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A variational approach for an inverse dynamical problem for composite beams

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This paper deals with a problem of nondestructive testing for a composite system formed by the connection of a steel beam and a reinforced concrete beam. The small vibrations of the composite beam are described by a differential system where a coupling takes place between longitudinal and bending motions. The motion is governed in space by two second order and two fourth order differential operators, which are coupled in the lower order terms by the shearing, k, and axial, μ, stiffness coefficients of the connection. The coefficients k and μ define the mechanical model of the connection between the steel beam and the concrete beam and contain direct information on the integrity of the system. In this paper we study the inverse problem of determining k and μ by mixed data. The inverse problem is transformed to a variational problem for a cost function which includes boundary measurements of Neumann data and also some interior measurements. By computing the Gateaux derivatives of the functional, an algorithm based on the projected gradient method is proposed for identifying the unknown coefficients. The results of some numerical simulations on real steel-concrete beams are presented and discussed.

1 Introduction

A steel-concrete composite beam is obtained by connecting two beams, a metallic one and a reinforced concrete beam, by means of small metallic elements (connectors) which are welded on the top flange of the metallic beam and immersed in the concrete, in order to hinder sliding on the concrete-steel interface, see Figure 1. The infinitesimal free vibrations of a steel-concrete composite beam are modelled by the following system of

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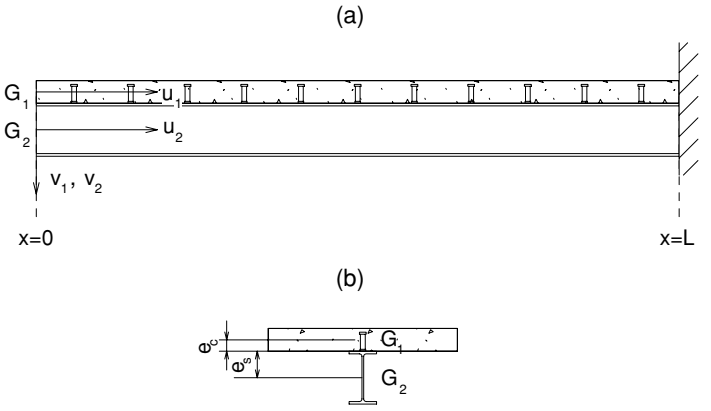


FIGURE 1. Steel-concrete composite beam with free left end and clamped right end: longitudinal view (a) and transversal cross-section (b).

partial differential equations:

$$\begin{cases} \frac{\partial}{\partial x}\left(a_1\frac{\partial u_1}{\partial x}\right)+k\left(u_2-u_1+\frac{\partial v_2}{\partial x}e_s\right)=\rho_1\frac{\partial^2 u_1}{\partial t^2} \quad (x,t)\in(0,L)\times(0,T),\\ \frac{\partial}{\partial x}\left(a_2\frac{\partial u_2}{\partial x}\right)-k\left(u_2-u_1+\frac{\partial v_2}{\partial x}e_s\right)=\rho_2\frac{\partial^2 u_2}{\partial t^2} \quad (x,t)\in(0,L)\times(0,T),\\ -\frac{\partial^2}{\partial x^2}\left(j_1\frac{\partial^2 v_1}{\partial x^2}\right)+\frac{\partial}{\partial x}\left(\frac{ke_c^2}{6}\left(2\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial x}\right)\right)\\ -\mu(v_1-v_2)=\rho_1\frac{\partial^2 v_1}{\partial t^2}, \qquad (x,t)\in(0,L)\times(0,T),\\ -\frac{\partial^2}{\partial x^2}\left(j_2\frac{\partial^2 v_2}{\partial x^2}\right)+\frac{\partial}{\partial x}\left(\frac{ke_c^2}{6}\left(2\frac{\partial v_2}{\partial x}+\frac{\partial v_1}{\partial x}\right)\right)\\ +\frac{\partial}{\partial x}\left(k\left(u_2-u_1+\frac{\partial v_2}{\partial x}e_s\right)e_s\right)+\mu(v_1-v_2)=\rho_2\frac{\partial^2 v_2}{\partial t^2},\\ (x,t)\in(0,L)\times(0,T), \end{cases} \tag{1.1}$$

see Dilenna & Morassi [5]. Under the assumption that the system is at rest at $t=0$, that is

$$u_1|_{t\leqslant 0}=u_2|_{t\leqslant 0}=v_1|_{t\leqslant 0}=v_2|_{t\leqslant 0}=0 \quad x\in(0,L), \tag{1.2}$$

we shall be concerned with the following Dirichlet boundary conditions at $x=0$ and $x=L$:

$$\begin{cases} u_1|_{x=L}=u_2|_{x=L}=v_1|_{x=L}=v_2|_{x=L}=\frac{\partial v_1}{\partial x}|_{x=L}=\frac{\partial v_2}{\partial x}|_{x=L}=0,\\ u_1|_{x=0}=\overline{u}_1(t), \quad u_2|_{x=0}=\overline{u}_2(t),\\ v_1|_{x=0}=\overline{v}_1(t), \quad v_2|_{x=0}=\overline{v}_2(t),\\ \frac{\partial v_1}{\partial x}|_{x=0}=\overline{\varphi}_1(t), \quad \frac{\partial v_2}{\partial x}|_{x=0}=\overline{\varphi}_2(t). \end{cases} \tag{1.3}$$

for $t\in(0,T)$. Hereinafter, the quantities relative to the concrete beam (the upper one in Figure 1) and the steel beam (the lower one) will be denoted by indices $i=1,2$, respectively. The functions $u_i=u_i(x,t)$ and $v_i=v_i(x,t)$ denote the longitudinal and transversal displacement, respectively, of the cross-section of abscissa x, evaluated at the moment of time t. In equations (1.1), the quantities $j_i=E_iI_i$ and $a_i=E_iA_i$ are the flexural and the axial stiffness of the cross-section, respectively, where I_i and A_i are the moment of

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inertia and the area of the transversal cross-section and E_i, $E_i > 0$, is the Young modulus of the ith material. The function $\rho_i = \rho_i(x)$ is the linear mass density of the ith beam, $\rho_i > 0$. Finally, e_s is the half-height of the steel beam and $e_c \equiv e - e_s$, where e is the distance between the axes of the two beams forming the system.

The two positive quantities $k = k(x)$, $\mu = \mu(x)$ express respectively the shearing and axial stiffness of the connection between the concrete slab and the steel beam. These coefficients define the mechanical properties of the connection and they contain direct information on its integrity. In particular, typical damage occurring in real steel-concrete systems involves a deterioration of the connection, causing a decrease in the stiffness coefficients k and μ. Since the inaccessibility of the connection from the exterior makes direct inspection difficult, an inverse problem interesting for applications consists in estimating the coefficients k, μ from suitable non destructive techniques.

In Morassi *et al.* [10], a diagnostic method based on dynamic data has been proposed for the simpler situation in which the coupling between bending and longitudinal motions is neglected. In this case, by formally taking $v_1 = v_2 = 0$ in the previous model, the system (1.1) simplifies into a two-velocity dynamical system. For this reduced problem it was proved that the shearing stiffness coefficient k can be uniquely determined from the measurement of the frequency response function of the composite system taken at one end of the beam. The strategy of the reconstruction procedure is based on a transformation of the equations governing the free longitudinal vibrations to an equivalent first order system and, subsequently, on the use of the progressive waves approach to reduce the local reconstruction of k to the resolution of a system of nonlinear Volterra integral equations. Finally, an iterative use of a layer stripping technique allows for a reconstruction, step by step, of the coefficient k on the whole interval $[0, L]$. We refer to Belishev & Ivanov [3] for an interesting application of the Boundary Control Method to solve this inverse problem when measurements are taken at both the ends of the beam.

All the above results have been obtained for the simplified model where the coupling between longitudinal and transversal motions is neglected and only longitudinal motions are present. In the engineering applications, see, for example, Johnson [7], it is important to examine the full complete coupled system (1.1), which includes two fourth order and two second order differential operators coupled on a term of low order. Unfortunately, it seems rather involved to extend the techniques presented in Morassi *et al.* [10] and Belishev & Ivanov [3] to this general case.

In this paper, we study the inverse problem of reconstructing the stiffness coefficients of a steel-concrete composite beam by using a different approach. More precisely, we propose a variational procedure based on dynamical measurements taken at the boundary and at some interior portions of the system. Our inspiration comes from a recent paper [6] in which the authors proposed a variational approach for identifying the coefficient of some second order evolution equation based on dynamical boundary measurements.

Let us introduce the set $\mathscr{C}$ of pairs of coefficients (k, μ):

$$\mathscr{C} = \{(k, \mu) \mid k \in C^1[0, L], \mu \in C^0[0, L],\ k(x) \geqslant 0,\ \mu(x) \geqslant 0 \text{ for } x \in [0, L]\}. \tag{1.4}$$

Let us denote by $\overline{Q}(t) \equiv Q[k, \mu](0, t)$ the vector of Neumann data for the system (1.1) evaluated at $x = 0$, $t \in [0, T]$ (see (3.3) and (3.6)), and by $\overline{v}_i(x, t) \equiv v_i[k, \mu](x, t)$ the

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transversal displacements on $[0, L] \times [0, T]$. Now, suppose we do not know the coefficients $k(x)$ and $\mu(x)$, but we are given the Neumann data $\overline{Q}(t)$ on $[0, T]$ and the displacements $\overline{v}_i(x, t)$ on $I \times [0, T]$, for T large enough. Here, I is an open interval of $[0, L]$. For every $\widetilde{k}$, $\widetilde{\mu} \in \mathscr{C}$ we define the following cost function:

$$J(\widetilde{k}, \widetilde{\mu}) := \int_0^T |Q[\widetilde{k}, \widetilde{\mu}](0, t) - \overline{Q}(t)|^2 \, dt + \int_0^T \int_I \sum_{i=1}^{2} (v_i[\widetilde{k}, \widetilde{\mu}](x, t) - \overline{v}_i(x, t))^2 \, dx \, dt, \tag{1.5}$$

where $Q[\widetilde{k}, \widetilde{\mu}](0, t)$ and $v_i[\widetilde{k}, \widetilde{\mu}](x, t)$ are respectively the Neumann data at $x = 0$ and the transversal displacements of the solution of (1.1) for $(k = \widetilde{k}, \mu = \widetilde{\mu})$.

The cost function J attains a global minimum when $(\widetilde{k} = k, \widetilde{\mu} = \mu)$ in $[0, L]$. Therefore, we expect to recover information on the unknown coefficients by minimizing $J(\widetilde{k}, \widetilde{\mu})$ on $\mathscr{C}$.

In this paper we present a projected gradient method which uses the analytical expressions of the first partial derivatives of J throughout the minimization process. More precisely, we give the complete form of the Gateaux partial derivatives of J with respect to the coefficients k and μ (see Theorem 3.1) and we present a numerical algorithm based on the complete form of the differential of J (see § 5). The results of the numerical simulation are encouraging. The identified coefficients are in good agreement with the exact ones and the method seems to be sufficiently stable and robust with respect to errors on the measured data.

The rest of the paper is organized as follows. In § 2 we discuss the well-posedness of the direct problem (1.1). The inverse problem is formulated in § 3 and the complete form of the Gateaux derivatives of J are given in Theorem 3.1. In § 4 we shall state some propositions which are useful in proving Theorem 3.1. In § 5 we present the numerical algorithm and we some numerical results.

2 Well-posedness of the direct problem

In this section we shall introduce some notations and we shall study the well-posedness of the direct problem (1.1).

We rewrite the dynamical system (1.1) governing the infinitesimal vibrations $w = (u_1(x, t), u_2(x, t), v_1(x, t), v_2(x, t))$ of a composite beam in the following compact form

$$\begin{cases} Cw_{,tt} - A_{k,\mu} w = 0 & (x, t) \in (0, L) \times (0, T), \\ w|_{t=0} = 0, \quad w_{,t}|_{t=0} = 0 & x \in (0, L), \\ Dw|_{x=0} = \overline{U} & t \in (0, T), \\ Dw|_{x=L} = 0 & t \in (0, T), \end{cases} \tag{2.1}$$

where C is the 4×4 diagonal matrix

$$C = \mathrm{diag}(\rho_1, \rho_2, \rho_1, \rho_2), \tag{2.2}$$

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$A_{k,\mu}$ is the spatial differential operator defined by

$$A_{k,\mu}w = \begin{Bmatrix} (a_1u_{1,x})_{,x} + k(u_2 - u_1 + v_{2,x}e_s) \\ (a_2u_{2,x})_{,x} - k(u_2 - u_1 + v_{2,x}e_s) \\ -(j_1v_{1,xx})_{,xx} + (\frac{ke_c^2}{6}(2v_{1,x} + v_{2,x}))_{,x} - \mu(v_1 - v_2) \\ -(j_2v_{2,xx})_{,xx} + (\frac{ke_c^2}{6}(2v_{2,x} + v_{1,x}))_{,x} + (k(u_2 - u_1 + v_{2,x}e_s)e_s)_{,x} \\ +\mu(v_1 - v_2) \end{Bmatrix} \tag{2.3}$$

and D is the operator given by

$$Dw = \{u_1, u_2, v_1, v_2, v_{1,x}, v_{2,x}\}^T. \tag{2.4}$$

In the above equation the function $v_{i,x}$, $i = 1, 2$, expresses the rotation of the cross-section of the ith beam.

The coefficients ρ_i, a_i, j_i, $i = 1, 2$, are assumed to be positive and regular in $[0, L]$. More precisely, for $i = 1, 2$, we assume:

$$\begin{cases} \rho_i \in C^0([0, L]), & \rho_i(x) \geqslant \rho_{i0} > 0 \text{ in } (0, L), \\ a_i \in C^1([0, L]), & a_i(x) \geqslant a_{i0} > 0 \text{ in } (0, L), \\ j_i \in C^2([0, L]), & j_i(x) \geqslant j_{i0} > 0 \text{ in } (0, L), \end{cases} \tag{2.5}$$

where ρ_{i0}, a_{i0} and j_{i0} are given constants.

Concerning the connection parameters, we shall assume:

$$\begin{cases} k \in C^1([0, L]), & k(x) \geqslant 0 \text{ in } [0, L], \\ \mu \in C^0([0, L]), & \mu(x) \geqslant 0 \text{ in } [0, L]. \end{cases} \tag{2.6}$$

The boundary data at $x = 0$ is assumed to be such that

$$\begin{aligned} &\overline{U}(t) := (\overline{u}_1(t), \overline{u}_2(t), \overline{v}_1(t), \overline{v}_2(t), \overline{\varphi}_1(t), \overline{\varphi}_2(t))^T, \\ &\qquad \overline{U}(t) \in (C^3([0, T]))^6, \quad \text{with } \overline{U}_{j,t^i}(t)|_{t=0} = 0, \quad i = 0, 1, 2, 3, \quad j = 1, \ldots, 6, \end{aligned} \tag{2.7}$$

where hereinafter we shall denote $f_{,t^i} := \frac{\partial^i f(t)}{\partial t^i}$, i integer and $i \geqslant 1$.

To study the well-posedness of the problem (2.1), we shall reduce it to an abstract evolution initial value problem with homogeneous boundary conditions by introducing a suitable inverse trace operator.

To this end, let us define the function $\Phi : [0, L] \times [0, T] \to \mathbb{R}^4$ such that

$$\Phi(x, t) = \begin{Bmatrix} \overline{u}_1(t)\vartheta(x) \\ \overline{u}_2(t)\vartheta(x) \\ \overline{\varphi}_1(t)\psi(x) + \overline{v}_1(t)\vartheta(x) \\ \overline{\varphi}_2(t)\psi(x) + \overline{v}_2(t)\vartheta(x) \end{Bmatrix}, \tag{2.8}$$

where

$$\psi(x) = x\left(1 - \left(\frac{x}{L}\right)\right)^2, \quad \vartheta(x) = 2\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^2 + 1. \tag{2.9}$$

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The function Φ belongs to $(C^3([0,T];C^\infty(0,L)))^4$ and verifies the Dirichlet boundary conditions

$$D\Phi|_{x=0} = \overline{U}, \quad D\Phi|_{x=L} = 0, \tag{2.10}$$

and, by (2.7), the initial conditions

$$\Phi(x,t)_{,t^i}|_{t=0} = 0 \quad x \in [0,L], \quad i = 0,1,2,3. \tag{2.11}$$

Therefore, the function $\hat{w} : [0,L] \times [0,T] \to \mathbb{R}$ defined as

$$\hat{w} := w - \Phi = (\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2)^T \tag{2.12}$$

satisfies the problem

$$\begin{cases} C\hat{w}_{,tt} - A_{k,\mu}\hat{w} = f & (x,t) \in (0,L) \times (0,T), \\ \hat{w}|_{t=0} = \hat{w}_0, \quad \hat{w}_{,t}|_{t=0} = \hat{w}_1 & x \in (0,L), \\ D\hat{w}|_{x=0} = D\hat{w}|_{x=L} = 0 & t \in (0,T), \end{cases} \tag{2.13}$$

where

$$f = A_{k,\mu}\Phi - C\Phi_{,tt} \qquad (x,t) \in (0,L) \times (0,T),$$

$$\begin{aligned} \hat{w}_0 &= -\Phi|_{t=0} = 0 \quad x \in (0,L), \\ \hat{w}_1 &= -\Phi_{,t}|_{t=0} = 0 \quad x \in (0,L). \end{aligned} \tag{2.14}$$

To study the well-posedness of the evolution system (2.13), we introduce the following function spaces

$$\begin{aligned} V = H_0^1(0,L) \times H_0^1(0,L) \times H_0^2(0,L) \times H_0^2(0,L) := (H_0^1(0,L))^2 \times (H_0^2(0,L))^2, \\ H = L^2_{\rho_1}(0,L) \times L^2_{\rho_2}(0,L) \times L^2_{\rho_1}(0,L) \times L^2_{\rho_2}(0,L). \end{aligned} \tag{2.15}$$

Here, $H_0^1(0,L)$ and $H_0^2(0,L)$ are the standard Hilbert spaces and $L^2_{\rho_i}(0,L)$ is the space $L^2(0,L)$ endowed with the usual scalar product $\langle f,g\rangle = \int_0^L \rho_i(x) f(x) g(x)\,dx$, $i = 1,2$. Let V' denote the dual of the space V.

The embedding $i : V \hookrightarrow H$ is continuous, injective, with image dense in H. Therefore, the dual $i' : H \hookrightarrow V'$ of i is continuous, injective and has a dense range. It follows that the triple (V,H,V') is a *Gelfand triple*.

Moreover, for our purposes it turns out to be useful to view a function $u = u(x,t)$ as a function $t \to u(t)$ with values $u(t) : x \mapsto u(x,t)$ in an appropriate Hilbert space of functions X. For example, for a measurable function u on $(0,L) \times (0,T)$, we will indicate

$$u \in H^k((0,T);X) \Leftrightarrow u_{,t^i} \in L^2((0,T);X) \quad 0 \leqslant i \leqslant k, \tag{2.16}$$

where k is a positive integer number and the differentiation in time is in the distributional sense. The norm of $u \in H^k((0,T);X)$ is given by $\|u\|^2_{H^k((0,T);X)} = \sum_{i=0}^k \int_0^T \|u_{,t^i}\|^2_X\,dt$.

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By multiplying equation (2.13) by any test function $m = (g_1, g_2, h_1, h_2)$, g_i and $h_i \in C_0^\infty([0,T];V)$, $i = 1,2$, and integrating by parts on $(0,L)\times(0,T)$ we find

$$\int_0^T (a_{k,\mu}(\hat{w},m) - c(\hat{w}',m'))\,dt = \int_0^T \langle f,m\rangle\,dt, \quad \forall\, m \in C_0^\infty((0,T);V), \tag{2.17}$$

where

$$a_{k,\mu}(\hat{w},m) := -(A_{k,\mu}\hat{w},m) = \alpha^{(0)}(\hat{w},m) + \alpha_k^{(1)}(\hat{w},m) + \alpha_\mu^{(2)}(\hat{w},m), \tag{2.18}$$

$$c(\hat{w}',m') = (C\hat{w}_{,t},m_{,t}) = \int_0^L \sum_{i=1}^{2} (\rho_i \hat{u}_{i,t} g_{i,t} + \rho_i \hat{v}_{i,t} h_{i,t})\,dx, \tag{2.19}$$

$$\langle f,m\rangle = \int_0^L \sum_{i=1}^{4} f_i m_i\,dx. \tag{2.20}$$

The functions $\alpha^{(0)}(\hat{w},m)$, $\alpha_k^{(1)}(\hat{w},m)$ and $\alpha_\mu^{(2)}(\hat{w},m)$ appearing in (2.18) are defined as

$$\begin{aligned}
\alpha^{(0)}(\hat{w},m) &= \int_0^L \sum_{i=1}^{2} (a_i \hat{u}_{i,x} g_{i,x} + j_i \hat{v}_{i,xx} h_{i,xx})\,dx,\\
\alpha_k^{(1)}(\hat{w},m) &= \int_0^L k(\hat{u}_2 - \hat{u}_1 + \hat{v}_{2,x} e_s)(g_2 - g_1 + h_{2,x} e_s)\,dx\\
&\quad + \int_0^L \frac{k e_c^2}{6}(2\hat{v}_{1,x}h_{1,x} + 2\hat{v}_{2,x}h_{2,x} + \hat{v}_{2,x}h_{1,x} + \hat{v}_{1,x}h_{2,x})\,dx,\\
\alpha_\mu^{(2)}(\hat{w},m) &= \int_0^L \mu(\hat{v}_1 - \hat{v}_2)(h_1 - h_2)\,dx.
\end{aligned} \tag{2.21}$$

The two symmetric sesquilinear forms $a_{k,\mu}(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are positive definite, that is

$$\begin{aligned}
&\exists\ \alpha > 0 \text{ s.t. } a_{k,\mu}(u,u) \geqslant \alpha \|u\|_V \ \forall u \in V,\\
&\exists\ \beta > 0 \text{ s.t. } c(u,u) \geqslant \beta \|u\|_H \ \forall u \in H.
\end{aligned} \tag{2.22}$$

Note that to derive the coercivity condition $(2.22)_1$ we have used the Poincaré inequality on $H_0^1(0,L)$ and on $H_0^2(0,L)$.

With the above definitions, the evolution system (2.13) can be written as the following Cauchy problem for an abstract evolution equation:

$$\begin{cases} C\hat{w}'' - A_{k,\mu}\hat{w} = f(t) & t \in (0,T),\\ \hat{w}(0) = \hat{w}_0, \quad \hat{w}(0) = \hat{w}_1, \end{cases} \tag{2.23}$$

where equation $(2.23)_1$ is understood in the weak sense of equation (2.17). To study (2.23) we will use the following well-known result, see, for example, Wloka [13].

Theorem 2.1 *Let us consider the evolution equation*

$$C\frac{d^2y(t)}{dt^2} - Ay(t) = f(t) \qquad t \in (0,T), \tag{2.24}$$

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with the initial conditions

$$y(0) = y_0, \quad \frac{dy}{dt}(0) = y_1. \tag{2.25}$$

Let V, H be two Hilbert spaces with $V \hookrightarrow H$ dense and V separable, such that (V, H, V') is a Gelfand triple. Let $a(\cdot,\cdot)$ be the symmetric sesquilinear form on V associated to the operator $-A$ and let $c(\cdot,\cdot)$ be the symmetric sesquilinear form on H associated to the operator C. Assume that there exist $\lambda \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$ such that

$$a(u,u) + \lambda \|u\|_H^2 \geqslant \alpha \|u\|_V^2 \qquad \forall u \in V \tag{2.26}$$

and

$$c(u,u) \geqslant \beta \|u\|_H^2 \qquad \forall u \in H. \tag{2.27}$$

Moreover, let us assume that

$$f \in H^{k-1}((0,T);H), \quad k \geqslant 1, \tag{2.28}$$

and

$$y_0 \in V \tag{2.29}$$

are such that the compatibility condition of degree $k-1$ is satisfied, that is

$$C\frac{d^j y}{dt^j}(0) \in V, \ j = 0, \ldots, k-1, \quad C\frac{d^k y}{dt^k}(0) \in H. \tag{2.30}$$

Then, the Cauchy problem (2.24), (2.25) *has a unique solution $y(t)$ such that*

$$y \in C^0([0,T];V), \quad \frac{dy(t)}{dt} \in C^0([0,T];H), \tag{2.31}$$

with y and $\frac{dy(t)}{dt}$ which depend linearly and continuously on $(f, y_0, y_1) \in L^2((0,T);H) \times V \times H$. Moreover, we have

$$y \in H^{k-1}((0,T);V), \ \frac{d^k y(t)}{dt^k} \in L^2((0,T);H), \ \frac{d^{k+1} y(t)}{dt^{k+1}} \in L^2((0,T);V'). \tag{2.32}$$

Remark 2.2 Differentiating equation (2.13) formally and substituting $t = 0$, we obtain

$$C\frac{d^{2j-1}y}{dt^{2j-1}}(0) = \frac{d^{2j-3}f}{dt^{2j-3}}(0) + A\frac{d^{2j-5}f}{dt^{2j-5}}(0) + \cdots + A^{j-2}\frac{df}{dt}(0) + A^{j-1}y_1, \tag{2.33}$$

$$C\frac{d^{2j}y}{dt^{2j}}(0) = \frac{d^{2j-2}f}{dt^{2j-2}}(0) + A\frac{d^{2j-4}f}{dt^{2j-4}}(0) + \cdots + A^{j-1}f(0) + A^j y_0. \tag{2.34}$$

These expressions give a more explicit form of the compatibility condition (2.30) in terms of (f, y_0, y_1).

We are now in position to consider the well-posedness of problem (2.1).

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Theorem 2.3 *Under the assumptions* (2.5), (2.6) *on the coefficients and on the assumptions* (2.7) *on the boundary data, the problem* (2.1) *has a unique solution*

$$w \in \left(H^2((0,T);H^1(0,L))\right)^2 \times \left(H^2((0,T);H^2(0,L))\right)^2,$$

such that

$$w_{,ttt} \in \left(L^2((0,T);L^2(0,L))\right)^4.$$

Moreover, we have

$$w \in \left(H^1((0,T);H^2(0,L))\right)^2 \times \left(H^1((0,T);H^4(0,L))\right)^2.$$

Proof The function $\hat{w} := w - \Phi$, where Φ is defined in (2.8), suppose to satisfy problem (2.13) with homogeneous initial data, i.e. $\hat{w}_0 \equiv 0$ and $\hat{w}_1 \equiv 0$ in $[0,L]$.

By the regularity of the coefficients and by the definition of Φ, from $(2.14)_1$ we have

$$f \in \left(C^1([0,T];L^2(0,L))\right)^4. \tag{2.35}$$

By the definition (2.15) of our spaces V and H, the triple (V,H,V') is clearly a Gelfand triple. By (2.22), the sesquilinear symmetric form $a(\cdot,\cdot)$ on V defined by (2.18) satisfies (2.26) with $\lambda = 0$ and $\alpha > 0$, where α only depends on (a_{i0}, j_{i0}), $i = 1,2$. Also, the sesquilinear symmetric form $c(\cdot,\cdot)$ on H defined in (2.19) satisfies (2.27) with $\beta > 0$ only depending on ρ_{i0}, $i = 1,2$.

Moreover, by (2.35) and recalling the initial conditions at $t = 0$, the inhomogeneous term f and the initial data $\hat{w}_0$, $\hat{w}_1$ satisfy the compatibility conditions (2.30) up to $k = 3$.

Therefore, by Theorem 2.1, the problem (2.23) has a unique solution

$$\hat{w} \in \{(H^2((0,T);H_0^1(0,L)))^2 \times (H^2((0,T);H_0^2(0,L)))^2\} \cap (H^3((0,T);L^2(0,L)))^4. \tag{2.36}$$

The regularity of $\hat{w}$ with respect to the space variable can be improved by observing that

$$A_{k,\mu}\hat{w} = C\frac{d^2\hat{w}}{dt^2} - f \in \left(H^1((0,T);L^2(0,L))\right)^4. \tag{2.37}$$

Therefore, by regularity results for solutions to elliptic equations under Dirichlet boundary conditions, we have also

$$\hat{w} \in \left(H^1((0,T);H^2(0,L))\right)^2 \times \left(H^1((0,T);H^4(0,L))\right)^2. \tag{2.38}$$

Finally, from the definition (2.12) of $\hat{w}$, the theorem follows. □

3 The main result

Let $w \in (H^2((0,T);H^2(0,L)))^2 \times (H^2((0,T);H^4(0,L)))^2$ be the solution of problem (2.1) (see Theorem 2.3) and let

$$m = (g_1, g_2, h_1, h_2) \in C_0^\infty\left([0,T];(H_L^1(0,L))^2 \times (H_L^2(0,L))^2\right),$$

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where H^k_L is the space of functions belonging to $H^k(0,L)$ such that the trace of their values and the values of their derivatives up to the order $k-1$, $k \geqslant 1$, vanish at $x = L$.

Integrating by parts, we have

$$a_{k,\mu}(w,m) = b_k(w,m) - (A_{k,\mu}w, m), \tag{3.1}$$

where the operator $A_{k,\mu}$ is defined in (2.3) and the boundary term at $x = 0$ can be expressed as

$$b_k(w,m) := -B_k w \cdot Dm|_{x=0} = -\sum_{i=1}^{2}(N_i g_i + T_i h_i - M_i h_i')|_{x=0}, \tag{3.2}$$

where the boundary operator B_k is such that

$$B_k w = \left\{ \begin{array}{l} a_1 u_{1,x} := N_1(0,t) \\ a_2 u_{2,x} := N_2(0,t) \\ -(j_1 v_{1,xx})_{,x} + \frac{k e_s^2}{6}(2v_{1,x} + v_{2,x}) := T_1(0,t) \\ -(j_2 v_{2,xx})_{,x} + \frac{k e_s^2}{6}(2v_{2,x} + v_{1,x}) + k(u_2 - u_1 + v_{2,x} e_s)e_s := T_2(0,t) \\ -j_1 v_{1,xx} := M_1(0,t) \\ -j_2 v_{2,xx} := M_2(0,t) \end{array} \right\}. \tag{3.3}$$

Note that the operator B_k does not depend explicitly on the coefficient μ.

Since $w \in (H^2((0,T)); H^2(0,L)))^2 \times (H^2((0,T)); H^4(0,L)))^2$, the operator B_k is well defined.

Let us denote by $\mathscr{C}$ the set of pairs of coefficients (k,μ):

$$\mathscr{C} = \{(k,\mu) \mid k \in C^1[0,L], \mu \in C^0[0,L],\ k(x) \geqslant 0,\ \mu(x) \geqslant 0 \text{ for } x \in [0,L]\}. \tag{3.4}$$

Moreover, let us introduce the following notations which will be useful in the sequel:

$$w[\widetilde{k},\widetilde{\mu}] := w[\widetilde{k},\widetilde{\mu}](x,t) \text{ solution of (2.1) for } k = \widetilde{k} \text{ and } \mu = \widetilde{\mu}, \quad \text{for given } \widetilde{k},\widetilde{\mu} \in \mathscr{C}; \tag{3.5}$$

$$Q[\widetilde{k},\widetilde{\mu}](x,t) := -B_{\widetilde{k}} w[\widetilde{k},\widetilde{\mu}](x,t), \quad \text{at } x = 0 \text{ or } x = L; \tag{3.6}$$

$$\overline{Q}(t) := Q[k,\mu](0,t), \quad \bar{v}_i(x,t) := v_i[k,\mu](x,t), \quad i = 1,2,\ x \in [0,L],\ t \in [0,T], \tag{3.7}$$

for some $k, \mu \in \mathscr{C}$.

For any $\widetilde{k}, \widetilde{\mu} \in \mathscr{C}$, we define the *cost function* $J(\widetilde{k},\widetilde{\mu})$ as

$$J(\widetilde{k},\widetilde{\mu}) := J_1(\widetilde{k},\widetilde{\mu}) + J_2(\widetilde{k},\widetilde{\mu}), \tag{3.8}$$

where

$$J_1(\widetilde{k},\widetilde{\mu}) := \int_0^T |Q[\widetilde{k},\widetilde{\mu}](0,t) - \overline{Q}(t)|^2\, dt, \tag{3.9}$$

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$$J_2(\widetilde{k},\widetilde{\mu}) := \int_0^T \int_I \sum_{i=1}^{2} (v_i[\widetilde{k},\widetilde{\mu}](x,t) - \overline{v}_i(x,t))^2 \, dx \, dt, \tag{3.10}$$

where I is an open interval such that $I \subseteq [0,L]$.

The cost function $J(\widetilde{k},\widetilde{\mu})$ attains a global minimum when $\widetilde{k}(x) = k(x)$ and $\widetilde{\mu}(x) = \mu(x)$ in $[0,L]$. Therefore, one reasonably expects to recover information on the unknown coefficients $k(x)$ and $\mu(x)$ on the interval $[0,L]$ by minimizing J on the set $\mathscr{C}$. We will study the inverse problem of reconstructing k and μ from the boundary measurements $\{\overline{Q}(t)\}$, $t \in [0,T]$, and from the interior measurement $\{\overline{v}_1(x,t), \overline{v}_2(x,t)\}$, $(x,t) \in I \times [0,T]$, by minimizing J via a projected gradient method.

In this section and next section we mainly provide a rigorous justification of the complete form of the Gateaux derivatives of J, which we use to give a numerical implementation scheme for minimizing J by the gradient method. We obtain the following result.

Theorem 3.1 *Let the coefficients of problem* (2.1) *satisfy* (2.5), (2.6) *and let the assumptions* (2.7) *on the boundary data be satisfied.*

Let $\widetilde{k}, \widetilde{\mu}, \delta k, \delta\mu \in \mathscr{C}$. *For any* $\epsilon, \gamma \in \mathbb{R}^+$ *such that* $\sqrt{\epsilon^2+\gamma^2} \to 0$ *we have*

$$J(\widetilde{k}+\epsilon\delta k, \widetilde{\mu}+\gamma\delta\mu) - J(\widetilde{k},\widetilde{\mu}) = \epsilon\partial_1 J(\widetilde{k},\widetilde{\mu})\delta k + \gamma\partial_2 J(\widetilde{k},\widetilde{\mu})\delta\mu + o(\sqrt{\epsilon^2+\gamma^2}), \tag{3.11}$$

where $\lim_{z\to 0^+} \frac{o(z)}{z} = 0$.

The Gateaux partial derivatives $\partial_1 J(\widetilde{k},\widetilde{\mu})$, $\partial_2 J(\widetilde{k},\widetilde{\mu})$ *of the functional* J, *evaluated at the point* $(\widetilde{k},\widetilde{\mu})$ *with respect to the first and the second variable respectively, are given by*

$$\begin{aligned} \partial_1 J(\widetilde{k},\widetilde{\mu})\delta k = &\int_0^T \alpha^{(1)}_{\delta k}(w[\widetilde{k},\widetilde{\mu}], V)\, dt + \langle C\partial_t U^{(\widetilde{k})}_{\delta k}(T), W\rangle \\ &+ \int_0^T \langle Z, F(\delta k) w[\widetilde{k},\widetilde{\mu}]\rangle \, dt, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \partial_2 J(\widetilde{k},\widetilde{\mu})\delta\mu = &\int_0^T \alpha^{(2)}_{\delta\mu}(w[\widetilde{k},\widetilde{\mu}], V)\, dt + \langle C\partial_t U^{(\widetilde{\mu})}_{\delta\mu}(T), W\rangle \\ &+ \int_0^T \langle Z, G(\delta\mu) w[\widetilde{k},\widetilde{\mu}]\rangle \, dt, \end{aligned} \tag{3.13}$$

where the bilinear forms $\alpha^{(1)}_{\delta k}(\cdot,\cdot)$ *and* $\alpha^{(2)}_{\delta\mu}(\cdot,\cdot)$ *are defined in* (2.21).

Here, $W \in (H^2(0,L))^2 \times (H^4(0,L))^2$ *is the strong solution of the elliptic problem*

$$\begin{cases} A_{\widetilde{k},\widetilde{\mu}} W = 0 \qquad x \in (0,L), \\ DW|_{x=0} = 2(Q[\widetilde{k},\widetilde{\mu}](0,T) - \overline{Q}(T)), \\ DW|_{x=L} = 0. \end{cases} \tag{3.14}$$

The function $V \in (C^0([0,T]; H^1(0,L)))^2 \times (C^0([0,T]; H^2(0,L)))^2$, *with* $V_{,t} \in (C^0([0,T]; L^2(0,L)))^2 \times (C^0([0,T]; L^2(0,L)))^2$, *is the weak solution of the abstract*

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evolution problem

$$\begin{cases} CV_{,tt} - A_{\widetilde{k},\widetilde{\mu}} V = 0 \qquad (x,t) \in (0,L) \times (0,T), \\ V|_{t=T} = W, \quad V_{,t}|_{t=T} = 0 \qquad x \in (0,L), \\ DV|_{x=0} = 2(Q[\widetilde{k},\widetilde{\mu}](0,t) - \overline{Q}(t)) \qquad t \in (0,T), \\ DV|_{x=L} = 0 \qquad t \in (0,T). \end{cases} \tag{3.15}$$

The function $U^{(\widetilde{k})} := U^{(\widetilde{k})}_{\delta k} \in (C^0([0,T]; H^1_0(0,L)))^2 \times (C^0([0,T]; H^2_0(0,L)))^2$, *with* $U^{(\widetilde{k})}{}_{,t} \in (C^0([0,T]; L^2(0,L)))^2 \times (C^0([0,T]; L^2(0,L)))^2$, *is the weak solution of the abstract evolution problem*

$$\begin{cases} CU^{(\widetilde{k})}_{,tt} - A_{\widetilde{k},\widetilde{\mu}} U^{(\widetilde{k})} = F(\delta k) w[\widetilde{k},\widetilde{\mu}] \qquad (x,t) \in (0,L) \times (0,T), \\ U^{(\widetilde{k})}|_{t=0} = 0, \quad U^{(\widetilde{k})}_{,t}|_{t=0} = 0 \qquad x \in (0,L), \\ DU^{(\widetilde{k})}|_{x=0} = 0, \quad DU^{(\widetilde{k})}|_{x=L} = 0 \qquad t \in (0,T). \end{cases} \tag{3.16}$$

The function $U^{(\widetilde{\mu})} := U^{(\widetilde{\mu})}_{\delta \mu} \in (C^0([0,T]; H^1_0(0,L)))^2 \times (C^0([0,T]; H^2_0(0,L)))^2$, *with* $U^{(\widetilde{\mu})}{}_{,t} \in (C^0([0,T]; L^2(0,L)))^2 \times (C^0([0,T]; L^2(0,L)))^2$, *is the weak solution of the abstract evolution problem*

$$\begin{cases} CU^{(\widetilde{\mu})}_{,tt} - A_{\widetilde{k},\widetilde{\mu}} U^{(\widetilde{\mu})} = G(\delta \mu) w[\widetilde{k},\widetilde{\mu}] \qquad (x,t) \in (0,L) \times (0,T), \\ U^{(\widetilde{\mu})}|_{t=0} = 0, \quad U^{(\widetilde{\mu})}_{,t}|_{t=0} = 0 \qquad x \in (0,L), \\ DU^{(\widetilde{\mu})}|_{x=0} = 0, \quad DU^{(\widetilde{\mu})}|_{x=L} = 0 \qquad t \in (0,T). \end{cases} \tag{3.17}$$

The function $Z \in (C^0([0,T]; H^1_0(0,L)))^2 \times (C^0([0,T]; H^2_0(0,L)))^2$, *with* $Z_{,t} \in (C^0([0,T]; L^2(0,L)))^2 \times (C^0([0,T]; L^2(0,L)))^2$, *is the weak solution of the abstract evolution problem*

$$\begin{cases} CZ_{,tt} - A_{\widetilde{k},\widetilde{\mu}} Z = 2(0,0,(v_1[\widetilde{k},\widetilde{\mu}] - \overline{v}_1)\chi_I, (v_2[\widetilde{k},\widetilde{\mu}] - \overline{v}_2)\chi_I)^T, \\ \qquad\qquad (x,t) \in (0,L) \times (0,T), \\ Z|_{t=T} = 0, \quad Z_{,t}|_{t=T} = 0 \qquad x \in (0,L), \\ DZ|_{x=0} = 0, \quad DZ|_{x=L} = 0 \qquad t \in (0,T), \end{cases} \tag{3.18}$$

where χ_I *is the characteristic function of the interval* I, *i.e.* $\chi_I(x) = 1$ *for* $x \in I$ *and* $\chi_I(x) = 0$ *for* $x \in [0,L] \setminus I$.

The operators $F(\delta k)$ *and* $G(\delta \mu)$ *appearing on the right hand side of* $(3.16)_1$ *and* $(3.17)_1$, *respectively, are defined as follows:*

$$F(\delta k) w[\widetilde{k},\widetilde{\mu}] = \begin{Bmatrix} \delta k(u_2 - u_1 + v_{2,x} e_s) \\ -\delta k(u_2 - u_1 + v_{2,x} e_s) \\ (\frac{\delta k e_c^2}{6}(2v_{1,x} + v_{2,x}))_{,x} \\ (\frac{\delta k e_c^2}{6}(2v_{2,x} + v_{1,x}))_{,x} + (\delta k(u_2 - u_1 + v_{2,x} e_s) e_s)_{,x} \end{Bmatrix}, \tag{3.19}$$

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$$G(\delta\mu)w[\widetilde{k},\widetilde{\mu}] = \begin{Bmatrix} 0 \\ 0 \\ -\delta\mu(v_1 - v_2) \\ \delta\mu(v_1 - v_2) \end{Bmatrix}, \tag{3.20}$$

where $w[\widetilde{k},\widetilde{\mu}] = (u_1, u_2, v_1, v_2)$ *is the solution of* (2.1) *when the coefficients* k, μ *are replaced by* $\widetilde{k}$, $\widetilde{\mu}$, *respectively.*

Concerning the well-posedness of the auxiliary problem (3.14), by Theorem 2.3, the value of $Q[\widetilde{k},\widetilde{\mu}](x,t)$ at $x = 0$ is well defined in the sense of trace's theory for $t \in [0,T]$. Moreover, since the solution w of (2.1) is continuous in time on the interval $[0,T]$, the value of $Q[\widetilde{k},\widetilde{\mu}](0,t)$ at $t = T$ is well-defined. Then, by the coercivity of the sesquilinear form $a_{\widetilde{k},\widetilde{\mu}}(\cdot,\cdot)$ associated to the operator $A_{\widetilde{k},\widetilde{\mu}}$ in $(H_L^1(0,L))^2 \times (H_L^2(0,L))^2$, we have $W \in (H^1(0,L))^2 \times (H^2(0,L))^2$.

4 Proof of Theorem 3.1

The proof of Theorem 3.1 is based on several preliminary results. In § 4.1 we shall present an evaluation of the increment of the functional J in terms of the variations of the coefficients. In the subsequent § 4.2 we shall recall some results of continuous dependence of the solution to an evolution problem on the coefficients of the equation. These results will be used in § 4.3 to justify the asymptotic behaviour (3.11).

We start by introducing some notations.

Definition 4.1 *Under the assumptions of Theorem 3.1 and with the notation of § 2, let* $w[\widetilde{k},\widetilde{\mu}] := w[\widetilde{k},\widetilde{\mu}](x,t)$ *be the weak solution of* (2.1) *in* $(0,L)\times(0,T)$ *when* $k(x) = \widetilde{k}(x)$ *and* $\mu(x) = \widetilde{\mu}(x)$ *in* $[0,L]$. *For any real numbers* $\epsilon, \gamma \in \mathbb{R}^+$ *small enough and given functions* $\delta k, \delta\mu \in \mathscr{C}$, *we define* $\delta w = \delta w(x,t)$ *and* $\delta Q = \delta Q(x,t)$ *as follows:*

$$\delta w := w[\widetilde{k} + \epsilon\cdot\delta k, \widetilde{\mu} + \gamma\cdot\delta\mu] - w[\widetilde{k},\widetilde{\mu}], \qquad (x,t) \in (0,L)\times(0,T), \tag{4.1}$$

$$\delta Q := Q[\widetilde{k} + \epsilon\cdot\delta k, \widetilde{\mu} + \gamma\cdot\delta\mu] - Q[\widetilde{k},\widetilde{\mu}], \qquad t \in (0,T),\ x = 0 \text{ or } x = L, \tag{4.2}$$

where Q *is defined in* (3.6).

4.1 Evaluation of the increment of J

In this section we shall prove the following lemma.

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Lemma 4.2 *Under the assumptions of Theorem 3.1 and within the above notations, we have*

$$\begin{aligned} J(\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu)-J(\widetilde{k},\widetilde{\mu}) &= \langle C\delta w_{,t}(T),W\rangle \\ &\quad+\epsilon\int_0^T \alpha^{(1)}_{\delta k}(w[\widetilde{k},\widetilde{\mu}],V)\,dt+\gamma\int_0^T \alpha^{(2)}_{\delta\mu}(w[\widetilde{k},\widetilde{\mu}],V)\,dt \\ &\quad+\int_0^T \langle Z,(C\delta w_{,tt}-A_{\widetilde{k},\widetilde{\mu}}\delta w)\rangle\,dt \\ &\quad+\epsilon\int_0^T \alpha^{(1)}_{\delta k}(\delta w,V)\,dt+\gamma\int_0^T \alpha^{(2)}_{\delta\mu}(\delta w,V)\,dt \\ &\quad+\int_0^T \delta Q(0,t)\cdot\delta Q(0,t)\,dt+\int_0^T\sum_{i=1}^{2}\langle\delta v_i,\delta v_i\rangle_I\,dt, \end{aligned} \tag{4.3}$$

where $\langle\delta v_i,\delta v_i\rangle_I := \int_I(\delta v_i)^2\,dx$, $i=1,2$, *and* W, V, Z *are the solutions of problems* (3.14), (3.15), (3.18), *respectively, and* $\alpha^{(1)}_{\delta k}$, $\alpha^{(2)}_{\delta\mu}$ *are defined in* (2.21).

Proof To simplify the presentation, we shall consider separately the functionals J_1 and J_2, see (3.8)–(3.10).

We begin with J_1. By the definition (3.9) we have

$$\begin{aligned} &J_1(\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu)-J_1(\widetilde{k},\widetilde{\mu}) \\ &\quad=\int_0^T (Q[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu](0,t)-Q[\widetilde{k},\widetilde{\mu}](0,t)) \\ &\qquad\cdot(Q[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu](0,t)+Q[\widetilde{k},\widetilde{\mu}](0,t)-2\overline{Q}(t))\,dt \\ &\quad=2\int_0^T \delta Q(0,t)(Q[\widetilde{k},\widetilde{\mu}](0,t)-\overline{Q}(t))\,dt+\int_0^T|\delta Q(0,t)|^2\,dt. \end{aligned} \tag{4.4}$$

The main idea consists in introducing the auxiliary problems (3.14), (3.15), (3.18) so as to replace the first term in the right hand side of (4.4) by some integral involving the solutions of (3.14), (3.15), (3.18) and the functions $w[\widetilde{k},\widetilde{\mu}]$ and δw.

Recalling that $w[\widetilde{k},\widetilde{\mu}]_{,t}|_{t=0}=0$ and $V|_{t=T}=W$, integrating by parts we get

$$\int_0^T \langle Cw[\widetilde{k},\widetilde{\mu}]_{,t},V_{,t}\rangle\,dt=\langle Cw[\widetilde{k},\widetilde{\mu}]_{,t}(T),W\rangle-\int_0^T\langle Cw[\widetilde{k},\widetilde{\mu}]_{,tt},V\rangle\,dt, \tag{4.5}$$

where we recall that $\langle f,g\rangle=\int_0^L f\cdot g\,dx$ for given vectors f and g.

By (3.1), with m replaced by V, and recalling the definition (3.6), we have

$$\int_0^T a_{\widetilde{k},\widetilde{\mu}}(w[\widetilde{k},\widetilde{\mu}],V)\,dt=\int_0^T(Q[\widetilde{k},\widetilde{\mu}](0,t)\cdot DV(0,t)-\langle A_{\widetilde{k},\widetilde{\mu}}w[\widetilde{k},\widetilde{\mu}],V\rangle)\,dt. \tag{4.6}$$

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By (4.6) from (4.5) and by considering that $Cw[\widetilde{k},\widetilde{\mu}]_{,tt} - A_{\widetilde{k},\widetilde{\mu}} w[\widetilde{k},\widetilde{\mu}] = 0$ for $(x,t) \in (0,L)\times(0,T)$, we have

$$\int_0^T (\langle Cw[\widetilde{k},\widetilde{\mu}]_{,t}, V_{,t}\rangle - a_{\widetilde{k},\widetilde{\mu}}(w[\widetilde{k},\widetilde{\mu}], V))\,dt = \langle Cw[\widetilde{k},\widetilde{\mu}]_{,t}(T), W\rangle - \int_0^T Q[\widetilde{k},\widetilde{\mu}](0,t)\cdot DV(0,t). \tag{4.7}$$

Arguing similarly for $w[\widetilde{k}+\epsilon\cdot\delta k, \widetilde{\mu}+\gamma\cdot\delta\mu]$, which is the solution of (2.1) for $k = \widetilde{k}+\epsilon\cdot\delta k$, $\mu = \widetilde{\mu}+\gamma\cdot\delta\mu$, we have

$$\int_0^T \langle Cw[\widetilde{k}+\epsilon\cdot\delta k, \widetilde{\mu}+\gamma\cdot\delta\mu]_{,t}, V_{,t}\rangle\,dt - \int_0^T a_{\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu}(w[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu], V)\,dt = \langle Cw[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu]_{,t}(T), W\rangle - \int_0^T Q[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu](0,t)\cdot DV(0,t). \tag{4.8}$$

By subtracting (4.7) from (4.8) we obtain

$$\int_0^T \langle C(w[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu] - w[\widetilde{k},\widetilde{\mu}])_{,t}, V_{,t}\rangle\,dt - \int_0^T (a_{\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu}(w[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu], V) - a_{\widetilde{k},\widetilde{\mu}}(w[\widetilde{k},\widetilde{\mu}], V))\,dt = \langle C(w[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu] - w[\widetilde{k},\widetilde{\mu}])_{,t}(T), W\rangle - \int_0^T (Q[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu](0,t) - Q[\widetilde{k},\widetilde{\mu}](0,t))\cdot DV(0,t),$$

that is

$$\int_0^T \langle C\delta w_{,t}, V_{,t}\rangle - (a_{\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu}(w[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu], V) - a_{\widetilde{k},\widetilde{\mu}}(w[\widetilde{k},\widetilde{\mu}], V))\,dt = \langle C\delta w_{,t}(T), W\rangle - \int_0^T \delta Q(0,t)\cdot DV(0)\,dt. \tag{4.9}$$

We have $\delta w(t=0) = 0$ and $V_{,t}(t=T) = 0$ in $(0,L)$. Then, the first integral on the left-hand side of (4.9) becomes

$$\int_0^T \langle C\delta w_{,t}, V_{,t}\rangle = -\int_0^T \langle \delta w, CV_{,tt}\rangle\,dt. \tag{4.10}$$

We consider now the second integral on the left hand side of (4.9). To simplify the notations, we temporarily set $w[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu] := \widetilde{w}$ and $w[\widetilde{k},\widetilde{\mu}] := w$. By the expression (2.18) of the sesquilinear form $a(\cdot,\cdot)$ we have

$$a_{\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu}(\widetilde{w}, V) - a_{\widetilde{k},\widetilde{\mu}}(w, V) = \alpha^{(0)}(\widetilde{w}, V) + \alpha^{(1)}_{\widetilde{k}+\epsilon\cdot\delta k}(\widetilde{w}, V) + \alpha^{(2)}_{\widetilde{\mu}+\gamma\cdot\delta\mu}(\widetilde{w}, V) - \left(\alpha^{(0)}(w, V) + \alpha^{(1)}_{\widetilde{k}}(w, V) + \alpha^{(2)}_{\widetilde{\mu}}(w, V)\right). \tag{4.11}$$

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By the expressions of the terms $\alpha^{(1)}(\cdot,\cdot)$ and $\alpha^{(2)}(\cdot,\cdot)$, see (2.18), we observe that

$$\begin{aligned}\alpha^{(1)}_{\widetilde{k}+\epsilon\cdot\delta k}(\widetilde{w},V)&=\alpha^{(1)}_{\widetilde{k}}(\widetilde{w},V)+\alpha^{(1)}_{\epsilon\cdot\delta k}(\widetilde{w},V),\\ \alpha^{(2)}_{\widetilde{\mu}+\gamma\cdot\delta\mu}(\widetilde{w},V)&=\alpha^{(2)}_{\widetilde{\mu}}(\widetilde{w},V)+\alpha^{(2)}_{\gamma\cdot\delta\mu}(\widetilde{w},V).\end{aligned}\tag{4.12}$$

Therefore, by (4.11) and (4.12) we have

$$a_{\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu}(\widetilde{w},V)-a_{\widetilde{k},\widetilde{\mu}}(w,V)=a_{\widetilde{k},\widetilde{\mu}}(\delta w,V)+\alpha^{(1)}_{\epsilon\cdot\delta k}(\widetilde{w},V)+\alpha^{(2)}_{\gamma\cdot\delta\mu}(\widetilde{w},V).\tag{4.13}$$

Let us consider $a_{\widetilde{k},\widetilde{\mu}}(\delta w,V)$. Integrating by parts on $(0,L)$ and recalling that $\delta w|_{x=0}=0$ and $DV|_{x=L}=0$ for $t\in(0,T)$, by (3.1), (3.2) we have

$$\int_0^T a_{\widetilde{k},\widetilde{\mu}}(\delta w,V)\,dt=-\int_0^T\langle\delta w,A_{\widetilde{k},\widetilde{\mu}}V\rangle\,dt.\tag{4.14}$$

By inserting (4.10), (4.13), (4.14) in (4.9) and recalling that $(CV_{,tt}-A_{\widetilde{k},\widetilde{\mu}}V)=0$ for $(x,t)\in(0,L)\times(0,T)$, we obtain

$$\begin{aligned}-\int_0^T\Big(\alpha^{(1)}_{\epsilon\cdot\delta k}(w[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu],V)+\alpha^{(2)}_{\gamma\cdot\delta\mu}(w[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu],V)\Big)\,dt\\ =\langle C\delta w_{,t}(T),W\rangle-\int_0^T\delta Q(0,t)\cdot DV(0)\,dt.\end{aligned}\tag{4.15}$$

By recalling the boundary conditions $(3.15)_3$ for V at $x=0$, i.e. $DV|_{x=0}=2(Q[\widetilde{k},\widetilde{\mu}](0,t)-\overline{Q}(t))$, for $t\in(0,T)$, and observing that

$$\begin{aligned}\alpha^{(1)}_{\epsilon\cdot\delta k}(w[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu],V)&=\epsilon\alpha^{(1)}_{\delta k}(\delta w,V)+\epsilon\alpha^{(1)}_{\delta k}(w[\widetilde{k},\widetilde{\mu}],V),\\ \alpha^{(2)}_{\gamma\cdot\delta\mu}(w[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu],V)&=\gamma\alpha^{(2)}_{\delta\mu}(\delta w,V)+\gamma\alpha^{(2)}_{\delta\mu}(w[\widetilde{k},\widetilde{\mu}],V),\end{aligned}\tag{4.16}$$

we can express the last integral of the right-hand side of (4.15) as

$$\begin{aligned}2\int_0^T\delta Q(0,t)\cdot(Q[\widetilde{k},\widetilde{\mu}](0,t)-\overline{Q}(t))\,dt&=\langle C\delta w_{,t}(T),W\rangle\\ &+\epsilon\int_0^T\Big(\alpha^{(1)}_{\delta k}(\delta w,V)+\alpha^{(1)}_{\delta k}(w[\widetilde{k},\widetilde{\mu}],V)\Big)\,dt\\ &+\gamma\int_0^T\Big(\alpha^{(2)}_{\delta\mu}(\delta w,V)+\alpha^{(2)}_{\delta\mu}(w[\widetilde{k},\widetilde{\mu}],V)\Big)\,dt.\end{aligned}\tag{4.17}$$

Finally, by using (4.17) in (4.4), we obtain the incremental form for J_1

$$\begin{aligned}&J_1(\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu)-J_1(\widetilde{k},\widetilde{\mu})\\ &\quad=\langle C\delta w_{,t}(T),W\rangle+\epsilon\int_0^T\alpha^{(1)}_{\delta k}(w[\widetilde{k},\widetilde{\mu}],V)\,dt+\gamma\int_0^T\alpha^{(2)}_{\delta\mu}(w[\widetilde{k},\widetilde{\mu}],V)\,dt\\ &\qquad+\epsilon\int_0^T\alpha^{(1)}_{\delta k}(\delta w,V)\,dt+\gamma\int_0^T\alpha^{(2)}_{\delta\mu}(\delta w,V)\,dt+\int_0^T\delta Q(0,t)\cdot\delta Q(0,t)\,dt,k\end{aligned}\tag{4.18}$$

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where, as it will be proved in Lemma 4.6, the last three terms are $o(\sqrt{\epsilon^2+\gamma^2})$ as $\sqrt{\epsilon^2+\gamma^2} \to 0$.

We come now to the functional J_2. Proceeding as we did for J_1, the increment of J_2 is given by

$$J_2(\widetilde{k}+\epsilon\cdot\delta k, \widetilde{\mu}+\gamma\cdot\delta\mu) - J_2(\widetilde{k},\widetilde{\mu}) = 2\int_0^T \sum_{i=1}^{2} \langle (v_i[\widetilde{k},\widetilde{\mu}] - \overline{v}_i), \delta v_i\rangle_I \, dt + \int_0^T \sum_{i=1}^{2} \langle \delta v_i, \delta v_i\rangle_I \, dt, \tag{4.19}$$

where $\delta v_i = v_i[\widetilde{k}+\epsilon\cdot\delta k, \widetilde{\mu}+\gamma\cdot\delta\mu] - v_i[\widetilde{k},\widetilde{\mu}]$, $i = 1,2$. Let us recall that v_i's are the components of $w[\widetilde{k},\widetilde{\mu}]$'s which solve the problem (2.1) for $k = \widetilde{k}$ and $\mu = \widetilde{\mu}$.

Since $\delta w|_{t=0} = 0$, $\delta w_{,t}|_{t=0} = 0$, $Z|_{t=T} = 0$, $Z_{,t}|_{t=T} = 0$, $C = C^T$, by recalling (3.18) and integrating by parts twice we have

$$\int_0^T \langle CZ_{,tt}, \delta w\rangle \, dt = \int_0^T \langle Z, C\delta w_{,tt}\rangle \, dt. \tag{4.20}$$

Now, since $D(\delta w)|_{x=0} = D(\delta w)|_{x=L} = 0$, $DZ|_{x=0} = DZ|_{x=L} = 0$ and $A_{\widetilde{k},\widetilde{\mu}} = A^T_{\widetilde{k},\widetilde{\mu}}$, by integrating by parts twice we get

$$\int_0^T \langle A_{\widetilde{k},\widetilde{\mu}} Z, \delta w\rangle \, dt = \int_0^T \langle Z, A_{\widetilde{k},\widetilde{\mu}}\delta w\rangle \, dt. \tag{4.21}$$

By inserting (4.20) and (4.21) in (4.19) we have

$$J_2(\widetilde{k}+\epsilon\cdot\delta k, \widetilde{\mu}+\gamma\cdot\delta\mu) - J_2(\widetilde{k},\widetilde{\mu}) = \int_0^T \langle Z, (C\delta w_{,tt} - A_{\widetilde{k},\widetilde{\mu}}\delta w\rangle \, dt + \int_0^T \sum_{i=1}^{2} \langle \delta v_i, \delta v_i\rangle_I \, dt. \tag{4.22}$$

By (4.18) and (4.22) the expression (4.3) follows. □

4.2 Continuous dependence on the coefficients

To study the behaviour of the first and fourth term on the right-hand side of (4.3) when $\sqrt{\epsilon^2+\gamma^2} \to 0$, as well as to show that the last four terms in the right-hand side of (4.3) are of higher order on $\sqrt{\epsilon^2+\gamma^2}$, we need to consider how the solutions of problem (2.1) depend on perturbations of the coefficients k, μ and on the source term.

In Theorem 4.3 below it is shown that if the coefficients and the source term are convergent in a suitable sense, then the corresponding solutions will converge to the solution of the limit problem. In particular, we state the result in a form that will be useful to us in § 4.3.

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To simplify the notations we define the perturbed operator in (2.1) as

$$\mathscr{A}_{\epsilon,\gamma} := C\partial_{tt} - A_{k+\epsilon\cdot\delta k,\mu+\gamma\cdot\delta\mu}, \tag{4.23}$$

for ϵ, $\gamma \in \mathbb{R}^+$ small numbers. Moreover, let us define

$$V = (H_0^1(0,L))^2 \times (H_0^2(0,L))^2, \qquad H = (L^2(0,L))^4. \tag{4.24}$$

We have the following result, see, for example, [11] (§2.8).

Theorem 4.3 *Let $v \in C^0([0,T];V) \cap C^1([0,T];H)$ be the solution to*

$$\begin{cases} \mathscr{A}v = f & (x,t) \in (0,L)\times(0,T), \\ v|_{t=0} = v_0, \quad v_{,t}|_{t=0} = v_1 & x \in (0,L), \\ Dv|_{x=0} = Dv|_{x=L} = 0 & t \in (0,T), \end{cases} \tag{4.25}$$

and let $v_{\epsilon,\gamma} \in C^0([0,T];V) \cap C^1([0,T];H)$ be the solution to

$$\begin{cases} \mathscr{A}_{\epsilon,\gamma} v_{\epsilon,\gamma} = f_{\epsilon,\gamma} & (x,t) \in (0,L)\times(0,T), \\ v_{\epsilon,\gamma}|_{t=0} = v_0, \quad v_{\epsilon,\gamma,t}|_{t=0} = v_1 & x \in (0,L), \\ Dv_{\epsilon,\gamma}|_{x=0} = Dv_{\epsilon,\gamma}|_{x=L} = 0 & t \in (0,T), \end{cases} \tag{4.26}$$

for small real numbers ϵ and γ. Assume that $\mathscr{A}$, $\mathscr{A}_{\epsilon,\gamma}$ are such that the sesquilinear forms $a(\cdot,\cdot)$, $a_{\epsilon,\gamma}(\cdot,\cdot)$, $c(\cdot,\cdot)$ defined in (2.18), (2.19) *satisfy the coercivity conditions* (2.22) *and the regularity conditions* (2.5), (2.6). *Moreover, let f, $f_{\epsilon,\gamma} \in C^0([0,T];H)$, $\int_0^L |f_\epsilon(x,t)|^2 dx \leqslant C_f$ for every $t \in [0,T]$, with the constant $C_f > 0$ independent of ϵ and γ, and $v_0 \in V$, $v_1 \in H$.*

Assume that

$$\begin{cases} \|(A_{\epsilon,\gamma} - A)\varphi\|_{V'} \to 0 & \text{for every } \varphi \in V \text{ as } \sqrt{\epsilon^2+\gamma^2} \to 0, \\ \|f_{\epsilon,\gamma} - f\|_{L^2(0,L)} \to 0 & \text{uniformly for } t \in [0,T] \text{ as } \sqrt{\epsilon^2+\gamma^2} \to 0. \end{cases} \tag{4.27}$$

Then we have

$$v_{\epsilon,\gamma} \to v \qquad \text{strongly in } C^0([0,T];V) \cap C^1([0,T];H) \text{ as } \sqrt{\epsilon^2+\gamma^2} \to 0. \tag{4.28}$$

Proof Let us denote the pair (ϵ,γ) by τ and define $|\tau| := \sqrt{\epsilon^2+\gamma^2}$. Let

$$E(t) := \frac{1}{2}(a(v,v) + c(v',v')), \qquad E_\tau(t) := \frac{1}{2}(a(v_\tau,v_\tau) + c(v'_\tau,v'_\tau)), \tag{4.29}$$

be the *energy* for the problems (4.25) and (4.26), respectively. We have

$$\frac{dE(t)}{dt} = \langle f, v'\rangle, \qquad \frac{dE_\tau(t)}{dt} = \langle f_\tau, v'_\tau\rangle, \quad t \in (0,T), \tag{4.30}$$

where $\langle f, v'\rangle := \int_0^L f v'\,dx$ and $\langle f_\tau, v'_\tau\rangle := \int_0^L f_\tau v'_\tau\,dx$. Since $\|f_\tau\|_{L^2(0,L)} \leqslant C_f$, with $C_f > 0$ constant independent from τ, by a standard argument based on Gronwall's inequality, we

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obtain

$$\begin{cases} E(t) \leqslant C(a(v_0, v_0) + c(v_1, v_1) + C_f), \\ E_\tau(t) \leqslant C'(a(v_0, v_0) + c(v_1, v_1) + C_f), \quad \forall\ |\tau| \text{ small}, t \in [0, T], \end{cases} \tag{4.31}$$

where C, C' are two positive constants only depending on the coercivity constants α, β and on T.

By $(4.31)_2$ and again by the coercivity conditions (2.22), it turns out that v_τ and v'_τ are uniformly bounded, with respect to τ, in $L^2((0, T); V)$ and in $L^2((0, T); H)$, respectively. Therefore, by Kakutani's theorem, there exists a subsequence $\tau_m = (\epsilon_m, \gamma_m)$ such that

$$\begin{cases} v_{\tau_m} \rightharpoonup \overline{v} \quad \text{weakly in } L^2((0, T); V), \\ v'_{\tau_m} \rightharpoonup \overline{v}' \quad \text{weakly in } L^2((0, T); H), \end{cases} \tag{4.32}$$

as $|\tau_m| \to 0$, where $\overline{v}'$ is to be understood in distributional sense.

We now prove that $\overline{v} = v$, that is $\overline{v}$ satisfies the differential equation $(4.25)_1$, the initial conditions $(4.25)_2$ and the Dirichlet boundary conditions $(4.25)_3$.

We start by showing that $\mathscr{A}\overline{v} = f$ (in the weak sense of (2.17)). Let $\varphi \in C_0^\infty([0, T]; V)$ be any test function and consider first the term involving the sesquilinear form $a_\tau(\cdot, \cdot)$. Recalling that $A = A^T$, we have

$$(A_{\tau_m} v_{\tau_m}, \varphi) = (A\overline{v}, \varphi) + (v_{\tau_m}, (A_{\tau_m} - A)\varphi) + (v_{\tau_m} - \overline{v}, A\varphi). \tag{4.33}$$

The second term tends to zero as $|\tau_m| \to 0$ because v_{τ_m} is bounded and $\|(A_{\tau_m} - A)\varphi\|_{V'} \to 0$ for every $\varphi \in V$ as $|\tau_m| \to 0$ by assumption (4.27). The third term of (4.33) tends to zero as $|\tau_m| \to 0$ since $v_{\tau_m} \rightharpoonup \overline{v}$ by $(4.32)_1$.

Similarly, under our assumptions, we have

$$\begin{aligned} (Cv_{\tau_m}, \varphi) &= (C\overline{v}, \varphi) + (v_{\tau_m} - \overline{v}, C\varphi) \to (C\overline{v}, \varphi), \\ \langle f_\tau, \varphi \rangle &= \langle f, \varphi \rangle + \langle f_\tau - f, \varphi \rangle \to \langle f, \varphi \rangle \quad \text{as } |\tau_m| \to 0. \end{aligned} \tag{4.34}$$

Therefore, $\overline{v}$ satisfies the differential equation $(4.25)_1$ in distributional sense.

To verify the initial condition for $\overline{v}$, let us consider a test function $\varphi = h\psi$, where $h \in V$ and $\psi \in C^\infty([0, T])$, with $\psi(T) = \psi'(T) = 0$. Integrating by parts we find

$$\int_0^T (v_{\tau_m, t}, \varphi) + (v_{\tau_m}, \varphi_{, t}) = \int_0^T (v_{\tau_m}, \varphi)_t = -(v_{\tau_m}, \varphi)|_{t=0} = -(v_0, \varphi)|_{t=0}. \tag{4.35}$$

The left hand side of (4.32) tends to $\int_0^T (\overline{v}, \varphi) = -(\overline{v}, \varphi)|_{t=0}$ by (4.32) and therefore, by the arbitrariness of h and ψ, we have $\overline{v}|_{t=0} = v_0$.

We can argue similarly for the initial conditions on $\overline{v}_{, t}$. Let φ be chosen as before. By multiplying (4.26) by φ and integrating by parts on $(0, L) \times (0, T)$ we obtain

$$-\int_0^T (\langle A_{\tau_m} v_{\tau_m}, \varphi \rangle + \langle Cv_{\tau_m, t}, \varphi_{, t} \rangle)\, dt = \int_0^T \langle f_{\tau_m}, \varphi \rangle\, dt + (Cv_{\tau_m, t}, \varphi)|_{t=0}. \tag{4.36}$$

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Taking the limit in (4.36) for $|\tau_m| \to 0$, recalling that the weak limit (4.32) of v_{τ_m} satisfies the equation $(4.25)_1$ and that $v_{\tau_m,t}\,|_{t=0} = v_1$, we have

$$\psi(0)\langle C(\overline{v}_{,t}\,|_{t=0} - v_1), h\rangle = 0 \qquad \forall \psi(0) \text{ and } \forall h \in V, \tag{4.37}$$

and then $\overline{v}_{,t}\,|_{t=0} = v_1$ in $(0, L)$. This shows that $\overline{v}$ satisfies problem (4.25). By the uniqueness of the solution we have $v = \overline{v}$.

We complete the proof by showing that, in fact, the convergence of v_{τ_m} to v is strong in $C([0, T]; V) \cap C^1([0, T]; H)$ as $|\tau| \to 0$.

Recall that $\frac{dE_{\tau_m}(t)}{dt} = \langle f_{\tau_m}, v'_{\tau_m}\rangle$ for $t \in [0, T]$. Since we know that $v'_{\tau_m} \rightharpoonup v$ weakly in $L^2((0, T); H)$ and $f_{\tau_m} \to f$ strongly in $L^2(0, L)$ (and uniformly for $t \in [0, T]$) as $|\tau_m| \to 0$, then $\langle f_{\tau_m}, v'_{\tau_m}\rangle \to \langle f, v'\rangle$ as $|\tau_m| \to 0$. Therefore, we have $\frac{dE_{\tau_m}(t)}{dt} \to \frac{dE(t)}{dt}$ in $[0, T]$ and, recalling that $E_{\tau_m}(0) = E(0)$, we have

$$E_{\tau_m}(t) \to E(t) \qquad t \in [0, T] \text{ as } |\tau| \to 0. \tag{4.38}$$

We compute:

$$a(v_{\tau_m} - v, v_{\tau_m} - v) + c(v'_{\tau_m} - v', v'_{\tau_m} - v') = 2E_m(t) + 2E(t) - 2(a(v_{\tau_m}, v) + c(v'_{\tau_m}, v')). \tag{4.39}$$

By (4.32) we have $(a(v_{\tau_m}, v) + c(v'_{\tau_m}, v')) \to 2E$ as $|\tau_m| \to 0$ and then the left hand side of (4.39) tends to zero as $|\tau_m| \to 0$. Finally, by the coercivity of the sesquilinear forms $a(\cdot, \cdot)$ and $c(\cdot, \cdot)$, we obtain

$$\begin{cases} v_{\tau_m} \to v & \text{strongly in } L^2((0, T); V), \\ v_{\tau_m,t} \to v_{,t} & \text{strongly in } L^2((0, T); H), \end{cases} \tag{4.40}$$

as $|\tau_m| \to 0$, which imply (4.28). □

4.3 Justification of the asymptotic expansion for J'

We begin by considering the asymptotic behaviour of the first term in the right hand side of (4.3).

Lemma 4.4 *Let the assumptions of Theorem 3.1 be satisfied. Let δw be defined as in* (4.1) *and let $W \in (H^2(0, L))^2 \times (H^4(0, L))^2$ be the solution of* (3.14). *For $\sqrt{\epsilon^2 + \gamma^2} \to 0$ we have*

$$\langle C\delta w_{,t}(T), W\rangle \;=\; \epsilon \left\langle C U^{(\tilde{k})}_{,t}(T), W \right\rangle \;+\; \gamma \left\langle C U^{(\tilde{\mu})}_{,t}(T), W \right\rangle \;+\; o(\sqrt{\epsilon^2 + \gamma^2}), \tag{4.41}$$

where $U^{(\tilde{k})} \in (C^0([0, T]; H_0^1(0, L))^2 \times (C^0([0, T]; H_0^2(0, L))^2$ is the solution of (3.16) *and $U^{(\tilde{\mu})} \in (C^0([0, T]; H_0^1(0, L))^2 \times (C^0([0, T]; H_0^2(0, L))^2$ is the solution of* (3.17).

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Proof To simplify the notation we define

$$\begin{cases} w := w[\widetilde{k},\widetilde{\mu}], \qquad w_{\epsilon,\gamma} := w[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu], \\ A := A_{\widetilde{k},\widetilde{\mu}}, \qquad A_{\epsilon,\gamma} := A_{\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu}, \\ \mathscr{A} := C\partial_{tt} - A, \qquad \mathscr{A}_{\epsilon,\gamma} := C\partial_{tt} - A_{\epsilon,\gamma}, \end{cases} \tag{4.42}$$

where w and $w_{\epsilon,\gamma}$ are the solutions of (2.1) for given elliptic operators A and $A_{\epsilon,\gamma}$, respectively.

By introducing a suitable trace operator Φ, see (2.8), as it was made in § 2, the initial boundary value problem (2.1) for w and $w_{\epsilon,\gamma}$ is reduced to an evolution problem of the form (2.23) with homogeneous Dirichlet boundary conditions in terms of the new functions

$$\widehat{w} := w - \Phi, \qquad \widehat{w}_{\epsilon,\gamma} := w_{\epsilon,\gamma} - \Phi. \tag{4.43}$$

More precisely, $\widehat{w}$ and $\widehat{w}_{\epsilon,\gamma}$, both belonging to $\{(H^2((0,T);H_0^1(0,L)))^2 \times (H^2((0,T);H_0^2(0,L)))^2\} \cap (H^3((0,T);L^2(0,L)))^4$, are respectively the solutions to

$$\begin{cases} \mathscr{A}\widehat{w} = -C\Phi_{,tt} + A\Phi := f \qquad (x,t) \in (0,L)\times(0,T), \\ \widehat{w}|_{t=0} = 0, \qquad \widehat{w}_{,t}\,|_{t=0} = 0 \quad x \in (0,L), \\ D\widehat{w}|_{x=0} = D\widehat{w}|_{x=L} = 0 \quad t \in (0,T), \end{cases} \tag{4.44}$$

and

$$\begin{cases} \mathscr{A}_{\epsilon,\gamma}\widehat{w}_{\epsilon,\gamma} = -C\Phi_{,tt} + A_{\epsilon,\gamma}\Phi := f_{\epsilon,\gamma} \qquad (x,t) \in (0,L)\times(0,T), \\ \widehat{w}_{\epsilon,\gamma}|_{t=0} = 0, \qquad \widehat{w}_{\epsilon,\gamma,t}\,|_{t=0} = 0 \quad x \in (0,L), \\ D\widehat{w}_{\epsilon,\gamma}|_{x=0} = D\widehat{w}_{\epsilon,\gamma}|_{x=L} = 0 \quad t \in (0,T), \end{cases} \tag{4.45}$$

where $f := f[\widetilde{k},\widetilde{\mu}]$, $f_{\epsilon,\gamma} := f[\widetilde{k}+\epsilon\cdot\delta k,\widetilde{\mu}+\gamma\cdot\delta\mu]$ are defined similarly to $(2.14)_1$, $f, f_{\epsilon,\gamma} \in (C^1([0,T];L^2(0,L)))^4$.

Since $\delta w := w_{\epsilon,\gamma} - w = \widehat{w}_{\epsilon,\gamma} - \widehat{w} =: \delta\widehat{w}$, we have

$$\begin{aligned} \mathscr{A}_{\epsilon,\gamma}(w_{\epsilon,\gamma} - w) = \mathscr{A}_{\epsilon,\gamma}(\widehat{w}_{\epsilon,\gamma} - \widehat{w}) = (\mathscr{A} - \mathscr{A}_{\epsilon,\gamma})\widehat{w} + (f_{\epsilon,\gamma} - f) = \\ = (A_{\epsilon,\gamma} - A)\widehat{w} + (A_{\epsilon,\gamma} - A)\Phi = (A_{\epsilon,\gamma} - A)w. \end{aligned} \tag{4.46}$$

By recalling the definition (2.3) of A and $A_{\epsilon,\gamma}$, $\delta\widehat{w}$ solves the evolution problem

$$\begin{cases} \mathscr{A}_{\epsilon,\gamma}\delta\widehat{w} = \epsilon F(\delta k)w + \gamma G(\delta\mu)w, \qquad (x,t) \in (0,L)\times(0,T), \\ \delta\widehat{w}|_{t=0} = 0, \qquad \delta\widehat{w}_{,t}\,|_{t=0} = 0 \quad x \in (0,L), \\ D\delta\widehat{w}|_{x=0} = D\delta\widehat{w}|_{x=L} = 0 \quad t \in (0,T), \end{cases} \tag{4.47}$$

where $F(\delta k)$ and $G(\delta\mu)$ are defined in (3.19) and (3.20), respectively.

By the regularity of $w = w[\widetilde{k},\widetilde{\mu}]$ (see Theorem 2.3), it turns out that $(\epsilon F(\delta k)w + \gamma G(\delta\mu)w) \in (H^2((0,T);L^2(0,L)))^4$ for every ϵ and γ small and then, by Theorem 2.3, $\delta\widehat{w} \in (H^2((0,T);H_0^1(0,L)))^2 \times (H^2((0,T);H_0^2(0,L)))^2$. Now, since $\|A_{\epsilon,\gamma} - A\|_{L^\infty(0,L)} \to 0$ and $\|\epsilon F(\delta k)w + \gamma G(\delta\mu)w\|_{L^2(0,L)} \to 0$ uniformly for $t \in [0,T]$ as $\sqrt{\epsilon^2+\gamma^2} \to 0$, by Theorem 4.3

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we have

$$\delta\widehat{w} \to 0 \quad \text{strongly in } C^0([0,T];V)\cap C^1([0,T];H) \text{ as } \sqrt{\epsilon^2+\gamma^2}\to 0. \tag{4.48}$$

To study the asymptotic behaviour of $\delta\widehat{w}$ for $\sqrt{\epsilon^2+\gamma^2}\to 0$, we introduce the functions $U^{(\widetilde{k})}\in (C^0([0,T];H_0^1(0,L))^2\times(C^0([0,T];H_0^2(0,L))^2$, $U^{(\widetilde{\mu})}\in (C^0([0,T];H_0^1(0,L))^2\times (C^0([0,T];H_0^2(0,L))^2$ solutions to the problems (3.16) and (3.17), respectively. Then, the function

$$\delta U_{\epsilon,\gamma} := \delta\widehat{w} - \epsilon U^{(\widetilde{k})} - \gamma U^{(\widetilde{\mu})} \tag{4.49}$$

solves the problem

$$\begin{cases} \mathscr{A}\delta U_{\epsilon,\gamma} = \epsilon F(\delta k)\delta\widehat{w} + \gamma G(\delta\mu)\delta\widehat{w}, & (x,t)\in(0,L)\times(0,T),\\ \delta U_{\epsilon,\gamma}|_{t=0} = 0, \qquad \delta U_{\epsilon,\gamma,t}\,|_{t=0} = 0 \quad x\in(0,L),\\ D\delta U_{\epsilon,\gamma}|_{x=0} = D\delta U_{\epsilon,\gamma}|_{x=L} = 0 \quad t\in(0,T). \end{cases} \tag{4.50}$$

The left-hand side of $(4.50)_1$ belongs to $(H^2((0,T);L^2(0,L)))^4$ for every ϵ, γ small and $\|\epsilon F(\delta k)\delta\widehat{w}+\gamma G(\delta\mu)\delta\widehat{w}\|_{L^2(0,L)}\leqslant C\sqrt{\epsilon^2+\gamma^2}r(\epsilon,\gamma)$ uniformly for $t\in[0,T]$ as $\sqrt{\epsilon^2+\gamma^2}\to 0$, where $r(\epsilon,\gamma)\to 0$ as $\sqrt{\epsilon^2+\gamma^2}\to 0$. Then, by Theorem 4.3, we have

$$\frac{\delta U_{\epsilon,\gamma}}{\sqrt{\epsilon^2+\gamma^2}}\to 0 \quad \text{strongly in } C^0([0,T];V)\cap C^1([0,T];H) \text{ as } \sqrt{\epsilon^2+\gamma^2}\to 0. \tag{4.51}$$

By (4.49) and recalling that both the two functions $U^{(\widetilde{k})}{}_{,t}$ and $U^{(\widetilde{\mu})}{}_{,t}$ are in $(C^0([0,T];L^2(0,L))^4$, we can compute

$$\delta w_{,t}\,|_{t=T} = \epsilon U^{(\widetilde{k})}{}_{,t}\,|_{t=T} + \gamma U^{(\widetilde{\mu})}{}_{,t}\,|_{t=T} + o(\sqrt{\epsilon^2+\gamma^2}) \quad \text{in H as } \sqrt{\epsilon^2+\gamma^2}\to 0, \tag{4.52}$$

with $o(\sqrt{\epsilon^2+\gamma^2})/\sqrt{\epsilon^2+\gamma^2}\to 0$ as $\sqrt{\epsilon^2+\gamma^2}\to 0$.

This immediately implies (4.41). □

Next Lemma clarifies the asymptotic behavior of the fourth term in the right-hand side of (4.3) when $\sqrt{\epsilon^2+\gamma^2}\to 0$.

Lemma 4.5 *Let the assumptions of Theorem 3.1 be satisfied. Let δw be defined as in* (4.1) *and let $Z\in(C^0([0,T];H_0^1(0,L))^2\times(C^0([0,T];H_0^2(0,L))^2$ be the weak solution of* (3.18). *For $\sqrt{\epsilon^2+\gamma^2}\to 0$ we have*

$$\int_0^T \langle Z, C\delta w_{,tt} - A_{\widetilde{k},\widetilde{\mu}}\delta w\rangle\,dt = \epsilon\int_0^T \langle Z, F(\delta k)w[\widetilde{k},\widetilde{\mu}]\rangle\,dt + \gamma\int_0^T \langle Z, G(\delta\mu)w[\widetilde{k},\widetilde{\mu}]\rangle\,dt + o(\sqrt{\epsilon^2+\gamma^2}), \tag{4.53}$$

where F and G are the operators defined in (3.19) *and* (3.20), *respectively.*

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Proof We shall use the notation introduced at the beginning of the proof of Lemma 4.4. The function $\delta w := w_{\epsilon,\gamma} - w = \widehat{w}_{\epsilon,\gamma} - \widehat{w} =: \delta\widehat{w}$ satisfies the problem

$$\begin{cases} C\delta\widehat{w}_{,tt} - A_{\widetilde{k},\widetilde{\mu}}\delta\widehat{w} = \epsilon F(\delta k)w_{\epsilon,\gamma} + \gamma G(\delta\mu)w_{\epsilon,\gamma} := f_{\epsilon,\gamma}, \ (x,t) \in (0,L)\times(0,T), \\ \delta\widehat{w}|_{t=0} = \delta\widehat{w}_{,t}\,|_{t=0} = 0 \qquad x \in (0,L), \\ D\delta\widehat{w}|_{x=0} = D\delta\widehat{w}|_{x=L} = 0 \qquad t \in (0,T), \end{cases} \tag{4.54}$$

where F and G are defined in (3.19) and (3.20), respectively.

Arguing as in Lemma 4.4, one can prove that

$$\delta\widehat{w} := \epsilon U^{(\widetilde{k})} + \gamma U^{(\widetilde{\mu})} + o(\sqrt{\epsilon^2+\gamma^2}), \tag{4.55}$$

as $\sqrt{\epsilon^2+\gamma^2} \to 0$, where the functions

$$U^{(\widetilde{k})} \in (C^0([0,T];H_0^1(0,L))^2 \times (C^0([0,T];H_0^2(0,L))^2,$$
$$U^{(\widetilde{\mu})} \in (C^0([0,T];H_0^1(0,L))^2 \times (C^0([0,T];H_0^2(0,L))^2$$

are solutions to the problems (3.16) and (3.17), respectively, and

$$o(\sqrt{\epsilon^2+\gamma^2})/\sqrt{\epsilon^2+\gamma^2} \to 0$$

as $\sqrt{\epsilon^2+\gamma^2} \to 0$. By $(4.54)_1$ and (4.55) the thesis of the Lemma follows. □

Finally, next Lemma shows that the last four terms on the right-hand side of (4.3) are of higher order with respect to $\sqrt{\epsilon^2+\gamma^2}$ when $\sqrt{\epsilon^2+\gamma^2} \to 0$.

Lemma 4.6 *Let the assumptions of Theorem 3.1 be satisfied. Let δw, δQ be defined as in* (4.1) *and* (4.2)*, respectively. Let $V \in (C^0([0,T];H^1(0,L))^2 \times (C^0([0,T];H^2(0,L))^2$ be the solution of* (3.15) *and let $\alpha^{(i)}$, $i = 1,2$, be defined as in* (2.18)*. We have*

$$\epsilon\int_0^T \alpha^{(1)}_{\delta k}(\delta w, V)\,dt + \gamma\int_0^T \alpha^{(2)}_{\delta\mu}(\delta w, V)\,dt + \int_0^T \delta Q(0,t)\cdot\delta Q(0,t)\,dt + \int_0^T \sum_{i=1}^{2}\langle \delta v_i, \delta v_i\rangle\,dt = o(\sqrt{\epsilon^2+\gamma^2}) \tag{4.56}$$

as $\sqrt{\epsilon^2+\gamma^2} \to 0$.

Proof Recalling that $\delta w := w_{\epsilon,\gamma} - w = \widehat{w}_{\epsilon,\gamma} - \widehat{w} := \delta\widehat{w}$, by Lemma 4.5 (estimate (4.55)), one can easily show that the first two terms and the last term on the left hand side of (4.56) are $o(\sqrt{\epsilon^2+\gamma^2})$ as $\sqrt{\epsilon^2+\gamma^2} \to 0$.

Concerning the third term, by the definition of the boundary operator (3.3), (3.6) and (4.2) we have

$$\delta Q := \delta Q(0,t) = -B_{\widetilde{k}+\epsilon\cdot\delta k}w_{\epsilon,\gamma} + B_{\widetilde{k}}w = -B_{\widetilde{k}}\delta w - (B_{\widetilde{k}+\epsilon\cdot\delta k} - B_{\widetilde{k}})w_{\epsilon,\gamma}, \tag{4.57}$$

where we have used the notation (4.42).

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By trace inequalities and (4.55) we have

$$\|B_{\tilde{k}}\delta w\|_{(L^2(0,T))^4} \leqslant C\|\delta w\|_{C^0([0,T];V)} \leqslant C\sqrt{\epsilon^2+\gamma^2} \qquad \text{as } \sqrt{\epsilon^2+\gamma^2}\to 0, \tag{4.58}$$

where the constant $C>0$ does not depend on ϵ and γ.

Finally, by the definition (3.3) of the boundary operator and the uniform boundedness of $\{w_{\epsilon,\gamma}\}$, we have

$$\|(B_{\tilde{k}+\epsilon\cdot\delta k}-B_{\tilde{k}})w_{\epsilon,\gamma}\|_{(L^2(0,T))^4} \leqslant C\sqrt{\epsilon^2+\gamma^2} \qquad \text{as } \sqrt{\epsilon^2+\gamma^2}\to 0, \tag{4.59}$$

where the constant $C>0$ does not depend on ϵ and γ.

By using inequalities (4.58), (4.59) in (4.57), we have that $\int_0^T \delta Q(0,t)\cdot\delta Q(0,t)\,dt = o(\sqrt{\epsilon^2+\gamma^2})$ as $\sqrt{\epsilon^2+\gamma^2}\to 0$. □

5 Numerical algorithm and experiments

In this section we shall present a projected gradient method for the minimization of the cost function $J = J(\tilde{k},\tilde{\mu})$ defined in (3.8)–(3.10). The numerical algorithm is based on the complete form of the differential of J determined in Theorem 3.1. The identification technique is tested on dynamic test data coming from a real specimen of steel-concrete composite beam studied in Dilena & Morassi [5]. A more comprehensive study of the proposed numerical algorithm for solving this inverse dynamical problem will be presented in a forthcoming paper.

5.1 Numerical algorithm

We shall formulate the minimization problem for the function J by adapting the *projected gradient method* presented in Daniel [4]. To apply the minimization algorithm, it is useful to represent the Gateaux partial derivatives $\partial_1 J$ and $\partial_2 J$, evaluated at the point $(\tilde{k},\tilde{\mu})$, as

$$\partial_1 J(\tilde{k},\tilde{\mu})\delta k = \langle \partial_k J(\tilde{k},\tilde{\mu}),\delta k\rangle, \tag{5.1}$$

$$\partial_2 J(\tilde{k},\tilde{\mu})\delta\mu = \langle \partial_\mu J(\tilde{k},\tilde{\mu}),\delta\mu\rangle, \tag{5.2}$$

for suitable operators $\partial_k J(\tilde{k},\tilde{\mu})$ and $\partial_\mu J(\tilde{k},\tilde{\mu})$, where $\delta k,\ \delta\mu\in\mathscr{C}$. The expressions on the left hand side of (5.1), (5.2) have been determined in Theorem 3.1, see equations (3.12) and (3.13). By integrating by parts the third term of (3.12), we have

$$\int_0^T \langle Z, F(\delta k)w[\tilde{k},\tilde{\mu}]\rangle\,dt = -\int_0^T \alpha^{(1)}_{\delta k}(w[\tilde{k},\tilde{\mu}],Z)\,dt. \tag{5.3}$$

Concerning the second term of (3.12), by the boundedness of the linear functional $C^1[0,\,L]\ni K\mapsto \langle C\partial_t U_K^{(\tilde{k})}(T),\,W\rangle\in\boldsymbol{R}$ and the Sobolev imbedding theorem, we can write

$$\langle d_k,\,\delta k\rangle = \langle C\partial_t U_{\delta k}^{(\tilde{k})}(T),\,W\rangle \tag{5.4}$$

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with some $d_k \in H^2(0, L)^* = H_0^{-2}(0, L)$. Therefore, by inserting (5.3) and (5.4) in (3.12), we have

$$\partial_1 J(\widetilde{k}, \widetilde{\mu})\, \delta k = \int_0^T \alpha_{\delta k}^{(1)}(w[\widetilde{k}, \widetilde{\mu}],\, V - Z)\, dt + \langle d_k,\, \delta k\rangle \tag{5.5}$$

and we can get the expression of $\partial_k J(\widetilde{k}, \widetilde{\mu})$ as follows

$$\begin{aligned}\partial_k J(\widetilde{k}, \widetilde{\mu}) = &\int_0^T (u_2 - u_1 + v_{2,x}e_s)\{(p_2 - p_1 + q_{2,x}e_s) - (z_2 - z_1 + Z_{2,x}e_s)\}\, dt \\ &+ \int_0^T \frac{e_c^2}{6}\{(2\,v_{1,x} + v_{2,x})(q_{1,x} - Z_{1,x}) + (2\,v_{2,x} + v_{1,x})(q_{2,x} - Z_{2,x})\}\, dt + d_k, \end{aligned} \tag{5.6}$$

where $V = (p_1,\, p_2,\, q_1,\, q_2)^T$ and $Z = (z_1,\, z_2,\, Z_1,\, Z_2)^T$.

By proceeding in a similar way for the partial derivative of J with respect to the variable μ, we have

$$\partial_\mu J(\widetilde{k}, \widetilde{\mu}) = \int_0^T (v_1 - v_2)\ \{(q_1 - q_2) - (Z_1 - Z_2)\}\ dt + d_\mu, \tag{5.7}$$

where $d_\mu \in H^1(0, L)^* = H_0^{-1}(0, L)$ is obtained by using the boundedness of the linear functional $C^0[0, L] \ni M \mapsto \langle C\partial_t U_M^{(\widetilde{\mu})}(T),\, W\rangle \in \boldsymbol{R}$ and the Sobolev imbedding theorem.

Since $L^2(0, L)$ is dense in $H_0^{-1}(0, L)$ and $H_0^{-2}(0, L)$, we can approximate them by L^2 functions. Hence, we have

$$\begin{aligned}\langle d_k, K\rangle &= \lim_{i_2' h_k \to d_k} (h_k, i_2 K)_{L^2} \quad \forall\ K \in H^2(0, L), \\ \langle d_\mu,\, M\rangle &= \lim_{i_1' h_\mu \to d_\mu} (h_\mu,\, i_1 M)_{L^2} \quad \forall\ M \in H^1(0, L),\end{aligned}$$

where $i_j : H^j(0, L) \to L^2(0, L)$ and $i_j' : L^2(0, L) \to H_0^{-j}(0, L)$ are continuous, injective embeddings $(j = 1, 2)$. Hence, in our numerical algorithm, we try to find the approximations $\widehat{d}_k$ and $\widehat{d}_\mu$ by solving the weak equations

$$\int_0^L K\, \widehat{d}_k\, dx = \langle C\partial_t U_K^{(\widetilde{k})}(T),\, W\rangle, \tag{5.8}$$

$$\int_0^L M\, \widehat{d}_\mu\, dx = \langle C\partial_t U_M^{(\widetilde{\mu})}(T),\, W\rangle \tag{5.9}$$

on the finite-dimensional subspaces of $H^2(0, L)$ and $H^1(0, L)$, respectively. These functions are used in (5.6) and (5.7) instead of the elements d_k and d_μ, respectively.

Now, to implement the projected gradient method we need to introduce a projection operator. Throughout this part, beside conditions (2.6), we assume that the coefficients $\widetilde{k}$ and $\widetilde{\mu}$ are bounded from above, *i.e.*

$$0 \leqslant \widetilde{k}(x) \leqslant \overline{k}, \quad 0 \leqslant \widetilde{\mu}(x) \leqslant \overline{\mu} \quad \text{for } x \in [0, L], \tag{5.10}$$

where $\overline{k}$ and $\overline{\mu}$ are given positive constants.

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Let P_k and P_μ be the *clip-off operators* such that

$$(P_k \widetilde{k})(x) = \begin{cases} 0 & (\widetilde{k}(x) < 0), \\ \widetilde{k}(x) & (0 \leqslant \widetilde{k}(x) \leqslant \bar{k}), \\ \bar{k} & (\widetilde{k}(x) > \bar{k}), \end{cases}$$

$$(P_\mu \widetilde{\mu})(x) = \begin{cases} 0 & (\widetilde{\mu}(x) < 0), \\ \widetilde{\mu}(x) & (0 \leqslant \widetilde{\mu}(x) \leqslant \bar{\mu}), \\ \bar{\mu} & (\widetilde{\mu}(x) > \bar{\mu}). \end{cases}$$

Then, we cannot guarantee that $P_k(\widetilde{k})$ belongs to $C^1[0, L]$ even if $\widetilde{k} \in C^1[0, L]$. In order to keep the smoothness of the function, we employ the mollifier as follows:

$$(S_\epsilon f)(x) := \int_{\boldsymbol{R}} s_\epsilon(y - x)\, \widehat{f}(y)\, dy$$

for bounded functions f defined on $(0, L)$, where the function $\widehat{f}$ is an extension of f to $(-c\epsilon, L + c\epsilon)$ $(c > 0)$, $s_\epsilon(x) = s(x/\epsilon)/\epsilon$ $(\epsilon > 0)$. The function s is a nonnegative C^1 function over $\boldsymbol{R}$ such that $\operatorname{supp} s = [-c, c]$ and

$$\int_{\boldsymbol{R}} s(x)\, dx = 1.$$

Then we have $S_\epsilon f \in C^1[0, L]$.

By using the above operators, we define an iterative descent procedure as follows:

$$\begin{pmatrix} \widetilde{k}_{l+1} \\ \widetilde{\mu}_{l+1} \end{pmatrix} = \begin{pmatrix} \widetilde{k}_l \\ \widetilde{\mu}_l \end{pmatrix} - \alpha_l \begin{pmatrix} s_l^{(\widetilde{k})} \\ s_l^{(\widetilde{\mu})} \end{pmatrix} \tag{5.11}$$

for $l = 0, 1, 2, \ldots$, where

$$s_l^{(\widetilde{k})} := \widetilde{k}_l - S_\epsilon \left(P_k \left(\widetilde{k}_l - \frac{\partial_k J(\widetilde{k}_l, \widetilde{\mu}_l)}{\|\partial_k J(\widetilde{k}_l, \widetilde{\mu}_l)\|} \right) \right), \tag{5.12}$$

$$s_l^{(\widetilde{\mu})} := \widetilde{\mu}_l - S_\epsilon \left(P_\mu \left(\widetilde{\mu}_l - \frac{\partial_\mu J(\widetilde{k}_l, \widetilde{\mu}_l)}{\|\partial_\mu J(\widetilde{k}_l, \widetilde{\mu}_l)\|} \right) \right). \tag{5.13}$$

Here α_l is a suitable step size which satisfy $0 < \alpha_l \leqslant 1$.

From the above discussion for the derivatives, the approximations of the derivatives $\partial_k J(\widetilde{k}, \widetilde{\mu})$ and $\partial_\mu J(\widetilde{k}, \widetilde{\mu})$ belong to $H^1(0, L)$ in our numerical experiment. Therefore we notice that, by the definition of the projections and mollifier, the updated coefficients belong to $\mathscr{C}$ and satisfy the conditions

$$0 \leqslant \widetilde{k}(x) \leqslant \bar{k}, \quad 0 \leqslant \widetilde{\mu}(x) \leqslant \bar{\mu} \quad \text{for } x \in [0, L]$$

in our experiments, whenever the initial guess $(\widetilde{k}_0, \widetilde{\mu}_0) \in \mathscr{C}$ satisfies the same bounds.

We can summarize the algorithm of our variational approach to the inverse problem for composite beams as follows.

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Numerical algorithm for coefficient identification

(1) *Set an initial guess* $(\widetilde{k}_0, \widetilde{\mu}_0)$ *which satisfies the a priori conditions* (5.10).

(2) *For* $l = 0, 1, 2, \ldots$:

(a) *Solve the evolution problem* (2.1) *with* $(k, \mu) = (\widetilde{k}_l, \widetilde{\mu}_l)$ *to find* w, $v_{i,x}$, $i = 1, 2$, *and* Q.

(b) *Solve the elliptic problem* (3.14) *with* $(\widetilde{k}_l, \widetilde{\mu}_l)$ *to get* W.

(c) *Solve the evolution problem* (3.15) *with* $(\widetilde{k}_l, \widetilde{\mu}_l)$ *to find* V *and* $q_{i,x}$, $i = 1, 2$.

(d) *Solve the evolution problem* (3.18) *to find* Z *and* $Z_{i,x}$, $i = 1, 2$.

(e) *Solve the weak equations* (5.8) *and* (5.9) *to obtain the functions* $\widehat{d}_k$ *and* $\widehat{d}_\mu$.

(f) *Calculate the derivatives* $\partial_k J(\widetilde{k}_l, \widetilde{\mu}_l)$ *and* $\partial_\mu J(\widetilde{k}_l, \widetilde{\mu}_l)$ *by* (5.6) *and* (5.7), *respectively*.

(g) *Set the search direction* $s_l^{(\widetilde{k})}$ *and* $s_l^{(\widetilde{\mu})}$ *by* (5.12) *and* (5.13), *respectively*.

(h) *Get the step size* α_l *by using the line search algorithm.*

(i) *Update the coefficients by* (5.11).

(j) *If the updated coefficients satisfy the condition*

$$\left| \frac{J(\widetilde{k}_{l+1}, \widetilde{\mu}_{l+1}) - J(\widetilde{k}_l, \widetilde{\mu}_l)}{J(\widetilde{k}_{l+1}, \widetilde{\mu}_{l+1})} \right| < \varepsilon,$$

for a small given control parameter ε, *then stop the iterations.*

5.2 Numerical experiments

In this section we shall present some numerical results obtained by applying the above variational approach to identify the stiffness coefficients k and μ of a composite beam. In the following applications we shall refer to the specimen $T1PR$ of steel-concrete beam analyzed in the experimental study [5]. In particular, the case of a damaged composite beam having initial constant elastic and inertial properties will be closely examined. This case is simple but rather meaningful for applications.

The physical constants of the *reference* configuration of the undamaged beam are summarized in Table 1. Here "d" means the physical dimension.
In our numerical experiments we suppose that the length of observation time is $T^d = 4.0 \times 10^{-3}$ s.

To implement the identification algorithm, it is useful to introduce non-dimensional values for the geometrical and mechanical quantities, that is

$$L = 1.0, \quad \rho_i = \frac{\rho_i^d}{\overline{\rho}^d}, \quad a_i = \frac{E_i^d A_i^d}{\overline{\rho}^d (\overline{\eta}^d)^2}, \quad j_i = \frac{E_i^d I_i^d}{\overline{\rho}^d (\overline{\eta}^d)^2 (L^d)^2},$$
$$e_s = \frac{e_s^d}{L^d}, \quad e_c = \frac{e_c^d}{L^d}, \quad T = \frac{\overline{\eta}^d}{L^d} T^d,$$

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Table 1. *Physical constants of the steel-concrete beam T1PR* [5]

Physical constants	Value
L^d	3.50 m
A_1^d	3.00×10^{-2} m^2
I_1^d	9.00×10^{-6} m^4
ρ_1^d	73.19 kg m^{-1}
E_1^d	4.2292×10^{10} N m^{-2}
A_2^d	1.64×10^{-3} m^2
I_2^d	5.41×10^{-6} m^4
ρ_2^d	12.90 kg m^{-1}
E_2^d	2.1×10^{11} N m^{-2}
e_s^d	0.07 m
e_c^d	0.03 m

for $i = 1, 2$, where

$$\overline{\rho}^d = \frac{\rho_1^d + \rho_2^d}{2} \text{ [kg m}^{-1}\text{]}, \quad \overline{\eta}_i^d = \sqrt{\frac{E_i^d A_i^d}{\rho_i^d}} \text{ [m s}^{-1}\text{]}, \quad \overline{\eta}^d = \frac{\overline{\eta}_1^d + \overline{\eta}_2^d}{2} \text{ [m s}^{-1}\text{]},$$

where $\overline{\eta}_i^d$ is the velocity of the longitudinal elastic waves propagating through the ith beam.

We assume that our target coefficients k and μ for the damaged configuration of the composite beam are defined by

$$k(x) = \frac{n_p L^d K(L^d x)}{\overline{\rho}^d (\overline{\eta}^d)^2}, \quad \mu(x) = \frac{n_p L^d E_c(L^d x) A_c(L^d x)}{e_c^d \overline{\rho}^d (\overline{\eta}^d)^2}$$

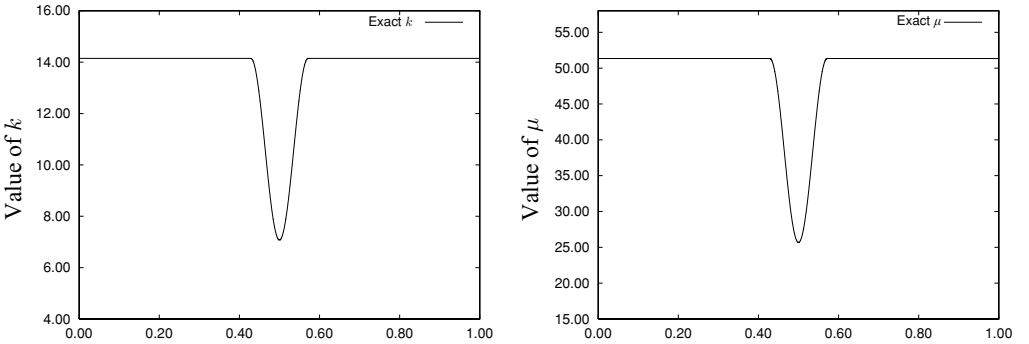
for $0 \leqslant x \leqslant L$ (see Figure 2), where

$$K(x^d) = \begin{cases} \{1.77525 + 0.59175 \cos 4\pi (x^d - 1.5)\} \times 10^8 \text{ [N m}^{-1}\text{]} \\ \qquad\qquad (1.5 < x^d < 2.0)\,, \\ 2.36700 \times 10^8 \text{ [N m}^{-1}\text{]} \qquad (\textit{otherwise}) \end{cases}$$

$$\frac{E_c(x^d) A_c(x^d)}{e_c^d} = \begin{cases} \{6.44250 + 2.14750 \cos 4\pi (x^d - 1.5)\} \times 10^8 \text{ [N m}^{-1}\text{]} \\ \qquad\qquad (1.5 < x^d < 2.0) \\ 8.59000 \times 10^8 \text{ [N m}^{-1}\text{]} \qquad (\textit{otherwise}) \end{cases}$$

for $0 \leqslant x^d \leqslant L^d$, and the number of connectors of the beam $n_p = 16$, see Dilenna & Morassi [5]. The present case corresponds to a composite beam with a localized damage at mid-span. In fact, it has been shown that real damages in the connection of a steel-concrete beam can be described as a rather abrupt reduction of the stiffness coefficients k and μ in the damaged region [5].

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(a) Exact k. (b) Exact μ.

FIGURE 2. Exact coefficients for the damaged beam.

Concerning the boundary data, in our experiments we assume to fix both the longitudinal displacements $u_i(0,t)$ and the rotations $v_{i,x}(0,t)$ of the left end of the beam, $i = 1, 2$, and to assign the transversal displacement $v_i(x = 0, t)$, $i = 1, 2$, at the same cross-section, see Figure 1. Therefore, the vector $\overline{U}(t)$ of the Dirichlet boundary data is given by

$$\overline{U}(t) = \left(0,\, 0,\, \frac{C_v B_5(t)}{L^d},\, \frac{C_v B_5(t)}{L^d},\, 0,\, 0\right)^T,$$

where $C_v = 1.5 \times 10^{-2}$ [m]. Here B_5 is a 5th order normalized B-spline function such that

$$B_5(t) = (\widetilde{t}_5 - \widetilde{t}_0) \sum_{j=0}^{5} \frac{(\widetilde{t}_j - t)_+^4}{\omega_6'(\widetilde{t}_j)},$$

where $\widetilde{t}_j = \overline{\eta}^d\,(8.0\,j \times 10^{-5})/L^d$ $(j = 0, 1, \ldots, 5)$, and

$$t_+ = \begin{cases} t & (t \geqslant 0) \\ 0 & (t < 0) \end{cases}, \quad \omega_6(t) = \prod_{j=0}^{5} (t - \widetilde{t}_j).$$

Then, $B_5 \in C^3(\mathbb{R})$ and $B_5 \equiv 0$ on $(-\infty, \widetilde{t}_0)$ and $(\widetilde{t}_5, +\infty)$, see Schumaker [12]. Therefore, the coefficients and the Dirichlet boundary data satisfy the assumption of Theorem 3.1.

The Neumann boundary data and the interior measured data are obtained by solving numerically the evolution problem (2.1) for the damaged beam with $\overline{U}$ given as above. To solve numerically the direct problem, we make use of the Newmark method for time integration, see, for example, Bathe & Wilson [2], and we introduce linear spline functions and cubic Hermite functions for approximating u_i and v_i in space, respectively, $i = 1, 2$, see Hilton & Owen [8]. Here, the intervals $[0, L]$ and $[0, T]$ are divided into 1200 and 7200 equally spaced sub-intervals, respectively. We use the numerical integration to get the mass and the stiffness matrices in this calculation, and then, we only use the values of coefficient functions at the quadrature points on each element. We denote by $\widehat{Q} = (\widehat{N}_1, \widehat{N}_2, \widehat{T}_1, \widehat{T}_2, \widehat{M}_1, \widehat{M}_2)$ and $\widehat{v}_i$, $i = 1, 2$, the calculated Neumann data at $x = 0$ and the transversal displacements of the beams, respectively.

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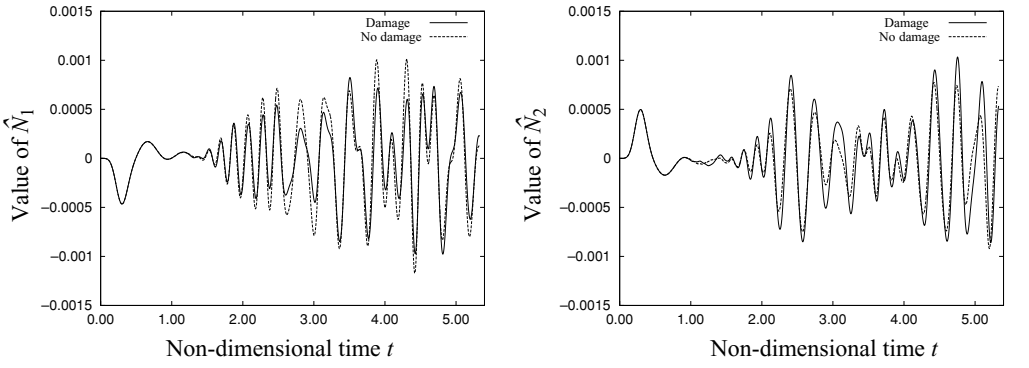


FIGURE 3. Calculated axial force $\widehat{N}_1$ and $\widehat{N}_2$ at $x = 0$.

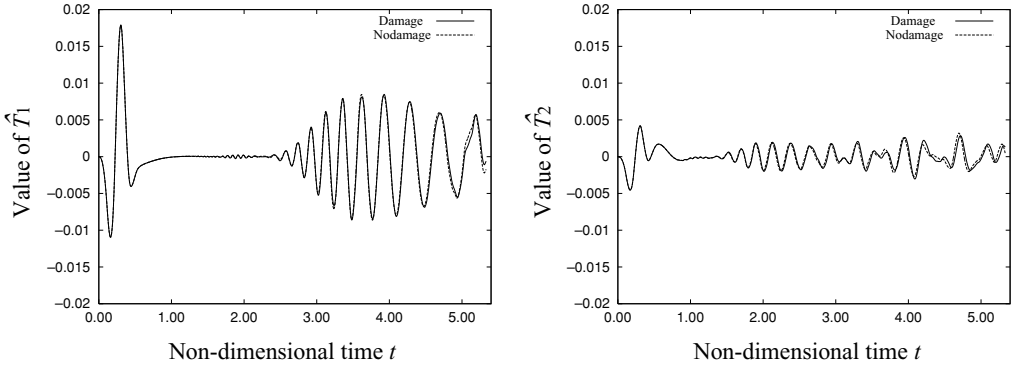


FIGURE 4. Calculated shear force $\widehat{T}_1$ and $\widehat{T}_2$ at $x = 0$.

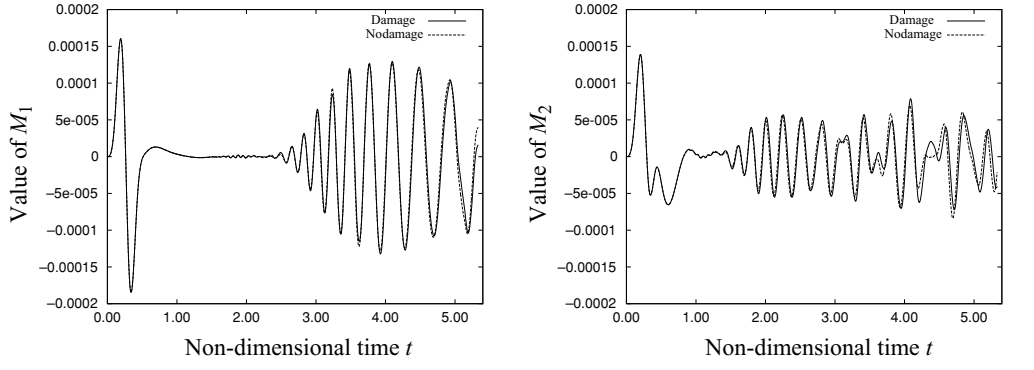


FIGURE 5. Calculated bending moment $\widehat{M}_1$ and $\widehat{M}_2$ at $x = 0$.

Figures 3, 4, and 5 show a comparison between the Neumann data evaluated at $x = 0$ for the damaged and the undamaged beam. The latter case corresponds to the solution of the evolution problem (2.1) with constant coefficients $\widetilde{k}_0$, $\widetilde{\mu}_0$ obtained by using $K \equiv 2.3670 \times 10^8$ Nm^{-1}, $E_c\, A_c / e_c^d \equiv 8.5900 \times 10^8$ Nm^{-1} in $[0, L]$, and with Dirichlet data $\overline{U}$ given at $x = 0$.

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To solve numerically the initial-boundary value problems in our inverse analysis, we also use the Newmark method for time integration with linear elements for approximating u_i and cubic Hermite elements for approximating v_i in space. The intervals $[0, L]$ and $[0, T]$ are divided coarsely into 600 and 3600 equally spaced, respectively. The values of updated coefficients on the quadrature points are employed to make the stiffness matrix on each step. Then, these quadrature points do not coincide with the points for generating the measured data. The second terms of the search directions are interpolated by the piecewise linear function before clipping and mollifying. We make use of the vertices and the quadrature points on each element to interpolate the second terms. To do approximately the mollifying, we use the Gauss-Legendre formula. Therefore, our target coefficients are identified approximately on a subspace of function space for calculating $\{\widehat{Q}, \widehat{v}_1, \widehat{v}_2\}$. Here we notice that the updated coefficients depend on the finite elements for the inverse analysis, because we employ the interpolation technique with the vertices and the quadrature points on the elements. On the other hand, the exact coefficients do no depend on the finite elements for calculating the measured data, because we only use its values. Hence, from the above settings, we can avoid an inverse crime for our calculation.

The upper bounds for the unknown coefficients appearing in (5.10) are supposed to be given by $\overline{k} \equiv \widetilde{k}_0$ and $\overline{\mu} \equiv \widetilde{\mu}_0$ in $[0, 1]$. The function for the mollifier is chosen as the normalized cubic B-spline function such that

$$s(x) = 4\sum_{j=0}^{4} \frac{(j-2-x)_+^3}{\omega_4'(j-2)}, \quad \omega_4(x) = \prod_{j=0}^{4}(x-j+2).$$

Then, we know that $\operatorname{supp} s = [-2, 2]$ and $s \in C^2(\boldsymbol{R})$, see Schumaker [12]. The parameter ϵ of our mollifier is picked as same value of the width of the finite elements in space, namely, $\epsilon = L/600$. Moreover, we fix the parameter of the convergence criterion as $\varepsilon = 1.0 \times 10^{-4}$. Finally, we employ the Armijo criterion [1] to get the step size α_l at every iteration.

We start by showing some numerical results when the interior data v_i, $i = 1, 2$, are measured in the whole interval $I = [0, L]$. We first assume that both the Neumann and the interior data are free of error, namely, $\overline{Q}(t) = \widehat{Q}(t)$ and $\overline{v}_i(x, t) = \widehat{v}_i(x, t)$ for $(x, t) \in [0, L] \times (0, T)$, $i = 1, 2$. The initial guess in the minimization procedure is set as $\widetilde{k}_0 \equiv \overline{k}$ and $\widetilde{\mu}_0 \equiv \overline{\mu}$ in $[0, L]$. After 108 steps the convergence criterion is satisfied and the iteration stops. Figures 6(a) and (b) show the graph of identified coefficients $\widetilde{k}_{108}$ and $\widetilde{\mu}_{108}$, respectively, and a good agreement with the exact values can be observed. These results seem to suggest that the minimization algorithm is effective in the ideal situation when the data are free of error.

Next, we perturb the Neumann data $\overline{Q}$ and the interior data $(\overline{v}_1, \overline{v}_2)$ by adding a random error of the Gaussian distribution with the mean value 0.0 and the variance $(\delta \cdot (\text{maximum value of each data})/100)^2$. We call this error $\delta\%$ measurement error. At first of this case, having fixed $\delta = 1.0\%$, we get the identified coefficients $\widetilde{k}_{99}$ and $\widetilde{\mu}_{99}$ shown in Figures 7(a) and (b), respectively. We notice that the identified coefficient $\widetilde{\mu}_{99}$ is in good agreement with the exact one. The accuracy of the identified coefficient k is less accurate, especially near $x = 0$ and around the real damaged area, but it is still acceptable. The above results suggest that the identification procedure is also effective when the data are affected by small measurement errors.

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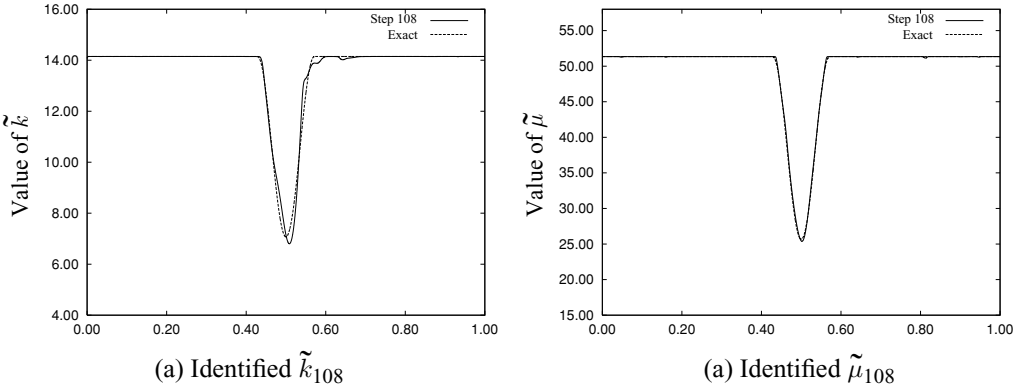


FIGURE 6. Calculated coefficients for $I = [0, L]$ without measurement error.

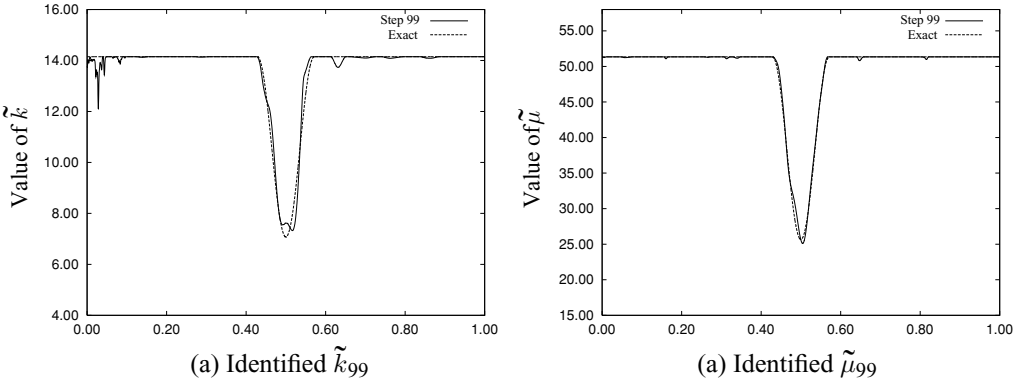


FIGURE 7. Calculated coefficients for $I = [0, L]$ with 1.0% measurement error.

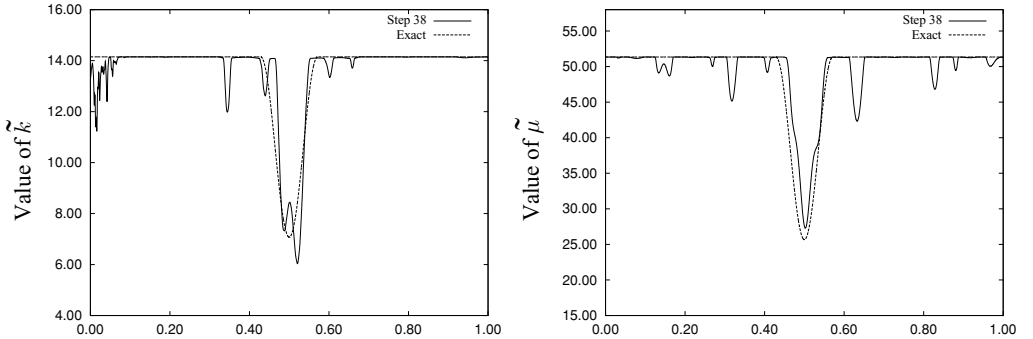
Second, having fixed $\delta = 5.0\%$, the identified coefficients $\widetilde{k}_{38}$ and $\widetilde{\mu}_{38}$ are obtained as shown in Figures 8(a) and (b), respectively. The reconstruction of the coefficients $\widetilde{k}$ and $\widetilde{\mu}$ is now less accurate. However, the results of identification can be still considered acceptable for a practical localization of damages in the connection of composite beams.

In several real applications, interior measurements are usually more difficult to make with respect to boundary measurements. Therefore, to discuss how the selection and the size of the interval I affect the results of identification, we have applied the variational algorithm for different choices of the interval I. Here, in particular, we present the results of identification for $I = [0.25, 0.75]$ and $I = [0, 0.25]$.

In the first case the interval I includes the damaged region of the beam. The initial guess for the coefficients and the data (free of error) are chosen as before. The identified values obtained after 131 iterations are shown in Figures 9(a) and (b). We can deduce that the accuracy of the identification is almost the same of the case in which the interior measurements were taken on the whole interval $[0, 1]$.

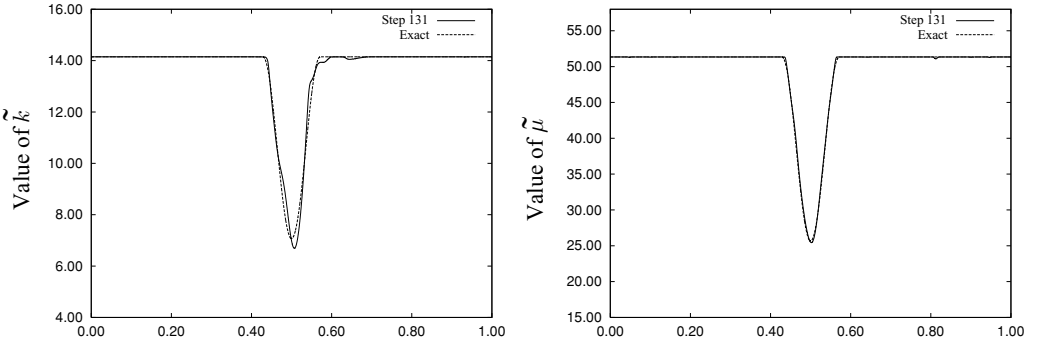
In the second case, the interval $I = [0, 0.25]$ does not include the damaged region in beam. By proceeding as before, after 173 iterations we obtain the optimal coefficients

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(a) Identified $\tilde{k}_{38}$ (a) Identified $\tilde{\mu}_{38}$

FIGURE 8. Calculated coefficients for $I = [0, L]$ with 5.0% measurement error.



(a) Identified $\tilde{k}_{131}$ (a) Identified $\tilde{\mu}_{131}$

FIGURE 9. Calculated coefficients for $I = [0.25, 0.75]$ without measurement error.

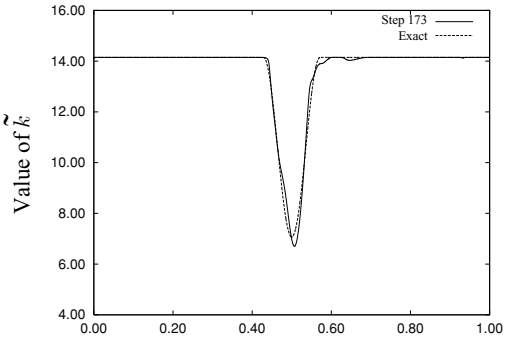
shown in Figures 10(a) and (b). These results seem to confirm that the results of the identification procedure are not so much affected by the choice of the interval I.

Finally, Figure 11 shows the graph of the functional value. We can notice that the selection of the interval I has an influence on the rate of decreasing of the cost function.

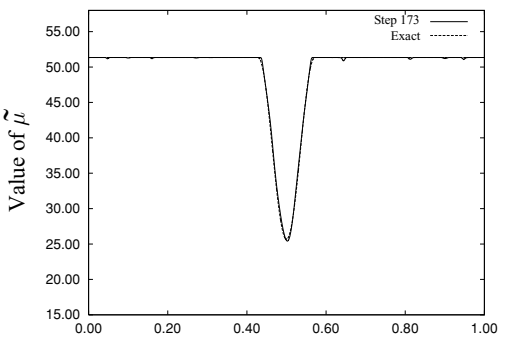
6 Concluding remarks

In this paper we have continued a line of research initiated in Morassi *et al.* [10] which aims to the identification of the stiffness coefficients of a steel-concrete composite beam from dynamical data. In Morassi *et al.* [10], a uniqueness result for the shearing stiffness of the connection was proved for the simpler situation in which the coupling between bending and longitudinal motions is neglected. Here, the full complete coupled system which includes two fourth order and two second order differential operators coupled on a term of lower order is examined. A variational procedure based on dynamic measurements taken at the boundary and, possibly, at some interior portions of the system is proposed for identifying the shearing and axial stiffness of the connection. A complete form of

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(a) Identified $\tilde{k}_{173}$



(a) Identified $\tilde{\mu}_{173}$

FIGURE 10. Calculated coefficients for $I = [0, 0.25]$ without measurement error.

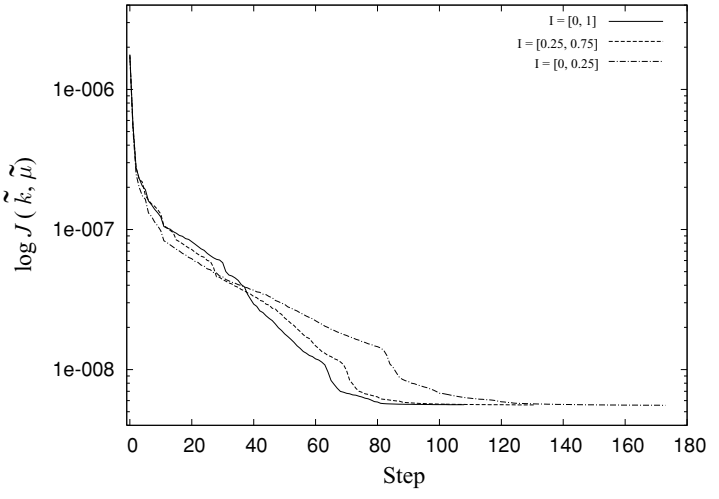


FIGURE 11. Functional value.

the Gateaux partial derivatives of the cost function is derived and used to implement a projected gradient method for solving iteratively the inverse problem. The method is tested on a composite beam with localized damage in the connection for exact data and for data perturbed by some random error. A concise account of numerical simulations has been presented in this paper. The results are encouraging and the method seems to be sufficiently stable with respect to errors on the data. However, real applications require a detailed investigation on the sensitivity of the procedure to the choices of various parameters affecting the identification, such as, for example, the selection of the initial point in the minimization process or the influence of boundary and interior measurements on the final results. This is an important direction of further investigation. Finally, we would like to remark that it remains an open question to establish what set of dynamic data is needed for the unique reconstruction of the stiffness coefficients for the

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full coupled system. New ideas and techniques are likely to be necessary for solving this inverse problem.

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References

[1] Armijo, L. (1966) Minimization of functions having Lipschitz continuous first partial derivatives, *Pacific J. Math.* **16**, 1–3.
[2] Bathe, K. J. & Wilson, E. L. (1976) *Numerical Methods in Finite Element Analysis*. Prentice-Hall.
[3] Belishev, M. I. & Ivanov, S. A. (2005) Recovering the parameters of the system of connected beams from dynamical boundary measurements. *Zapiski Nauch. Semin. POMI* (in Russian), **324**, 20–42.
[4] Daniel, J. W. (1971) *The Approximate Minimization of Functionals*, Prentice-Hall.
[5] Dilena, M. & Morassi, A. (2003) A damage analysis of steel-concrete composite beams via dynamic methods: Part II. Analytical models and damage detection. *J. Vibration & Control*, **9**(5), 529–565.
[6] Choi, C., Nakamura, G. & Shirota, K. (2005) Variational approach for identifying the coefficient of wave equation. Preprint.
[7] Johnson, R. P. (1994) *Composite Structures of Steel and Concrete*. Blackwell Scientific.
[8] Hilton, E. & Owen, D. R. J. (1979) *An Introduction to Finite Element Computations*. Pineridge Press.
[9] Lions, J. L. & Magenes, E. (1972) *Non-Homogeneous Boundary Value Problems and Applications*. Springer-Verlag, Volume I.
[10] Morassi, A., Nakamura, G. & Sini, M. (2005) An inverse dynamical problem for connected beams. *Euro. J. Appl. Math.* **16**, 83–109.
[11] Stolk, C. C. (2000) On the modeling and inversion of seismic data. PhD Thesis.
[12] Schumaker, L. L. (1981) *Spline functions: Basic theory*. Wiley.
[13] Wloka, J. *Partial Differential Equations*. Cambridge University Press.

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