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Dynamical properties of spatial discretizations of a generic homeomorphism

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Abstract. This paper concerns the link between the dynamical behaviour of a dynamical system and the dynamical behaviour of its numerical simulations. Here, we model numerical truncation as a spatial discretization of the system. Some previous works on well-chosen examples (such as Gambaudo and Tresser [Some difficulties generated by small sinks in the numerical study of dynamical systems: two examples. *Phys. Lett.* A **94**(9) (1983), 412–414]) show that the dynamical behaviours of dynamical systems and of their discretizations can be quite different. We are interested in generic homeomorphisms of compact manifolds. So our aim is to tackle the following question: can the dynamical properties of a generic homeomorphism be detected on the spatial discretizations of this homeomorphism? We will prove that the dynamics of a single discretization of a generic conservative homeomorphism does not depend on the homeomorphism itself, but rather on the grid used for the discretization. Therefore, dynamical properties of a given generic conservative homeomorphism cannot be detected using a single discretization. Nevertheless, we will also prove that some dynamical features of a generic conservative homeomorphism (such as the set of the periods of all periodic points) can be read on a sequence of finer and finer discretizations.

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1. *Introduction*

When one tries to simulate a discrete time dynamical system with a computer, calculations are performed with a finite numerical precision. Numerical errors made at each iteration may add up, so that after a while, the numerically calculated orbit of a point will have nothing in common with the actual one. Nevertheless, a numerically calculated orbit is close to an actual orbit at any time, thus one can hope that the collective behaviour of numerically calculated orbits provides information about the collective behaviour of actual orbits. Interesting properties of dynamical systems are often *asymptotic* (i.e. concern the behaviour of the system in the long term) on the one hand, and *collective* (i.e. concern the collective behaviour of orbits or even the action of the system on open sets and probability measures) on the other. The purpose of our study is to tackle the question: can such properties be detected on numerical simulations?

We will model numerical truncation as a spatial discretization†. Consider a dynamical system whose phase space is a manifold X and whose evolution law is given by a map f from X into itself. Roughly speaking, a numerical simulation of the system with a precision 10^{-N} replaces the continuous phase space X by a discrete space E_N consisting of points of X whose coordinates are decimal numbers with at most N decimal places, and replaces the map f by its discretization $f_N : E_N \to E_N$ that maps $x \in E_N$ to the point (or one of the points) of E_N nearest $f(x)$. We can clarify a bit the question raised in the previous paragraph.

Question. Which dynamical properties of a homeomorphism f can be read on the dynamics of its discretizations $(f_N)_{N\geq 0}$?

Questions about discretizations of dynamical systems have mainly been studied empirically, using numerical simulations of well-chosen examples (see, for example, [**Bin92**, **HYG87**, **Lan98**, **NY88**, **SB86**]). For instance, particular attention has been paid to obtaining physical measures by numerical calculations (see, for example, [**BCG+78**, **BCG+79**, **Boy86**, **GB88**, **Mie06**]). These papers explain why in some cases absolutely continuous invariant measures of the initial system can be

† See page 12 on the accuracy of this model.

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obtained from numerical simulations, for example when the system is uniquely ergodic. However, there are situations where (serious) problems arise: in a short article, Gambaudo and Tresser [**GT83**] give the examples of two simple homeomorphisms of $[0, 1]^2$ which both have attractors that attract most of the orbits. Nevertheless, those attractors are undetectable in practice, simply because the connected components of their basins are much too small. Thus, the actual dynamics of these homeomorphisms does not have much in common with that observed on simulations.

In the early 1990s, significant work was done by a group of researchers including Diamond, Kloeden, Kozyakin, Mustard and Pokrovskii. Their point of view lies between theory and practice: for instance, in [**DKP96**, **DKKP95a**, **DKKP95b**, **DKP94**] they define some dynamical properties that are robust under discretization. Nevertheless, the precision where one can detect these properties on discretizations might be huge, so that these notions could be unusable in practice. On the other hand, in [**DKKP96**, **DKKP97**, **DKPV95**, **DP96**, **DSKP96**] the authors empirically remark that some quantities related to the discretizations of dynamical systems in dimension one, such as the proportion of points in the basin attraction of the fixed point 0, the distribution of the lengths of cycles etc., are similar to the same quantities for random applications with an attractive centre.

From a more theoretical point of view, Luzzatto and Pilarczyk recently conducted in [**LP11**] a quite interesting study of the modelling of a computer discretization by multivalued maps (see also [**DKP96**] and [**Mro96**]). Note that this modelling is very different from that used by most authors and which is described above. Their paper includes a discussion about the relationship between the continuous and the discrete: at first glance, properties of the original system can be deduced from a finite number of discretizations only when these properties are robust, which is relatively rare. To overcome this difficulty the authors define what they call finite resolution properties; in particular, such properties can be verified in finite time by computational methods.

In [**Ghy94**], Ghys made a complete study of the dynamical behaviour of discretizations of the Anosov automorphism of the torus $\mathbf{T}^2$ given by $(x, y) \mapsto (y, x + y)$ (see also [**DF92**]). Ghys notes that the dynamics of these discretizations (he shows that they are permutations with remarkably small orders) has nothing to do with the actual dynamics of the automorphism, which is a paradigm of chaotic dynamics. The very specific arithmetical properties of this example allow one to hope that such a difference between the actual dynamics of a system and the dynamics of its discretizations might be exceptional. This hope is supported by more recent results of Hayashi in [**Hay12**]; roughly speaking, he shows that every C^1-diffeomorphism of a compact manifold with no positive Lyapunov exponent can be approximated by another diffeomorphism whose attractors are 'observable in practise'. Furthermore, Miernowski made a fairly comprehensive study of discretizations of circle homeomorphisms [**Mie06**]. He basically shows that the dynamical properties of a typical (generic or prevalent) circle homeomorphism/diffeomorphism can be read on the dynamics of its discretizations. All these results suggest tackling the above question not for *arbitrary* homeomorphisms but rather for *typical* homeomorphisms; this is the point of view we will adopt in this paper.

We study dynamical properties of discretizations of generic homeomorphisms in the sense of Baire. We establish properties for both *dissipative*, i.e. arbitrary homeomorphisms

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of X, and *conservative* homeomorphisms, i.e. homeomorphisms of X that preserve a given good probability measure.

Our results concern generic homeomorphisms of a compact manifold (with boundary) of dimension $n \geq 2$. In this introduction we state the main results for the torus $\mathbf{T}^n$ and for 'uniform' discretizations on $\mathbf{T}^n$. The general framework in which the results are valid is defined in §2.

Let $n \geq 2$. For all $N \in \mathbb{N}$, let E_N be the finite subset of the torus $\mathbf{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ consisting of the points whose coordinates are decimal numbers with at most N decimal places:

$$E_N = \left\{ \left(\frac{i_1}{10^N}, \ldots, \frac{i_n}{10^N} \right) \in \mathbf{T}^n \;\middle|\; (i_1, \ldots, i_n) \in \mathbb{Z}^n \right\}.$$

Let P_N be a projection of $\mathbf{T}^n$ on E_N minimizing the Euclidian distance: for $x \in \mathbf{T}^n$, $P_N(x)$ is one of the points of E_N nearest x. Let $f_N : E_N \to E_N$ be the discretization of the homeomorphism $f : \mathbf{T}^n \to \mathbf{T}^n$ according to E_N defined by $f_N = P_N \circ f$.

We will prove many results, concerning various aspects of the dynamics of the discretizations. In the rest of this introduction, we try to organize those results according to some 'lessons'.

(1) *The dynamics of a single discretization of a generic homeomorphism has in general nothing to do with the dynamics of the initial homeomorphism.*

Given a finite set E, the dynamics of any finite map $\sigma : E \to E$ is quite simple: for any $x \in E$, the orbit $(\sigma^k(x))_k$ is preperiodic. Therefore, the union of periodic orbits of σ, which we will denote by $\Omega(\sigma)$ and call the *maximal invariant set* of σ, is exactly the union of the ω-limits sets of points of E. To study the dynamics of σ one can focus on quantities such as the cardinality of $\Omega(\sigma)$, the stabilization time of σ (i.e. the smallest $t \in \mathbb{N}$ such that $\sigma^t(E) = \Omega(\sigma)$), the number of orbits of $\sigma_{|\Omega(\sigma)}$, their lengths, the period of $\sigma_{|\Omega(\sigma)}$, ...

Another interesting dynamical quantity for a finite map $\sigma : E \to E$ is the *recurrence rate*: it is the ratio between the cardinalities of $\Omega(\sigma)$ and that of E. The starting point of our paper is a question from Ghys (see [**Ghy94**, §6]): for a generic conservative homeomorphism of the torus, what is the asymptotical behaviour of the sequence of recurrence rates of f_N? A partial answer to this question was obtained by Miernowski in the second chapter of his thesis.

THEOREM. (Miernowski) *For a generic conservative homeomorphism*† *f, there are infinitely many integers N such that the discretization f_N is a cyclic permutation.*

To prove this theorem, Miernowski combines a genericity argument with a quite classical technique in generic dynamics of homeo- (and auto)morphisms: approximation by permutations (see, for example, [**Alp78**, **Hal56**, **KS70**]), and more precisely Lax's theorem (see [**Lax71**]; see also [**DKP93**, **KM97**] for a generalization and some simulations in dimension one), which states that any conservative homeomorphism of the torus can be approximated by cyclic permutations of the discretization grids. In fact this proof can be generalized to obtain many results about discretizations. We will establish some variants

† That is, there is a G_δ dense subset of the set of conservative homeomorphisms of the torus on which the conclusion of the theorem holds.

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of Lax theorem; each of them, combined with a genericity argument, leads to a result for discretizations of generic homeomorphisms. For instance, see the following theorem.

THEOREM. *For a generic conservative homeomorphism f, there exists $C > 0$ such that there are infinitely many integers N such that the cardinality of $\Omega(f_N)$ is smaller than C.*

Note that the combination of these two theorems answers Ghys's question: for a generic homeomorphism f, the sequence of the recurrence rate of f_N accumulates on both 0 and 1; one can even show that it accumulates on the whole segment [0, 1].

Let us move on to the behaviour on the maximal invariant set $\Omega(f_N)$. Another variation of Lax's theorem leads to a theorem that throws light on the behaviour of the discretizations on their maximal invariant set.

THEOREM. *For a generic conservative homeomorphism f, there are infinitely many integers N such that the discretization f_N is a cyclic permutation and, for all $M \in \mathbb{N}$, there are infinitely many integers N such that f_N is a permutation of E_N having at least M periodic orbits.*

To summarize, generically, infinitely many discretizations are cyclic permutations, but also infinitely many discretizations are highly non-injective or else permutations with many cycles. In particular, it implies that for all $x \in X$, there exist infinitely many integers N such that the orbit of x_N under f_N does not shadow the orbit of x under f: in this sense, generically, the dynamics of discretizations does not reflect that of the homeomorphism.

(2) *A dynamical property of a generic homeomorphism cannot be deduced from the frequency with which it appears on discretizations either.*

The previous theorems express that the dynamics of a single discretization does not reflect the actual dynamics of the homeomorphism. However, one might reasonably expect that the properties of the homeomorphism are transmitted to many discretizations. More precisely, one may hope that given a property (P) about discretizations, if there are many integers N such that the discretization f_N satisfies (P), then the homeomorphism satisfies a similar property. It is not so; see, for instance, the following theorem.

THEOREM. *For a generic conservative homeomorphism f, when M goes to infinity, the proportion of integers N between* 1 *and M such that f_N is a cyclic permutation accumulates on both* 0 *and* 1.

In fact, for most of the properties considered in the previous paragraph, the frequency with which they appear on discretizations of orders smaller than M accumulates on both 0 and 1 when M goes to infinity.

(3) *Some important dynamical features of a generic homeomorphism can be detected by looking at some dynamical features of* all *the discretizations.*

We have observed that one cannot detect the dynamics of a generic homeomorphism when looking at the dynamics of its discretizations, or even at the frequency with which some dynamics appears on discretizations. Nevertheless, some dynamical features can be

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deduced from the analogous dynamical features of *all* the discretizations. This idea of convergence of the dynamics when looking at arbitrary large precisions can be related to the work of Diamond *et al* (see page p. 3). For instance, the periods of periodic orbits of a homeomorphism can be read on the periods of periodic orbits of its discretizations.

PROPOSITION. *A generic homeomorphism f has a periodic orbit with period p if and only if there exist infinitely many integers N such that f_N has a periodic orbit with period p.*

We will also prove a theorem in the same vein for invariant measures, which expresses that the set of invariant measures of the homeomorphism can be deduced from the sets of invariant measures of its discretizations. More precisely, we prove the following theorem.

THEOREM. *Let $\mathcal{M}_N$ be the set of probability measures on E_N that are invariant under f_N. For a generic conservative homeomorphism f, the upper limit over N (for Hausdorff topology) of the sets $\mathcal{M}_N$ is exactly the set of probability measures that are invariant under f.*

(4) *Physical measures of a generic homeomorphism cannot be detected on discretizations.*

Given $x \in \mathbf{T}^n$, the Birkhoff limit of x is defined (when it exists) as the limit of $(1/m)\sum_{i=0}^{m-1} f_*^i \delta_x$ when m goes to infinity; the *basin* of a measure μ is the set of points x whose Birkhoff limit coincides with μ. Heuristically, the basin of a measure is the set of points that can see the measure. A Borel measure is said *physical* if its basin has positive Lebesgue measure (see, for example, [**You02**]). The heuristic idea underlying this concept is that physical measures are the invariant measure which can be detected 'experimentally' (since many initial conditions lead to these measures). Indeed, some experimental results on specific examples of dynamical systems show that they are actually the measures that are detected in practice (see, for example, [**BCG+78**, **BCG+79**] or [**Boy86**, **GB88**]); moreover, if the dynamical system is uniquely ergodic then the invariant measure appears naturally on discretizations (see [**Mie06**]).

According to this heuristic and these results, one could expect physical measures to be the only invariant measures that can be detected on discretizations of generic conservative homeomorphisms. This is not the case: for a generic conservative homeomorphism, there exists a unique physical measure, say Lebesgue measure (it is the famous Oxtoby–Ulam's theorem [**OU41**]). According to the previous theorem, invariant measures of the discretizations accumulate on all the invariant measures of the homeomorphism and not only on Lebesgue measure.

However, one can still hope to distinguish the physical measure from other invariant measures. For this purpose, we define the canonical physical measure μ_N^f associated to a discretization f_N: it is the limit in the sense of Cesàro of the images of the uniform measure on E_N by the iterates of f_N. If λ_N is the uniform measure on E_N, then

$$\mu_N^f = \lim_{M\to\infty} \frac{1}{M} \sum_{m=0}^{M-1} (f_N^m)_* \lambda_N.$$

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This measure is supported by the maximal invariant set $\Omega(f_N)$; it is uniform on every periodic orbit and the total weight of a periodic orbit is proportional to the size of its basin of attraction. The following theorem expresses that these measures accumulate on the whole set of f-invariant measures: physical measures cannot be distinguished from other invariant measures on discretizations, at least for generic homeomorphisms.

THEOREM. *For a generic conservative homeomorphism f, the set of accumulations points of the sequence $(\mu_N^f)_{N\in\mathbb{N}}$ is the set of all f-invariant measures.*

(5) *The dynamics of discretizations of a generic dissipative homeomorphism tends to that of the initial homeomorphism.*

We then study properties of discretizations of generic *dissipative* homeomorphisms†. The basic tool is the *shredding lemma* of Abdenur and Andersson [**AA13**], which implies that a generic homeomorphism has an 'attractor dynamics'. This easily transmits to discretizations, for example the basins of attraction of the homeomorphism can be seen on all the fine enough discretizations. Moreover, there is *convergence* of the dynamics of discretizations f_N to that of f.

THEOREM. *For a generic dissipative homeomorphism f, for all $\varepsilon > 0$ and all $\delta > 0$, there is a full measure dense open subset O of $\mathbf{T}^n$ such that for all $x \in O$, all $\delta > 0$ and all N large enough, the orbit of x_N under f_N δ-shadows the orbit‡ of x under f.*

Thus, one can detect on discretizations the dynamics of a generic dissipative homeomorphism, which is mainly characterized by position of the attractors and of the corresponding basins of attraction. Note that this behaviour is in the opposite of the conservative case, where the individual behaviour of discretizations does not indicate anything about the actual dynamics of the homeomorphism.

(6) *In practice, one cannot deduce the dynamics of a conservative homeomorphism from its discretizations, and the dynamics of a dissipative homeomorphism can be detected on discretizations only if the basins are large enough.*

Finally, we compare our theoretical results with the reality of numerical simulations. Indeed, it is not clear that the behaviour predicted by our results can be observed on computable discretizations of a homeomorphism defined by a simple formula. On the one hand, all our results are valid 'for *generic* homeomorphisms'; nothing indicates that these results apply to practical examples of homeomorphisms defined by simple formulas. On the other hand, results such as 'there are infinitely many integers N such that the discretization of order $N\ldots$' provide no control over the integers N involved; they may be so large that the associated discretizations are not calculable in practice.

For conservative homeomorphisms, our numerical simulations produce mixed results. From a quantitative viewpoint, the behaviour predicted by our theoretical results cannot be observed on our numerical simulations. For example, we cannot observe any discretization whose recurrence rate is equal to 1 (i.e. which is a permutation). This is nothing but

† Without assumption of preservation of a given measure.
‡ That is, for all $k \in \mathbb{N}$, $d(f_N^k(x_N),\ f^k(x)) < \delta$.

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surprising: the events pointed out by the theorems are *a priori* very rare. For instance, there is a very little proportion of bijective maps among maps from a given finite set into itself. From a more qualitative viewpoint, the behaviour of the simulations is quite in accordance with the predictions of the theoretical results. For example, for a given conservative homeomorphism, the recurrence rate of a discretization depends a lot on the size of the grid used for the discretization. Similarly, the canonical invariant measure associated with a discretization of a homeomorphism f does depend a lot on the size of the grid used for the discretization.

We also carried out simulations of dissipative homeomorphisms. The results of discretizations of a small perturbation of identity (in C^0 topology) may seem disappointing at first sight: the attractors of the initial homeomorphisms cannot be detected, and there is little difference from the conservative case. This behaviour is similar to that highlighted by Gambaudo and Tresser in [**GT83**] (see p. 3). That is why it seemed to us useful to test a homeomorphism which is C^0 close to the identity, but whose basins are large enough. In this case the simulations point to a behaviour that is very similar to that described by theoretical results, namely that the dynamics converges to the dynamics of the initial homeomorphism. In fact, one has actually observed behaviours as described by theorems only for examples of homeomorphisms with a very small number of attractors.

We carried out many other numerical simulations, which can be found on the web page: http://www.math.u-psud.fr/∼guiheneu/Simulations.html.

In §2 we will present the framework. Section 4 concerns conservative homeomorphisms. More precisely, the results concerning the behaviour of a single discretization are set out in §4.3, and those concerning the average behaviour of discretizations in §4.4. We then come to the results about all the discretizations in §4.5 and about physical measures in §4.6. The behaviour of discretizations of generic dissipative homeomorphisms is established in §5.2. Finally, the results of numerical simulations are presented in §6.

2. *Framework*

The manifold X and the measure λ. The results stated in the introduction for the torus $\mathbf{T}^n$ and the Lebesgue measure Leb actually extend to any smooth connected manifold X with dimension $n \geq 2$, compact and possibly with boundary, endowed with a Riemannian metric d. *We fix once and for all such a manifold X endowed with the metric d.* In the general case, Lebesgue measure on $\mathbf{T}^n$ can be replaced by a *good measure* λ on X.

Definition 1. A Borel probability measure λ on X is called a *good measure*, or an *Oxtoby–Ulam* measure if it is non-atomic, it has total support (it is positive on non-empty open sets) and it is zero on the boundary of X.

We fix once and for all a good measure λ on X.

Notation 2. We denote by $\mathrm{Homeo}(X)$ the set of homeomorphisms of X, endowed by the metric d defined by

$$d(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

We denote by $\mathrm{Homeo}(X, \lambda)$ the subset of $\mathrm{Homeo}(X)$ consisting of the homeomorphisms that preserve the measure λ, endowed with the same metric d. Elements of $\mathrm{Homeo}(X)$

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will be called *dissipative* homeomorphisms, and elements of Homeo(X, λ) *conservative* homeomorphisms.

Generic properties in Homeo(X) *and* Homeo(X, λ). The topological spaces Homeo(X) and Homeo(X, λ) are Baire spaces (see [**Gui12**]), i.e. the intersection of every countable collection of dense open sets is dense. We call G_δ a countable intersection of open sets; a property satisfied on at least a G_δ dense set is called *generic*. Note that in a Baire space, generic properties are stable under intersection.

We will sometimes use the phrase 'for a generic homeomorphism $f \in$ Homeo(X) (respectively, Homeo(X, λ)), we have the property (P)'. By that we will mean that 'the property (P) is generic in Homeo(X) (respectively, Homeo(X, λ))', i.e. 'there exists a G_δ dense subset G of Homeo(X) (respectively, Homeo(X, λ)), such that every $f \in G$ satisfies property (P)'.

Discretization grids, discretizations of a homeomorphism. We now define a more general notion of discretization grid. Some of the assumptions about these grids will be useful later (see also the next subsection).

Definition 3. (Discretization grids) A *sequence of discretization grids* on X is a sequence $(E_N)_{N\in\mathbb{N}}$ of discrete subsets of $X \setminus \partial X$, such that the grids are more and more precise: for all $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, the grid E_N is ε-dense. We denote by q_N the cardinality of E_N.

We fix once and for all a sequence $(E_N)_{N\in\mathbb{N}}$ *of discretization grids on* X. We can now define discretizations associated to these grids.

Notation 4. (Discretizations) Let P_N be a projection of X on E_N (the projection of $x_0 \in X$ on E_N is some $y_0 \in E_N$ minimizing the distance $d(x_0, y)$ when y runs through E_N). Such a projection is uniquely defined outside the set E'_N consisting of the points $x \in X$ for which there exist at least two points minimizing the distance between x and E_N. On E'_N the map P_N is chosen arbitrarily (nevertheless measurably). For $x \in X$ we denote by x_N the *discretization of order* N of x, defined by $x_N = P_N(x)$. For $f \in$ Homeo(X) we denote by $f_N : E_N \to E_N$ the *discretization of order* N of f, defined by $f_N = P_N \circ f$.

Let $\mathcal{D}_N$ be the set of homeomorphisms g such that $g(E_N) \cap E'_N = \emptyset$.

If $\sigma : E_N \to E_N$ and $f \in$ Homeo(X), we denote by $d_N(f, \sigma)$ the distance between $f_{|E_N}$ and σ, considered as maps from E_N into X.

Remark 5. One might wonder why the points of the discretization grids are supposed to be *inside* X. The reason is simple: a homeomorphism f of X sends ∂X to ∂X. Putting points of some grids on the edge could perturb the dynamics of discretizations f_N. In particular, it would introduce at least one orbit† with length smaller than Card$(E_N \cap \partial X)$.

Remark 6. As the exponential map is a local diffeomorphism, the sets E'_N are closed and have empty interior for every N large enough. Subsequently, we will implicitly suppose that the union $\bigcup_{N\in\mathbb{N}} E'_N$ is an F_σ with empty interior. This is not a limiting assumption as

† Recall that orbit means forward orbit.

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we will focus only on the behaviour of the discretizations for N going to $+\infty$. It will allow us to restrict the study to the G_δ dense set $\bigcap_{N\in\mathbb{N}} \mathcal{D}_N$, which is the set of homeomorphisms whose Nth discretization is uniquely defined for all $N \in \mathbb{N}$.

Probability measures on X. From §4.5, we will be interested in *ergodic* properties of discretizations of f. Denote by $\mathcal{P}$ the set of Borel probability measures on X endowed with the weak-star topology: a sequence $(\nu_m)_{m\in\mathbb{N}}$ of $\mathcal{P}$ tends to $\nu \in \mathcal{P}$ (denoted by $\nu_m \rightharpoonup \nu$) if for every continuous function $\varphi : X \to \mathbb{R}$,

$$\lim_{m\to\infty} \int_X \varphi \, d\nu_m = \int_X \varphi \, d\nu.$$

Under these conditions the space $\mathcal{P}$ is metrizable and compact, therefore separable (Prohorov, Banach–Alaoglu–Bourbaki theorem).

To study ergodic properties of homeomorphisms and their discretizations, we define natural invariant probability measures associated with these maps.

Definition 7. For any non-empty open subset U of X, we denote by λ_U the normalized restriction of λ on U, i.e. $\lambda_U = (1/\lambda(U))\lambda_{|U}$. We also denote by $\lambda_{N,U}$ the uniform probability measure on $E_N \cap U$ and $\lambda_N = \lambda_{N,X}$. For $x \in X$ we denote by the *Birkhoff limit* of x (when the limit exists):

$$\mu_x^f = \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} f_*^i \delta_x,$$

and similarly for f_N. When it is well defined, we set

$$\mu_U^f = \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} f_*^i \lambda_U$$

and

$$\mu_{N,U}^f = \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} (f_N)_*^i \, \lambda_{N,U}.$$

Finally, we note that $\mu_N^f = \mu_{N,X}^f$.

We define two types of invariant measures: one from a point x, the other from the uniform measure λ. The link between them is stated in the following proposition.

PROPOSITION 8. *When U is an open set almost every point in which admits a Birkhoff limit, the measure μ_U^f is well defined and satisfies, for every continuous map $\varphi : X \to \mathbb{R}$,*

$$\int_X \varphi \, d\mu_U^f = \int_U \left(\int_X \varphi \, d\mu_x^f \right) d\lambda_U.$$

Similarly,

$$\int_X \varphi \, d\mu_{N,U}^f = \int_U \left(\int_X \varphi \, d\mu_{N,x}^f \right) d\lambda_{N,U}.$$

Proof of Proposition 8. This follows easily from the dominated convergence theorem. □

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Hypothesis on discretization grids. Previously, we have given a very general definition of the concept of sequence of discretization grids. In some cases, we will need additional technical assumptions about these sequences of grids. Of course all of them will be satisfied by the uniform discretization grids on the torus (as defined in the introduction).

The first assumption is useful for proving Lax's theorem (Theorem 20), and therefore necessary only in the part concerning conservative homeomorphisms.

Definition 9. (Well-distributed and well-ordered grids) We say that a sequence of discretization grids $(E_N)_{N\in\mathbb{N}}$ is *well distributed* if one can associate to each $x \in E_N$ a subset $C_{N,x}$ of X, which will be called a *cube of order* N, such that:

- for all N and all $x \in E_N$, $x \in C_{N,x}$;
- for all N, $\{C_{N,x}\}_{x\in E_N}$ is a measurable partition of X: $\bigcup_{x\in E_N} C_{N,x}$ is full measure and for x, y two distinct points of E_N, the intersection $C_{N,x} \cap C_{N,y}$ is null measure;
- for a fixed N, all the cubes $C_{N,x}$ have the same measure;
- the diameter of the cubes of order N tends to 0: $\max_{x\in E_N} \operatorname{diam}(C_{N,x}) \underset{N\to+\infty}{\longrightarrow} 0$.

If $(E_N)_{N\in\mathbb{N}}$ is well distributed and if $\{C_{N,x}\}_{N\in\mathbb{N},x\in E_N}$ is a family of cubes as below, we will say that $(E_N)_{N\in\mathbb{N}}$ is *well ordered* if, for a fixed N, the cubes $\{C_{N,x}\}_{x\in E_N}$ can be indexed by $\mathbb{Z}/q_N\mathbb{Z}$ such that two consecutive cubes (in $\mathbb{Z}/q_N\mathbb{Z}$) are close to each other: $\max_{i\in\mathbb{Z}/q_N\mathbb{Z}} \operatorname{diam}(C_N^i \cup C_n^{i+1}) \underset{N\to+\infty}{\longrightarrow} 0$ (in particular, this is true when the boundaries of two consecutive cubes overlap).

At first glance, it can seem surprising that there is no link between the cubes and the projections. In fact, the existence of such cubes expresses that the grids 'fit' the measure λ.

The two following definitions describe assumptions that will be useful especially for obtaining properties on average.

Definition 10. (Refining grids) We say that a sequence of discretization grids *refines* if for all N, $N' \in \mathbb{N}$ such that $N \le N'$, one has $E_N \subset E_{N'}$.

Definition 11. (Self-similar grids) We say that a sequence of discretization grids $(E_N)_{N\in\mathbb{N}}$ is *self-similar* if for all $\varepsilon > 0$, there exist N_0, $N_1 \in \mathbb{N}$ such that for all $N \ge N_1$, the set E_N contains disjoint subsets $\widetilde{E}_N^1, \ldots, \widetilde{E}_N^{\alpha_N}$ whose union fills a proportion greater than $1-\varepsilon$ of E_N, and such that for all j, $\widetilde{E}_N^j$ is the image of the grid E_{N_0} by a bijection h_j which is ε-close to identity.

We say that a sequence of discretization grids $(E_N)_{N\in\mathbb{N}}$ is *strongly self-similar* if it is self-similar and for all $N > N_0$, one of the h_j is equal to the identity.

Remark 12. One can easily verify that the hypothesis 'being strongly self-similar' implies both 'being self-similar' and 'refining'.

3. *Some examples of discretization grids*

In the previous section we set properties on discretizations—namely being well distributed, being well ordered, refining, being strongly self-similar or self-similar—for later use. In this section we give some examples of grids that satisfy some of these assumptions.

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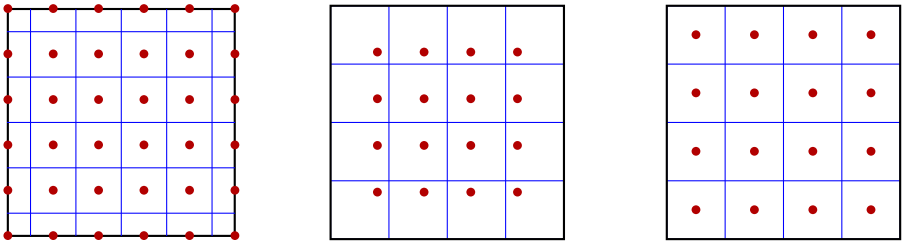


FIGURE 1. Uniform discretization grids of order five on the torus $\mathbf{T}^2$ (E^N, left) and on the cube I^2 (E_N^0, middle, and E_N^1, right) and their associated cubes.

Uniform discretization grids on the torus. The simplest example, which will be used for the simulations, is that of the torus $\mathbf{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ of dimension $n \geq 2$ endowed with discretizations called *uniform* discretizations, defined from the fundamental domain $I^n = [0, 1]^n$ of $\mathbf{T}^n$: take an increasing sequence of integers $(k_N)_{N\in\mathbb{N}}$ and set

$$E_N = \left\{ \left(\frac{i_1}{k_N}, \ldots, \frac{i_n}{k_N} \right) \in I^n \;\middle|\; \forall j,\ 0 \leq i_j \leq k_N - 1 \right\},$$

$$C_{N,(i_1/N,\ldots,i_n/N)} = \prod_{j=1}^{n} \left[\frac{i_j}{k_N} - \frac{1}{2k_N}, \frac{i_j}{k_N} + \frac{1}{2k_N} \right].$$

We easily verify that this sequence of grids is well distributed, well ordered and self-similar. If we further assume that for any $N \in \mathbb{N}$, k_N divides k_{N+1} (which is true when $k_N = p^N$ with $p \geq 2$), then the sequence is strongly self-similar (therefore refines). When $k_N = p^N$ with $p = 2$ (respectively, $p = 10$) the discretization performs what one can expect from a numerical simulation: doing a binary (respectively, decimal) discretization at order N is the same as truncating each binary (respectively, decimal) coordinate of the point $x \in I^n$ to the Nth digit, i.e. working with a fixed digital precision†.

Uniform discretization grids on the cube. Similarly, one can define uniform discretizations on the cube $I^n = [0, 1]^n$ by setting

$$E_N^0 = \left\{ \left(\frac{i_1}{k_N}, \ldots, \frac{i_n}{k_N} \right) \in I^n \;\middle|\; \forall j,\ 1 \leq i_j \leq k_N - 1 \right\},$$

still with an increasing sequence $(k_N)_{N\in\mathbb{N}}$ of integers, to which are associated the cubes‡ (see Figure 1)

$$C_{N,(i_1/N,\ldots,i_n/N)} = \prod_{j=1}^{n} \left[\frac{i_j - 1}{k_N - 1}, \frac{i_j}{k_N - 1} \right].$$

As before, one easily verifies that this sequence of grids is well distributed, well ordered and self-similar (see Figure 2). If we further assume that for any $N \in \mathbb{N}$, k_N divides k_{N+1}, then the sequence is strongly self-similar (therefore refines).

† In practice the computer works in floating point format, so that the number of decimal places is not the same when the number is close to 0 or not.

‡ Be careful: these cubes have their vertices on the grid of order $k_N - 1$.

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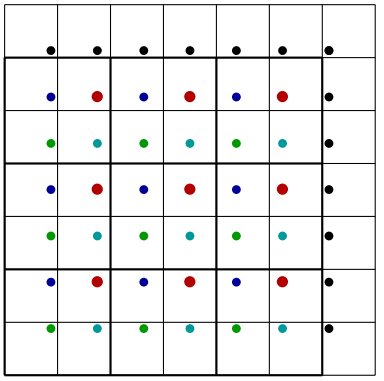


FIGURE 2. Self-similarity of grids E_N^0.

One can also take discretizations according to the centres of the cubes:

$$E_N^1 = \left\{ \left(\frac{i_1 + 1/2}{k_N}, \ldots, \frac{i_n + 1/2}{k_N} \right) \in I^n \;\middle|\; \forall j, 0 \le i_j \le k_N - 1 \right\}.$$

This time the cubes are much more natural (see Figure 1):

$$C_{N,(i_1/N,\ldots,i_n/N)} = \prod_{j=1}^{n} \left[\frac{i_j}{k_N}, \frac{i_j + 1}{k_N} \right].$$

Again, we easily verify that this sequence of grids is well distributed, well ordered and self-similar (but does not refine).

Discretization grids on an arbitrary manifold. In fact, discretizations E_N^0 on I^n can be generalized to an arbitrary manifold X with the Oxtoby–Ulam theorem (see [**OU41**]).

THEOREM 13. (Oxtoby and Ulam) *Under the assumptions that have been made on X and λ, there exists a map $\phi : I^n \to X$ such that:*

(1) *ϕ is surjective;*
(2) *$\phi_{|\mathring{I}^n}$ is a homeomorphism on its image;*
(3) *$\phi(\partial I^n)$ is a closed subset of X with empty interior and disjoint from $\phi(\mathring{I}^n)$;*
(4) *$\lambda(\phi(\partial I^n)) = 0$;*
(5) *$\phi^*(\lambda) = \mathrm{Leb}$, where* Leb *is Lebesgue measure.*

This theorem allows us to define discretization grids on X from uniform discretizations on the cube. We obtain the following informal proposition.

PROPOSITION 14. *If there is a sequence $(E_N)_N$ of grids on I^n whose elements are not on the edge of the cube, then its image by ϕ defines a sequence of grids on X which satisfy the same properties as the initial grid on the cube.*

Remark 15. The example also includes the case where $X = I^n$, $\lambda = \mathrm{Leb}$ and where the grids are the images of the grids E_N by a unique homeomorphism of X preserving Lebesgue measure.

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4. *Part 1. Discretizations of a generic conservative homeomorphism*

Recall that we have fixed once and for all a manifold X, a good measure λ on X and a sequence of discretization grids $(E_N)_{N\in\mathbb{N}}$ (see Definition 3). It is further assumed that this sequence of grids is well distributed and well ordered (see Definition 9). In this part of the paper we will focus on discretizations of elements of $\mathrm{Homeo}(X, \lambda)$, so homeomorphisms will always be supposed conservative.

4.1. *Dense types of approximations.*

To begin with we define the notion of *type of approximation*.

Definition 16. A *type of approximation* $\mathcal{T} = (\mathcal{T}_N)_{N\in\mathbb{N}}$ is a sequence of subsets of the set $\mathcal{F}(E_N, E_N)$ consisting of applications from E_N into itself.

Let $\mathcal{U}$ be an open subset of $\mathrm{Homeo}(X, \lambda)$. A type of approximation $\mathcal{T}$ is said to be *dense* in $\mathcal{U}$ if for all $f \in \mathcal{U}$, $\varepsilon > 0$ and all $N_0 \in \mathbb{N}$, there exist $N \geq N_0$ and $\sigma_N \in \mathcal{T}_N$ such that $d_N(f, \sigma_N) < \varepsilon$ (recall that d_N is the distance between $f_{|E_N}$ and σ_N considered as maps from E_N into X).

The goal of this section is to obtain the following result: every dense type of approximation appears on infinitely many discretizations of a generic homeomorphism.

THEOREM 17. *Let $\mathcal{U}$ be an open subset† of* $\mathrm{Homeo}(X, \lambda)$ *and $\mathcal{T}$ be a type of approximation which is dense in $\mathcal{U}$. Then for a generic homeomorphism $f \in \mathcal{U}$ and for all $N_0 \in \mathbb{N}$, there exists $N \geq N_0$ such that $f_N \in \mathcal{T}_N$.*

When we try to obtain properties of discretizations of generic homeomorphisms, this theorem will be the second step of the proof, the first being to show that the approximation type we are interested in is dense. In §6 we will try to establish as many corollaries of this theorem as possible, showing that the approach 'prove that such approximation is dense then apply Theorem 17' is a general method in the study of properties of discretizations‡.

To prove Theorem 17 we start from a dense type of approximations—in other words, discrete applications—and we want to get properties of homeomorphisms. The tool that allows us to restore a homeomorphism from a finite map $\sigma_N : E_N \to E_N$, i.e. do the opposite of a discretization, is the finite map extension proposition.

PROPOSITION 18. (Finite map extension) *Let $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ be two k-tuples of pairwise distinct points in $X \setminus \partial X$ (i.e. for all $i \neq j$, $x_i \neq x_j$ and $y_i \neq y_j$). Then there exists $f \in \mathrm{Homeo}(X, \lambda)$ such that for all i, $f(x_i) = y_i$. Moreover, f can be chosen such that if $\max_i d(x_i, y_i) < \delta$ then $d(f, \mathrm{Id}) < \delta$.*

The idea of the proof is quite simple: it suffices to compose homeomorphisms whose support is smaller in size than ε and which are central symmetries within this support. Then

† We will often take $\mathcal{U} = \mathrm{Homeo}(X, \lambda)$.

‡ Some properties can be proved in different ways: for instance, Corollary 38 can be proved by inserting horseshoes in the dynamics of a given homeomorphism, using the local modification theorem (Theorem 31; for a presentation of the technique in another context, see [**Gui12**, §3.3]); also a variation of Corollary 26 can be shown by perturbing any given homeomorphism such that it has a periodic orbit whose distance to the grid E_N (grids are assumed to refine) is smaller than the modulus of continuity of f, so that the actual periodic orbit and the discrete orbit fit together.

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one constructs a sequence $(z_j)_{1\le j\le k}$ such that $z_1 = x_1$, $y_1 = z_k$ and $d(z_j, z_{j+1}) < \varepsilon/10$. Composing $k-1$ homeomorphisms as above, such that each one sends z_j to z_{j+1}, one obtains a conservative homeomorphism which sends x_0 to y_0. Implementing these remarks is then essentially technical. A detailed proof can be found in [**Gui12**, §2.2].

With this proposition, we can construct homeomorphisms from a dense type of approximation.

LEMMA 19. *Let $\mathcal{U}$ be an open subset of* Homeo(X, λ) *and $\mathcal{T} = (\mathcal{T}_N)_{N\in\mathbb{N}}$ be a type of approximation which is dense in $\mathcal{U}$. Then for all $N_0 \in \mathbb{N}$, the set of homeomorphisms f such that there exists $N \ge N_0$ such that the discretization f_N belongs to $\mathcal{T}_N$ is dense in $\mathcal{U}$.*

Proof of Lemma 19. Let $f \in \mathcal{U}$, $N_0 \in \mathbb{N}$ and $\varepsilon > 0$. Since $\mathcal{T}$ is a type of approximation which is dense in $\mathcal{U}$, there exist $N \ge N_0$ and $\sigma_N \in \mathcal{T}_N$ such that $d_N(f, \sigma_N) < \varepsilon$. Let $x_1, \dots, x_{q_N}$ be the elements of E_N. Then for all ℓ, $d(f(x_\ell), \sigma_N(x_\ell)) < \varepsilon$. We modify σ_N into a bijection $\sigma'_N : E_N \to X$ whose discretization on E_N equals σ_N. Set $\sigma'_N(x_1) = \sigma(x_1)$; σ'_N is defined by induction by choosing $\sigma'_N(x_\ell)$ such that $\sigma'_N(x_\ell)$ is different from $\sigma'_N(x_i)$ for $i \le \ell$, such that $P_N(\sigma'_N(x_\ell)) = P_N(\sigma_N(x_\ell))$ and that $d(f(x_\ell), \sigma'_N(x_\ell)) < \varepsilon$.

Since σ'_N is a bijection Proposition 18 can be applied to $f(x_\ell)$ and $\sigma'_N(x_\ell)$; this gives a measure-preserving homeomorphism φ such that $\varphi(f(x_\ell)) = \sigma'_N(x_\ell)$ for all ℓ and such that $d(\varphi, \mathrm{Id}) < \varepsilon$. Set $f' = \varphi \circ f$; we have $d(f, f') \le \varepsilon$ and $f'(x_\ell) = \sigma'_N(x_\ell)$ for all ℓ, and therefore $f_N = \sigma_N$. □

Proof of Theorem 17. Let $(x_{N,\ell})_{1\le\ell\le q_N}$ be the elements of E_N and consider the set

$$\bigcap_{N_0\in\mathbb{N}} \bigcup_{\substack{N\ge N_0\\ \sigma_N\in\mathcal{T}_N}} \left\{ f\in\mathcal{U} \;\middle|\; \forall \ell,\, d_N(f(x_{N,\ell}), \sigma_N(x_{N,\ell})) < \frac{\rho_N}{2} \right\}$$

where ρ_N is the minimal distance between two distinct points of E_N. This set is clearly a G_δ set and its density follows directly from Lemma 19. Moreover, we can easily see that its elements satisfy the conclusions of the theorem. □

4.2. *Lax's theorem.* Now that we have shown Theorem 17, we have to obtain dense types of approximation. Systematic use will be made of Lax's theorem, and specifically its improvement as stated by Alpern [**Alp76**] which allows us to approach any homeomorphism by a cyclic permutation of the elements of a discretization grid.

THEOREM 20. (Lax and Alpern) *Let $f \in$ Homeo(X, λ) and $\varepsilon > 0$. Then there exists $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$, there exists a cyclic permutation σ_N of E_N such that $d(f, \sigma_N) < \varepsilon$.*

As a compact metric set can be seen as a 'finite set up to ε', Lax's theorem allows us to see a homeomorphism as a 'cyclic permutation up to ε'. In practice this theorem is used to 'break' homeomorphisms into small pieces. Genericity of transitivity in Homeo(X, λ) follows easily from Lax's theorem together with the finite map extension proposition (see [**AP00**] or [**Gui12**, §2.4]). In our case, applying Theorem 17, we deduce that infinitely many discretizations of a generic homeomorphism are cyclic permutations. The purpose

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of §4.3 is to find variations of Lax's theorem (which are at the same time corollaries of it) concerning other properties of discretizations.

We give briefly the beautiful proof of Lax's theorem, which is essentially combinatorial and based on the marriage lemma and on a lemma of approximation of permutations by cyclic permutations due to Alpern. Readers wishing to find proofs of these lemmas may consult [**Gui12**, §2.1]. It is here that we use the fact that discretizations are well distributed and well ordered.

LEMMA 21. (Marriage lemma) *Let E and F be two finite sets and $\approx$ a relation between elements of E and F. Suppose that the number of elements of any subset $E' \subset E$ is smaller than the number of elements in F that are associated with an element of E':*

$$\text{for all } E' \subset E, \quad \text{Card}(E') \leq \text{Card}\{f \in F \mid \exists e \in E' : e \approx f\}.$$

Then there exists an injection $\Phi : E \to F$ such that for all $e \in E$, $e \approx \Phi(e)$.

LEMMA 22. (Cyclic approximations in $\mathfrak{S}_q$, [**Alp76**]) *Let $q \in \mathbb{N}^*$ and $\sigma \in \mathfrak{S}_q$ ($\mathfrak{S}_q$ is seen as the permutation group of $\mathbb{Z}/q\mathbb{Z}$). Then there exists $\tau \in \mathfrak{S}_q$ such that $|\tau(k) - k| \leq 2$ for all k (where $|\cdot|$ is the distance in $\mathbb{Z}/q\mathbb{Z}$) and such that the permutation $\tau\sigma$ is cyclic.*

Proof of Theorem 20. Let $f \in \text{Homeo}(X, \lambda)$ and $\varepsilon > 0$. Consider $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, the diameter of the cubes of order N (given by the hypothesis 'being well distributed') and their images by f is smaller than ε. We define a relation $\approx$ between cubes of order $N - 1$: $C \approx C'$ if and only if $f(C) \cap C' \neq \emptyset$. Since f preserves λ, the image of the union of ℓ cubes intersects at least ℓ cubes, so the marriage lemma (Lemma 21) applies: there exists an injection Φ_N from the set of cubes of order N into itself (thus a bijection) such that for every cube C, $f(C) \cap \Phi_N(C) \neq \emptyset$. Let σ_N be the application that maps the centre of any cube C to the centre of the cube $\Phi_N(C)$. We obtain:

$$d_N(f, \sigma_N) \leq \sup_C(\text{diam}(C)) + \sup_C(\text{diam}(f(C))) \leq 2\varepsilon.$$

It remains to show that σ_N can be chosen as a cyclic permutation. Increasing N if necessary, and using the hypothesis that the grids are well ordered, we number the cubes such that the diameter of the union of two consecutive cubes is smaller than ε. Then we use Lemma 22 to obtain a cyclic permutation σ'_N whose distance to σ_N is smaller than ε. Thus we have found $N_0 \in \mathbb{N}$ and for all $N \geq N_0$ a cyclic permutation σ'_N of E_N whose distance to f is smaller than 3ε. □

4.3. *Individual behaviour of discretizations.* Once we have set Theorems 17 and 20, we can establish results concerning the behaviour of discretizations of a generic conservative homeomorphism. Here we study *individual* behaviour of discretizations, i.e. properties about only one order of discretization. As has already been said, applying Theorem 17, it suffices to find dense types of approximation to obtain properties about discretizations. In practice, these dense types of approximations are obtained from variations of Lax's theorem (Theorem 20).

Recall that the sequence $(E_N)_{n\in\mathbb{N}}$ of discretization grids is well distributed and well ordered (see Definition 9); we denote by f_N the discretization of a homeomorphism f and by $\Omega(f_N)$ the maximal invariant set of f_N (i.e. the union of periodic orbits of f_N).

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We will try to show that for simple dynamical properties (P) about finite maps and for a generic conservative homeomorphism f, infinitely many discretizations f_N satisfy (P) and infinitely many discretizations satisfy its contrary. For instance, for a generic homeomorphism f, the maximal invariant set $\Omega(f_N)$ is sometimes as large as possible, i.e. $\Omega(f_N) = E_N$ (Corollary 23), sometimes very small (Corollary 26), and, even better, sometimes the number of elements of the image of E_N is small (Corollary 38). In the same way stabilization time† is sometimes zero (Corollary 23, for example), sometimes around $\mathrm{Card}(E_N)$ (Corollary 28). Finally, concerning the dynamics of $f_{N|\Omega(f_N)}$, sometimes it is a cyclic permutation (Corollary 23) or a bicyclic permutation (Remark 24, see also Corollary 36), sometimes it has many orbits (Corollary 40).

Firstly, we deduce directly from Lax's theorem that cyclic permutations of sets E_N form a dense type of approximation in $\mathrm{Homeo}(X, \lambda)$. Combining this with Theorem 17, we obtain the following corollary.

COROLLARY 23. (Miernowski [**Mie05**]) *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$, for every $N_0 \in \mathbb{N}$, there exists $N \geq N_0$ such that f_N is a cyclic permutation*‡.

Remark 24. The same result is obtained for with bicyclic permutations, which are permutations having exactly two orbits whose lengths are relatively prime (see [**Gui12**]).

The first variation of Lax's theorem concerns the approximation of applications whose maximal invariant set is small. It is the contrary to what happens in Corollary 23.

PROPOSITION 25. (First variation of Lax's theorem) *Let $f \in \mathrm{Homeo}(X, \lambda)$. Then for all $\varepsilon, \varepsilon' > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, there exists a map $\sigma_N : E_N \to E_N$ such that $d_N(f, \sigma_N) < \varepsilon$ and*

$$\frac{\mathrm{Card}(\Omega(\sigma_N))}{\mathrm{Card}(E_N)} = \frac{\mathrm{Card}(\Omega(\sigma_N))}{q_N} < \varepsilon',$$

and such that E_N consists of a unique (pre-periodic) orbit of σ_N.

Proof of Proposition 25. Let $f \in \mathrm{Homeo}(X, \lambda)$, $\varepsilon > 0$ and x a recurrent point of f. There exists $\tau \in \mathbb{N}^*$ such that $d(x, f^\tau(x)) < \varepsilon/8$. This inequality remains true for sufficiently fine discretizations: there exists $N_1 \in \mathbb{N}$ such that if $N \geq N_1$, then

$$d(x, x_N) < \frac{\varepsilon}{8}, \quad d\big(f^\tau(x), f^\tau(x_N)\big) < \frac{\varepsilon}{8} \quad \text{and} \quad \frac{\tau}{q_N} < \varepsilon'.$$

Using the modulus of continuity of f^τ and Lax's theorem (Theorem 20), we obtain an integer $N_0 \geq N_1$ such that for all $N \geq N_0$, there exists a cyclic permutation σ_N of E_N such that $d_N(f, \sigma_N) < \varepsilon/2$ and $d_N(f^\tau, \sigma_N^\tau) < \varepsilon/8$. Then

$$\begin{aligned} d(x_N, \sigma_N^\tau(x_N)) &\leq d(x_N, x) + d(x, f^\tau(x)) + d(f^\tau(x), f^\tau(x_N)) \\ &\quad + d(f^\tau(x_N), \sigma_N^\tau(x_N)) \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

† That is, the smallest integer k such that $f_N^k(E_N) = \Omega(f_N)$.

‡ In fact, Miernowski proves 'permutation' but his arguments, combined with Alpern's version of Lax's theorem, show 'cyclic permutation'.

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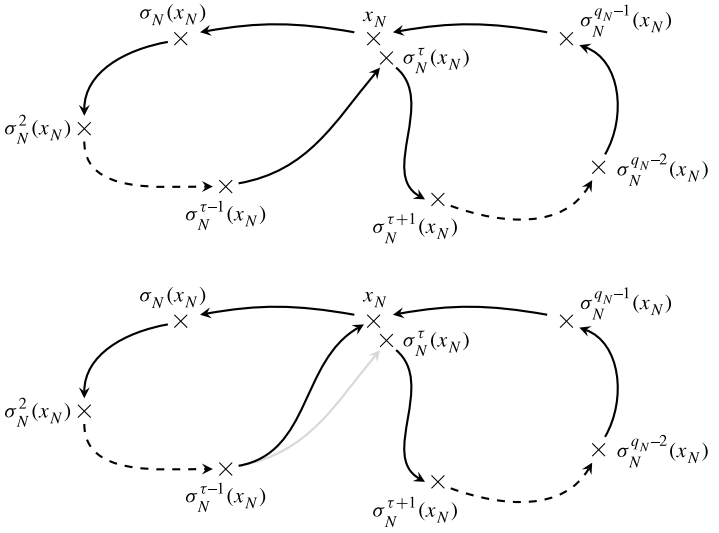


FIGURE 3. Modification of the cyclic permutation σ_N to σ'_N.

We then compose σ_N by the (non-bijective) application mapping $\sigma_N^{\tau-1}(x_N)$ on x_N and any other point $y_N \in E_N$ on itself (see Figure 3); in other words, we consider the application

$$\sigma'_N(x) = \begin{cases} x_N & \text{if } x = \sigma_N^{\tau-1}(x_N), \\ \sigma_N(x) & \text{otherwise.} \end{cases}$$

The map σ'_N has a unique injective orbit whose associated periodic orbit $\Omega(\sigma'_N)$ has length τ (it is $(x_N, \sigma_N(x_N), \ldots, \sigma_N^{\tau-1}(x_N))$). Since $d(f, \sigma'_N) < \varepsilon$, the map σ'_N verifies the conclusion of the proposition. □

A direct application of Theorem 17 gives the following result.

COROLLARY 26. *For a generic homeomorphism* $f \in \mathrm{Homeo}(X, \lambda)$,

$$\underline{\lim}_{N\to+\infty} \frac{\mathrm{Card}(\Omega(f_N))}{\mathrm{Card}(E_N)} = 0.$$

Specifically, for all $\varepsilon > 0$ *and all* $N_0 \in \mathbb{N}$, *there exists* $N \geq N_0$ *such that* $\mathrm{Card}(\Omega(f_N))/\mathrm{Card}(E_N) < \varepsilon$ *and such that* E_N *consists of a unique (pre-periodic) orbit of* f_N.

The same kind of idea leads to the following proposition.

PROPOSITION 27. *Let* $f \in \mathrm{Homeo}(X, \lambda)$ *have at least one periodic point of period* p. *Then for all* $\varepsilon > 0$, *there exists* $N_0 \in \mathbb{N}$ *such that for all* $N \geq N_0$, *there exists an application* $\sigma_N : E_N \to E_N$ *with* $\mathrm{Card}(\Omega(\sigma_N)) = p$ *and* $d_N(f, \sigma_N) < \varepsilon$, *such that* E_N *consists of a unique (pre-periodic) orbit of* σ_N.

Proof of Proposition 27. Simply replace the recurrent point by a periodic point of period p in the proof of Proposition 25. □

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We will show that possessing a periodic point is a generic property in $\mathrm{Homeo}(X, \lambda)$ (see also [**DF00**] or [**Gui12**, §3.2]), then a proper application of Baire's theorem leads to a refinement of Corollary 26.

COROLLARY 28. *Let $\varepsilon > 0$. A generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$ has a periodic point whose orbit is ε-dense. Moreover, f_N has a unique periodic orbit for infinitely many integers N, whose period does not depend on $N \in \mathbb{N}$ and equals the smallest period of periodic points of f which are ε-dense; moreover, E_N is covered by a single (pre-periodic) orbit of f_N.*

To prove Corollary 28 we have to introduce the concept of *persistent point*.

Definition 29. Let $f \in \mathrm{Homeo}(X)$. A periodic point x of f with period p is said *persistent* if for every neighbourhood U of x, there exists a neighbourhood $\mathcal{V}$ of f in $\mathrm{Homeo}(X)$ such that every $\widetilde{f} \in \mathcal{V}$ has a periodic point $\widetilde{x} \in U$ with period p.

Example 30. The endomorphism $h = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ of $\mathbb{R}^n$, with $\prod \lambda_i = 1$ and $\lambda_i \neq 1$ for all i, is measure-preserving and has a persistent fixed point at the origin (see, for example, [**KH95**, p. 319]). Let s be a reflection of $\mathbb{R}^n$; the application $h \circ f$ is also measure-preserving and has a persistent fixed point at the origin.

Finally, we use the theorem of local modification of conservative homeomorphisms, which allows us to replace locally one homeomorphism by another. Although it is geometrically 'obvious' and it has an elementary proof in dimension two, it is easily deduced from the (difficult) *annulus theorem*. For more details one may refer to [**DF00**] or [**Gui12**, §3.1].

THEOREM 31. (Local modification) *Let σ_1, σ_2, τ_1 and τ_2 be four bicollared embeddings† of $\mathbf{S}^{n-1}$ in $\mathbb{R}^n$, such that σ_1 is in the bounded connected component‡ of σ_2 and τ_1 is in the bounded connected component of τ_2. Let A_1 be the bounded connected component of $\mathbb{R}^n \setminus \sigma_1$ and B_1 the bounded connected component of $\mathbb{R}^n \setminus \tau_1$; Σ be the connected component of $\mathbb{R}^n \setminus (\sigma_1 \cup \sigma_2)$ having $\sigma_1 \cup \sigma_2$ as boundary and Λ the connected component of $\mathbb{R}^n \setminus (\tau_1 \cup \tau_2)$ having $\tau_1 \cup \tau_2$ as boundary; A_2 be the unbounded connected component of $\mathbb{R}^n \setminus \sigma_2$ and B_2 the unbounded connected component of $\mathbb{R}^n \setminus \tau_2$ (see Figure 4).*

Suppose that $\mathrm{Leb}(A_1) = \mathrm{Leb}(B_1)$ and $\mathrm{Leb}(\Sigma) = \mathrm{Leb}(\Lambda)$. Let $f_i : A_i \to B_i$ be two measure-preserving homeomorphisms such that either each one preserves the orientation, or each one reverses it. Then there exists a measure-preserving homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ whose restriction to A_1 equals f_1 and restriction to A_2 equals f_2.

Since this theorem is local, it can be applied to an open space O instead of $\mathbb{R}^n$, or, even better, together with the Oxtoby–Ulam theorem (Theorem 13), to any coordinate neighbourhood of a manifold X instead of $\mathbb{R}^n$ and measure λ instead of Lebesgue measure.

† An embedding i of a manifold M in $\mathbb{R}^n$ is said *bicollared* if there exists an embedding $j : M \times [-1, 1] \to \mathbb{R}^n$ such that $j_{M \times \{0\}} = i$.

‡ By the Jordan–Brouwer theorem the complement of a set which is homeomorphic to $\mathbf{S}^{n-1}$ has exactly two connected components: one bounded and one unbounded.

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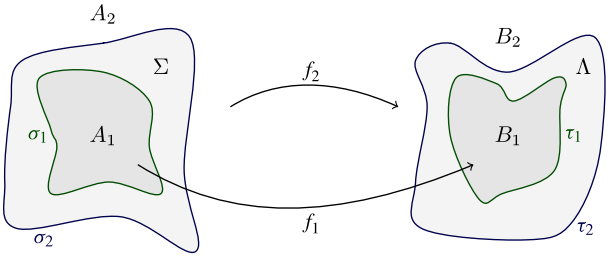


FIGURE 4. Local modification technique.

Proof of Corollary 28. Let $\varepsilon > 0$ and $\mathcal{U}_p$ the set of homeomorphisms that have at least one periodic persistent point with period p whose orbit is ε-dense (strictly), but no periodic point with period less than p whose orbit is ε-dense (not strictly). The sets $\mathcal{U}_p$ are open and pairwise disjoints. Moreover, their union is dense. Indeed, for $f \in \mathrm{Homeo}(X, \lambda)$, if f has at least one periodic point whose orbit is ε-dense, then we choose a periodic point x whose orbit is ε-dense and whose period p is minimal. Then we perturb f such that this point is persistent, avoiding the creation of periodic points of smaller periods: we choose a neighbourhood D of x such that the sets $D, f(D), \ldots, f^{p-1}(D)$ are pairwise disjoints and we locally replace f by the map $h \circ f^{-(p-1)}$ (where h is one of the two maps of the Example 30, depending of whether $f^{-(p-1)}$ preserves orientation or not) in the neighbourhood of $f^{-1}(x)$, using the theorem of local modification (Theorem 31), such that f does not change outside the union of the sets $f^i(D)$. In contrast, if f does not have any periodic point, we pick a recurrent point and perturb f such that this point is periodic, and then we apply the previous case.

Let $p \in \mathbb{N}^*$. Proposition 27 indicates that the set of maps whose longest periodic orbit has length p is a dense type of approximation in $\mathcal{U}_p$. So, by Theorem 17, there exist open sets $\mathcal{O}_{p,N}$ such that $\bigcap_{N\in\mathbb{N}} \mathcal{O}_{p,N}$ is a dense G_δ set of $\mathcal{U}_p$, consisting of homeomorphisms of $\mathcal{U}_p$ whose infinite number of discretizations have only one injective orbit whose associated cycle has length p and is ε-dense.

Set $\mathcal{O}_N = \bigcup_{p\in\mathbb{N}^*} \mathcal{O}_{p,N}$. Since the sets $\mathcal{U}_p$ are pairwise disjoints,

$$\bigcap_{N\in\mathbb{N}} \mathcal{O}_N = \bigcap_{N\in\mathbb{N}} \bigcup_{p\in\mathbb{N}^*} \mathcal{O}_{p,N} = \bigcup_{p\in\mathbb{N}^*} \bigcap_{N\in\mathbb{N}} \mathcal{O}_{p,N}.$$

The right-hand side of the equation forms a dense subset of $\mathrm{Homeo}(X, \lambda)$, so the left-hand side is a dense G_δ set of $\mathrm{Homeo}(X, \lambda)$. It consists of homeomorphisms f such that for infinitely many $N \in \mathbb{N}$, the grid E_N is covered by a single forward orbit of f_N whose associated periodic orbit is ε-dense and whose length is the shortest period of periodic points of f whose orbit are ε-dense. □

Remark 32. The shortest period of periodic points of generic homeomorphisms has no global upper bound in $\mathrm{Homeo}(X, \lambda)$: for example, for all $p \in \mathbb{N}$, there is a open set of homeomorphisms of the torus without periodic point of period less than p (e.g. the neighbourhood of a non-trivial rotation) and this property remains true for discretizations.

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We now want to obtain a discrete analogue to topological weak mixing (which is generic in Homeo(X, λ), see, for example, [**AP00**] or [**Gui12**, §2.4]).

Definition 33. A homeomorphism f is said to be *topologically weakly mixing* if for all non-empty open sets $(U_i)_{i\leq M}$ and $(U'_i)_{i\leq M}$, there exists $m \in \mathbb{N}$ such that $f^m(U_i) \cap U'_i$ is non-empty for all $i \leq M$.

The proof of the genericity of topological weak mixing starts with an approximation of every conservative homeomorphism by another having ε-dense periodic orbits whose lengths are relatively prime. The end of the proof primarily involves the use of Baire's theorem and Bezout's identity. In the discrete case, the notion of weak mixing is replaced by the following.

Definition 34. Let $\varepsilon > 0$. A finite map σ_N is said to be *ε-topologically weakly mixing* if for all $M \in \mathbb{N}$ and all balls $(B_i)_{i\leq M}$ and $(B'_i)_{i\leq M}$ with diameter ε, there exists $m \in \mathbb{N}$ such that for all i,

$$\sigma_N^m(B_i \cap E_N) \cap (B'_i \cap E_N) \neq \emptyset.$$

The first step of the proof is replaced by the following variation of Lax's theorem.

PROPOSITION 35. (Second variation of Lax's theorem) *Let $f \in$ Homeo(X, λ) be a homeomorphism whose iterates are all topologically transitive. Then for all $\varepsilon > 0$ and all $M \in \mathbb{N}^*$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, there exists $\sigma_N : E_N \to E_N$ which has M ε-dense periodic orbits whose lengths are pairwise relatively prime, and such that $d_N(f, \sigma_N) < \varepsilon$.*

Proof of Proposition 35. We prove the proposition in the case where $M = 2$, the other cases being easily obtained by induction. Let $\varepsilon > 0$ and f be a homeomorphism whose iterates are all topologically transitive. Then there exist $x_0 \in X$ and $p \in \mathbb{N}^*$ such that $\{x_0, \dots, f^{p-1}(x_0)\}$ is ε-dense and $d(x_0, f^p(x_0)) < \varepsilon/2$. Since transitive points of f^p form a dense G_δ subset of X, while the orbit of x_0 forms a F_σ set with empty interior, the set of points whose orbit under f^p is dense and disjoint from that of x_0 is dense. So we can pick such a transitive point y_0. Set $y_1 = f(y_0)$. Then there exists a multiple q_1 of p such that the orbit $\{y_1, \dots, f^{q_1-1}(y_1)\}$ is ε-dense and $d(y_1, f^{q_1}(y_1)) < \varepsilon/2$. Again, by density, we can choose a transitive point y_2 whose orbit is disjoint from that of x_0 and y_1, with $d(y_1, y_2) < \varepsilon/2$ and $d(y_0, y_1) - d(y_0, y_2) > \varepsilon/4$. Then there exists a multiple q_2 of p such that $d(y_2, f^{q_2}(y_2)) < \varepsilon/2$. And so on. Continuing in this way, we construct a sequence $(y_m)_{1\leq m\leq \ell}$ such that (see Figure 5):

(i) for all m, there exists $q_m > 0$ such that $p|q_m$ and $d(y_m, f^{q_m}(y_m)) < \varepsilon/2$;
(ii) the orbits $\{x_0, \dots, f^{p-1}(x_0)\}$ and $\{y_m, \dots, f^{q_m-1}(y_m)\}$ (m going from 0 to $\ell - 1$) are pairwise disjoint;
(iii) for all m, $d(y_m, y_{m+1}) < \varepsilon/2$ and $d(y_0, y_m) - d(y_0, y_{m+1}) > \varepsilon/4$;
(iv) $y_\ell = y_0$.

Let σ_N be a finite map given by Lax's theorem. For all N large enough, σ_N satisfies the same properties (i)–(iii) as f. Changing σ_N at the points $\sigma_N^{q_m-1}((y_m)_N)$ and $\sigma_N^{p-1}((x_0)_N)$, we obtain a finite map σ'_N such that $\sigma'^{q_N}_N((y_m)_N) = (y_{m+1})_N$ and $\sigma'^{p}_N((x_0)_N) = (x_0)_N$. Thus the orbit of $(x_0)_N$ under σ'_N is 2ε-dense and has period p and the orbit of $(y_0)_N$

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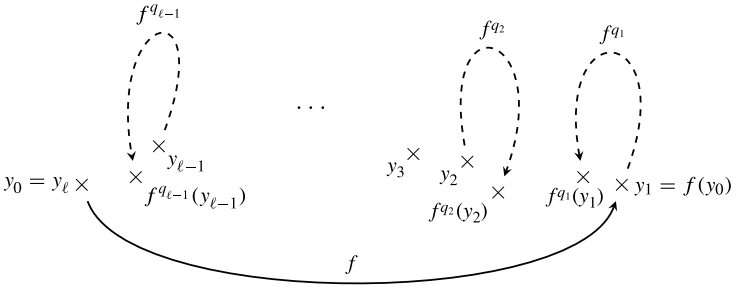


FIGURE 5. Construction of the sequence $(y_m)_{1\le m\le \ell}$.

under σ'_N is 2ε-dense, disjoint from that of $(x_0)_N$ and has period $1 + q_1 + \cdots + q_{\ell-1}$ relatively prime to p. □

COROLLARY 36. *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$, for all $\varepsilon > 0$ and all $N_0 \in \mathbb{N}$, there exists $N \ge N_0$ such that f_N is ε-topologically weakly mixing.*

Proof of Corollary 36. Again, we prove the corollary in the case where $M = 2$, other cases being easily obtained by induction. Let $\varepsilon > 0$ and $N_0 \in \mathbb{N}$. All iterates of a generic homeomorphism f are topologically transitive: this is an easy consequence of the genericity of transitivity (see, for example, Corollary 23 or [**Gui12**, Theorem 2.11]); we pick such a homeomorphism. Combining Theorem 17 and Proposition 35, we obtain $N \ge N_0$ such that f_N has two $\varepsilon/3$-dense periodic orbits whose lengths p and q are coprime. We now have to prove that f_N is ε-topologically weakly mixing. Let B_1, B_2, B'_1 and B'_2 be balls with diameter ε. Since each one of these orbits is $\varepsilon/3$-dense, there exists $x_N \in X$ which is in the intersection of the orbit whose length is p and B_1, and $y_N \in X$ which is in the intersection of the orbit whose length is q and B_2. Similarly, there exist two integers a and b such that $f_N^a(x_N) \in B'_1$ and $f_N^b(y_N) \in B'_2$.

Recall that we want to find a power of f_N which sends both x_N to B'_1 and y_N to B'_2. It suffices to pick $m \in \mathbb{N}$ such that $m = a + \alpha p = b + \beta q$. Bezout's identity states that there exist two integers α and β such $\alpha p - \beta q = b - a$. Setting $m = a + \alpha p$, adding a multiple of pq if necessary, we can suppose that m is positive. Thus $f_N^m(x_N) \in B'_1$ and $f_N^m(y_N) \in B'_2$. □

We now establish a new variation of Lax's theorem. It asserts that every homeomorphism can be approximated by a finite map whose image has a small cardinality, unlike what happens in Lax's theorem.

PROPOSITION 37. (Third variation of Lax's theorem) *Let $f \in \mathrm{Homeo}(X, \lambda)$ and $\vartheta : \mathbb{N} \to \mathbb{R}_+^*$ be a map which tends to $+\infty$ at $+\infty$. Then for all $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $N \ge N_0$, there exists a map $\sigma_N : E_N \to E_N$ such that $\mathrm{Card}(\sigma_N(E_N)) < \vartheta(N)$ and $d_N(f, \sigma_N) < \varepsilon$.*

Proof of Proposition 37. Let $f \in \mathrm{Homeo}(X, \lambda)$, $\vartheta : \mathbb{N} \to \mathbb{R}_+^*$ be a map which tends to $+\infty$ at $+\infty$ and $\varepsilon > 0$. By Lax's theorem (Theorem 20) there exists $N_1 \in \mathbb{N}$ such that

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for all $N \geq N_1$, there exists a cyclic permutation $\sigma_N : E_N \to E_N$ whose distance to f is smaller than $\varepsilon/2$. For $N \geq N_1$, set $\sigma'_N = P_{N_1} \circ \sigma_N$. Increasing N_1 if necessary, we have $d(f, \sigma'_N) < \varepsilon$, regardless of N. Moreover, $\mathrm{Card}(\sigma'_N(E_N)) \leq q_{N_1}$; if we choose N_0 large enough such that for all $N \geq N_0$ we have $q_{N_1} < \vartheta(N)$, then $\mathrm{Card}(\sigma'_N(E_N)) \leq \vartheta(N)$. We have shown that the map σ'_N satisfies the conclusions of the proposition for all $N \geq N_0$. □

COROLLARY 38. *Let $\vartheta : \mathbb{N} \to \mathbb{R}^*_+$ be a map which tends to $+\infty$ at $+\infty$. Then for a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$,*

$$\underline{\lim}_{N \to +\infty} \frac{\mathrm{Card}(f_N(E_N))}{\vartheta(N)} = 0.$$

In particular, generically, $\underline{\lim}(\mathrm{Card}(f_N(E_N))/\mathrm{Card}(E_N)) = 0$.

Proof of Corollary 38. Remark that if we replace $\vartheta(N)$ by $\sqrt{\vartheta(N)}$, it suffices to prove that for a generic homeomorphism, $\underline{\lim}(\mathrm{Card}(f_N(E_N))/\vartheta(N)) \leq 1$. This is easily done by combining Theorem 17 and Proposition 37. □

So far all variations of Lax's theorem have constructed finite maps with a small number of orbits. With the additional assumption that the sequence of grids is self-similar, we final a last variation of Lax's theorem approaching every homeomorphism with a finite map with a large number of orbits.

PROPOSITION 39. (Fourth variation of Lax's theorem) *Assume that the sequence of grids $(E_N)_{N \in \mathbb{N}}$ is self-similar. Let $f \in \mathrm{Homeo}(X, \lambda)$ and $\vartheta : \mathbb{N} \to \mathbb{R}$ be such that $\vartheta(N) = o(q_N)$. Then for all $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for all $N \geq N_1$, there exists a permutation σ_N of E_N such that $d_N(f, \sigma_N) < \varepsilon$ and that the number of cycles of σ_N is greater than $\vartheta(N)$. Moreover, these $\vartheta(N)$ cycles of σ_N are conjugated to a cyclic permutation of E_{N_0} by bijections whose distance to identity is smaller than ε.*

Proof of Proposition 39. Let $\varepsilon > 0$, for all $N_0 \in \mathbb{N}$ large enough, Lax's theorem gives us a cyclic permutation σ'_{N_0} of E_{N_0} whose distance to f is smaller than ε. Since the grids are self-similar, there exists $N_1 \in \mathbb{N}$ such that for all $N \geq N_1$, the set E_N contains $q_N/q_{N_0} \geq \vartheta(N)$ disjoint subsets $\widetilde{E}^j_N$, each one conjugated to a grid E_{N_0} by a bijection h_j whose distance to identity is smaller than ε. On each $\widetilde{E}^j_N$, we define σ_N as the conjugation of σ'_{N_0} by h_j; outside these sets we just pick σ_N such that $d_N(f, \sigma_N) < \varepsilon$. Since the distance between h_j and identity is smaller than ε, we have $d_N(f, \sigma_N) < 2\varepsilon$. Moreover, σ_N has at least $\vartheta(N)$ cycles. This completes the proof. □

The application of Theorem 17 gives us the following result.

COROLLARY 40. *We still assume that the sequence of grids $(E_N)_{N \in \mathbb{N}}$ is self-similar. Let $\vartheta : \mathbb{N} \to \mathbb{R}$ be such that $\vartheta(N) = o(q_N)$. Then for a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$ and for infinitely many integers N, the discretization f_N of f has at least $\vartheta(N)$ cycles which are pairwise conjugated.*

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4.4. *Average behaviour of discretizations.* We now want to study the average behaviour of discretizations of a generic homeomorphism. For example, one could imagine that even if for a generic homeomorphism f, the event 'f_N is a cyclic permutation' appears for infinitely many orders N, it is still quite rare. More precisely, we study the frequency of occurrence of properties related to the discretizations of generic homeomorphisms in the Cesàro sense: given a property (P) concerning discretizations, what is the behaviour of the proportion between 1 and M of discretizations satisfying the property (P), when M goes to infinity? For this study, we assume that the sequence of discretization grids refines (which is true, for example, for discretizations upon uniform grids of orders powers of an integer; see §3). This prevents us from having to deal with tricky arithmetic problems about overlay of grids.

Definition 41. Let $f \in \mathrm{Homeo}(X, \lambda)$. We say that a property (P) about discretizations is *satisfied on average* if for all $N_0 \in \mathbb{N}$ and all $\varepsilon > 0$, there exists $N \geq N_0$ such that the proportion of integers $M \in \{0, \ldots, N\}$ such that f_M satisfies (P) is greater than $1 - \varepsilon$, i.e.

$$\overline{\lim_{N \to +\infty}} \frac{1}{N+1} \mathrm{Card}\left\{M \in \{0, \ldots, N\} \mid f_M \text{ satisfies } (P)\right\} = 1.$$

We will show that most of the dynamical properties studied in the previous section are actually satisfied on average for generic homeomorphisms. To start with we set out a technical lemma.

LEMMA 42. *Let $\mathcal{T}$ be a dense type of approximation in* $\mathrm{Homeo}(X, \lambda)$. *Then for a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$, for all $\varepsilon > 0$ and all $\alpha > 0$, the property (P) : 'E_N contains at least α disjoints subsets which fills a proportion greater than $1 - \varepsilon$ of E_N, each one stabilized by f_N and such that the restriction of f_N to each one is conjugated to a map of $\mathcal{T}$ by a bijection whose distance to identity is smaller than ε' is satisfied on average.*

In practice, this lemma provides many properties satisfied on average, for instance:

- quantitative properties on discretizations, such as possessing at least M periodic orbits;
- properties of existence of sub-dynamics on discretizations, such as possessing at least one dense periodic orbit.

Proof of Lemma 42. Let us consider the set

$$\mathcal{C} = \bigcap_{\substack{\varepsilon > 0 \\ N_0 \in \mathbb{N}}} \bigcup_{N \geq N_0} \left\{ f \in \mathrm{Homeo}(X, \lambda) \;\middle|\; \frac{1}{N+1} \mathrm{Card}\{M \in \{0, \ldots, N\} \mid f_M \text{ satisfies } (P)\} > 1 - \varepsilon \right\}.$$

We want to show that $\mathcal{C}$ contains a dense G_δ subset of $\mathrm{Homeo}(X, \lambda)$. The set $\mathcal{C}$ is a G_δ of the generic set $\bigcap_{N \in \mathbb{N}} \mathcal{D}_N$, it suffices to prove that it is dense in $\bigcap_N \mathcal{D}_N$. Let $f \in \mathrm{Homeo}(X, \lambda)$, $N_0 \in \mathbb{N}$, $\delta > 0$ and $\varepsilon > 0$. For our proof we want to find a homeomorphism g whose distance to f is smaller than δ and an integer $N \geq N_0$ such that

$$\frac{1}{N+1} \mathrm{Card}\left\{M \in \{0, \ldots, N\} \mid g_M \text{ satisfies } (P)\right\} > 1 - \varepsilon.$$

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This is done simply by combining the density of the type of approximation $\mathcal{T}$ and the fact that the grids refine. □

This corollary allows us to obtain some properties about the average behaviour of discretizations. For instance, there is an improvement of Corollary 28.

COROLLARY 43. *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$, the property '$f_N$ has an ε-dense periodic orbit and the cardinality of $\Omega(f_N)$ is smaller than $\vartheta(N) = o(q_N)$' is satisfied on average.*

There is also an improvement of Corollary 36; see the following result.

COROLLARY 44. *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$ and for all $\varepsilon > 0$, the property 'f_N is ε-topologically weakly mixing (see Definition 34)' is satisfied on average.*

There is even an improvement of Corollary 38; see the following result.

COROLLARY 45. *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$ and for all $\varepsilon > 0$, the property '*$\mathrm{Card}(f_N(E_N))/\mathrm{Card}(E_N) < \varepsilon$*' is satisfied on average.*

Finally, there is an improvement of Corollary 40; see the following result.

COROLLARY 46. *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$ and for all $M \in \mathbb{N}$, the property 'f_N has at least M periodic orbits' is satisfied on average.*

However, note that the simplest property of discretizations, i.e. being a cyclic permutation, cannot be proved by using Lemma 42. To do this, we need a slightly more precise result, which we will not prove here.

PROPOSITION 47. *For a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$, the property '$f_N$ is a cyclic permutation' is satisfied on average.*

Remark 48. However, the property of approximation by bicyclic permutations on average cannot be proved with this technique.

Remark 49. A simple calculation shows that everything that has been done in this section also applies to the behaviour of discretizations on average of Cesàro average, on average of average of Cesàro average etc., i.e. when studying quantities

$$\frac{1}{N_2+1}\sum_{N_1=0}^{N_2}\frac{1}{N_1+1}\,\mathrm{Card}\left\{M \in \{0,\dots,N_1\} \mid f_M \text{ satisfies } (P)\right\},$$

$$\frac{1}{N_3+1}\sum_{N_2=0}^{N_3}\frac{1}{N_2+1}\sum_{N_1=0}^{N_2}\frac{1}{N_1+1}\,\mathrm{Card}\left\{M \in \{0,\dots,N_1\} \mid f_M \text{ satisfies } (P)\right\},\dots.$$

4.5. *Behaviour of all the discretizations.* In the previous sections we showed that the dynamical behaviour of discretizations seems very chaotic, depending on the order of discretization. In contrast, the dynamics of a generic homeomorphism is well known (see, for example, [**Gui12**]) and independent of the homeomorphism. One even has a 0–1 law on $\mathrm{Homeo}(X, \lambda)$ (see [**GK98**] or the final chapter of [**Gui12**]) which states that either

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a given ergodic property on conservative homeomorphisms is generic, or its contrary is generic. Thus, the dynamics of a generic homeomorphism and that of its discretizations seem quite of independent. In fact one can deduce some dynamical features of a generic homeomorphism from the corresponding dynamical features of *all* its discretizations. For instance, the following property can be easily deduced from Corollary 28.

PROPOSITION 50. *Let f be a generic homeomorphism of* Homeo(X, λ) *and p be an integer. Then f has a periodic orbit with period p if and only if there exist infinitely many integers N such that f_N has a periodic orbit with period p.*

Proof of Proposition 50. An easy variation of Corollary 28 shows that if p is a period of a periodic orbit of a generic homeomorphism f, then there exist infinitely many discretizations f_N such that f_N has a periodic orbit with period p. The other implication of the proposition follows easily from a compactness argument. □

We now try to obtain information about invariant measures of a generic homeomorphism from invariant measures of its discretizations. More precisely, given all the invariant measures of discretizations of a generic homeomorphism, what can be deduced about invariant measures of the initial homeomorphism? A first step in this study was taken in 2006 by Miernowski [**Mie06**, §8].

PROPOSITION 51. (Miernowski) *Let $f : X \to X$ be a uniquely ergodic homeomorphism and μ^f its unique invariant probability measure. For all $N \in \mathbb{N}$, let $\gamma_N \subset E_N$ be a periodic cycle of f_N and ν_N the uniform probability measure on γ_N. Then the weak convergence $\nu_N \rightharpoonup \mu^f$ occurs independently of the choice of the cycles γ_N.*

The proof of this proposition essentially involves an appropriate application of Prokhorov's theorem which expresses the compactness of the set of probability measures on X. We now set out a theorem of this kind for *generic* homeomorphisms. Recall that μ_N^f is the limit in the sense of Cesàro of the pushforwards by iterates of f_N of uniform measure on E_N (see Definition 7):

$$\mu_N^f = \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} (f_N)_*^i \lambda_N.$$

The measure μ_N^f is supported by the maximal invariant set of f_N; it is uniform on every periodic orbit and the total weight of a periodic orbit is proportional to the size of its basin of attraction. The following theorem expresses that we can obtain all the invariant measures of a generic homeomorphism from all the invariant measures of its discretizations.

THEOREM 52. *Let $f \in$ Homeo(X, λ) be a generic homeomorphism and suppose that the sequence of grids $(E_N)_{N\in\mathbb{N}}$ is self-similar. Let $\mathcal{M}_N$ be the set of probability measures on E_N that are invariant under f_N. Then the upper limit over N (for Hausdorff metric) of the sets $\mathcal{M}_N$ is exactly the set $\mathcal{M}$ of probability measures that are invariant under f.*

Before giving a detailed proof of Theorem 52 let us give its main arguments. To show that the upper limit of sets $\mathcal{M}_N$ is a subset of $\mathcal{M}$, by a compactness argument,

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we only have to prove that limit points of a sequence $(\nu_N^f)_{N\in\mathbb{N}}$, such that every measure ν_N^f is invariant under f_N, are invariant under f. This can be easily verified by using uniform convergence of f_N to f and equicontinuity of the measures ν_N^f. Then (again by a compactness argument) we have to show that generically every invariant measure is a limit point of a sequence of discretizations. An *ad hoc* application of Baire's theorem reduces the proof to that of the following variation of Lax's theorem†.

LEMMA 53. (Ergodic variation of Lax's theorem) *Suppose that the sequence of grids $(E_N)_{N\in\mathbb{N}}$ is self-similar. For all $f \in \mathrm{Homeo}(X, \lambda)$, for every collection of f-invariant measures $\nu_1, \ldots, \nu_\ell$, for all $\varepsilon > 0$ and $k_0, N_0 \in \mathbb{N}$:*

(1) *there exist $g \in \mathrm{Homeo}(X, \lambda)$ and $N \geq N_0$ such that $d(f, g) < \varepsilon$, $d(\nu_1, \mu_N^g) < 1/k_0$ and μ_N^g is g-invariant;*

(2) *there exist $g \in \mathrm{Homeo}(X, \lambda)$ and $N \geq N_0$ such that $d(f, g) < \varepsilon$, and for all $i \leq \ell$ there exists a measure $\nu_{i,N}^g$ such that $d(\nu_i, \nu_{i,N}^g) < 1/k_0$ and that $\nu_{i,N}^g$ is invariant under g and g_N.*

Suppose first that $\ell = 1$ and that ν_1 is ergodic. For this purpose we apply Birkhoff's theorem to f, ν_1, φ and a recurrent point x: for all M large enough,

$$\frac{1}{M}\sum_{k=0}^{M-1} \varphi \circ f^k(x) \simeq \int \varphi \, d\nu_1.$$

Since x is recurrent we can choose an integer M large enough such that $x \simeq f^M(x)$. First we approximate f by a cyclic permutation σ_N given by Lax's theorem, then we slightly modify σ_N into a map σ_N' by choosing $\sigma_N'(\sigma_N^M(x_N)) = x_N$, as in Proposition 25. The measure $\nu_N^{\sigma_N'}$ is the uniform measure on the orbit $x_N, \ldots, \sigma_N^{M-1}(x_N)$, so it is close to ν_1 with respect to the test function φ. This property can be obtained for every test function by a diagonal-like method. The proof of the lemma when ν_1 is not ergodic but only invariant is obtained by approximating the invariant measure by a finite convex combination of ergodic measures; this proves the first point of the lemma. To obtain the latter, it suffices to use the self-similarity hypothesis: on at least ℓ sub-grids we apply the first point to each measure ν_i.

Proof of Theorem 52. For all $\delta > 0$, for all N large enough, $\mathcal{M}_N \subset B(\mathcal{M}, \delta)$. For a set A and $\delta > 0$, we denote by $B(A, \delta)$ the set of elements whose distance to A is smaller than δ. This inclusion follows easily from the upper semi-continuity of the application $g \mapsto \mathcal{M}^g$ (where $\mathcal{M}^g$ denotes the set of Borel probability measures which are invariant under g) and the compactness of $\mathcal{P}$.

Generically, for all $\delta > 0$ and all $N_0 \in \mathbb{N}$, there exists $N \geq N_0$ such that $\mathcal{M} \subset B(\mathcal{M}_N, \delta)$. Recall that we denote by $\mathcal{P}$ the set of all Borel probability measures on X. Let d be a distance on $\mathcal{P}$ defining the weak-star topology. Subsequently homeomorphisms will be taken in the generic set

$$\bigcap_{N\in\mathbb{N}} \mathcal{D}_N,$$

† The first point of this lemma is in some sense a weak version of the latter. It will be useful in the next section.

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consisting of homeomorphisms whose Nth discretization is uniquely defined for all N. Consider

$$\mathcal{A} = \bigcap_{(N_0,k_0)\in\mathbb{N}^2} \mathcal{O}_{N_0,k_0},$$

where

$$\mathcal{O}_{N_0,k_0} = \left\{ f \in \bigcap_{N\in\mathbb{N}} \mathcal{D}_N \;\middle|\; \exists N \geq N_0 \text{ s.t. } \mathcal{M} \subset B\left(\mathcal{M}_N, \frac{3}{k_0}\right) \right\}.$$

Trivially, if $f \in \mathcal{A}$, then the upper limit of the sets $\mathcal{M}_N$ contains $\mathcal{M}$.

To show that the sets $\mathcal{O}_{N_0,k_0}$ are open, it suffices to remark that if $M \subset B(\mathcal{M}_N, 3/k_0)$ for a given N, then it is also true on a neighbourhood of f since $g \mapsto \mathcal{M}_N^g$ is constant on a neighbourhood of f and $g \mapsto \mathcal{M}^g$ is upper semi-continuous.

It remains to show that the sets $\mathcal{O}_{N_0,k_0}$ are dense; this follows from the second point of the variation of Lax's theorem (Lemma 53): by upper semi-continuity of $g \mapsto \mathcal{M}^g$, one has $\mathcal{M}^g \subset B(\mathcal{M}^f, 1/k_0)$ for all g close enough to f. Moreover, by compactness, there exist f-invariant measures $\nu_1, \ldots, \nu_\ell$ such that $\mathcal{M}^f \subset \bigcup_i B(\nu_i, 1/k_0)$. So it suffices to find g close to f such that for an $N \geq N_0$, there exist g_N-invariant measures $\nu_{1,N}^g, \ldots, \nu_{\ell,N}^g$ such that $d(\nu_i, \nu_{i,N}^g) < 1/k_0$ for all i, but this is exactly the conclusion of the lemma. □

Proof of Lemma 53. To begin with we prove the first point of the lemma. Suppose first that the measure $\nu = \nu_1$ is ergodic. We want to show that there exist a homeomorphism g whose distance to f is smaller than ε, and an integer $N \geq N_0$ such that $d(\nu, \mu_N^g) < 2/k_0$. Let $(\varphi_j)_{j\in\mathbb{N}}$ be a sequence which is dense in the set of continuous maps from X into $\mathbb{R}$. By Prokhorov's theorem, there exists $i \in \mathbb{N}$ such that if

$$\left| \int \varphi_j \, d\nu - \int \varphi_j \, d\mu \right| \leq \frac{1}{i}$$

for all $j \leq i$, then $d(\nu, \mu) < 1/k_0$.

Since ν is ergodic, for every continuous map φ, by Birkhoff's theorem,

$$\frac{1}{M} \sum_{m=0}^{M-1} \varphi \circ f^m(x) \underset{M\to+\infty}{\longrightarrow} \int \varphi \, d\nu \tag{1}$$

for ν-almost every x. Let $x \in X$ be a recurrent point for f satisfying equation (1) for all φ_j (such points form a ν-full measure set). For all j there exists $M_j \in \mathbb{N}$ such that for all $M \geq M_j$,

$$\left| \frac{1}{M} \sum_{m=0}^{M-1} \varphi_j \circ f^m(x) - \int \varphi_j \, d\nu \right| \leq \frac{1}{2i}. \tag{2}$$

Let $M_i' = \max_{j\leq i} M_j$. Since x is recurrent, there exists $\tau \geq M_i'$ such that $d(x, f^\tau(x)) \leq \varepsilon/4$. Let σ_N be a map from E_N into itself given by Lax's theorem: it is a cyclic permutation and its distance to f is smaller than $\varepsilon/2$. For all N large enough the orbit $(\sigma_N^m(x_N))_{0\leq m\leq\tau}$ shadows the orbit $(f^m(x))_{0\leq m\leq\tau}$, thus $d(x_N, \sigma_N^\tau(x_N)) < \varepsilon/2$. Then we 'close' the orbit of x_N between the points x_N and $\sigma_N^\tau(x_N)$, i.e. we set (as in Proposition 25; see also Figure 3)

$$\sigma_N'(y_N) = \begin{cases} x_N & \text{if } y_N = \sigma_N^{\tau-1}(x_N), \\ \sigma_N(y_N) & \text{otherwise.} \end{cases}$$

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Then $d_N(\sigma'_N, f) < \varepsilon$ and $(x_N, \dots, \sigma'^{\tau-1}_N(x_N))$ is a periodic orbit for σ'_N whose basin of attraction is the whole set E_N.

Since the periodic orbit $(\sigma'^m_N(x))_{0\le m\le\tau}$ attracts E_N, for all M' large enough and $y_N \in E_N$ we have

$$\frac{1}{M'}\sum_{m=0}^{M'-1}\varphi_j\circ\sigma'^m_N(y_N) \underset{M'\to\infty}{\longrightarrow} \frac{1}{\tau}\sum_{m=0}^{\tau-1}\varphi_j\circ\sigma'^m_N(x_N). \tag{3}$$

With Proposition 18, the same way as in Lemma 19, we construct a homeomorphism g^N from the map σ'_N such that the discretization $(g^N)_N$ and σ'_N fit together, and such that g and the restriction of $(g^N)_N$ to the orbit of x_N fit together. Since $d_N(f, \sigma'_N) < \varepsilon$, we can furthermore assume that $d(f, g^N) < \varepsilon$. Thus, equation (3) implies that

$$\mu^{g^N}_N = \frac{1}{\tau}\sum_{m=0}^{\tau-1}\delta_{\sigma'^m_N(x_N)}. \tag{4}$$

Since φ_j is continuous, increasing N (if necessary) such that the orbit $(\sigma^m_N(x_N))_{0\le m\le\tau}$ shadows $(f^m(x))_{0\le m\le\tau}$ better, we have, for all $j \le i$:

$$\left|\frac{1}{\tau}\sum_{m=0}^{\tau-1}\varphi_j\circ f^m(x) - \frac{1}{\tau}\sum_{m=0}^{\tau-1}\varphi_j\circ\sigma'^m_N(x_N)\right| \le \frac{1}{2i}. \tag{5}$$

We now have all the necessary estimates to compute the distance between $\mu^{g^N}_N$ and ν: by equation (4) and triangle inequality,

$$\begin{aligned}\left|\int\varphi_j\,d\mu^{g^N}_N - \int\varphi_j\,d\nu\right| &= \left|\frac{1}{\tau}\sum_{m=0}^{\tau-1}\varphi_j\circ\sigma'^m_N(x_N) - \int\varphi_j\,d\nu\right| \\ &\le \left|\frac{1}{\tau}\sum_{m=0}^{\tau-1}\varphi_j\circ\sigma'^m_N(x_N) - \frac{1}{\tau}\sum_{m=0}^{\tau-1}\varphi_j\circ f^m(x_N)\right| \\ &\quad + \left|\frac{1}{\tau}\sum_{m=0}^{\tau-1}\varphi_j\circ f^m(x_N) - \int\varphi_j\,d\nu\right|.\end{aligned}$$

Hence, with equation (2) applied to $M = \tau$ and equation (5),

$$\left|\int\varphi_j\,d\mu^{g^N}_N - \int\varphi_j\,d\nu\right| \le \frac{1}{i}.$$

We obtain $d(\mu^{g^N}_N, \nu) < 1/k_0$. Since $\mu^{g^N}_N$ is invariant under g^N (g^N is a cyclic permutation of the orbit of x_N), g^N is the desired homeomorphism.

In the general case the measure ν is only invariant (and not ergodic). It reduces to the ergodic case by the fact that the set of invariant measures is a compact convex set whose extremal points are exactly the ergodic measures: by the Krein–Milman theorem, for all $M \ge 1$ there exists an f-invariant measure ν' which is a finite convex combination of ergodic measures,

$$\nu' = \sum_{j=0}^{r}\lambda_j\nu^e_j,$$

and whose distance to ν is smaller than $1/k_0$.

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Then we use the hypothesis of self-similarity of the grids: for all $\varepsilon > 0$ and all $P \in \mathbb{N}$ there exists a map σ_N whose distance to f is smaller than ε and which has at least $p \geq P$ cycles that are conjugated to a permutation of a coarser subdivision, via the composition of a fixed homothety by a translation. Moreover, the union on these p cycles fills the set E_N with a proportion of at least $1 - 1/(2k_0)$. Take $P \in \mathbb{N}$ such that $2r/P \leq 1/(2k_0)$. For all j between 1 and r we set

$$\lambda'_j = \frac{\lfloor \lambda_j p \rfloor}{p}$$

and

$$\lambda'_0 = 1 - \sum_{j=1}^{r} \lambda'_j.$$

Then for all $j > 0$ we have $|\lambda_j - \lambda'_j| \leq 1/P$ and $|\lambda_0 - \lambda'_0| \leq r/P$. Since $\sum \lambda'_j = 1$ we can associate bijectively a set Λ_j consisting of $p\lambda'_j$ cycles to each integer j; the union of the sets Λ_j fills the set E_N with a proportion of at least $1 - 1/(2k_0)$. We do the previous construction concerning ergodic measures for each j: increasing N if necessary, the cycles of each Λ_j can be changed such that each of them carries a (unique) invariant probability measure $\nu_j^{e'}$ satisfying $d(\nu_j^e, \nu_j^{e'}) \leq 1/2k_0$. These modifications transform the map σ_N into a map σ'_N satisfying $d(\sigma'_N, \sigma_N) < \varepsilon$. Then the measure $\mu_N^{\sigma'_N}$ associated with the map σ'_N satisfies

$$d\left(\mu_N^{\sigma'_N}, \sum_{j=0}^{r} \lambda'_j \nu_j^{e'}\right) \leq \frac{1}{2k_0},$$

and we have

$$\begin{aligned} d(\nu', \mu_N^{\sigma'_N}) &\leq \sum_{j=0}^{r} d(\lambda_j \nu_j^e, \lambda'_j \nu_j^{e'}) + \frac{1}{2k_0} \\ &\leq \sum_{j=0}^{r} d(\lambda_j \nu_j^e, \lambda'_j \nu_j^e) + \sum_{j=0}^{r} d(\lambda'_j \nu_j^e, \lambda'_j \nu_j^{e'}) + \frac{1}{2k_0} \\ &\leq \sum_{j=0}^{r} |\lambda_j - \lambda'_j| + \sum_{j=0}^{r} \lambda'_j d(\nu_j^e, \nu_j^{e'}) + \frac{1}{2k_0} \\ &\leq \frac{2r}{P} + \frac{1}{2k_0} + \frac{1}{2k_0} \leq \frac{3}{2k_0}. \end{aligned}$$

Thus $d(\nu, \mu_N^{\sigma'_N}) \leq 2/k_0$. As above, the desired homeomorphism is obtained by extending σ'_N to a homeomorphism of X. We have proven the first point of the lemma.

We now turn to the second point of the lemma. Here, it suffices to apply the first point on ℓ different sub-grids of a given grid E_{N_0}. On the ith sub-grid, define the discretization g_N such that it has a single invariant measure which is close to ν_i. This proves the lemma. □

There is a kind of duality between invariant measures and invariant compact sets, thus the previous theorem is also true for invariant compact sets.

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PROPOSITION 54. *Let $f \in \mathrm{Homeo}(X, \lambda)$ be a generic homeomorphism and suppose that the sequence of grids $(E_N)_{N\in\mathbb{N}}$ is self-similar. Let $\mathcal{I}_N$ be the set of subsets of E_N that are invariant under f_N. Then the upper limit over N (for the Hausdorff metric) of the sets $\mathcal{I}_N$ is exactly the set of compacts that are invariant under f.*

Sketch of proof of Proposition 54. To show that all limit points K of a sequence (K_N) of compact invariant sets form a compact invariant set, it suffices to calculate $d_H(K, f(K))$. For the other inclusion, as in the case of invariant measures, it follows from an approximation lemma whose proof is the same as for invariant measures. □

LEMMA 55. (Compact variation of Lax's theorem) *Suppose that the sequence of grids $(E_N)_{N\in\mathbb{N}}$ is self-similar. For all $f \in \mathrm{Homeo}(X, \lambda)$, for every collection of f-invariant compacts $K_1, \ldots, K_\ell$, for all $\varepsilon > 0$ and $k_0, N_0 \in \mathbb{N}$:*

(1) *there exist $g \in \mathrm{Homeo}(X, \lambda)$ and $N \geq N_0$ such that $d(f, g) < \varepsilon$, $d_H(K_1, \Omega(f_N)) < 1/k_0$ and $\Omega(f_N)$ is g-invariant;*
(2) *there exist $g \in \mathrm{Homeo}(X, \lambda)$ and $N \geq N_0$ such that $d(f, g) < \varepsilon$, and for all $i \leq \ell$, there exists a compact $K^g_{i,N}$ such that $d_H(K_i, K^g_{i,N}) < 1/k_0$ and that $K^g_{i,N}$ is invariant under g and g_N.*

4.6. *Physical measures.* Above we proved that, generically, the upper limit of the sets of invariant measures of discretizations is the set of invariant measures of the initial homeomorphism. This expresses the notion that the sets of invariant measures of discretizations behave 'as chaotically as possible'. However, one might expect that physical measures (Borel measures μ such that $\mu = \mu^f_x$ is satisfied on a set of points x with λ positive measure; see, for example, [**You02**]) play a specific part: their definition expresses that they are the measures that can be observed in practice, because they govern the ergodic behaviour of λ-almost every point. So one can hope that the natural invariant measures μ^f_N of f_N, which can be seen as the physical measures of f_N, converge to the physical measures of f. This expectation is supported by the following variation of Proposition 51, obtained by replacing ν_N by μ^f_N: *if f is uniquely ergodic, then the measures μ^f_N converge weakly to the only measure μ^f that is invariant under f.*

In this section we show that this is not at all the case: the sequence of measures (μ^f_N) accumulates on the whole set of f-invariant measures†. More precisely, one has the following theorem.

THEOREM 56. *If the sequence of grids $(E_N)_{N\in\mathbb{N}}$ is self-similar, for a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$, the set of limit points of the sequence $(\mu^f_N)_{N\in\mathbb{N}}$ is exactly the set of f-invariant measures.*

This theorem can be seen as a discretized version of the following conjecture.

† It is easy to see that if $(\nu_N)_{N\in\mathbb{N}}$ is a sequence of measures, with ν_N invariant under f_N for all N, then its limit set is included in the set of f-invariant measures. Again, this reflects the fact that the behaviour of discretizations of a generic homeomorphism is 'as chaotic as possible'.

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CONJECTURE 57. (Abdenur and Andersson [**AA13**]) *A homeomorphism f which is generic in the set of homeomorphisms of X (without measure-preserving hypothesis) that do not have any open trapping set is* wicked*, i.e. it is not uniquely ergodic and the measures*

$$\frac{1}{m}\sum_{k=0}^{m-1} f_*^k(\mathrm{Leb})$$

accumulate on the whole set of invariant measures under f.

The behaviour described in this conjecture is the opposite of that involving possessing a physical measure.

Sketch of proof of Theorem 56. The proof is similar to which of Theorem 52: the set $\mathcal{A}$ is replaced by

$$\mathcal{A}' = \bigcap_{(\ell,N_0,k_0)\in\mathbb{N}^3} \mathcal{O}_{\ell,N_0,k_0},$$

where

$$\mathcal{O}_{\ell,N_0,k_0} = \left\{ f \in \bigcap_{N\in\mathbb{N}} \mathcal{D}_N \;\middle|\; \begin{array}{l} \left(\exists \nu f\text{-inv.} : d(\nu, \tilde{\nu}_\ell) \le \dfrac{1}{k_0}\right) \Longrightarrow \\ \left(\exists N \ge N_0 : d(\tilde{\nu}_\ell, \mu_N^f) < \dfrac{2}{k_0}\right) \end{array} \right\},$$

and the second point of Lemma 53 is replaced by the first. □

Remark 58. Resuming the proof of the theorem, if we set $x \notin \bigcap_N E'_N$, then generically the measures $\mu_{N,x}^f$ accumulate on the whole set of f-invariant measures. This seems to contradict the empirical observations made by Boyarsky in 1986 (see [**Boy86**] or [**GB88**]): when a homeomorphism f has a unique ergodic measure μ^f which is absolutely continuous with respect to Lebesgue measure 'most of' the measures $\mu_{N,x}^f$ tend to measure μ^f. However, the author does not specify in what sense he means 'most of the points', or if his remark is based on a tacit assumption of regularity for f.

Note that as in the previous section, we have a compact counterpart of Theorem 56.

PROPOSITION 59. *If the sequence of grids $(E_N)_{N\in\mathbb{N}}$ is self-similar, for a generic homeomorphism $f \in \mathrm{Homeo}(X, \lambda)$, the set of limit points of the sequence $(\Omega(f_N))_{N\in\mathbb{N}}$ is exactly the set of f-invariant compact sets.*

5. *Part 2. Discretizations of a generic dissipative homeomorphism*

Throughout this part of the paper, we fix a manifold X and a good measure λ as defined in §2. Now homeomorphisms are no longer supposed conservative. More formally, recalling that we denote by $\mathrm{Homeo}(X)$ the set of all *dissipative* homeomorphisms of X (without assuming conservation of a given measure), we are interested in properties of discretizations of generic elements of $\mathrm{Homeo}(X)$. The manifold X is equipped with a sequence $(E_N)_{N\in\mathbb{N}}$ of discretization grids.

Generic topological properties of dissipative homeomorphisms have been discussed in a survey of Akin, Hurley and Kennedy [**AHK03**]. However, ergodic properties of

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generic homeomorphisms had not been studied until the work of Abdenur and Andersson in [**AA13**], in which the authors are interested in the typical behaviour of Birkhoff averages. For this purpose they establish a technical lemma they call the *shredding lemma* which allows them to show that a generic homeomorphism is *weird*.

Definition 60. A homeomorphism f is said to be *weird* if almost every point $x \in X$ (for λ) has a Birkhoff's limit μ_x^f, and if f is *totally singular* (i.e. there exists a Borel set with total measure whose image by f is null measure) and does not admit any physical measure.

In opposition to the conservative case, the lack of conservation of a given measure allows us to perturb any diffeomorphism to create attractors whose basins of attraction cover almost the entire space X. This is an open property, so such basins of attraction can be seen on discretizations.

With what has just been said, we can already note that properties of discretizations of generic dissipative and conservative homeomorphisms are quite different. If before (almost) all possible dynamical properties are observed on the discretizations of a homeomorphism, the dissipative case reveals a much more regular behaviour: 'the dynamics of discretizations of a homeomorphism f converges to the dynamics of f'. For example, the measures μ_N^f accumulate on a single measure, namely μ_X^f, and not on all the invariant measures under f.

5.1. *The shredding lemma and its discrete counterpart.* In the article [**AA13**] Abdenur and Andersson try to identify the generic ergodic properties of continuous maps and homeomorphisms of compact manifolds. More precisely, they study the behaviour of Birkhoff limits μ_x^f for a generic homeomorphism $f \in \mathrm{Homeo}(X)$ and almost every point x for Lebesgue measure. To do this they define some interesting behaviours of homeomorphisms related to Birkhoff limits, including one they call *weird* (see Definition 60). This definition is supported by their proof, based on the shredding lemma, that a generic homeomorphism is weird. We outline an improvement of this lemma, whose main consequence is that a generic homeomorphism has many open attractive sets, all of small measure and decomposable into a small number of small diameter open sets.

LEMMA 61. (Shredding lemma, Abdenur and Andersson [**AA13**]) *For all homeomorphisms $f \in \mathrm{Homeo}(X)$, for all $\varepsilon, \delta > 0$, there exists a family of regular pairwise disjoints open sets† $U_1, \ldots, U_\ell$ such that for all $\varepsilon' > 0$, there exists $g \in \mathrm{Homeo}(X)$ such that $d(f, g) < \delta$ and:*

(i) $g(\overline{U_j}) \subset U_j$;
(ii) $\lambda(U_j) < \varepsilon$;
(iii) $\lambda(\bigcup_{j=1}^{\ell} U_j) > 1 - \varepsilon$;
(iv) $\lambda(g(U_j)) < \varepsilon\, \lambda(U_j)$;
(v) *there exist open sets $W_{j,1}, \ldots, W_{j,\ell_j}$ such that:*
  (a) $\mathrm{diam}(W_{j,i}) < \varepsilon'$ *for all* $i \in \{1, \ldots, \ell_j\}$;
  (b) $g(\overline{W_{j,i}}) \subset W_{j,i+1}$; *for every* $i \in \{1, \ldots, \ell_j - 1\}$ *and* $g(\overline{W_{j,\ell_j}}) \subset W_{j,1}$;
  (c) $\overline{U_j} \subset \bigcup_{m \geq 0} g^{-m}(\bigcup_{i=1}^{\ell_j} W_{j,i})$.

† An open set is said to be *regular* if it is equal to the interior of its closure.

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The lemma as stated by Abdenur and Andersson in their paper can be obtained by equating $\varepsilon = \varepsilon'$. To get this improvement, it suffices to consider the sets $W_{j,i}$ given by the weak version of the lemma and to crush them with a contraction mapping, as the authors do in the first part of the proof of the lemma.

Remark 62. It can be easily seen that the lemma remains true on a neighbourhood of the homeomorphism g.

Remark 63. We can further make the following assumptions:

(1) The open sets $W_{j,i}$ have disjoint attractive sets, i.e. for all $j \in \mathbb{N}$ and all $i \neq i'$, we have†
$$\left(\bigcap_{m\geq 0}\bigcup_{m'\geq m} f^{m'}(W_{j,i})\right)\cap\left(\bigcap_{m\geq 0}\bigcup_{m'\geq m} f^{m'}(W_{j,i'})\right)=\emptyset.$$
For this it suffices to replace $W_{j,i}$ by $W'_{j,i}$ defined inductively:
$$\begin{aligned} W'_{j,1} &= W_{j,1},\\ W'_{j,i+1} &= W_{j,i+1}\setminus\left(\bigcup_{i'\leq i}\bigcap_{m\geq 0}\bigcup_{m'\geq m} f^{m'}(W_{j,i'})\right).\end{aligned}$$

(2) Each set $W_{j,i}$ contains a periodic Lyapunov stable point. Indeed, for all $\varepsilon > 0$, there exist $x \in W_{j,i}$ and $m > 0$ such that $d(x, f^m(x)) < \varepsilon$. Then we perturb f in order to make x m-periodic and attractive, and the Lyapunov stability remains true in a neighbourhood of f.

(3) The set $g(\overline{U_j})$ is 'independent' of the choice of ε': there exist compact sets V_j such that $g(\overline{U_j}) \subset V_j$ and $V_j \subset U_j$, $\lambda(V_j) < \varepsilon\,\lambda(U_j)$.

This lemma tells us a lot about the dynamics of a generic homeomorphism, which becomes quite clear: there are many attractors whose basins of attraction are small and attract almost all the manifold X. Moreover, there is convergence of the attractive sets of the shredding lemma to the closure of the Lyapunov stable periodic points of f.

Notation 64. For a homeomorphism $f \in \mathrm{Homeo}(X)$, we denote by A_0 the set of Lyapunov stable periodic points of f, i.e. the sets of periodic points x such that for all $\delta > 0$, there exists $\eta > 0$ such that if $d(x, y) < \eta$, then $d(f^m(x), f^m(y)) < \delta$ for all $m \in \mathbb{N}$.

COROLLARY 65. *Let $f \in \mathrm{Homeo}(X)$ satisfy the conclusions of the shredding lemma for all $\varepsilon = \varepsilon' > 0$ and $W_{j,i,\varepsilon}$ be the corresponding open sets. Such homeomorphisms form a G_δ dense subset of* $\mathrm{Homeo}(X)$. *Then the sequence of sets*
$$A_\varepsilon = \overline{\bigcup_{j,i} W_{j,i,\varepsilon}}$$
converges to a closed set for Hausdorff distance when ε tends to 0. *This is a null set‡ which coincides generically with the closure of the set A_0.*

† These two sets are decreasing intersections of compact sets, so they are compact and non-empty.
‡ And better, if we are given a countable family $(\lambda_m)_{m\in\mathbb{N}}$ of good measures, generically $\lambda_m(A_0) = 0$.

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Proof of Corollary 65. Let f satisfy the hypothesis of the corollary. We want to show that the sets A_ε tend to A_0 when ε goes to 0. This is equivalent to showing that for all $\delta > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $A_0 \subset B(A_\varepsilon, \delta)$ and $A_\varepsilon \subset B(A_0, \delta)$ (where $B(A, \delta)$ denotes the set of points of X whose distance to A is smaller than δ). Subsequently we will denote by $U_{j,\varepsilon}$ and by $W_{i,j,\varepsilon}$ the open sets given by the shredding lemma for the parameters $\varepsilon = \varepsilon'$.

Let $\delta > 0$. We start by taking $x \in X$ whose orbit is periodic and Lyapunov stable. Then there exists $\eta > 0$ such that if $d(x, y) < \eta$, then $d(f^m(x), f^m(y)) < \delta/2$ for all $m \in \mathbb{N}$. Let $O = B(x\eta)$. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[$, there exists $j \in \mathbb{N}$ such that the intersection between O and $U_{j,\varepsilon}$ is non-empty. Let y be an element of this intersection. By compactness, there exists a subsequence of $(f^m(y))_{m\in\mathbb{N}}$ which tends to x_0; moreover, $f^m(y) \in \bigcup_i W_{j,i}$ eventually and $d(x, x_0) < \delta/2$. We deduce that $x \in B\big(\overline{\bigcup_{j,i} W_{j,i,\varepsilon_0}}, \delta/2\big)$ for all ε_0 small enough. Since $\overline{A_0}$ is compact, it is covered by a finite number of balls of radius $\delta/2$ centred at some points x_i whose orbits attract non-empty open sets. Taking ε_0' the minimum of all the ε_0 associated to the x_i, the inclusion $A_0 \subset B(A_\varepsilon, \delta)$ occurs for all $\varepsilon < \varepsilon_0'$.

Conversely, let $\delta > 0$, $\varepsilon < \delta$ and focus on the set $W_{j,i,\varepsilon}$. By point (2) of Remark 63, we can suppose that there exists $x \in W_{j,i,\varepsilon}$ whose orbit is periodic and Lyapunov stable. Thus $x \in A_0$ and since the diameter of $W_{j,i,\varepsilon}$ is smaller than δ, $W_{j,i,\varepsilon} \subset B(A_0, \delta)$. The corollary is proved. □

Now we establish a discrete counterpart of the shredding lemma. This result shows that the general dynamics of a generic dissipative homeomorphism is quite simple; thus the dynamics of the discretizations of these homeomorphisms is also quite simple. More precisely, since having basins of attraction is stable by perturbation, we have a similar statement for discretizations of a generic homeomorphism. In what follows these arguments are developed.

To each point $x_N \in E_N$, we associate a closed set $\overline{P_N^{-1}(\{x_N\})}$, consisting of the points in X one of whose projections on E_N is x_N. The closed sets $\overline{P_N^{-1}(\{x_N\})}$ form a basis of the topology of X when N runs through $\mathbb{N}$ and x_N runs through E_N. Let $f \in \mathrm{Homeo}(X)$, $N \in \mathbb{N}$ and $\vartheta : \mathbb{N} \to \mathbb{R}_+^*$ be a function that tends to $+\infty$ at $+\infty$. Let $\delta, \varepsilon > 0$ and U_j be the sets obtained by the shredding lemma for f, δ and ε.

For all $j \le \ell$ we denote by $\widetilde{U}_j^N$ the union over $x_N \in E_N$ of the closed sets $\overline{P_N^{-1}(\{x_N\})}$ whose intersection with U_j is non-trivial. Then $\widetilde{U}_j^N$ tends to U_j for the metric $d(A, B) = \lambda(A \Delta B)$, and for the Hausdorff metric. By point (3) of Remark 63, these convergences are independent of the choice of ε'. Thus, for all k big enough, properties (i)–(iii) of the shredding lemma remain true for the discretizations g_N (for arbitrary g satisfying the properties of the lemma). Taking ε' small enough and modifying g slightly if necessary, there exist sets $W_{j,i}$ and $g \in \mathrm{Homeo}(X)$ such that $d(f, g) < \delta$, $W_{j,i} \subset \overline{P_N^{-1}(\{x_N\})}$ and $\mathrm{Card}(\bigcup_{j,i} W_{j,i} \cap E_N) \le \vartheta(N)$. The others estimates of the sizes of the sets involved in the lemma are obtained similarly. Finally, we have the following lemma.

LEMMA 66. (Discrete shredding lemma) *For all $f \in \mathrm{Homeo}(X)$, for all $\varepsilon, \delta > 0$ and every function $\vartheta : \mathbb{N} \to \mathbb{R}_+^*$ that tend to $+\infty$ at $+\infty$, there exists $N_0 \in \mathbb{N}$ such that for all*

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$N \geq N_0$, there exists a family of subsets $U_1^N, \ldots, U_\ell^N$ of E_N and $g \in \mathrm{Homeo}(X)$ such that $d(f, g) < \delta$ and:

(i) $g_N(U_j^N) \subset U_j^N$;

(ii) $\mathrm{Card}(U_j^N) < \varepsilon q_N$;

(iii) $\mathrm{Card}(\bigcup_{j=1}^{\ell} U_j^N) > (1-\varepsilon) q_N$;

(iv) $\mathrm{Card}(g_N(U_j^N)) < \varepsilon\, \mathrm{Card}(U_j^N)$;

(v) *for all j, there exists $w_{j,1}^N, \ldots, w_{j,\ell_j}^N \in E_N$ such that:*

  (a) $\mathrm{Card}(\bigcup_{j,i}\{w_{j,i}\}) \leq \vartheta(N)$;

  (b) $g_N(w_{j,i}^N) = w_{j,i+1}^N$, *for every $i \in \{1, \ldots, \ell_j - 1\}$ and* $g_N(w_{j,\ell_j}^N) = w_{j,1}^N$;

  (c) $U_j^N \subset \bigcup_{m \geq 0} g_N^{-m}(\bigcup_{i=1}^{k_j} w_{j,i}^N)$;

(vi) *for all j and all i, there exist sets U_j and $W_{j,i}$ satisfying properties* (i)–(v) *of the shredding lemma such that $U_j \subset P_N^{-1}(U_j^N)$ and $W_{j,i} \subset P_N^{-1}(w_{j,i}^N)$;*

(vii) *if $N \geq N_0$, then for all j and all i, we have*

$$\sum_j d_H(\overline{U_j}, \overline{P_N^{-1}(U_j^N)}) < \varepsilon \quad \text{and} \quad \sum_{i,j} d_H(\overline{W_{j,i}}, \overline{P_N^{-1}(w_{j,i}^N)}) < \varepsilon$$

(where d_H is the Hausdorff metric), and

$$\sum_j \lambda(U_j \Delta P_N^{-1}(U_j^N)) < \varepsilon \quad \text{and} \quad \sum_{i,j} \lambda(W_{j,i} \Delta P_N^{-1}(w_{j,i}^N)) < \varepsilon.$$

Remark 67. Properties (i)–(v) are discrete counterparts of properties (i)–(v) of the continuous shredding lemma, but properties (vi) and (vii) reflect the convergence of the dynamics of discretizations to that of the original homeomorphism.

This lemma implies that we can theoretically deduce the behaviour of a generic homeomorphism from the dynamics of its discretizations. The next section details this remark.

5.2. *Dynamics of discretizations of a generic homeomorphism.* To begin with, we deduce from the shredding lemma that the dynamics of discretizations f_N tends to that of the homeomorphism f. More precisely, almost all orbits of the homeomorphism are δ-shadowed by the orbits of the corresponding discretizations.

Definition 68. Let f and g be two maps from a metric space X into itself, $x, y \in X$ and $\delta > 0$. We say that the orbit of x by f *δ-shadows* the orbit of y by g if for all $m \in \mathbb{N}$, $d(f^m(x), g^m(y)) < \delta$.

COROLLARY 69. *For a generic homeomorphism $f \in \mathrm{Homeo}(X)$, for all $\varepsilon > 0$ and all $\delta > 0$, there exist an open set A such that $\lambda(A) > 1 - \varepsilon$ and $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ and all $x \in A$, the orbit of $x_N = P_N(x)$ by f_N δ-shadows that of x by f.*

Therefore, for a generic homeomorphism f, there exists a full measure dense open set O such that for all $x \in O$, all $\delta > 0$ and all N large enough, the orbit of x_N by f_N δ-shadows that of x by f.

Proof of Corollary 69. This easily follows from the discrete shredding lemma, and especially from the fact that the points $w_{j,i}^N$ tend to the sets $W_{j,i}$ for Hausdorff metric. □

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This statement is a bit different from the genericity of shadowing (see [**PP99**]): here the start point is not a pseudo-orbit but a point $x \in X$; Corollary 69 expresses that we can 'see' the dynamics of f on that of f_N, with arbitrarily high precision, provided that N is large enough. Among other things, this allows us to observe the basins of attraction of the Lyapunov stable periodic points of f on discretizations. Better yet, to each family of attractors $(W_{j,i})_i$ of the basin U_j of the homeomorphism correspond a unique family of points $(w_{j,i}^N)_i$ that are permuted cyclically by f_N and attract a neighbourhood of U_j. Thus attractors are shadowed by cyclic orbits of f_N and we can detect the 'period' of the attractor (i.e. the integer k_j) on discretizations. This behaviour is the opposite of what happens in the conservative case, where discretized orbits and true orbits are very different for most points.

Again, in order to show that the dynamics of discretizations converges to that of the initial homeomorphism, we establish the convergence of attractive sets of f_N to that of f. Recall that A_0 is the closure of the Lyapunov periodic points of f (see Notation 64).

PROPOSITION 70. *For a generic homeomorphism $f \in \mathrm{Homeo}(X)$, the sets $\Omega(f_N)$ tend weakly to A_0 in the following sense: for all $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, there exists a subset $\widetilde{E}_N$ of E_N, stabilized by f_N such that, denoting by $\widetilde{\Omega}(f_N)$ the corresponding maximal invariant set, we have $\mathrm{Card}(\widetilde{E}_N)/\mathrm{Card}(E_N) > 1 - \varepsilon$ and $d_H(A_0, \widetilde{\Omega}(f_N)) < \varepsilon$.*

Proof of Proposition 70. Let $\varepsilon > 0$. For all $N \in \mathbb{N}$, let $\widetilde{E}_N$ be the union of the sets U_j^N of Lemma 66 for the parameter ε. This lemma ensures that $\widetilde{E}_N$ is stable by f_N and fills a proportion greater than $1 - \varepsilon$ of E_N. We also denote by $\widetilde{\Omega}(f_N)$ the associated maximal invariant set:

$$\widetilde{\Omega}(f_N) = \bigcup_{x \in \bigcup_{j=1}^{\ell} \widetilde{U}_j^N} \omega_{f_N}(x).$$

Property (vii) of Lemma 66 ensures that

$$\overline{\lim_{N \to +\infty}} d_H(A_\varepsilon, \widetilde{\Omega}(f_N)) < \varepsilon.$$

To conclude, it suffices to observe that $A_\varepsilon \to A_0$ for Hausdorff distance. □

We conclude this section with a final consequence of Lemma 66 which reflects that the ratio between the cardinality of the image of discretizations and that of the grid is smaller and smaller:

COROLLARY 71. *For a generic homeomorphism $f \in \mathrm{Homeo}(X)$,*

$$\lim_{N \to +\infty} \frac{\mathrm{Card}(f_N(E_N))}{\mathrm{Card}(E_N)} = 0;$$

particularly for all $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that if $N \geq N_0$, then the cardinality of each orbit of f_N is smaller than $\delta\,\mathrm{Card}(E_N)$.

This corollary can be seen as a discrete analogue of the fact that a generic homeomorphism is totally singular, i.e. that there exists a Borel set of full measure

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whose image under f is zero measure [**AA13**]. Again, it reflects the regularity of the behaviour of the discretizations of a dissipative homeomorphism: generically, we can describe the behaviour of *all* (sufficiently fine) discretizations. This is very different from the conservative case, where sometimes $f_N(E_N) = E_N$ and sometimes $\mathrm{Card}(f_N(E_N)) \leq \vartheta(N)$ where $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+^*$ is a given map that tends to $+\infty$ at $+\infty$. Moreover, we have the following estimate on the maximal invariant set (easily deduced from the discrete shredding lemma).

COROLLARY 72. *Let $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+^*$ be a function that tends to $+\infty$ at $+\infty$. Then for a generic homeomorphism $f \in \mathrm{Homeo}(X)$,*

$$\lim_{N \to +\infty} \mathrm{Card}(\Omega(f_N)) \leq \vartheta(N).$$

It remains to study the behaviour of measures $\mu^{f_{N,U}}$ (see Definition 7). Again, the results are very different from the conservative case: for any open set U, the measures $\mu^f_{N,U}$ tend to a single measure, say μ^f_U.

THEOREM 73. *For a generic homeomorphism $f \in \mathrm{Homeo}(X)$ and an open subset U of X, the measure μ^f_U is well defined† and is supported by the closure of the set of attractors A_0. Moreover, the measures $\mu^f_{N,U}$ tend weakly to μ^f_U.*

Sketch of proof of Theorem 73. The proof of this theorem is based on the shredding lemma: the set of homeomorphisms which satisfy the conclusions of the lemma is G_δ dense, so it suffices to prove that such homeomorphisms f satisfy the conclusion of the proposition. Let U be an open subset of X and $\varphi : X \to \mathbb{R}$ be a continuous function.

We want to show that on the one hand the integral $\int_X \varphi \, \mathrm{d}\mu^f_U$ is well defined, i.e. that the Birkhoff limits for the function φ,

$$\lim_{m \to +\infty} \frac{1}{m} \sum_{i=0}^{m-1} \varphi(f^i(x)),$$

are well defined for almost every $x \in U$; and on the other hand we have the convergence

$$\int_X \varphi \, d\mu^f_{N,U} \underset{N \to +\infty}{\longrightarrow} \int_X \varphi \, d\mu^f_U.$$

For the first step, the idea of the proof is that the iterates of most of the points (for λ) eventually belong to a set $W_{j,i}$. Since the iterates of the sets $W_{j,i}$ have small diameter, by uniform continuity, the function φ is almost constant on the sets $f^m(W_{j,i})$. Thus the measure μ^f_x is well defined and almost constant on the set of points whose iterates eventually belong to $W_{j,i}$. And by the same construction, since the dynamics of f_N converge to that of f, and in particular the sets U^N_j and $\{w^N_{j,i}\}$ converge to the sets U_j and $W_{j,i}$, the measures $\mu^f_{N,U}$ tend to the measures μ^f_U. □

† In other words, a generic homeomorphism is weird; see also [**AA13**].

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6. *Part 3. Numerical simulations*

6.1. *Conservative homeomorphisms.* We now present the results of numerical simulations of conservative homeomorphisms we conducted. We were interested in whether it is possible to observe behaviours such as those obtained in §6 or that described by Theorem 56 on actual simulations: it is not clear that the orders of discretization described by these results can be reached in practice, or if simple examples of homeomorphisms behave the same way as generic homeomorphisms.

Recall that we simulate homeomorphisms $f(x, y) = P \circ Q \circ P(x, y)$, where P and Q are both homeomorphisms of the torus that modify only one coordinate,

$$P(x, y) = (x, y + p(x, y)) \quad \text{and} \quad Q(x, y) = (x + q(x, y), y),$$

and that we discretize according to the uniform grids on the torus,

$$E_N = \left\{ \left(\frac{i_1}{N}, \dots, \frac{i_n}{N} \right) \in \mathbf{T}^n \;\middle|\; \forall j,\, 0 \le i_j \le N - 1 \right\}.$$

To obtain conservative homeomorphisms, we choose $p(x, y) = p(x)$ and $q(x, y) = q(y)$. We perform simulations of two conservative homeomorphisms which are small perturbations of identity or of the standard Anosov automorphism $A : (x, y) \mapsto (x + y, x + 2y)$.

- To begin with we study $f_1 = P \circ Q \circ P$, with
$$p(x) = \tfrac{1}{259} \cos(2\pi \times 227x) + \tfrac{1}{271} \sin(2\pi \times 253x),$$
$$q(y) = \tfrac{1}{287} \cos(2\pi \times 241y) + \tfrac{1}{263} \sin(2\pi \times 217y).$$
This conservative homeomorphism is a small C^0 perturbation of identity. Experience shows that even dynamical systems with fairly simple definitions behave chaotically (see, for example, [**GT83**]). We can expect that a homeomorphism such as f_1 has a complex dynamical behaviour and even behaves essentially as a generic homeomorphism, at least for small orders of discretization. Note that we choose coefficients that have (virtually) no common divisors to avoid arithmetical phenomena such as periodicity or resonance.
- We also simulate f_2, the composition of f_1 with the standard Anosov automorphism A, say $f_2 = P \circ Q \circ P \circ A$. This makes it a small C^0-perturbation of A. Thus f_2 is semi-conjugated to A but not conjugated: to each periodic orbit of A corresponds many periodic orbits of f_2. As for f_1, we define f_2 in the hope that the behaviour of its discretizations is fairly close to that of discretizations of a generic homeomorphism.

Note that the homeomorphisms f_1 and f_2 have at least one fixed point (for f_1, simply make simultaneously $p(x)$ and $q(y)$ equal to 0; for f_2, note that 0 is a persistent fixed point of A). Thus, the theoretical results indicate that for a generic homeomorphism f which has a fixed point, infinitely many discretizations have a unique fixed point; moreover, a subsequence of $(\mu_N^f)_N$ tends to an invariant measure under f supported by a fixed point. We can test if this is true on simulations.

From a practical point of view, we restricted ourselves to grids of sizes smaller than $2^{15} \times 2^{15}$: the initial data quickly become very large, and the algorithm creates temporary variables that are of the order of five times the size of the initial data. For example, for a grid $2^{15} \times 2^{15}$, the algorithm requires between 25 and 30 GB of RAM.

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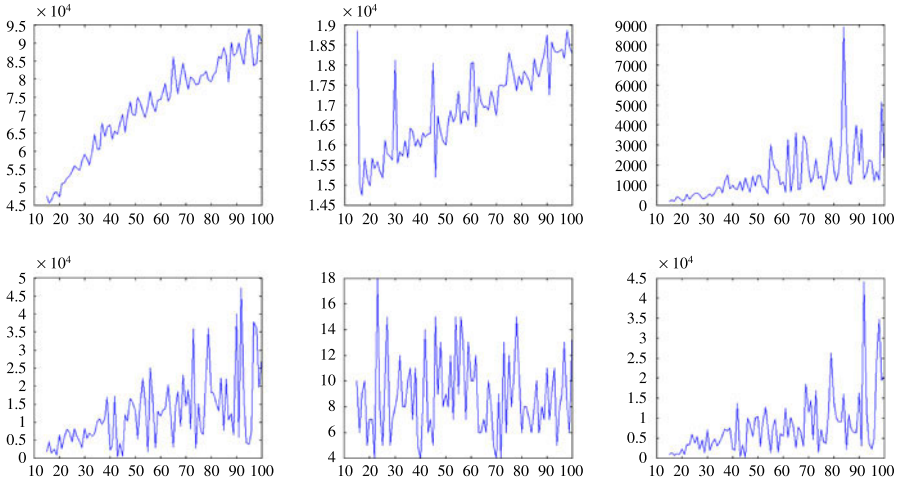


FIGURE 6. Size of the maximal invariant set $\Omega((f_i)_N)$ (left), number of periodic orbits of $(f_i)_N$ (middle) and length of the largest periodic orbit of $(f_i)_N$ (right) depending on N, for f_1 (top) and f_2 (bottom), on the grids E_N with $N = 128k$, $k = 15, \ldots, 100$.

6.1.1. *Combinatorial behaviour.* As a first step, we are interested in some quantities related to the combinatorial behaviour of discretizations of homeomorphisms. These quantities are:

- the cardinality of the maximal invariant set $\Omega(f_N)$;
- the number of periodic orbits of the map f_N;
- the maximal size of a periodic orbit of f_N.

Recall that according to Corollaries 23, 28 and 40, for a generic homeomorphism, the ratio $\mathrm{Card}(\Omega(f_N))/q_N$ must fluctuate between 0 and 1 depending on N, the number of orbits must fluctuate between 1 and (for example) $\sqrt{q_N}$, and the maximal size of a periodic orbit must fluctuate between 1 and q_N.

We calculated these quantities for discretizations of orders $128k$, for k from 1 to 100 and have represented them graphically. For information, if $N = 128 \times 100$, then $q_N \simeq 1.6 \times 10^8$.

We begin with the cardinality of the maximal invariant set $\Omega(f_N)$ (Figure 6, left). Contrary to what the theoretical results provide for a generic homeomorphism, for all simulations the ratio $\mathrm{Card}(\Omega(f_N))/q_N$ tends to 0 as N increases. More specifically, the cardinality of $\Omega(f_N)$ evolves much more regularly for f_1 than for f_2: for f_1 the value of this cardinality seems to be the sum of a smooth increasing function and a random noise, but for f_2 this value seems to be the product of a smooth increasing function with a random noise. We have no explanation for the parabolic shape of the curve for f_1: it reflects the fact that the cardinality of $\Omega((f_1)_N)$ evolves as $\sqrt[4]{q_N}$ (whereas for a random map of a finite set with q elements into itself it evolves as $\sqrt{q}$). Finally, it is interesting to note that for f_2, the size of the maximal invariant set is distributed more or less around the size of the maximal invariant set of a random map of a set with q_N elements into itself, which depends (asymptotically) linearly on N and has a value of about 16 000 for $N = 128 \times 100$ (see [**Bol01**]).

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According to the results of the previous sections, for a generic homeomorphism f, the number of periodic orbits of f_N must fluctuate between 1 and (for example) $\sqrt{q_N}$. This is clearly not the case for the simulations. In addition, its behaviour is not the same for f_1 and for f_2 (Figure 6, middle): for f_1 the number of orbits quickly reaches a value around 2×10^4 to increase slightly thereafter, while for f_2 it oscillates between 1 and 18, regardless the order of discretization. These graphics can be compared with those representing the size of the maximal invariant set $\Omega((f_i)_N)$: the number of periodic orbits and the size of the maximal invariant set are of the same order of magnitude for f_1 (up to a factor 5), which means that the average period of a periodic orbit is small (which is not surprising, since f_1 is a small perturbation of identity). In contrast, they differ by a factor roughly equal to 10^3 for f_2, which means that in this instance the average period of a periodic orbit is very large. This can be explained partly by the fact that the standard Anosov automorphism tends to mix what happens in the neighbourhood of identity. The fact remains that these simulations (such as the size of the maximal invariant set $\Omega((f_i)_N)$) suggest that the behaviour in the neighbourhood of identity and of the standard Anosov automorphism are quite different, at least for reasonable orders of discretization.

Regarding the maximum size of a periodic orbit of f_N (Figure 6, right), again its behaviour does not correspond to that of a generic homeomorphism: it should oscillate between 1 and q_N, which is not the case. However, it varies widely depending on N, especially when N is large. The qualitative behaviours are similar for both simulations, but there are some quantitative differences: the maximum of the maximal length of a periodic orbit is greater for f_2 than for f_1.

6.1.2. *Behaviour of invariant measures.* We also simulated the measure $\mu_N^{f_i}$ for the two examples of conservative homeomorphisms f_1 and f_2 as defined on p. 39. Our purpose is to test whether phenomena as described by Theorem 56 can be observed in practice or not. We obviously cannot expect to see the sequence $(\mu_N^{f_i})_{N\in\mathbb{N}}$ accumulating on *all* the invariant probability measures of f, since these measures generally form an infinite-dimensional convex set, but we can still test if it seems to converge or not.

We present images of sizes 128×128 pixels representing on the logarithmic scale the density of the measures μ_N^f: each pixel is coloured according to the measure carried by the set of points of E_N it covers. Blue corresponds to a pixel with very small measure, and red to a pixel with very high measure. Scales on the right of each image correspond to the measure of one pixel on the log 10 scale: if the green corresponds to -3, then a green pixel will have measure 10^{-3} for μ_N^f. For information, when Lebesgue measure is represented, all the pixels have a value of about -4.2.

Theoretically, the maximal value of the measure μ_N (Figure 7) should oscillate between $1/q_N$ and 1, but this is not the case for these examples. Again the behaviour is quite different for f_1 and for f_2: for f_1 it seems that the values of μ_N are well distributed between the null function and a linear function of N, while for f_2 we observe peaks—there are a few values of N for which μ_N is much higher than elsewhere. For f_1 the maximal value of μ_N seems to grow globally with N, but with a quite irregular behaviour, apparently confirming Theorem 56. Finally, we note that the maximum value of μ_N is much lower for f_2 than for f_1.

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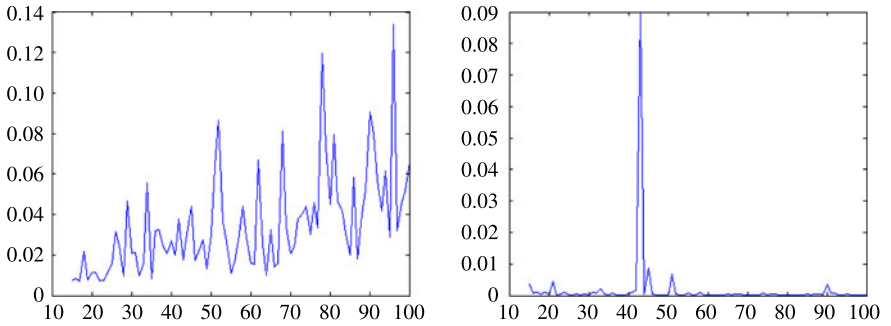


FIGURE 7. Maximal value of the measure μ_N on E_N depending on N for f_1 (left) and f_2 (right), on the grids E_N with $N = 128k$, $k = 15, \ldots, 100$.

The results of simulations of invariant measures of discretizations of f_1 (which is a C^0 conservative perturbation of identity) are quite positive: they agree with theoretical results about discretizations of generic conservative homeomorphisms, in particular with Theorem 56. When f_1 is discretized, we observe that the measure is first fairly well distributed. When the step of the discretization improves we can observe places where the measure accumulates; moreover, these places change a lot when the orders of discretizations vary (see Figure 8). This agrees fairly well with what happens in the C^0 generic case, where the measure μ_N^f depends very much on the order of discretization rather than on the homeomorphism itself. There is also another phenomenon: when the size of the grid is large enough (around $10^{12} \times 10^{12}$), areas uniformly charged by the measure μ_N^f appear; their size seems to be inversely proportional to the common mass of the pixels of the area.

For the discretizations of f_2, which is a C^0 conservative perturbation of the standard Anosov automorphism, the simulations on grids of size $2^k \times 2^k$ might suggest that the measures $\mu_N^{f_2}$ tend to Lebesgue measure. In fact, having carried out a large number of simulations, we realize that there are also strong variations in the behaviour of measures (Figure 9): the measure is often well distributed in the torus and sometimes quite singular with respect to Lebesgue measure (as in Figure 7). For example when we discretize on the grid of size $20\,010 \times 20\,010$ (middle right of Figure 9), we observe an orbit with length 369 with mass 84% of the total measure. In fact the behaviour of discretizations looks the same as in the neighbourhood of identity, modulo the fact that the standard Anosov automorphism tends to spread the attractive periodic orbits of discretizations on the entire torus: for many values of N composing by A spreads the behaviour of the measure $\mu_N^{f_2}$, but sometimes (in fact, seldom) a fixed point of $(f_1)_N$ which attracts a large part of E_N is located around one of the few periodic points of small period for A. This then creates a periodic orbit for $(f_2)_N$ with a big measure for $\mu_N^{f_2}$.

6.2. *Dissipative homeomorphisms.* Let us finish by presenting the results of some numerical simulations of dissipative homeomorphisms. Again, our aim is to compare the theoretical results with the reality of numerical simulations: for simple homeomorphisms

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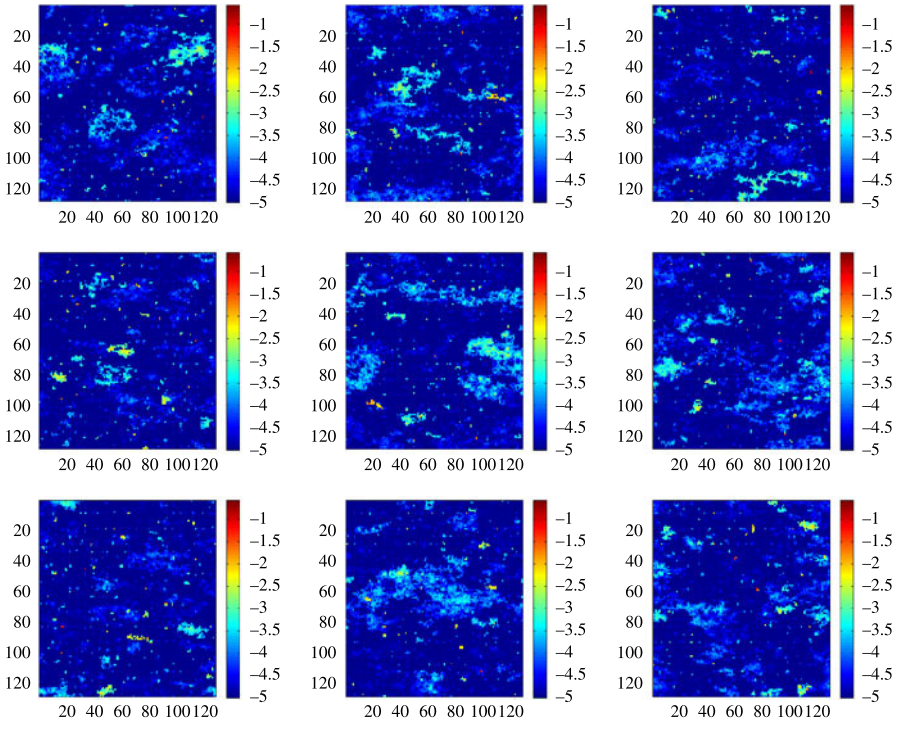


FIGURE 8. Simulations of invariant measures $\mu_N^{f_1}$ on grids of size $N \times N$, with $N = 20\,000, \ldots, 20\,008$ (from left to right and top to bottom).

and reasonable orders of discretization, does one have convergence of the dynamics of discretizations to that of the homeomorphism, as suggested by the above theorems?

Recall that we simulate homeomorphisms such that

$$f(x, y) = Q \circ P(x, y) \quad \text{or} \quad f(x, y) = P \circ Q \circ P(x, y),$$

where P and Q are two homeomorphisms of the torus that modify only one coordinate:

$$P(x, y) = (x, y + p(x, y)) \quad \text{and} \quad Q(x, y) = (x + q(x, y), y).$$

Unlike the conservative case, p and q depend on both x and y. Again, we tested two homeomorphisms.

- To begin with we studied $f_3 = Q \circ P$, with

$$p(x, y) = \tfrac{1}{259} \cos(2\pi \times 227y) + \tfrac{1}{271} \sin(2\pi \times 233x),$$
$$q(x, y) = \tfrac{1}{287} \cos(2\pi \times 241y) + \tfrac{1}{263} \sin(2\pi \times 217y) + \tfrac{1}{263} \cos(2\pi \times 271x).$$

  This dissipative homeomorphism is a small C^0 perturbation of identity, whose derivative has many oscillations whose amplitudes are close to 1. That creates many fixed points which are attractors, sources or saddles.
- It also seemed to us useful to simulate a homeomorphism close to the identity in C^0 topology, but with a small number of attractors—indeed, as explained heuristically

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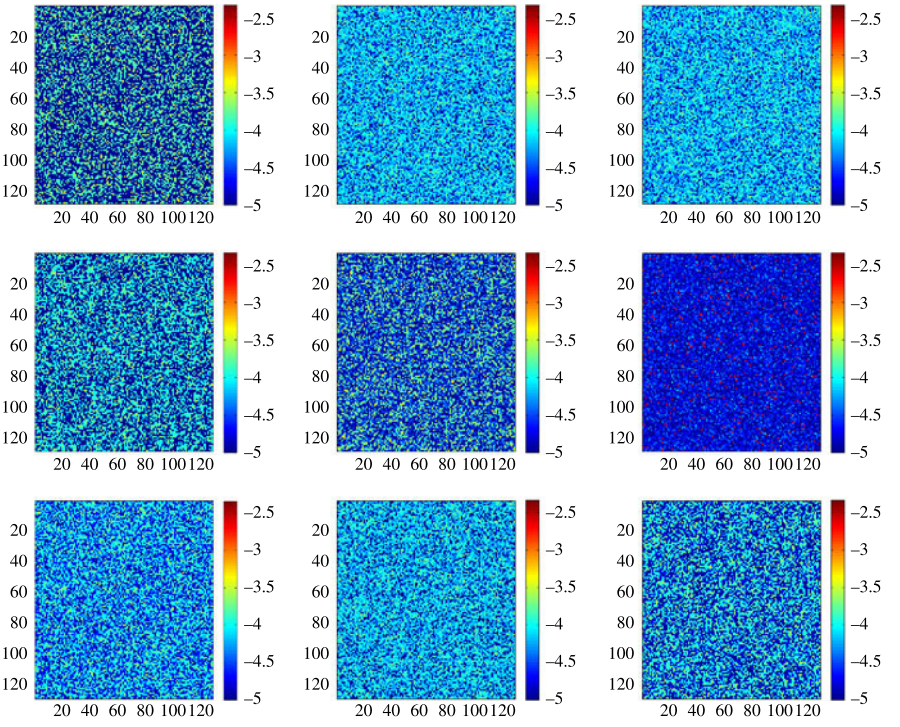


FIGURE 9. Simulations of invariant measures $\mu_N^{f_2}$ on grids of size $N \times N$, with $N = 20\,005, \ldots, 20\,013$ (from left to right and top to bottom)

by Gambaudo and Tresser in [**GT83**], a homeomorphism such as f_3 can have a large number of attractors whose basins of attraction are small. It turns out that the dissipative behaviour of f_3 cannot be detected in orders of discretization that can be achieved in practice. We therefore defined another homeomorphism close to the identity in C^0 topology, but with much less attractors, say $f_4 = P \circ Q \circ P$, with

$$\begin{aligned} p(x, y) &= \tfrac{1}{259}\tanh(50\cos(2\pi \times y)) + \tfrac{1}{271}\tanh(50\cos(2\pi \times 5x)), \\ q(x, y) &= \tfrac{1}{287}\tanh(50\cos(2\pi \times y)) + \tfrac{1}{263}\tanh(50\cos(2\pi \times 7y)) \\ &\quad + \tfrac{1}{263}\tanh(50\cos(2\pi \times 3x)). \end{aligned}$$

6.2.1. *Combinatorial behaviour.* We simulated some quantities related to the combinatorial behaviour of discretizations of homeomorphisms, namely:

- the cardinality of the maximal invariant set $\Omega(f_N)$;
- the number of periodic orbits of f_N;
- the maximal size of a periodic orbit of f_N.

We calculated these quantities for discretizations of orders $128k$ for k from 1 to 100 and represented them graphically (see Figure 10). For information, if $N = 128 \times 100$, then $q_N \simeq 1.6 \times 10^8$.

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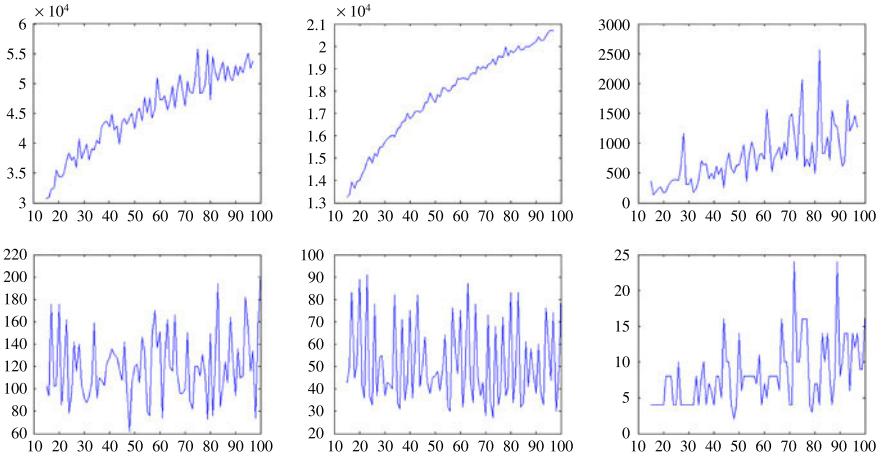


FIGURE 10. Size of the maximal invariant set $\Omega((f_i)_N)$ (left), number of periodic orbits of $(f_i)_N$ (middle) and length of the largest periodic orbit of $(f_i)_N$ (right) depending on N, for f_3 (top) and f_4 (bottom), on the grids E_N with $N = 128k$, $k = 15, \ldots, 100$

Theoretically, the ratio between the cardinality of $\Omega(f_N)$ and q_N must tend to 0; this is what can be seen in simulations. This is not really surprising: we have already observed it for discretizations of conservative homeomorphisms. In this context it is interesting to compare the behaviour of $\Omega(f_N)$ in the conservative and the dissipative case. The result is a little disappointing: for f_3, the graphics in the case of conservative and dissipative homeomorphisms are as alike as two peas in a pod, while in theory they should be very different. Still, for f_4, there are very few attractors, and around each one there are just a few attractive points of the discretization (each attractor is 'sharp') so that the cardinality of $\Omega((f_4)_N)$ is more or less constant. For this case, the behaviour of discretizations seems clearly typical of the dissipative case.

The behaviour of the number of periodic orbits of f_N is not provided by the theoretical study. However, we can hope that it reflects the fact that the dynamics converges to that of the initial homeomorphism: among others, we can test if it is of the same order as the number of attractors of the homeomorphism. In practice, its behaviour is similar to that of the cardinality of the maximal invariant set $\Omega(f_N)$: its evolution is very regular for f_3 but oscillates (from $N = 15$) between 30 and 90 for f_4. As in the conservative case, the comparison with the graphics in the left of Figure 10 indicates that the average length of a periodic cycle is about 3 for f_3 and f_4.

Since the dynamics of discretizations is assumed to converge to that of the initial homeomorphism, one could expect that the length of the longest periodic orbit of discretizations $(f_i)_N$ is almost always a multiple of that of an attractive periodic orbit of f_i. The graphic of this length for f_3 looks like the conservative case, so one can say that the dissipative behaviour of this homeomorphism is not detected for reasonable orders of discretizations. Despite this, the values of the lengths of the longest periodic orbit are much smaller (up to a factor of 10) than in the conservative case. It is not obvious if it is more a dissipative effect than a coincidence. For f_4, the length of the longest orbit is smaller than 25, which is normal because we are supposed to detect the attractors of the

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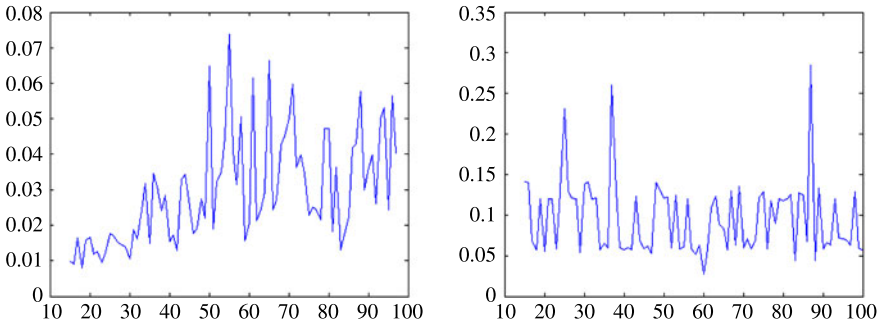


FIGURE 11. Maximal value of μ_N on E_N depending on N for f_3 (left) and f_4 (right), on the grids E_N with $N = 128k, k = 15, \ldots, 100$.

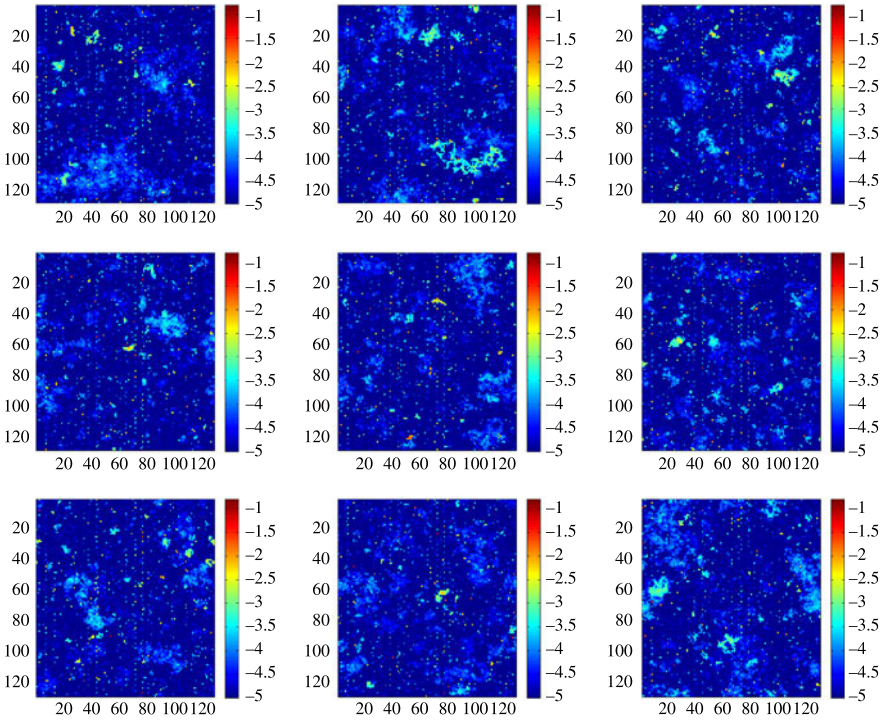


FIGURE 12. Simulations of $\mu_N^{f_3}$ on grids E_N, with $N = 20\,000, \ldots, 20\,008$ (from left to right and top to bottom).

homeomorphism. The changes of the length of the longest orbit are probably due to errors of discretization in the neighbourhood of the attractors of f_4. Furthermore, we note that some values of the size of the longest orbit are more common: for example, for 30 different integers N the longest periodic orbit has length 4, for 19 different integers N this length is 8 and for 10 different integers N it is 10. This is certainly due to the shadowing property of periodic orbits of f_4 by those of the discretizations: see the work of Blank for a discussion about the phenomenon of multiplication of the period [**Bla84**, **Bla86**, **Bla89**, **Bla94**].

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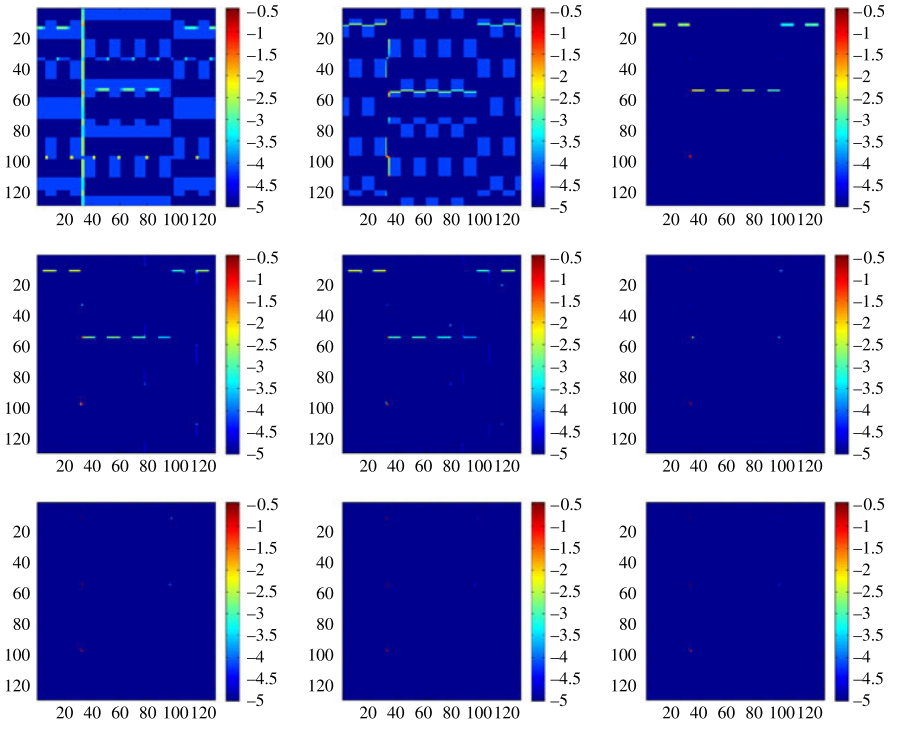


FIGURE 13. Simulations af $\mu_N^{f_4}$ on the grids E_N, with $N = 2^k$ and $k = 6, \ldots, 14$ (from left to right and top to bottom).

6.2.2. *Behaviour of invariant measures.* As in the conservative case we calculated the invariant measures $\mu_N^{f_i}$ of dissipative homeomorphisms f_3 and f_4 as defined on p. 43. Our aim is to test whether Theorem 73 applies in practice or if there are technical constraints such that this behaviour cannot be observed on these examples. For a presentation of the representations of the measures, see p. 1514.

There are no clear theoretical results concerning the behaviour of the maximal value of the measure μ_N^f (Figure 11), but we can at least observe that convergence of this would be a good indication that the homeomorphism considered is dissipative. As in the conservative case (in the neighbourhood of identity), the behaviour of $\mu_N^{f_3}$ seems very chaotic. While $\mu_N^{f_3}$ is always smaller than 0.07, the maximum value of $\mu_N^{f_4}$ is very high and fluctuates between 0.02 and 0.3, with many values around 0.06 and 0.12 and some values around 0.25. The obvious arithmetic relationships between these values can be explained by the fact that some attractors of discretizations merge for some values of N. Thus, there are orbits which attract many points; it seems to reflect the typical behaviour of generic homeomorphisms.

The behaviour of invariant measures for f_3, which is a small C^0 dissipative perturbation of identity, is relatively similar to that of invariant measures for f_1, i.e. the corresponding conservative case: when the order discretization is large enough, there is a strong variation of the measure $\mu_N^{f_3}$. Moreover, this measure has a significant absolutely continuous

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component with respect to Lebesgue measure. Nevertheless, there are differences compared to the conservative case: the maximum value of $\mu_N^{f_3}$ is very large, much more than for f_1, and there are orbits which attract many points. On the other hand, one can observe the following phenomenon (we hope that it is anecdotal): these orbits are located on vertical stripes. This is probably related to specific arithmetic phenomena due to the specific form of the homeomorphism f_3.

It is normal not to be able to observe all the attractors of f_3 on these simulations: as noted by Gambaudo and Tresser in [**GT83**], the size of these attractors can be very small compared to the numbers involved in the definition of f_3. So even in orders of discretization such as 2^{15}, the dissipative nature of the homeomorphism cannot be detected on discretizations. For this reason, it looks as if the discretizations of f_3 are very similar to those of f_1.

Recall that what happens for f_3 is rather close to what happens for f_1. For their part, simulations of invariant measures for f_4 on grids of size $2^k \times 2^k$ highlight what we expect from a generic dissipative homeomorphism: the measures $\mu_N^{f_4}$ tend quickly to a single measure (that is also observed on a series of simulations), which is carried by the attractors of f_4. The fact that f_4 has few attractors allows the discretizations of reasonable orders (typically 2^{11}) to find the actual attractors of the initial homeomorphism, contrary to what we observed for f_3.

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