = 0$ for all $j = 1,\dots,NT_N$. It remains to be shown that no other term is of greater order than $O\left((NT_N)^{q/2}\right)$. So assume $\min_{1\leq j\leq NT_N}\alpha_j \geq 2$.[18] Consider the set A of all vectors α with s entries different from 0 and larger than or equal to 2 with nonzero values $a_1,\dots,a_s$ and $a_1+\cdots+a_s = q$.

$$\begin{aligned}
&\mathbb{E}\left(\sum_{\alpha\in A}\prod_{j=1}^{NT_N}(x_{jk}\epsilon_j)^{\alpha_j}\right)\\
&\quad= \mathbb{E}\left(\sum_{j_1=1}^{NT_N}\sum_{j_2\neq j_1}^{NT_N}\cdots\sum_{j_s\notin\{j_1,\dots,j_{s-1}\}}^{NT_N}(x_{j_1k}\epsilon_{j_1})^{a_1}(x_{j_2k}\epsilon_{j_2})^{a_2}\cdot\cdots\cdot(x_{j_sk}\epsilon_{j_s})^{a_s}\right)\\
&\quad= K\mathbb{E}\left(\sum_{j_1=1}^{NT_N}\sum_{j_2\neq j_1}^{NT_N}\cdots\sum_{j_s\notin\{j_1,\dots,j_{s-1}\}}^{NT_N}(x_{j_1k})^{a_1}(x_{j_2k})^{a_2}\cdot\cdots\cdot(x_{j_sk})^{a_s}\right)\\
&\quad\leq K(NT_N)^{q/2}\mathbb{E}\left(\sum_{j_1=1}^{NT_N}\sum_{j_2\neq j_1}^{NT_N}\cdots\sum_{j_s\notin\{j_1,\dots,j_{s-1}\}}^{NT_N}\left[\frac{x_{j_1k}^2}{NT_N}\right]^{a_1/2}\left[\frac{x_{j_2k}^2}{NT_N}\right]^{a_2/2}\cdots\cdots\left[\frac{x_{j_sk}^2}{NT_N}\right]^{a_s/2}\right)\\
&\quad= K(NT_N)^{q/2}\mathbb{E}\left(\sum_{j_1=1}^{NT_N}\left[\frac{x_{j_1k}^2}{NT_N}\right]^{a_1/2}\sum_{j_2\neq j_1}^{NT_N}\left[\frac{x_{j_2k}^2}{NT_N}\right]^{a_2/2}\cdots\cdots\sum_{j_s\notin\{j_1,\dots,j_{s-1}\}}^{NT_N}\left[\frac{x_{j_sk}^2}{NT_N}\right]^{a_s/2}\right)\\
&\quad\leq K(NT_N)^{q/2},
\end{aligned}$$

where the second equality used $\mathbf{X}_N \perp\!\!\!\perp \epsilon_N$ and $K := \mathbb{E}(\epsilon_1^{a_1})\cdot\cdots\cdot\mathbb{E}(\epsilon_1^{a_s})$. The first estimate follows from the nonnegativity of all terms on the right-hand side. The last estimate uses that for any subset S of $\{1,\dots,NT_N\}$, $\sum_{j\in S}\left[x_{jk}^2/(NT_N)\right]^{a_i/2}$ $\leq \sum_{j=1}^{NT_N}\left[x_{jk}^2/(NT_N)\right]^{a_i/2} \leq \sum_{j=1}^{NT_N}\left[x_{jk}^2/(NT_N)\right] = 1$ because $\left[x_{jk}^2/(NT_N)\right] \leq 1$ and so $\left[x_{jk}^2/(NT_N)\right]^{a_i/2} \leq \left[x_{jk}^2/(NT_N)\right]$ for $a_i \geq 2$. Because s was arbitrary

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and there are only finitely many terms of the preceding kind this establishes the lemma. ■

Proof of Theorem 4. Let $\epsilon_N = (\epsilon_1, \ldots, \epsilon_{NT_N})'$ and recall that, as stated in Section 4, $\xi_{Nk} = 1/(NT_N)\sum_{j=1}^{NT_N} \mathbf{w}_j'\beta_{10}x_{jk}$. Then,

$$
\begin{aligned}
U_N(\beta) &= \sum_{k=1}^{p_N}\sum_{j=1}^{NT_N}\left(y_j - x_{jk}\beta_k\right)^2 + \lambda_N\sum_{k=1}^{p_N}|\beta_k|^\gamma \\
&= \sum_{k=1}^{p_N}\left[\sum_{j=1}^{NT_N} y_j^2 + NT_N\beta_k^2 - 2\sum_{j=1}^{NT_N} y_j x_{jk}\beta_k\right] + \lambda_N\sum_{k=1}^{p_N}|\beta_k|^\gamma \\
&= \sum_{k=1}^{p_N}\left[\sum_{j=1}^{NT_N}\epsilon_j^2 + \sum_{j=1}^{NT_N}(\mathbf{w}_j'\beta_{01})^2 + 2\sum_{j=1}^{NT_N}(\mathbf{w}_j'\beta_{01})\epsilon_j + NT_N\beta_k^2\right. \\
&\qquad \left. -2\sum_{j=1}^{NT_N}(\epsilon_j + \mathbf{w}_j'\beta_{01})x_{jk}\beta_k\right] + \lambda_N\sum_{k=1}^{p_N}|\beta_k|^\gamma \\
&= \sum_{k=1}^{p_N}\left[\sum_{j=1}^{NT_N}\epsilon_j^2 + \sum_{j=1}^{NT_N}(\mathbf{w}_j'\beta_{01})^2 + 2\sum_{j=1}^{NT_N}(\mathbf{w}_j'\beta_{01})\epsilon_j + NT_N\beta_k^2\right. \\
&\qquad \left. -2(\epsilon_N'\mathbf{x}_k + NT_N\xi_{Nk})\beta_k\right] + \lambda_N\sum_{k=1}^{p_N}|\beta_k|^\gamma .
\end{aligned}
$$

So minimizing $U_N(\beta)$ is equivalent to minimizing $\sum_{k=1}^{p_N}\left[NT_N\beta_k^2 - 2(\epsilon_N'\mathbf{x}_k + NT_N\xi_{Nk})\beta_k + \lambda_N|\beta_k|^\gamma\right]$. Because $0 < \gamma < 1$ it follows from Lemma A of Knight and Fu (2000) that $\beta_k = 0$ if and only if

$$\lambda_N/(NT_N) > |\epsilon_N'\mathbf{x}_k/(NT_N) + \xi_{Nk}|^{2-\gamma}\, c_\gamma ,$$

where $c_\gamma = (2/(2-\gamma))((2(2-\gamma))/(2-\gamma))^{1-\gamma}$. Defining $w_N = c_\gamma^{-1/(2-\gamma)}(\lambda_N/(NT_N)^{\gamma/2})^{1/(2-\gamma)}$ the preceding inequality is equivalent to

$$w_N > (NT_N)^{-1/2}|\epsilon_N'\mathbf{x}_k + (NT_N)\xi_{Nk}|.$$

So to prove the theorem it is enough to show

$$P\left(w_N > (NT_N)^{-1/2}\max_{k\in J_N}\left|\epsilon_N'\mathbf{x}_k + (NT_N)\xi_{Nk}\right|\right) \to 1 \tag{7.6}$$

and

$$P\left(w_N > (NT_N)^{-1/2}\min_{k\in K_N}\left|\epsilon_N'\mathbf{x}_k + (NT_N)\xi_{Nk}\right|\right) \to 0. \tag{7.7}$$

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We first prove (7.6). On $A_N = \left\{ \left| \sum_{j=1}^{NT_N} x_{jk} x_{jl} \right| \Big/ ((NT_N)^{1/2}) \le c_0,\ k \in K_N,\ l \in J_N \right\}$, and under Assumption (B6)

$$\begin{aligned}\max_{l\in J_N}(NT_N)^{1/2}|\xi_{Nl}| &= \max_{l\in J_N}\frac{1}{(NT_N)^{1/2}}\left|\sum_{j=1}^{NT_N}\sum_{k=1}^{k_N} x_{jk}\beta_{10k}x_{jl}\right| \\ &= \max_{l\in J_N}\frac{1}{(NT_N)^{1/2}}\left|\sum_{k=1}^{k_N}\beta_{10k}\sum_{j=1}^{NT_N} x_{jk}x_{jl}\right| \\ &\le \max_{l\in J_N}\sum_{k=1}^{k_N}|\beta_{10k}|\left|\frac{1}{(NT_N)^{1/2}}\sum_{j=1}^{NT_N} x_{jk}x_{jl}\right| \le b_1 c_0 k_N.\end{aligned}$$

Hence,

$$\begin{aligned}&P\left(w_N > (NT_N)^{-1/2}\max_{k\in J_N}|\epsilon_N'\mathbf{x}_k + (NT_N)\xi_{Nk}|\right) \\ &\quad\ge P\left(w_N > (NT_N)^{-1/2}\max_{k\in J_N}|\epsilon_N'\mathbf{x}_k| + (NT_N)^{1/2}\max_{k\in J_N}|\xi_{Nk}|,\ A_N\right) \\ &\quad\ge P\left(w_N > (NT_N)^{-1/2}\max_{k\in J_N}|\epsilon_N'\mathbf{x}_k| + b_1c_0k_N,\ A_N\right) \\ &\quad\ge P\left(w_N > (NT_N)^{-1/2}\max_{k\in J_N}|\epsilon_N'\mathbf{x}_k| + b_1c_0k_N\right) + P(A_N) - 1,\end{aligned}$$

where the last estimate follows from the inclusion-exclusion principle. By Lemma 4 for every even q,

$$\begin{aligned}P\left((NT_N)^{1/2}(w_N - b_1c_0k_N) > \max_{k\in J_N}|\epsilon_N'\mathbf{x}_k|\right) &\ge 1 - \frac{Km_N(NT_N)^{q/2}}{\left((NT_N)^{1/2}[w_N - b_1c_0k_N]\right)^q} \\ &= 1 - \frac{Km_N/w_N^q}{(1 - b_1c_0k_N/w_N)^q}.\end{aligned}$$

Furthermore, by Assumption (B4) we may choose q sufficiently large such that

$$m_N/w_N^q \in O(1)\frac{m_N}{\left(\lambda_N(NT_N)^{-\gamma/2}\right)^{q/(2-\gamma)}} \to 0,$$

and by Assumption (B3)

$$b_1c_0k_N/w_N \in O(1)\frac{k_N}{\left(\lambda_N(NT_N)^{-\gamma/2}\right)^{1/(2-\gamma)}} \to 0.$$

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Finally, $P(A_N)$ can be made arbitrarily close to 1 by Assumption (B5), which establishes (7.6). Next we verify (7.7).

$$P\left(w_N > (NT_N)^{-1/2} \min_{k\in K_N} |\epsilon_N' \mathbf{x}_k + (NT_N)\xi_{Nk}|\right) \tag{7.8}$$

$$= P\left(\bigcup_{k\in K_N} \left\{w_N > |(NT_N)^{-1/2}\epsilon_N' \mathbf{x}_k + (NT_N)^{1/2}\xi_{Nk}|\right\}\right)$$

$$\leq \sum_{k\in K_N} P\left(\left\{w_N > |(NT_N)^{-1/2}\epsilon_N' \mathbf{x}_k + (NT_N)^{1/2}\xi_{Nk}|\right\}\right). \tag{7.9}$$

Because $\min_{k\in K_N} |\xi_{Nk}| \geq \xi_0 > 0$ by Assumption (B1) we may write

$$\begin{aligned}
&P\left(\left\{w_N > |(NT_N)^{-1/2}\epsilon_N' \mathbf{x}_k + (NT_N)^{1/2}\xi_{Nk}|\right\}\right) \\
&\quad \leq P\left(\left\{w_N > (NT_N)^{1/2}|\xi_{Nk}| - (NT_N)^{-1/2}|\epsilon_N' \mathbf{x}_k|\right\}\right) \\
&\quad \leq P\left(\left\{w_N > (NT_N)^{1/2}\xi_0 - (NT_N)^{-1/2}|\epsilon_N' \mathbf{x}_k|\right\}\right) \\
&\quad = P\left(\left\{(NT_N)^{-1/2}|\epsilon_N' \mathbf{x}_k| > (NT_N)^{1/2}\xi_0 - w_N\right\}\right) \\
&\quad = 1 - P\left(\left\{(NT_N)^{1/2}\xi_0 - w_N \geq (NT_N)^{-1/2}|\epsilon_N' \mathbf{x}_k|\right\}\right) \\
&\quad \leq 1 - P\left(\left\{(NT_N)^{1/2}\xi_0 - w_N > (NT_N)^{-1/2}|\epsilon_N' \mathbf{x}_k|\right\}\right).
\end{aligned}$$

By Lemma 4,

$$\begin{aligned}
P\left(\left\{(NT_N)^{1/2}\left((NT_N)^{1/2}\xi_0 - w_N\right) > |\epsilon_N' \mathbf{x}_k|\right\}\right) &\geq 1 - \frac{K(NT_N)^{q/2}}{\left((NT_N)^{1/2}[(NT_N)^{1/2}\xi_0 - w_N]\right)^q} \\
&= 1 - \frac{K}{((NT_N)^{1/2}\xi_0 - w_N)^q}.
\end{aligned}$$

And so for all $k \in K_N$

$$P\left(\left\{w_N > |(NT_N)^{-1/2}\epsilon_N' \mathbf{x}_k + (NT_N)^{1/2}\xi_{Nk}|\right\}\right) \leq \frac{K}{((NT_N)^{1/2}\xi_0 - w_N)^q}.$$

Inserting this into (7.9) yields

$$\begin{aligned}
P\left(w_N > (NT_N)^{-1/2} \min_{k\in k_N} |\epsilon_N' \mathbf{x}_k + (NT_N)\xi_{Nk}|\right) &\leq \frac{Kk_N}{((NT_N)^{1/2}\xi_0 - w_N)^q} \\
&= \frac{Kk_N/(NT_N)^{q/2}}{\left(\xi_0 - w_N/(NT_N)^{1/2}\right)^q}.
\end{aligned}$$

By Assumption (B2),

$$\frac{w_N}{(NT_N)^{1/2}} \in O(1)\left(\frac{\lambda_N(NT_N)^{-\gamma/2}}{(NT_N)^{(2-\gamma)/2}}\right)^{1/(2-\gamma)} = O(1)\left(\lambda_N/(NT_N)\right)^{1/(2-\gamma)} \subseteq o(1).$$

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Furthermore, for any $q \geq 1$, $k_N/(NT_N)^{q/2} \to 0$ by (B3). Hence,

$$P\left(w_N > (NT_N)^{-1/2} \min_{k \in K_N} |\epsilon_N' \mathbf{x}_k + (NT_N)\xi_{Nk}|\right) \to 0.$$

■

NOTES

1. In principle it suffices for most purposes that ratio σ_c/σ is known, but it is hard to imagine a situation where the ratio is known whereas σ_c and σ are not. I thank the co-editor for pointing this out.

2. The sole reason for multiplying $\Omega^{-1/2}$ by σ is that (FE3′) and (RE3′) become identical except for the dimension of the covariance matrix. Because (FE3′) and (RE3′) are the assumptions used in the proofs this indicates that the proofs only have to be carried out in either the fixed or the random effects setting.

3. Because we allow for random covariates some of our assumptions must hold in an almost sure sense and the equivalent assumptions in Huang et al. (2008) must hold surely.

4. More generally, if $\mathrm{Var}(w_{1tl}w_{1tm}) \in O(g(t))$ for all $1 \leq l, m \leq k$ for some positive increasing function g it suffices that $g(T_N)/N \to 0$.

5. It is sufficient to assume $\tilde{\mathbf{X}}_{iN} \perp\!\!\!\perp \tilde{\epsilon}_{iN}$ for all $1 \leq i \leq N$ but for comparison with (FE3) we refrain from this (see also the comment after (FE3)). Furthermore, as was the case in the $p_N < NT_N$ setting in Sections 2 and 3 $(\tilde{\epsilon}_{i1}, \ldots, \tilde{\epsilon}_{iT_N})$ may have any covariance structure as long as it is known so that it can be handled by an appropriate transformation. The most common choice is a diagonal matrix.

6. Let $\psi(x)$ be a nondecreasing, convex function with $\psi(0) = 0$. Then for a random variable X, the Orlicz norm is defined as $\inf\left\{C > 0 : \mathbb{E}\psi\left(\frac{|X|}{C}\right) \leq 1\right\}$. For more details on Orlicz norms and maximal inequalities based on these see Van der Vaart and Wellner (1996).

7. If x_{j1} and x_{j2} are perfectly correlated and (assume for simplicity) have an empirical mean of 0 $x_{j2} = bx_{j1}$ almost surely for some constant b. Then $\left(\sum_{j=1}^{NT_N} x_{j1}x_{j2}\right)\Big/(NT_N)^{1/2} = b(NT_N)^{1/2}$, which violates (B5).

8. A random variable X is said to be sub-Gaussian if there exist positive constants C and K such that $P(|X| \geq x) \leq C\exp(-Kx^2)$.

9. A model with 100 observations and 2,500 potential explanatory variables takes between 0.2 and 0.3 seconds to estimate on a 2.66 GHz i7 processor.

10. The Bridge estimator was also tried in the second-step but did not outperform least squares and was considerably slower.

11. The author thanks an anonymous referee for suggesting this.

12. The presence of empty columns is due to the fact that the oracle results of the Bridge Estimator are not applicable in the $p_N > NT_N$ setting. BIC and OLS including all variables are not applicable either because no more than NT_N variables can be included at a time.

13. Note that

$$\begin{aligned}
&\left\{\|\hat{\beta}_N - \beta_0\|/r_N > 2^M\right\} \\
&\quad \subseteq \bigcup_{l>M}\left\{\inf_{\beta \in S_{l,N}} (L_N(\beta) - L_N(\beta_0)) \leq 0\right\} \\
&\quad \subseteq \left(\bigcup_{l>M}\left\{\inf_{\beta \in S_{l,N}} (L_N(\beta) - L_N(\beta_0)) \leq 0\right\} \cap \left\{\|\hat{\beta}_N - \beta_0\| \leq \delta\right\}\right) \cup \left\{\|\hat{\beta}_N - \beta_0\| > \delta\right\}
\end{aligned}$$

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$$= \left(\bigcup_{l>M} \left\{ \inf_{\beta \in S_{l,N}} (L_N(\beta) - L_N(\beta_0)) \leq 0 \right\} \cap \left\{ \|\hat{\beta}_N - \beta_0\| / r_N \leq \delta / r_N \right\} \right) \cup \left\{ \|\hat{\beta}_N - \beta_0\| > \delta \right\}$$

$$\subseteq \bigcup_{\substack{l>M \\ 2^{l-1} < \delta / r_N}} \left\{ \inf_{\beta \in S_{l,N}} (L_N(\beta) - L_N(\beta_0)) \leq 0 \right\} \cup \left\{ \|\hat{\beta}_N - \beta_0\| > \delta \right\}$$

and conclude using the subadditivity of P.

14. All limits are taken elementwise in the matrices.

15. The uniform integrability of $\left\{ U_{1N}^2 \right\}_{N=1}^{\infty}$ implies that $\left\{ U_{1N}^2 \mathbf{1}_{\left\{ |U_{1N}| > K\sqrt{N} \right\}} \right\}_{N=1}^{\infty}$ is uniformly integrable, which is what is needed to utilize the well-known fact that convergence in measure plus uniform integrability is equivalent to convergence in L^1.

16. The partial averages of a uniformly integrable sequence are themselves uniformly integrable.

17. Finite sums of uniformly integrable sequences are themselves uniformly integrable.

18. Here we follow the convention of setting $t^0 = 1$ even when $t = 0$, and so we only have to consider the case $\min_{1 \leq j \leq NT_N} \alpha_j \geq 2$.

REFERENCES

Arellano, M. (2003) *Panel Data Econometrics*. Oxford University Press.

Candes, E. & T. Tao (2007) The Dantzig selector: Statistical estimation when p is much larger than n. *Annals of Statistics* 35, 2313–2351.

Fan, J. & J. Lv (2008) Sure independence screening for ultrahigh dimensional feature space. *Journal of the Royal Statistical Society, Series B* 70, 849–911.

Hoffmann-Jørgensen, J. (1994) *Probability with a View Toward Statistics*, vol. 1. Chapman & Hall/CRC.

Huang, J., J. Horowitz, & S. Ma (2008) Asymptotic properties of bridge estimators in sparse high-dimensional regression models. *Annals of Statistics* 36, 587–613.

Hunter, D. & R. Li (2005) Variable selection using MM algorithms. *Annals of Statistics* 33, 1617–1642.

Knight, K. & W. Fu (2000) Asymptotics for lasso-type estimators. *Annals of Statistics* 28, 1356–1378.

Meinshausen, N. & B. Yu (2009) Lasso-type recovery of sparse representations for high-dimensional data. *Annals of Statistics* 37, 246–270.

Stoyanov, J. (1997) *Counterexamples in Probability*, 2nd ed. Wiley.

Tibshirani, R. (1996) Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society, Series B* 73, 267–288.

Van der Vaart, A. & J. Wellner (1996) *Weak Convergence and Empirical Processes*. Springer-Verlag.

Zou, H. (2006) The adaptive lasso and its oracle properties. *Journal of the American Statistical Association* 101, 1418–1429.

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