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Cup-length estimates for leaf-wise intersections

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Abstract

We prove that on a restricted contact type hypersurface the number of leaf-wise intersections is bounded from below by a certain cup-length.

1. *Introduction*

Let (M, ω) be a symplectic manifold and $\Sigma \subset M$ be a hypersurface. Then Σ is foliated by the characteristic foliation induced by the line bundle $\ker \omega|_\Sigma \to \Sigma$. We denote by L_x the leaf through $x \in \Sigma$. Let $\psi \in \mathrm{Ham}(M)$ be a Hamiltonian diffeomorphisms. Then a leaf-wise intersection is a point $x \in \Sigma$ with the property that $\psi(x) \in L_x$.

Definition 1·1. Let $\iota : \Sigma \hookrightarrow M$ be the inclusion map.

(1) We denote by

$$\mathrm{cl}(\Sigma, M) := \max\{k \mid \exists\, a_1, ..., a_k \in \mathrm{H}^{\geqslant 1}(M; \mathbb{Z}/2) \text{ with } \iota^*\big(a_1 \cup \ldots \cup a_k\big) \neq 0\} \quad (1{\cdot}1)$$

the relative cup-length of Σ in M.

(2) Suppose $\omega = d\lambda$ and that Σ is of restricted contact type, i.e. $\alpha := \iota^*\lambda$ is a contact form on Σ. Then we denote by $\wp(\Sigma, \alpha) > 0$ the minimal period of a Reeb orbit of (Σ, α) which is contractible in M. If there exists no such Reeb orbit we set $\wp(\Sigma, \alpha) = \infty$.

(3) We denote by $\mathrm{Ham}_c(M)$ the space of Hamiltonian diffeomorphisms generated by compactly supported, time dependent Hamiltonian functions and by $\|\psi\|$ the Hofer norm.

(4) (M, ω) is called convex at infinity if it is isomorphic to the symplectization of a compact contact manifold at infinity.

THEOREM 1. *Let $\Sigma \subset (M, d\lambda)$ be a closed, bounding, restricted contact type hypersurface and $(M, d\lambda)$ be convex at infinity. If $\psi \in \mathrm{Ham}_c(M)$ satisfies $\|\psi\| < \wp(\Sigma, \alpha)$ then*

$$\nu_{\mathrm{leaf}}(\psi) := \#\{\textit{leaf-wise intersections of } \psi\} \geqslant \mathrm{cl}(\Sigma, M) + 1. \quad (1{\cdot}2)$$

Remark 1·2. The search for leaf-wise intersections was initiated by Moser in [**Mos78**] and pursued further in [**Ban80**, **Hof90**, **EH89**, **Gin07**, **Dra08**, **AF10a**, **Zil08**, **AF08**, **AM09**,

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Gür09, **Kan09**, **AF10b**, **AF10c**, **Mer10**]. We refer to [**AF08**] for a brief history of the problem.

Remark 1·3. The example $S^1 \subset \mathbb{C}$ shows that if $\|\psi\| > \wp(\Sigma, \alpha)$ then $\nu_{\text{leaf}}(\psi) = 0$ is possible.

Remark 1·4. The relative cup-length is smaller than the cup-length of M and Σ. In the case $\Sigma = S^*B \subset T^*B$ is a unit cotangent bundle we have $\text{cl}(S^*B, T^*B) \geqslant \text{cl}(B) - 1$ by examining the Gysin sequence.

Remark 1·5. For simplicity we shall use $\mathbb{Z}/2$-coefficients for all (co-)homology theories in this paper. We expect that Theorem 1 continues to hold with $\mathbb{Z}$-coefficients.

Remark 1·6. Cup-length estimates have been established for Lagrangian intersections by Floer [**Flo89**], Hofer [**Hof88**], and Liu [**Liu05**] in terms of the cup-length of the Lagrangian submanifold and for fixed points of Hamiltonian diffeomorphisms by Schwarz [**Sch98**] in terms of the quantum cup-length of the symplectic manifold.

2. *Leaf-wise intersections and the Rabinowitz action functional*

Let (M, ω) be a symplectic manifold and $f \in C^\infty(M)$ an autonomous Hamiltonian function. Since energy is preserved the hypersurface $\Sigma := f^{-1}(0)$ is invariant under the Hamiltonian flow ϕ_f^t of f. The Hamiltonian flow ϕ_f^t is generated by the Hamiltonian vector field X_f which is uniquely defined by the equation $\omega(X_f, \cdot) = df$. If 0 is a regular value of f the hypersurface is a coisotropic submanifold which is foliated by 1-dimensional isotropic leaves, see [**MS98**, section 3·3]. If we denote by L_x the leaf through $x \in \Sigma$ we have the equality

$$L_x = \bigcup_{t \in \mathbb{R}} \phi_f^t(x). \tag{2·1}$$

Given a time-dependent Hamiltonian function $H : [0, 1] \times M \to \mathbb{R}$ with Hamiltonian flow ϕ_H^t we are interested in points $x \in \Sigma$ with the property

$$\phi_H^1(x) \in L_x. \tag{2·2}$$

This notion was introduced and studied by Moser in [**Mos78**]. Such points are called leaf-wise intersections. We recall some notions from [**AF10a**].

Definition 2·1. A leaf-wise intersection $x \in \Sigma$ is called periodic if the leaf L_x is a closed orbit of the flow ϕ_f^t.

Definition 2·2. A pair $\mathfrak{M} = (F, H)$ of Hamiltonian functions $F, H : S^1 \times M \to \mathbb{R}$ is called a Moser pair if it satisfies

$$F(t, \cdot) = 0 \quad \forall t \in \left[0, \tfrac{1}{2}\right] \qquad \text{and} \qquad H(t, \cdot) = 0 \quad \forall t \in \left[\tfrac{1}{2}, 1\right], \tag{2·3}$$

and F is of the form $F(t, x) = \rho(t) f(x)$ for some smooth map $\rho : S^1 \to [0, \infty)$ with $\int_0^1 \rho(t) dt = 1$ and $f : M \to \mathbb{R}$.

Definition 2·3. We set

$$\mathcal{H} := \left\{ H \in C^\infty(S^1 \times M) \mid H \text{ has compact support and } H(t, \cdot) = 0 \quad \forall t \in \left[\tfrac{1}{2}, 1\right] \right\} \tag{2·4}$$

Remark 2·4. It is easy to see that the $\text{Ham}(M, \omega) \equiv \{\phi_H^1 \mid H \in \mathcal{H}\}$, e.g. [**AF10a**].

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Let $(M, \omega = d\lambda)$ be an exact symplectic manifold. Then for a Moser pair $\mathfrak{M} = (F, H)$ the perturbed Rabinowitz action functional is defined by

$$\begin{aligned} \mathcal{A}^{\mathfrak{M}} : \mathscr{L}_M \times \mathbb{R} &\longrightarrow \mathbb{R} \\ (v, \eta) &\longmapsto -\int_{S^1} v^*\lambda - \int_0^1 H(t, v)\, dt - \eta \int_0^1 F(t, v)\, dt \end{aligned} \tag{2·5}$$

where $\mathscr{L}_M := C^\infty(S^1, M)$. A critical point (v, η) of $\mathcal{A}^{\mathfrak{M}}$ is a solution of

$$\left.\begin{aligned} \partial_t v = \eta X_F(t, v) + X_H(t, v) \\ \int_0^1 F(t, v)\, dt = 0 \end{aligned}\right\}. \tag{2·6}$$

In [**AF10a**] it is proved that critical points of $\mathcal{A}^{\mathfrak{M}}$ give rise to leaf-wise intersections.

PROPOSITION 2·5 ([**AF10a**]). *Let (v, η) be a critical point of $\mathcal{A}^{\mathfrak{M}}$. Then $x := v(0) \in f^{-1}(0)$ and*

$$\phi_H^1(x) \in L_x \tag{2·7}$$

thus, x is a leaf-wise intersection.

Moreover, the map $\mathrm{Crit}\mathcal{A}^{\mathfrak{M}} \to \{$*leaf-wise intersections*$\}$ *is injective unless there exists a periodic leaf-wise intersection (see Definition* 2·1*).*

Definition 2·6. A Moser pair $\mathfrak{M} = (F, H)$ is of restricted contact type if the following four conditions hold.

(1) 0 is a regular value of f.
(2) df has compact support.
(3) The hypersurface $f^{-1}(0)$ is a closed restricted contact type hypersurface of (M, λ).
(4) The Hamiltonian vector field X_f restricts to the Reeb vector field on $f^{-1}(0)$.

Remark 2·7. If $\Sigma \subset (M, d\lambda)$ is a closed, bounding, restricted contact type hypersurface then it is easy to construct a restricted contact type Moser pair $\mathfrak{M}_0 = (F, 0)$ with $\Sigma = f^{-1}(0)$. We fix such a Moser pair for the rest of the paper. Critical points (v, η) of the unperturbed Rabinowitz action functional $\mathcal{A}^{(F,0)}$ are η-periodic Reeb orbits in Σ or if $\eta = 0$ constant loops in Σ. Their critical values are $\mathcal{A}^{(F,0)}(v, \eta) = -\eta$.

In [**CF09**] Cieliebak and Frauenfelder construct a Floer homology for the unperturbed Rabinowitz action functional. This has been extended to the perturbed case by the first author and Frauenfelder in [**AF10a**]. The corresponding Floer equation for maps $u : \mathbb{R} \times S^1 \to M$ and $\eta : \mathbb{R} \to \mathbb{R}$ is

$$\left.\begin{aligned} \partial_s u + J(s, t, u)\big(\partial_t u - \eta X_F(t, u) - X_{H_s}(t, u)\big) = 0 \\ \partial_s \eta - \int_0^1 F(t, u)\, dt = 0. \end{aligned}\right\} \tag{2·8}$$

Here J is a smooth (s, t)-dependent family of compatible almost complex structures and $H_s : S^1 \times M \to \mathbb{R}$ is a smooth s-dependent family of functions. Counting solutions of the s-independent equation modulo $\mathbb{R}$-action defines the boundary operator in Rabinowitz Floer homology. In this paper we do not need the full machinery of Rabinowitz Floer homology.

We set $w = (u, \eta)$ for solutions of (2·6). We will think of w also as a map $w : \mathbb{R} \to \mathscr{L}_M \times \mathbb{R}$.

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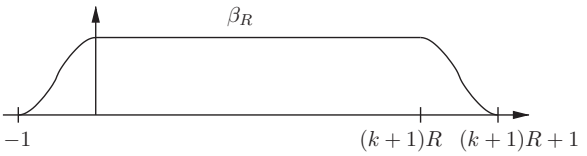


Fig. 1. The function β_R.

Definition 2·8. The energy of a map $w = (u, \eta)$ is defined as

$$E(w) := \int_{-\infty}^{\infty}\int_0^1 \|\partial_s u\|^2\, dt\, ds + \int_{-\infty}^{\infty} |\partial_s \eta|^2\, ds. \tag{2·9}$$

The following has been established in [**AF10a**].

LEMMA 2·9 ([**AF10a**, lemma 2·7]). *Let w solve* (2·8) *with limits*

$$w(\pm\infty) := \lim_{s\to\pm\infty} w(s) \in \mathrm{Crit}\mathcal{A}^{(F,H_\pm)} \tag{2·10}$$

then we have

$$E(w) \leqslant \mathcal{A}^{(F,H_-)}(w(-\infty)) - \mathcal{A}^{(F,H_+)}(w(+\infty)) - \int_{-\infty}^{\infty}\int_0^1 \min_M \partial_s H_s(t,\cdot)\, dt\, ds \tag{2·11}$$

and equality holds if $\partial_s H_s = 0$.

THEOREM 2·10. *Let $w_n = (u_n, \eta_n)$ be a sequence of solutions of* (2·8) *for which there exists $a < b$ such that*

$$a \leqslant \mathcal{A}^{(F,H)}(w_n(s)) \leqslant b \quad \forall s \in \mathbb{R}. \tag{2·12}$$

Then for every reparametrisation sequence $(\sigma_n) \subset \mathbb{R}$ the sequence $u_n(\cdot + \sigma_n)$ has a subsequence which converges in $C^\infty_{\mathrm{loc}}(\mathbb{R}\times S^1, M)$ and similarly for $\eta_n(\cdot+\sigma_n)$.

3. *Moduli spaces*

Let $k \geqslant 1$ be a natural number. We choose a smooth family of functions $\beta_R \in C^\infty(\mathbb{R}, [0,1])$ satisfying:

(1) for $R \geqslant 1$: $\beta_R'(s)\cdot s \leqslant 0$ for all $s \in \mathbb{R}$, $\beta_R(s) = 1$ for $s \in [0, (k+1)R]$, and $\beta_R(s) = 0$ for $s \leqslant -1$ and $s \geqslant (k+1)R + 1$;
(2) for $R \leqslant 1$: $\beta_R(s) \leqslant R$ for all $s \in \mathbb{R}$ and $\mathrm{supp}\beta_R \subset [-1, k+2]$;
(3) $\lim_{R\to\infty} \beta_R(s) =: \beta^+_\infty(s)$ and $\lim_{R\to\infty} \beta_R(s + (k+1)R) =: \beta^-_\infty(s)$ exists, where the limit is taken with respect to the C^∞_{loc} topology.

We fix a Hamiltonian function $H : [0,1]\times M \to \mathbb{R}$ with $H(t,\cdot) = 0$ for all $t \in [\frac{1}{2}, 1]$ and set $K_R(s,t,x) := \beta_R(s)H(t,x)$ and

$$\mathfrak{M}_R := (F, K_R(s,t,x)). \tag{3·1}$$

For every $R \geqslant 0$ we define

$$\mathcal{M}(R) := \left\{ w = (u,\eta) \in C^\infty(\mathbb{R}, \mathscr{L}_M \times \mathbb{R}) \;\middle|\; \begin{array}{l} w \text{ solves (2·8) with } \mathfrak{M}_R \\ \lim_{s\to\pm\infty} u(s) \in \Sigma \end{array} \right\}, \tag{3·2}$$

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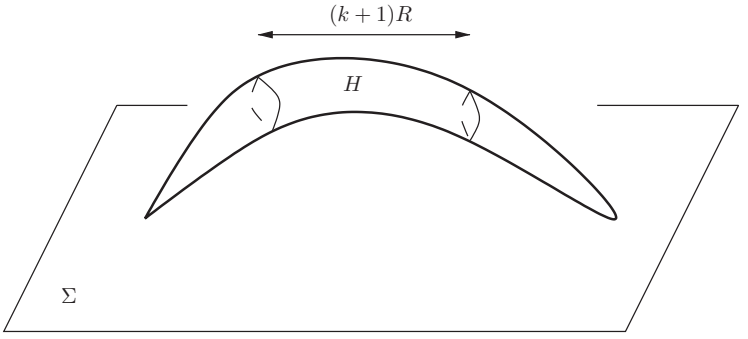


Fig. 2. An element of the moduli space $\mathcal{M}(R)$.

and

$$\mathcal{M}[0,R] := \{(r,w) \mid 0 \leqslant r \leqslant R \text{ and } w \in \mathcal{M}(r)\}, \tag{3·3}$$

see Figure 2.

LEMMA 3·1. *For $w \in \mathcal{M}(R)$ we have*

$$E(w) \leqslant \|H\|. \tag{3·4}$$

Moreover, for $R = 0$ we have $E(w) = 0$. Furthermore,

$$-2\int_0^1 \max_M |H(t,\cdot)|dt \leqslant \mathcal{A}^{\mathfrak{M}_R(s)}(w(s)) \leqslant 2\int_0^1 \max_M |H(t,\cdot)|dt \tag{3·5}$$

holds.

Proof. We compute using $\partial_s w + \nabla \mathcal{A}^{\mathfrak{M}_R(s)} = 0$

$$\begin{aligned}
\underbrace{\mathcal{A}^{\mathfrak{M}_0}(w(-\infty))}_{=0} - \mathcal{A}^{\mathfrak{M}_R(s)}(w(s)) &= -\int_{-\infty}^{s} \frac{d}{ds}\mathcal{A}^{\mathfrak{M}_R(s)}(w(s))\,ds \\
&= -\int_{-\infty}^{s} d\mathcal{A}^{\mathfrak{M}_R(s)}(w(s))\cdot\partial_s w ds - \int_{-\infty}^{s}\frac{\partial \mathcal{A}^{\mathfrak{M}_R(s)}}{\partial s}(w(s))\,ds \\
&= -\int_{-\infty}^{s}\langle \underbrace{\nabla \mathcal{A}^{\mathfrak{M}_R(s)}}_{=-\partial_s w}, \partial_s w\rangle ds + \int_{-\infty}^{s}\int_0^1 \frac{\partial K_R}{\partial s}(s,t,u(s,t))\,dt\,ds \\
&= \int_{-\infty}^{s}\|\partial_s w(s)\|^2\,ds + \int_{-\infty}^{s}\int_0^1 \frac{\partial K_R}{\partial s}(s,t,u(s,t))\,dt\,ds \\
&\geqslant \int_{-\infty}^{s}\left(\frac{\partial}{\partial s}\beta_R(s)\right)\int_0^1 H(t,u(s,t))\,dt\,ds \\
&\geqslant \underbrace{\int_{-\infty}^{0}\left(\frac{\partial}{\partial s}\beta_R(s)\right)ds}_{=1}\cdot\int_0^1 -\max_M |H(t,\cdot)|dt \\
&\quad + \underbrace{\int_0^{\infty}\left(\frac{\partial}{\partial s}\beta_R(s)\right)ds}_{=-1}\cdot\int_0^1 \max_M |H(t,\cdot)|dt \\
&\geqslant -2\int_0^1 \max_M |H(t,\cdot)|dt.
\end{aligned}$$

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The other inequality is proved by replacing $\int_{-\infty}^{s}$ by $\int_{s}^{\infty}$. The first two assertions follow from Lemma 2·9 together with the observation that $\mathcal{A}^{(F,\beta_R H)}(w(\pm\infty)) = 0$ and

$$\begin{aligned} -\int_{-\infty}^{\infty}\int_0^1 \min_M \partial_s K_R(s,t,\cdot)\,dt\,ds &= -\int_{-\infty}^{\infty}\int_0^1 \min_M\{\partial_s\beta_R(s)H(t,\cdot)\}\,dt\,ds \\ &= -\int_{-\infty}^{0}\int_0^1 \partial_s\beta_R(s)\min_M H(t,\cdot)\,dt\,ds \\ &\quad -\int_{0}^{+\infty}\int_0^1 \partial_s\beta_R(s)\max_M H(t,\cdot)\,dt\,ds \\ &= \beta_R(0)\|H\| \leqslant \|H\|. \end{aligned} \tag{3·6}$$

PROPOSITION 3·2. *For any $R \geqslant 0$ the moduli spaces $\mathcal{M}(R)$ and $\mathcal{M}[0,R]$ are compact. Moreover, $\mathcal{M}(0) \cong \Sigma$.*

Proof. This follows easily as in the proof of [**AF10a**, theorem A] as follows. Using Lemma 3·1 we can apply Theorem 2·10 to extract C^{∞}_{loc}-convergent subsequences of any sequence $w_n(s-\sigma_n)$ where $w_n \in \mathcal{M}(R)$ and $(\sigma_n) \subset \mathbb{R}$. Then it is proved in [**AF10a**] that if the sequence does not converge to an element in $\mathcal{M}(R)$ there has to exist a non-constant gradient flow line v of $\mathcal{A}^{(F,0)}$ with one asymptotic end on Σ. Therefore, there exists a Reeb orbit x of period η on (Σ,α) which is contractible in M. We conclude

$$E(v) \leqslant \limsup E(w_n) \leqslant \|H\| < \wp(\Sigma,\alpha). \tag{3·7}$$

On the other hand we can compute $E(v)$ by Lemma 2·9:

$$E(v) = |\eta| \geqslant \wp(\Sigma,\alpha) \tag{3·8}$$

where the inequality follows from the definition of $\wp(\Sigma,\alpha)$. This contradiction shows that $\mathcal{M}(R)$ is compact. That $\mathcal{M}[0,R]$ is compact follows in the same way.

If $R = 0$ then, according to, Lemma 3·1 $E(w) = 0$ for all $w \in \mathcal{M}(0)$. So $\partial_s w(s) = 0$ and $w(s) = (p,0) \in \mathrm{Crit}\mathcal{A}^{(F,0)}$ with $p \in \Sigma$ being the constant loop; thus, $\mathcal{M}(0) \cong \Sigma$.

Remark 3·3. The moduli space $\mathcal{M}(R)$ is the zero-set of a Fredholm section $\mathcal{F}(R)$ of a Banach space bundle. The Fredholm index of $\mathcal{F}(R)$ equals $\mathrm{ind}\,\mathcal{F}(R) = \dim\Sigma$. The moduli space $\mathcal{M}(0)$ contains only constant gradient flow lines with Lagrange multiplier $\eta = 0$, i.e. $\mathcal{M}(0) \cong \Sigma$. As shown in [**AF10a**] the Morse–Bott property of the defining function F implies that the Fredholm section is transverse to the zero-section for $R = 0$. In particular, $\mathcal{M}(0)$ is a smooth manifold. Since $\mathcal{M}(R)$ is compact it is a smooth manifold for R sufficiently small since transversality is an open property.

4. *Cohomology operations*

We fix a natural number $k \geqslant 1$. The moduli space $\mathcal{M}(R)$ carries an evaluation map

$$\begin{aligned} \mathrm{ev}_R : \mathcal{M}(R) &\longrightarrow M^k = M\times\cdots\times M \\ w = (u,\eta) &\longmapsto (u(R,0), u(2R,0), \ldots, u(kR,0)). \end{aligned} \tag{4·1}$$

For $R = 0$ the evaluation map is under the identification $\mathcal{M}(0) \cong \Sigma$ the diagonal embedding $\Sigma \hookrightarrow \Delta_\Sigma \subset M^k$. Similarly, $\mathcal{M}[0,R]$ has an evaluation map $\mathrm{EV}(r,w) := \mathrm{ev}_r(w)$ to M^k.

Next, we define a Morse theoretic realization of the cohomology operation

$$\begin{aligned} \Theta : \mathrm{H}^*(M)\otimes\cdots\otimes\mathrm{H}^*(M)\otimes\mathrm{H}_*(\Sigma) &\longrightarrow \mathrm{H}_*(\Sigma) \\ a_1\otimes\cdots\otimes a_k\otimes b &\longmapsto \iota^*(a_1\cup\ldots\cup a_k)\cap b \end{aligned} \tag{4·2}$$

where $\iota : \Sigma \hookrightarrow M$ is the inclusion map.

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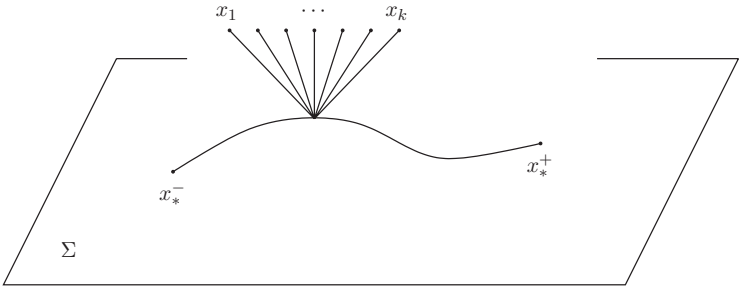


Fig. 3. An element of the moduli space $\mathcal{M}(0, x_1, \ldots, x_k, x_*^-, x_*^+)$.

Remark 4·1. Since the symplectic manifold (M, ω) is exact it either has boundary or is non-compact. In the following we will choose Morse functions on M in order to model singular (co-)homology by Morse (co-)homology. For this we need to restrict to a certain class of Morse functions. We are always considering the negative gradient flow. If M has boundary then we assume that the (positive) gradient of the Morse function points outward along the boundary of M. If M is non-compact we assume that the Morse function is proper and bounded from below. Under these assumptions standard Morse (co-)homology can be defined and is isomorphic to singular (co-)homology. From now on we assume whenever we choose Morse functions on M they are in the just prescribed class.

For that we choose Morse functions $f_1, \ldots, f_k : M \to \mathbb{R}$ and $f_* : \Sigma \to \mathbb{R}$ and Riemannian metrics $g_1, \ldots, g_k, g_*$. We set for critical points $x_j \in \mathrm{Crit}(f_j)$ and $x_*^\pm \in \mathrm{Crit}(f_*)$

$$\begin{aligned}&\mathcal{M}(0, x_1, \ldots, x_k, x_*^-, x_*^+)\\ &:= \left\{ w = (u, \eta) \in \mathcal{M}(0) \;\middle|\; \begin{array}{l} u(-\infty) \in W^u(x_*^-, f_*),\ u(+\infty) \in W^s(x_*^+, f_*) \\ \mathrm{ev}_0(u) \in W^s(x_1, f_1) \times \cdots \times W^s(x_k, f_k) \end{array} \right\},\end{aligned}$$

see Figure 3. In particular, $\mathcal{M}(0, x_1, \ldots, x_k, x_*^\pm) = \varnothing$ unless $\bigcap W^s(x_i, f_i) \cap \Sigma \neq \varnothing$. We denote by $\mathrm{CM}^*(f)$ the Morse chain complex associated to a Morse-Smale pair (f, g).

PROPOSITION 4·2. *For generic Morse functions f_i and Riemannian metrics g_i the moduli space $\mathcal{M}(0, x_1, \ldots, x_k, x_*^-, x_*^+)$ is a smooth manifold. Moreover, the map defined by*

$$\begin{aligned}\theta_0 : \mathrm{CM}^*(f_1) \otimes \cdots \otimes \mathrm{CM}^*(f_k) \otimes \mathrm{CM}_*(f_*) &\longrightarrow \mathrm{CM}_*(f_*)\\ x_1 \otimes \cdots \otimes x_k \otimes x_*^- &\longmapsto \sum_{x_*^+} \#_2 \mathcal{M}(0, x_1, \ldots, x_k, x_*^-, x_*^+) \cdot x_*^+\end{aligned} \tag{4·3}$$

defines a chain map which on homology agrees with the cohomology operation Θ.

Remark 4·3. $\#_2\mathcal{M}$ denotes the parity of the set $\mathcal{M}$ if it is finite and zero otherwise.

Proof. It follows from standard Morse theory that for generic Morse functions f_i and Riemannian metrics g_i we have

$$W^s(x_1, f_1) \times \cdots \times W^s(x_k, f_k) \pitchfork \Delta_\Sigma \subset M^k \tag{4·4}$$

where $\Delta_\Sigma \subset M^k$ is the diagonal embedding of Σ. Since for $R = 0$ the evaluation map $\mathrm{ev}_0 : \mathcal{M}(0) \to M^k$ is this diagonal embedding we conclude

$$W^s(x_1, f_1) \times \cdots \times W^s(x_k, f_k) \pitchfork \mathrm{ev}_0. \tag{4·5}$$

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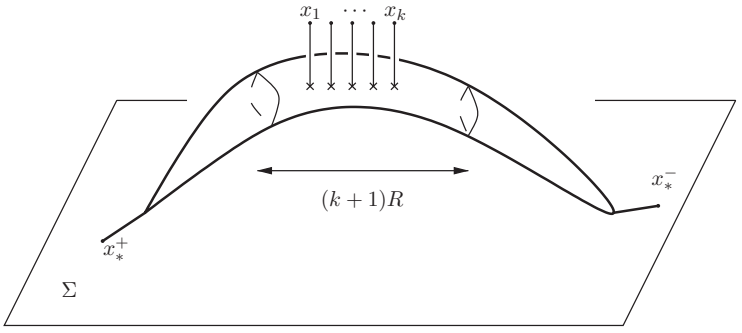


Fig. 4. An element of the moduli space $\widetilde{\mathcal{M}}(R, x_1, \ldots, x_k, x_*^-, x_*^+)$.

Finally, choosing f_* and g_* generic we see that the moduli space $\mathcal{M}(0, x_1, \ldots, x_k, x_*^-, x_*^+)$ is smooth. That $\theta_0 = \Theta$ after identifying Morse homology with singular homology is again standard Morse theory, see [**Sch93**].

Since the moduli space $\mathcal{M}(R)$ is the zero-set of a Fredholm section $\mathcal{F}(R)$ and since $\mathcal{M}(R)$ is compact we can choose an arbitrarily small abstract perturbation of the Fredholm section $\mathcal{F}(R)$ such that the zero-set $\widetilde{\mathcal{M}}(R)$ of the perturbed Fredholm section $\widetilde{\mathcal{F}}(R)$ is a smooth compact finite-dimensional manifold. Since $\mathcal{M}(R)$ is already transverse for sufficiently small R we can arrange that $\widetilde{\mathcal{M}}(R) = \mathcal{M}(R)$ for small R. We point out that $\mathcal{M}[0, R]$ has a natural projection to $[0, R]$. The same abstract perturbation procedure gives rise to smooth perturbed moduli spaces $\widetilde{\mathcal{M}}[0, R]$ where the perturbation can be chosen with fixed ends, i.e., the fibers over 0 resp. R are $\mathcal{M}(0)$ resp. $\widetilde{\mathcal{M}}(R)$. The perturbed moduli space $\widetilde{\mathcal{M}}(R)$ resp. $\widetilde{\mathcal{M}}[0, R]$ still carries an evaluation map $\widetilde{\mathrm{ev}}(R) : \widetilde{\mathcal{M}}(R) \to M^k$ resp. $\widetilde{\mathrm{EV}} : \widetilde{\mathcal{M}}[0, R] \to M^k$.

For Morse functions $f_1, \ldots, f_k : M \to \mathbb{R}$ and $f_* : \Sigma \to \mathbb{R}$, Riemannian metrics $g_1, \ldots, g_k, g_*$, and critical points $x_j \in \mathrm{Crit}(f_j)$ and $x_*^\pm \in \mathrm{Crit}(f_*)$ we set

$$\widetilde{\mathcal{M}}(R, x_1, \ldots, x_k, x_*^-, x_*^+) := \left\{ w = (u, \eta) \in \widetilde{\mathcal{M}}(R) \;\middle|\; \begin{array}{l} u(-\infty) \in W^u(x_*^-, f_*),\ u(+\infty) \in W^s(x_*^+, f_*) \\ \widetilde{\mathrm{ev}}_R(u) \in W^s(x_1, f_1) \times \cdots \times W^s(x_k, f_k) \end{array} \right\},$$

see Figure 4.

PROPOSITION 4·4. *For generic Morse functions f_i and Riemannian metrics g_i the moduli space $\widetilde{\mathcal{M}}(R, x_1, \ldots, x_k, x_*^-, x_*^+)$ is a smooth manifold. Moreover, the map defined by*

$$\begin{aligned} \theta_R : \mathrm{CM}^*(f_1) \otimes \cdots \otimes \mathrm{CM}^*(f_k) \otimes \mathrm{CM}_*(f_*) &\longrightarrow \mathrm{CM}_*(f_*) \\ x_1 \otimes \cdots \otimes x_k \otimes x_*^- &\longmapsto \sum_{x_*^+} \#_2 \widetilde{\mathcal{M}}(R, x_1, \ldots, x_k, x_*^-, x_*^+) \cdot x_*^+ \end{aligned} \tag{4·6}$$

defines a chain map which is chain homotopic to the cohomology operation θ_0.

Proof. For generic Morse functions f_i, generic Riemannian metrics g_i, and generic perturbation of the Fredholm section we have

$$W^s(x_1, f_1) \times \cdots \times W^s(x_k, f_k) \pitchfork \widetilde{\mathrm{ev}}_R. \tag{4·7}$$

For generic f_* and g_* the stable and unstable manifolds are transversal to the evaluation

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maps $u \mapsto u(\pm\infty)$ and thus the moduli space $\widetilde{\mathcal{M}}(R, x_1, \ldots, x_k, x_*^-, x_*^+)$ is smooth. The map θ_R is a chain map by standard Morse theory since $\widetilde{\mathcal{M}}(R)$ is compact.

To prove that θ_R is chain homotopic to θ_0 we recall that $\widetilde{\mathcal{M}}[0, R]$ is perturbed while keeping the ends $\mathcal{M}(0)$ and $\widetilde{\mathcal{M}}(R)$ fixed. Perturbing further (with fixed ends), the evaluation map $\widetilde{\mathrm{EV}} : \widetilde{\mathcal{M}}[0, R] \to M^k$ will be transverse to all products of unstable manifolds. Thus, the moduli space $\widetilde{\mathcal{M}}[0, R]$ together with $\widetilde{\mathrm{EV}}$ induces a cobordism between the moduli spaces $\widetilde{\mathcal{M}}(R, x_1, \ldots, x_k, x_*^{\pm})$ and $\widetilde{\mathcal{M}}(0, x_1, \ldots, x_k, x_*^{\pm})$. This give rise to a chain homotopy operator between θ_0 and θ_R.

5. *Proof of theorem* 1

We assume that ψ has only finitely many leaf-wise intersections since otherwise we are done. We set $k := \mathrm{cl}(\Sigma, M)$. It has been proved in [**AF10a**] that $\nu_{\mathrm{leaf}}(\psi) \geqslant 1$ if $\|\psi\| < \wp(\Sigma, \alpha)$. Thus, we may assume that $k \geqslant 1$. We choose a Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$ with $H(t, \cdot) = 0$ for $t \in [\frac{1}{2}, 1]$ such that $\phi_H^1 = \psi$. Let $\mathcal{LI} \subset \Sigma$ be the set of leaf-wise intersections of ψ.

We choose Morse functions $f_1, \ldots, f_k$ and Riemannian metrics $g_1, \ldots, g_k$ on M and f_*, g_* on Σ with the following properties:

(1) for $x_i \in \mathrm{Crit}(f_i)$ with Morse index $\mu_{\mathrm{Morse}}(x_i, f_i) \neq 0$ we have $W^s(x_i, f_i) \cap \mathcal{LI} = \varnothing$;
(2) for all $n \in \mathbb{N}$ the evaluation maps $\widetilde{\mathrm{ev}}_n : \widetilde{\mathcal{M}}(n) \to M^k$ are transverse to the products of stable manifolds of the (f_i, g_i);
(3) for all $n \in \mathbb{N}$ the evaluation maps at $\pm\infty$ are transverse to all stable and unstable manifolds of (f_*, g_*).

The second and third property holds for a fixed n for a Baire set of Morse functions and Riemannian metrics as explained in the proofs of Propositions 4·2 and 4·4. Intersecting these Baire sets over $n \in \mathbb{N}$ we see (2) and (3) is a generic property. Avoiding finitely many unstable manifolds not of top dimension is clearly also a generic condition.

By definition of k we find cohomology classes $a_1, \ldots, a_k \in \mathrm{H}^{\geqslant 1}(M)$ with $\iota^*(a_1 \cup \ldots \cup a_k) \neq 0 \in \mathrm{H}^*(\Sigma)$. Thus, the cohomology operation Θ, see equation (4·2), is non-zero. According to Propositions 4·2 and 4·4 the cohomology operations θ_R then have to be non-zero for all $R \geqslant 0$. Thus, we can find critical points $x_i \in \mathrm{Crit}(f_i)$ with Morse index $\mu_{\mathrm{Morse}}(x_i, f_i) \neq 0$ and $x_*^{\pm} \in \mathrm{Crit}(f_*)$ such that for all $n \in \mathbb{N}$ the moduli spaces

$$\widetilde{\mathcal{M}}(n, x_1, \ldots, x_k, x_*^-, x_*^+) \neq \varnothing. \tag{5·1}$$

In fact, the choice of $x_i \in \mathrm{Crit}(f_i)$ will depend on n in general. For notational convenience we suppress the n. In fact, there exists a subsequence n_k for which we can choose x_i fixed. We conclude

$$\widetilde{\mathcal{M}}(n) \neq \varnothing. \tag{5·2}$$

If the unperturbed moduli space $\mathcal{M}(n)$, see equation (3·2), were empty then for sufficiently small perturbations of the Fredholm section also the moduli space $\widetilde{\mathcal{M}}(n)$ would be empty. Indeed, if $\mathcal{M}(n) = \varnothing$ then the corresponding Fredholm section is transverse. Since $\mathcal{M}(n)$ is compact a sufficiently small perturbation of the Fredholm section remains empty and thus transverse. Therefore, we conclude that

$$\mathcal{M}(n) \neq \varnothing. \tag{5·3}$$

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Now we can choose a sequence $w_n \in \mathcal{M}(n)$, that is, w_n solves equation (2·8) with Hamiltonian perturbation $\beta_n(s)H(t,x)$. Since $\widetilde{\mathcal{M}}(n) \cap_{\widetilde{\mathrm{ev}}_n} (W^s(x_1, f_1) \times \cdots \times W^s(x_k, f_k)) \neq \varnothing$ and all evaluation maps are transverse a similar argument as above allows us to conclude that

$$\mathrm{ev}_n(w_n) \in W^s(x_1, f_1) \times \cdots \times W^s(x_k, f_k). \tag{5·4}$$

By Lemma 3·1 the action $\mathcal{A}^{\mathfrak{M}_n}(w_n(s))$ is uniformly bounded and we can apply Theorem 2·10. We consider the following sequences

$$w_n(s + jn), \quad j = 0, \ldots, k+1. \tag{5·5}$$

For $j = 1, \ldots, k$ these sequences converge (after choosing a subsequence) to solutions $\varpi^j = (u^j, \eta^j) : \mathbb{R} \to \mathscr{L}_M \times \mathbb{R}$ of

$$\left.\begin{aligned} \partial_s u^j + J(s,t,u^j)\big(\partial_t u^j - \eta^j X_F(t,u) - X_H(t,u^j)\big) = 0 \\ \partial_s \eta^j - \int_0^1 F(t,u^j)dt = 0 \end{aligned}\right\} \tag{5·6}$$

of energy

$$E(\varpi^j) \leqslant \limsup E(w_n) \leqslant \|H\|. \tag{5·7}$$

The sequence $w_n(s)$ converges to a solution $\varpi^0 = (u^0, \eta^0)$ of

$$\left.\begin{aligned} \partial_s u^0 + J(s,t,u^0)\big(\partial_t u^0 - \eta^0 X_F(t,u) - \beta_\infty^+(s) X_H(t,u^0)\big) = 0 \\ \partial_s \eta^0 - \int_0^1 F(t,u^0)dt = 0 \end{aligned}\right\} \tag{5·8}$$

and the sequence $w_n(s + (k+1)n)$ converges to a solution $\varpi^{k+1} = (u^{k+1}, \eta^{k+1})$ of

$$\left.\begin{aligned} \partial_s u^{k+1} + J(s,t,u^{k+1})\big(\partial_t u^{k+1} - \eta^{k+1} X_F(t,u) - \beta_\infty^-(s) X_H(t,u^{k+1})\big) = 0 \\ \partial_s \eta^{k+1} - \int_0^1 F(t,u^{k+1})dt = 0 \end{aligned}\right\} \tag{5·9}$$

where $\beta_\infty^\pm$ are defined at the beginning of Section 3. In particular,

$$(y_\pm^j, \eta_\pm^j) := \varpi^j(\pm\infty) \in \mathrm{Crit}\mathcal{A}^{(F,H)}, \quad j = 1, \ldots, k, \tag{5·10}$$

and

$$(y_+^0, \eta_+^0) := \varpi^0(+\infty), \ (y_-^{k+1}, \eta_-^{k+1}) := \varpi^{k+1}(-\infty) \in \mathrm{Crit}\mathcal{A}^{(F,H)} \tag{5·11}$$

are critical points. Thus, $y_\pm^j(0)$ are leaf-wise intersections, see Proposition 2·5. Moreover, they are ordered by action as follows

$$\begin{aligned} \mathcal{A}^{(F,H)}(y_+^0, \eta_+^0) \geqslant \mathcal{A}^{(F,H)}(y_-^1, \eta_-^1) \geqslant \mathcal{A}^{(F,H)}(y_+^1, \eta_+^1) \geqslant \mathcal{A}^{(F,H)}(y_-^2, \eta_-^2) \geqslant \cdots \\ \cdots \geqslant \mathcal{A}^{(F,H)}(y_+^k, \eta_+^k) \geqslant \mathcal{A}^{(F,H)}(y_-^{k+1}, \eta_-^{k+1}). \end{aligned} \tag{5·12}$$

This follows directly from the choice of the sequences $w_n(s + jn)$. If $\mathcal{A}^{(F,H)}(\varpi_-^j) = \mathcal{A}^{(F,H)}(\varpi_+^j)$ then, by Lemma 2·9, $E(\varpi^j) = \mathcal{A}^{(F,H)}(\varpi_-^j) - \mathcal{A}^{(F,H)}(\varpi_+^j) = 0$ and thus $\partial_s \varpi^j(s) = 0$. We recall that $\mathrm{ev}_n(w_n) \in W^s(x_1, f_1) \times \cdots \times W^u(x_k, f_k)$. By C^∞_{loc}-convergence of $w_n(s + jn) \to \varpi^j(s) = (u^j, \eta^j)$ we conclude

$$u^j(0,0) \in \overline{W^s(x_j, f_j)} \tag{5·13}$$

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where $\varpi^j = (u^j, \eta^j)$. Now, since $\partial_s \varpi^j(s) = 0$, i.e. $\varpi^j(s) = (y_-^j, \eta_-^j) = (y_+^j, \eta_+^j)$, we have

$$u^j(0,0) = y_-^j(0) = y_+^j(0) \in \overline{W^s(x_j, f_j)}. \tag{5·14}$$

But this contradicts assumption (1) that $W^s(x_j, f_j) \cap \mathcal{LI} = \varnothing$ since $\partial\overline{W^s(x_j, f_j)}$ is composed out of stable manifolds of critical points of higher indices. Therefore, $\partial_s \varpi^j \neq 0$ for all $j = 1, \ldots, k$ and we conclude

$$\mathcal{A}^{(F,H)}(y_-^j, \eta_-^j) > \mathcal{A}^{(F,H)}(y_+^j, \eta_+^j), \quad j = 1, \ldots, k. \tag{5·15}$$

If we denote $(y_l, \eta_l) := (y_-^l, \eta_-^l)$ for $l = 1, \ldots, k$ and $(y_{k+1}, \eta_{k+1}) := (y_+^k, \eta_+^k)$ we have

$$\mathcal{A}^{(F,H)}(y_+^0, \eta_+^0) \geqslant \mathcal{A}^{(F,H)}(y_1, \eta_1) > \cdots > \mathcal{A}^{(F,H)}(y_{k+1}, \eta_{k+1}) \geqslant \mathcal{A}^{(F,H)}(y_-^{k+1}, \eta_-^{k+1}). \tag{5·16}$$

In particular, all (y_l, η_l) are different as critical points of $\mathcal{A}^{(F,H)}$. To finally conclude that the leaf-wise intersections $y_1(0), \ldots, y_{k+1}(0)$ are all distinct we recall from Proposition 2·5 that two critical points of $\mathcal{A}^{(F,H)}$ can give rise to the same leaf-wise intersection only if some $y_j(0)$ is a periodic leaf-wise intersection. If $y_l(0) = y_{l'}(0)$ then as in the proof of [**AF10a**, lemma 2·19] we conclude that

$$\left|\mathcal{A}^{(F,H)}(y_l, \eta_l) - \mathcal{A}^{(F,H)}(y_{l'}, \eta_{l'})\right| \geqslant \wp(\Sigma, \alpha) > \|H\|. \tag{5·17}$$

On the other hand we estimate for $(y_+^0, \eta_+^0) = \varpi^0(+\infty)$

$$\begin{aligned}
&\mathcal{A}^{(F,H)}(\varpi^0(+\infty)) - \mathcal{A}^{(F,0)}(\varpi^0(-\infty)) \\
&= \int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}^{(F,\beta_\infty^+(s)H)}(\varpi^0(s)) ds \\
&= \int_{-\infty}^{\infty} d\mathcal{A}^{(F,H)}(\varpi^0) \cdot \partial_s \varpi^0 \, ds + \int_{-\infty}^{\infty} \frac{\partial \mathcal{A}^{(F,\beta_\infty^+(s)H)}}{\partial s}(\varpi^0) \, ds \\
&= -\int_{-\infty}^{\infty} \|\partial_s \varpi^0\|^2 ds - \int_{-\infty}^{\infty} \int_0^1 \frac{\partial(\beta_\infty^+(s)H)}{\partial s}(t, u^0) \, dt \, ds \\
&\leqslant -\int_{-\infty}^{\infty} \Big(\underbrace{\frac{\partial}{\partial s}\beta_\infty^+(s)}_{\geqslant 0} \Big) \int_0^1 H(t, u^0(s,t)) \, dt \, ds \\
&\leqslant -\underbrace{\int_{-\infty}^{\infty} \left(\frac{\partial}{\partial s}\beta_\infty^+(s) \right) ds}_{=1} \cdot \int_0^1 \min_M H(t, \cdot) \, dt \\
&= -\int_0^1 \min_M H(t, \cdot) \, dt.
\end{aligned}$$

The critical point $\varpi^0(-\infty)$ of $\mathcal{A}^{(F,0)}$ corresponds either to a Reeb orbit or constant loop in Σ. Since $E(\varpi^0) \leqslant \|H\|$ the same argument as in Proposition 3·2 rules out the former case. In particular, $\mathcal{A}^{(F,0)}(\varpi^0(-\infty)) = 0$ and therefore

$$\mathcal{A}^{(F,H)}(y_+^0, \eta_+^0) \leqslant -\int_0^1 \min_M H(t, \cdot) \, dt. \tag{5·18}$$

Analogously, for $(y_-^{k+1}, \eta_-^{k+1}) = \varpi^{k+1}(-\infty)$

$$\mathcal{A}^{(F,H)}(y_-^{k+1}, \eta_-^{k+1}) \geqslant -\int_0^1 \max_M H(t, \cdot) \, dt \tag{5·19}$$

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is derived. Thus, if we assume $y_l(0) = y_{l'}(0)$ we have the following inequalities

$$\begin{aligned} \|H\| &= -\int_0^1 \min_M H(t,\cdot)dt + \int_0^1 \max_M H(t,\cdot)dt \\ &\geqslant \mathcal{A}^{(F,H)}(y_+^0, \eta_+^0) - \mathcal{A}^{(F,H)}(y_-^{k+1}, \eta_-^{k+1}) \\ &\geqslant |\mathcal{A}^{(F,H)}(y_l, \eta^l) - \mathcal{A}^{(F,H)}(y_{l'}, \eta^{l'})| \\ &\geqslant \wp(\Sigma, \alpha) \\ &> \|H\| \end{aligned} \tag{5·20}$$

where the second inequality follows from (5·16) and the last two from (5·17). This contradiction shows that the the leaf-wise intersection points $y_1(0), \ldots, y_{k+1}(0)$ are all distinct. This finishes the proof of Theorem 1.

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