NOTE ON DIOPHANTINE APPROXIMATION

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A well-known theorem on Diophantine approximation, due to Hurwitz, states that for any irrational number θ there are infinitely many pairs of integers x, y, with $y \neq 0$, for which

$$|(x-\theta y)y| < 1/\sqrt{5},$$

the constant on the right being the best possible. Prasad has recently shown (2) that if only one solution is demanded, instead of infinitely many, the appropriate constant is less than $1/\sqrt{5}$, and is in fact $\frac{1}{2}(3-\sqrt{5})$.

Another well-known theorem, due to Minkowski, states that if θ is any irrational number and α is any real number, not of the form $x-\theta y$, there are infinitely many pairs of integers x, y, with $y \neq 0$, for which

$$|(x-\theta y-\alpha)y| < \tfrac{1}{4}, \tag{1}$$

the constant on the right being again the best possible.

We show that the inequality (1) remains the best possible of its kind, even if only one solution is demanded instead of an infinity. The proof depends on using a modified form of an example given by Davenport ((1), Lemma 3).

Let k be a positive integer, and let

$$\theta = k+\sqrt{(k^2+1)}, \quad \alpha = \tfrac{1}{2}(\theta-1). \tag{2}$$

We prove the following theorem, which establishes the result just stated.

THEOREM. *If δ is any fixed positive number, and k is sufficiently large, then*

$$|(x-\theta y-\alpha)y| > \tfrac{1}{4}-\delta \tag{3}$$

for all integers x, y, with $y \neq 0$.

Proof. We use two lower bounds for $|x-\theta y-\alpha|$. The first, which is useful when $|y|$ is small, is as follows. Since $\theta = 2k+\theta^{-1}$, we have

$$\begin{aligned} x-\theta y-\alpha &= x+\tfrac{1}{2}-\theta(y+\tfrac{1}{2}) \\ &= x-(2y+1)k+\tfrac{1}{2}-\theta^{-1}(y+\tfrac{1}{2}). \end{aligned}$$

Hence

$$|x-\theta y-\alpha| \geqslant \tfrac{1}{2}-\theta^{-1}|y+\tfrac{1}{2}|$$

for any integers x, y. Since $\theta^{-1} < (2k)^{-1}$, it follows that

$$|x-\theta y-\alpha| > \tfrac{1}{4}$$

if

$$|y| < \tfrac{1}{2}(k-1). \tag{4}$$

Since $|y| \geqq 1$, the inequality (3) holds when y satisfies (4).

The second lower bound comes from the lemma of Davenport already mentioned, which asserts that

$$|X^2-2kXY-Y^2| \geqslant 2k$$

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for any odd integers X, Y. Since

$$x-\theta y-\alpha = \tfrac{1}{2}\{(2x+1)-\theta(2y+1)\},$$

the lemma implies that $$|(x-\theta y-\alpha)(x-\theta' y-\alpha')| \geqslant \tfrac{1}{2}k \tag{5}$$

for all integers x, y, where θ' and α' denote the algebraic conjugates of θ and α. In proving (3), we can suppose without loss of generality that $x = \theta y+\alpha+\xi$, where $|\xi| < \frac{1}{4}$. Using (5), we obtain

$$|(x-\theta y-\alpha)y| \geqslant \tfrac{1}{2}k\left|\frac{y}{x-\theta' y-\alpha'}\right| \geqslant \tfrac{1}{2}k\,\frac{|y|}{(\theta-\theta')|y|+(\alpha-\alpha')+\frac{1}{4}}.$$

Since $\theta-\theta' = 2\sqrt{(k^2+1)}$ and $\alpha-\alpha' = \sqrt{(k^2+1)}$, the conclusion (3) follows if

$$|y| \geqslant \tfrac{1}{2}(k-1)$$

and k is sufficiently large.

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A NOTE ON THE INTERSECTION THEOREM FOR IDEALS

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Let R be a commutative ring (with a unit element) in which the ascending chain condition for ideals is satisfied, and let $\mathfrak{a}$ be an ideal. The following theorem is well known.

Theorem. *In order that* $\bigcap_{i=1}^{\infty} \mathfrak{a}^i = (0)$ *it is necessary and sufficient that no zero divisor shall be congruent modulo* $\mathfrak{a}$ *to the unit element.*

A special case of this important theorem was proved by Krull ((3); see also (2)), and in the form stated above it has been established by Chevalley ((1), Lemma 2, p. 692). We now give two refinements which appear to be new.

Theorem 1. *In order that* $x \in \bigcap_{i=1}^{\infty} \mathfrak{a}^i$ *it is necessary and sufficient that* $x = xa$ *for at least one element* $a \in \mathfrak{a}$.

Proof. We may suppose that $\mathfrak{a} \neq (1)$. Let S consist of all elements of the form $1-a$ where $a \in \mathfrak{a}$, then S is multiplicatively closed. Further, we have $x = xa$ for at least one $a \in \mathfrak{a}$, if, and only if, $cx = 0$ for at least one $c \in S$. Denote the set of all such x by $\mathfrak{n}$, then $\mathfrak{n}$ is an ideal, and, since $\mathfrak{a} \neq (1)$, $\mathfrak{n}$ does not contain the unit element and so is a proper ideal. Theorem 1 states that $\bigcap_{i=1}^{\infty} \mathfrak{a}^i = \mathfrak{n}$. Now if $x \in \mathfrak{n}$ then $x = ax$ for some $a \in \mathfrak{a}$; hence

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