Ergod. Th. & Dynam. Sys. (2000), **20**, 1553–1556
Printed in the United Kingdom © *2000 Cambridge University Press*

BOOK REVIEW

Geodesic Flows

By G. PATERNAIN
Progress in Mathematics, 180, Birkhaüser. ISBN 3-7643-4144-0.

Ergodic theory has been described as the study of the qualitative properties of actions of groups on spaces. In classical cases the group is $\mathbb{R}$ (flows) or $\mathbb{Z}$ (diffeomorphisms), and more recently the actions of $\mathbb{R}^k$ and $\mathbb{Z}^k$ have been studied. The past twenty years have brought important developments in the ergodic theory of actions by nonabelian Lie groups and their lattices, particularly when the group is semisimple [**Z, F**]. Ergodic theory intersects geometry when the group acting preserves some geometric structure such as a Riemannian metric or a symplectic 2-form. Many interesting phenomena have arisen from this mixture of ideas and methods.

Geodesic flows, the subject and title of this book, have linked ergodic theory and Riemannian geometry for many years. Traditionally, geodesic behavior has been studied in the context of compact Riemannian manifolds with sectional curvature of fixed sign [**GKM, CE, BGS, B, E**]. In particular, both geometers and ergodic theorists have devoted much attention to the case that the sectional curvature is strictly negative, which implies that the geodesic flow is uniformly hyperbolic and hence Bernoulli [**A**]. This book sets a different course, by disregarding curvature and focusing instead on the symplectic and entropy properties of the geodesic flow $\{g_t\}$ in the unit tangent bundle SM of an arbitrary compact Riemannian manifold M.

The book consists of five chapters. The first chapter presents basic material on geodesic flows $\{g_t\}$ of the tangent bundle TM and the unit tangent bundle SM, including the contact 1-form, symplectic 2-form and the description of the differential maps $\{dg_t\}$ in terms of Jacobi vector fields on the geodesics of M. The second chapter discusses the definition and basic properties of the bundle $\Lambda(SM)$ of Lagrangian subspaces of SM, including the action $\{g_t^*\}$ on $\Lambda(SM)$ induced by the geodesic flow $\{g_t\}$ on SM. This chapter includes a discussion of the Maslov index and a proof of Mañé's theorem on continuous invariant Lagrangian sub-bundles of $T(SM)$, stated below. In chapter 3 the topological entropy of the geodesic flow and related concepts are discussed, including the counting function $n_T(x, y)$, which for a positive number T is the number of geodesics of length at most T that join two points x and y of M. Recall that the geodesics from x to y are precisely the critical points of the arc length functional on the path space $\Omega(x, y)$ of all piecewise differentiable curves in M from x to y. For fixed $T > 0$ the function $n_T(x, y)$ is integrable on $M \times M$ and locally constant on a dense open subset of $M \times M$. In chapter 4, Mañé's formulae for the topological entropy of geodesic flows and for convex billiards are proved.

https://doi.org/10.1017/S0143385700000845 Published online by Cambridge University Press

In the final chapter, topological entropy and loop space homology are related and some open problems are posed.

We now give an account of some of the main results. Roughly speaking, the book has two main directions. The first concerns the properties of the action induced by the geodesic flow on the bundle $\Lambda(SM)$ of Lagrangian subspaces of $T(SM)$. The culmination of this discussion is a proof in dimension two of the following result of Mañé [**Ma1**].

THEOREM. *Let M be a compact Riemannian manifold that admits a continuous Lagrangian sub-bundle E of $\Lambda(SM)$ that is invariant under the flow $\{g_t^*\}$ of $\Lambda(SM)$ induced by the geodesic flow $\{g_t\}$ on SM. Let $V = \text{kernel}\,(d\pi)$ denote the vertical subbundle of $T(SM)$, where $\pi : SM \to M$ is the projection. Then $E(v) \cap V(v) = \{0\}$ for all $v \in SM$, and the geodesic flow of SM has no conjugate points.*

As an application, if the geodesic flow $\{g_t\}$ in SM is Anosov, then the stable bundle $E = E^s$ (or the unstable bundle $E = E^u$) is a continuous invariant Lagrangian sub-bundle of $T(SM)$. We conclude from the theorem that the geodesic flow of SM has no conjugate points, a result first proved by Klingenberg [**K**]. The geodesic flow is said to have no conjugate points in SM if for every $\xi \in T(SM)$ the orbit $\{dg_t(\xi)\}$ intersects the vertical sub-bundle V of $T(SM)$ at most once. Equivalently, any Jacobi vector field on a geodesic of M vanishes at most once.

The second main direction of the book is a discussion of the topological entropy $h_{\text{top}}(g)$ of the geodesic flow in SM of a compact manifold M with a Riemannian metric g. The discussion splits into two themes, of which we give only a flavor here. The first theme concerns the formulae and inequalities for the geodesic flow in terms of the asymptotic behavior of other geometric quantities, such as the geodesic counting function $n_T : M \times M \to \mathbb{Z}^+$ or vol $(B_T(x))$, the volume of the closed metric ball with radius T and center x in the universal cover $\tilde{M}$. The main result, due to Mañé [**Ma2**], consists of four geometric integral formulae for $h_{\text{top}}(g)$, one of which is $h_{\text{top}}(g) = \lim_{T\to\infty} T^{-1} \log \int_{M\times M} n_T(x, y)\, dx\, dy$. The book also discusses an analogous formula of Mañé for convex billiards. In [**Man1**], Manning considered the asymptotic growth rate of metric balls in the universal cover $\tilde{M}$, namely, $\lambda = \lim_{T\to\infty} T^{-1} \int_{\tilde{M}} \text{vol}(B_T(x))\, dx$. Manning showed that $\lambda \leq h_{\text{top}}(g)$ with equality if the geodesic flow in SM has no conjugate points. In [**Man2**], Manning also showed that $h_{\text{top}}(g) = \lambda = \lim_{T\to\infty} T^{-1} n_T(x, y)$ for all points x, y in M if the geodesic flow in SM has no conjugate points. The text also contains proofs of these two results of Manning.

The second main theme of the discussion of topological entropy concerns the existence of topological conditions on a compact differentiable manifold M that ensure that $h_{\text{top}}(g)$ is positive for any C^∞ Riemannian metric g on M. One of the oldest is a result of Dinaburg [**D**] which states that $h_{\text{top}}(g)$ is positive for any Riemannian metric g on M if the fundamental group of M has exponential growth. For example, if M admits a metric g whose sectional curvature is strictly negative, then the fundamental group of M has exponential growth [**Mi**]. At the opposite extreme one may consider topological conditions on compact simply connected differentiable manifolds M. In [**FH**] Félix and Halperin show that for an n-dimensional manifold M of this sort, either the homotopy

https://doi.org/10.1017/S0143385700000845 Published online by Cambridge University Press

groups $\pi_i(M)$ are finite for all $i > 2n-1$ (rationally elliptic manifolds) or the integers $\rho_i = \sum_{j \le i} \dim(\pi_j(M) \otimes \mathbb{Q})$ grow exponentially in i (rationally hyperbolic manifolds).

For compact, simply connected manifolds M, the rationally elliptic manifolds are scarce although they include all homogeneous spaces such as spheres, complex projective spaces and universal covers of compact Lie groups with finite center. In dimension four there are only five homeomorphism types: S^4, $S^2 \times S^2$, $\mathbb{C}P^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ [**P1**]. In [**P2**] the author shows that if M is compact, simply connected and rationally hyperbolic, or more generally if M is the compact total space of a fiber bundle over a compact, simply connected rationally hyperbolic manifold X, then $h_{\text{top}}(g)$ is positive for any C^∞ Riemannian metric g on M. As a corollary, Paternain uses the list of rationally elliptic 4-manifolds above to show that for every finitely presented group Γ there exists an 8-manifold M with fundamental group Γ such that $h_{\text{top}}(g)$ is positive for any C^∞ Riemannian metric g on M.

The rational homotopy type of a compact, simply connected manifold M also has a powerful effect on the behavior of the geodesics of an arbitrary Riemannian metric g on M. See, for example, the discussions in [**G, GH, GHV**].

The book concludes with a discussion of the relationships between other entropies, normalized by volume or diameter over all Riemannian metrics, and the simplicial and minimal volumes of Gromov.

Open problems and exercises are sprinkled throughout the book, and at the end one may find hints and answers for some of the exercises. This book will be useful both to ergodic theorists and geometers, and a valuable addition to the more traditional curvature-oriented treatments of geodesic flows.

References

[A] D. V. Anosov. Geodesic flows on Riemannian manifolds of negative curvature. *Proc. Steklov Instit. Math.* **90** (1969).

[B] W. Ballmann. *Lectures on Spaces of Nonpositive Curvature (DMV Seminar, 25)*. Birkhaüser, Basel, 1995.

[BGS] W. Ballmann, M. Gromov and V. Schroeder. *Manifolds of Nonpositive Curvature (Progress in Mathematics, 61)*. Birkhaüser, Boston, 1985.

[CE] J. Cheeger and D. Ebin. *Comparison Theorems in Riemannian Geometry*. North-Holland, Amsterdam, 1975.

[D] E. I. Dinaburg. On the relation of various entropy characteristics of dynamical systems. *Math. USSR Izv.* **5** (1971), 337–378.

[E] P. Eberlein. *Geometry of Nonpositively Curved Manifolds (Chicago Lectures in Mathematics)*. University of Chicago Press, 1996.

[F] R. Feres. *Dynamical Systems and Semisimple Groups, An Introduction*. Cambridge University Press, Cambridge, 1998.

[FH] Y. Félix and S. Halperin. Rational Lusternik Schnirelmann category and its applications. *Trans. Amer. Math. Soc.* **273** (1982), 1–37.

[G] K. Grove. *The Isometry-Invariant Geodesics Problem: Closed And Open (Lecture Notes in Mathematics, 1167)*. Springer, 1985, pp. 125–140.

[GH] K. Grove and S. Halperin. Contributions of rational homotopy theory to global problems in geometry. *Publ. I.H.E.S.* **56** (1982, 171–177.

https://doi.org/10.1017/S0143385700000845 Published online by Cambridge University Press

[GHV] K. Grove, S. Halperin and M. Vigué-Poirrier. The rational homotopy theory of certain path spaces with applications to geodesics. *Acta Math.* **140** (1978), 277–303.

[GKM] D. Gromoll, W. Klingenberg and W. Meyer. *Riemannsche Geometrie im Grossen (Lecture Notes in Mathematics, 55)*. Springer, Heidelberg, 1968.

[K] W. Klingenberg. Riemannian manifolds with geodesic flows of Anosov type. *Ann. Math.* **99** (1974), 1–13.

[Ma1] R. Mañé. On a theorem of Klingenberg. *Dynamical Systems and Bifurcation Theory (Pitman Research Notes in Mathematics, 160)*. Eds. M. Camacho, M. Pacifico and F. Takens. Longman Scientific and Technical, Harlow; Wiley, New York, 1987, pp. 319–345.

[Ma2] R. Mañé. On the topological entropy of geodesic flows. *J. Diff. Geom.* **45** (1997), 74–93.

[Man1] A. Manning. Topological entropy for geodesic flows. *Ann. Math.* **110** (1979), 567–573.

[Man2] A. Manning. More topological entropy for geodesic flows. *Dynamical Systems and Turbulence, Warwick, 1980 (Lecture Notes in Mathematics, 898)*. Eds. D. Rand and L.-S. Young. Springer, 1981, pp. 243–249.

[Mi] J. Milnor. A note on curvature and fundamental group. *J. Diff. Geom.* **2** (1968), 1–7.

[P1] G. Paternain. Topological entropy for geodesic flows on fibre bundles over rationally hyperbolic manifolds. *Proc. Amer. Math. Soc.* **125** (1997), 2759–2765.

[P2] G. Paternain. Differentiable structures with zero entropy on simply connected 4-manifolds. *Bol. Soc. Brasil Mat. (N.S.)* **31**(1) (2000), 1–8.

[Z] R. Zimmer. *Ergodic Theory and Semisimple Groups*. Birkhaüser, Basel, 1984.

PATRICK EBERLEIN
Mathematics Department
CB# 3250 Phillips Hall
The University of North Carolina at Chapel Hill
Chapel Hill
North Carolina 27599-3250
USA
(e-mail: pbe@math.unc.edu)

https://doi.org/10.1017/S0143385700000845 Published online by Cambridge University Press