$t^4 - 6t^2 + 1 = 0$, so that $t^2 = 3 \pm 2\sqrt{2}$, impossible since $t \in \mathbb{Q}$. If $t = 0$, then (7) forces $s^2 = -1$, impossible. If $t^2 = 1$, then (6) forces $x = 1$. Then (5) forces $u^2 = 2$, impossible since $u \in \mathbb{Q}$. So there are no points on the line EF with rational distances to A, B, C, D. The proof is complete.

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106.03 Real-rooted polynomials and a generalised Hermite-Sylvester theorem

A polynomial is *real-rooted* if all of its roots are real. A real-valued function $\Phi(x_1, \ldots, x_n)$ is *positive semidefinite* if $\Phi(x_1, \ldots, x_n) \geqslant 0$ for all vectors $(x_1, \ldots, x_n) \in \mathbb{R}^n$. An n-ary m-adic form is a polynomial in n variables that is homogeneous of degree m.

Let $f(t)$ be a polynomial of degree $n \geqslant 1$ with real coefficients, and let $\lambda_1, \ldots, \lambda_n$ be the (not necessarily distinct) roots of $f(t)$. Hermite and Sylvester [1, 2] proved that the polynomial $f(t)$ is real-rooted if, and only if, the quadratic form in n variables

$$\Phi_2(x_1, \ldots, x_n) = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \sum_{\ell=1}^{n} \lambda_\ell^{j_1+j_2-2} x_{j_1} x_{j_2}$$

is positive semidefinite. This Note generalises the Hermite-Sylvester theorem. We define, for every positive integer m, the following n-ary m-adic form:

$$\Phi_m(x_1, \ldots, x_n) = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_m=1}^{n} \sum_{\ell=1}^{n} \lambda_\ell^{\sum_{k=1}^{m}(j_k-1)} x_{j_1} x_{j_2} \cdots x_{j_m}. \qquad (1)$$

We shall prove that the polynomial $f(t)$ is real-rooted if, and only if, $\Phi_m(x_1, \ldots, x_n)$ is positive semidefinite for all positive even integers m.

Lemma 1: For all positive integers m,

$$\Phi_m(x_1, \ldots, x_n) = \sum_{\ell=1}^{n} p(\lambda_\ell)^m \qquad (2)$$

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where

$$p(t) = \sum_{j=1}^{n} x_j t^{j-1}. \tag{3}$$

Proof: We have

$$\begin{aligned}
\Phi_m(x_1, \ldots, x_n) &= \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_m=1}^{n} \sum_{\ell=1}^{n} \lambda_\ell^{\sum_{k=1}^{m}(j_k-1)} x_{j_1} x_{j_2} \cdots x_{j_m} \\
&= \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_m=1}^{n} \sum_{\ell=1}^{n} \prod_{k=1}^{m} x_{j_k} \lambda_\ell^{j_k-1} \\
&= \sum_{\ell=1}^{n} \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_m=1}^{n} \prod_{k=1}^{m} x_{j_k} \lambda_\ell^{j_k-1} \\
&= \sum_{\ell=1}^{n} \left(\sum_{j=1}^{n} x_j \lambda_\ell^{j-1} \right)^m \\
&= \sum_{\ell=1}^{n} p(\lambda_\ell)^m
\end{aligned}$$

where $p(t)$ is the polynomial defined by (3).

Lemma 2: Let $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r\}$ be a non-empty finite set of r complex numbers that is closed under complex conjugation. Suppose that $\lambda_1 \in \Lambda$ is not real, and that $\lambda_2 = \overline{\lambda_1} \in \Lambda$. Let $\omega \in \mathbf{C}$, with complex comjugate $\overline{\omega}$. There exists a polynomial $p(t) \in \mathbf{R}[t]$ of degree at most $r - 1$ such that

$$p(\lambda_1) = \omega \qquad p(\lambda_2) = \overline{\omega} \tag{4}$$

and

$$p(\lambda_j) = 0 \text{ for all } j \in \{3, 4, \ldots, r\}. \tag{5}$$

Proof: This was proved in Nathanson [1] in the special case $\omega = i$, but the proof applies unchanged for every complex number ω.

Theorem 1: The following are equivalent:

(a) The polynomial $f(t)$ has real roots.

(b) The n-ary m-adic form $\Phi_m(x_1, \ldots, x_n)$ is positive semidefinite for all positive even integers m.

(c) The n-ary m-adic form $\Phi_m(x_1, \ldots, x_n)$ is positive semidefinite for some positive even integer m.

Proof: Let m be a positive even integer. Let Φ_m be the n-ary m-adic form defined by (1), and let $p(t)$ be the polynomial defined by (3). By Lemma 1,

$$\Phi_m(x_1, \ldots, x_n) = \sum_{\ell=1}^{n} p(\lambda_\ell)^m.$$

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For all vectors $(x_1, \ldots, x_n) \in \mathbf{R}^n$, the polynomial $p(t)$ has real coefficients. Because m is even, if the root λ_ℓ of the polynomial $f(t)$ is real, then $p(\lambda_\ell)^m \geqslant 0$. It follows that if every root of the polynomial $f(t)$ is real, then the form Φ_m is positive semidefinite for all positive even integers m. Thus, (a) implies (b). Also, (b) implies (c).

Suppose that $f(t)$ has a non-real root. The non-real roots of a polynomial with real coefficients occur in complex conjugate pairs. Renumbering the roots $\lambda_1, \ldots, \lambda_n$ of $f(t)$, we can assume that $\lambda_1, \ldots, \lambda_r$ are the distinct roots of $f(t)$, that λ_1 is non-real, and that $\lambda_2 = \overline{\lambda_1}$. If μ is the multiplicity of the root λ_1, then μ is also the multiplicity of the root λ_2.

Let m be a positive even integer. The method of Lemma 2 of Nathanson [1] produces a polynomial $p(t) = \sum_{j=1}^{n} x_j t^{j-1} \in \mathbf{R}[t]$ of degree at most $r - 1 \leqslant n - 1$ such that

$$p(\lambda_1) = e^{\pi i/m}, \qquad p(\lambda_2) = e^{-\pi i/m},$$

and

$$p(\lambda_\ell) = 0 \text{ for } \ell = 3, 4, \ldots, r.$$

The polynomial $p(t)$ and the vector $(x_1, \ldots, x_n) \in \mathbf{R}^n$ depend on m. We have

$$p(\lambda_1)^m = p(\lambda_2)^m = -1$$

and

$$\begin{aligned} \Phi_m(x_1, \ldots, x_n) &= \sum_{\ell=1}^{n} p(\lambda_\ell)^m \\ &= \mu p(\lambda_1)^m + \mu p(\lambda_2)^m \\ &= -2\mu < 0. \end{aligned}$$

Thus, if $f(t)$ has a non-real root, then the form Φ_m is not positive semidefinite. Equivalently, if Φ_m is positive semidefinite for some positive even integer m, then the polynomial $f(t)$ is real-rooted. Thus, (c) implies (a). This completes the proof.

Here is an example. Let $f(t)$ be a quadratic polynomial with roots λ_1 and λ_2. We have $p(t) = x_1 + x_2 t$ and, for every positive integer m, the binary m-adic form

$$\begin{aligned} \Phi_m(x_1, x_2) &= p(\lambda_1)^m + p(\lambda_2)^m = (x_1 + \lambda_1 x_2)^m + (x_1 + \lambda_2 x_2)^m \\ &= \sum_{k=0}^{m} \binom{m}{k} (\lambda_1^k + \lambda_2^k) x_1^{m-k} x_2^k. \end{aligned}$$

For the polynomial $f(t) = t^2 - 1$ with real roots $\lambda_1 = 1$ and $\lambda_2 = -1$, we have

$$\lambda_1^k + \lambda_2^k = \begin{cases} 2 & \text{if } k \equiv 0 \pmod 2 \\ 0 & \text{if } k \equiv 1 \pmod 2 \end{cases}$$

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and

$$\Phi_m(x_1, x_2) = 2 \sum_{\substack{k=0 \\ k \equiv 0 \pmod 2}}^{m} \binom{m}{k} x_1^{m-k} x_2^k.$$

Thus,

$$\begin{aligned}
\Phi_1(x_1, x_2) &= 2x_1 \\
\Phi_2(x_1, x_2) &= 2(x_1^2 + x_2^2) \\
\Phi_3(x_1, x_2) &= 2x_1(x_1^2 + 3x_2^2) \\
\Phi_4(x_1, x_2) &= 2(x_1^4 + 6x_1^2x_2^2 + x_2^4) \\
\Phi_5(x_1, x_2) &= 2x_1(x_1^4 + 10x_1^2x_2^2 + 5x_2^4) \\
\Phi_6(x_1, x_2) &= 2(x_1^6 + 15x_1^4x_2^2 + 15x_1^2x_2^4 + x_2^6).
\end{aligned}$$

For the polynomial $t^2 + t + 1$ with non-real roots $\lambda_1 = -1/2 + i\sqrt{3}/2$ and $\lambda_2 = -1/2 - i\sqrt{3}/2$, we have

$$\lambda_1^k + \lambda_2^k = \begin{cases} 2 & \text{if } k \equiv 0 \pmod 3 \\ -1 & \text{if } k \equiv \pm 1 \pmod 3 \end{cases}$$

and

$$\Phi_m(x_1, x_2) = 2 \sum_{\substack{k=0 \\ k \equiv 0 \pmod 3}}^{m} \binom{m}{k} x_1^{m-k} x_2^k - \sum_{\substack{k=0 \\ k \equiv \pm 1 \pmod 3}}^{m} \binom{m}{k} x_1^{m-k} x_2^k.$$

Thus,

$$\begin{aligned}
\Phi_1(x_1, x_2) &= 2x_1 - x_2 \\
\Phi_2(x_1, x_2) &= 2x_1^2 - 2x_1x_2 - x_2^2 \\
\Phi_3(x_1, x_2) &= 2x_1^3 - 3x_1^2x_2 - 3x_1x_2^2 + 2x_2^3 \\
\Phi_4(x_1, x_2) &= 2x_1^4 - 4x_1^3x_2 - 6x_1^2x_2^2 + 8x_1x_2^3 - x_2^4 \\
\Phi_5(x_1, x_2) &= 2x_1^5 - 5x_1^4x_2 - 10x_1^3x_2^2 + 20x_1^2x_2^3 - 5x_1x_2^4 - x_2^5 \\
\Phi_6(x_1, x_2) &= 2x_1^6 - 6x_1^5x_2 - 15x_1^4x_2^2 + 40x_1^3x_2^3 - 15x_1^2x_2^4 - 6x_1x_2^5 + 2x_2^6.
\end{aligned}$$

Theorem 1 suggests the following problems.

Problem 1: Let $f(t) \in \mathbf{R}[t]$ be a polynomial of degree n, and let $(\Phi_m)_{m=1}^{\infty}$ be the sequence of n-ary m-adic forms defined by (1). What does the subsequence $(\Phi_{2m+1})_{m=0}^{\infty}$ of forms of odd degree tell us about the polynomial $f(t)$?

Problem 2: Does the sequence of polynomials $(\Phi_m)_{m=1}^{\infty}$ constructed by (1) satisfy a nice recursion relation?

Problem 3: Let $f(t)$ be a quadratic polynomial. Is there a binary cubic form $\Psi_3(x_1, x_2)$ associated with $f(t)$ such that Ψ_3 determines if $f(t)$ has real roots?

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Problem 4: More generally, given a polynomial $f(t)$ of degree n, does there exist a homogeneous n-ary form Ψ_{2m+1} of odd degree $2m + 1$ associated with the polynomial such that the form Ψ_{2m+1} determines if the polynomial has real roots?

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106.04 A geometric interpretation of Cramer's rule

A homogeneous system $A\mathbf{x} = \mathbf{0}$ consists of a set of equations

$$\alpha_i \cdot \mathbf{x}^T = 0$$

where α_i are the rows of the matrix A and "$\cdot$" denotes inner product. Solutions to $A\mathbf{x} = \mathbf{0}$ are exactly vectors perpendicular to all the row vectors of A. If the system contains n variables, then we have

$$\dim \ker A = n - \operatorname{rank} A.$$

An inhomogeneous system $A\mathbf{x} = \mathbf{b}$ could be converted to a 'homogeneous' one

$$[A, -\mathbf{b}]\,[\mathbf{x}, 1]^T = \mathbf{0}.$$

The price we pay is that the last coordinate of the solution should be set equal to 1. As a scheme, we could try to solve first the homogeneous system $[A, -\mathbf{b}]\,\mathbf{y} = \mathbf{0}$, then normalise the last coordinate to 1 by scaling. The first n coordinates of the scaled vector will serve as a solution of the original system.

When A is invertible, this scheme works very well. As $\operatorname{rank}[A, -\mathbf{b}] = \operatorname{rank} A = n$, the dimension of the solution set is $(n + 1) - n = 1$. We are left to find the direction perpendicular to all the rows of $[A, -\mathbf{b}]$.

Exterior product will do the job. In $\mathbb{R}^3$, the exterior product of two vectors could be written as a formal determinant. That $\beta_1 \times \beta_2$ is perpendicular to β_i $(i = 1, 2)$ is algebraically reflected in the vanishing of

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