Proc. Camb. Phil. Soc. (1964), **60**, 21
Printed in Great Britain

Groups in which normality is a transitive relation

By DEREK J. S. ROBINSON
Trinity College, Cambridge

(*Received* 19 *December* 1962)

1. Introduction and notation

1·1. *Introduction.* A group is said to have the *property T* or to be a *T-group* if every subnormal subgroup is normal. Thus the class of T-groups is just the class of all groups in which normality is a transitive relation. Finite T-groups have been studied by Best and Taussky (1), Gaschütz (4) and Zacher (11). Gaschütz has shown that if G is a finite soluble T-group and G/L is the unique maximal nilpotent quotient group of G, then G/L is Abelian or Hamiltonian and L is an Abelian group of odd order prime to $|G:L|$ ((4), Satz 1). Our aim is to study infinite T-groups and more especially infinite soluble T-groups with a view to extending Gaschütz's results. One of the simplest results on soluble T-groups is

Theorem 2·3·1. *Every soluble T-group is metabelian.*

Periodic soluble T-groups are considered in §4. In structure they resemble finite soluble T-groups and we are able to prove a generalization of Gaschütz's Satz 1 for these groups (Theorem 4·2·2). In §5 certain splitting properties of periodic soluble T-groups are obtained and it is shown how to construct and solve the isomorphism problem for a class of periodic soluble T-groups which includes all those that are countable.

Non-periodic soluble T-groups have a rather different structure. They are studied in §§3 and 4: we will quote one result of §3,

Theorem 3·3·1. *A finitely generated soluble T-group is finite or Abelian.*

1·2. *Notation.* We use the familiar notation $H \lhd K$ for 'H is a normal subgroup of the group K'. If X is a non-empty subset of a group G, $C_G(X)$ is the centralizer of X in G. If A is an Abelian group and m is any positive integer, A^m is the subgroup formed by all the mth powers of elements of A.

If x and y are group elements, x^y is the conjugate of x, $y^{-1}xy$, and $[x, y]$ is the commutator, $x^{-1}y^{-1}xy = x^{-1}x^y$. Let H be a subgroup and let X, Y be non-empty subsets of a group. By $[H, X]$ or $[X, H]$ we mean the subgroup generated by all commutators $[h, x]$, $h \in H$, $x \in X$; $[H, X, Y]$ is defined to be $[[H, X], Y]$. It is easy to show that $[H, X] \lhd \{H, [H, X]\}$ and that if $H = H^x$ ($\equiv x^{-1}Hx$), then $[H, \{x\}] = [H, x]$.

For each subgroup H and each non-empty subset X of a group we define X^H to be the subgroup generated by all the conjugates x^h, $h \in H$, $x \in X$. Thus X^H is the *normal closure* of X in $\{X, H\}$, i.e. the smallest normal subgroup of $\{X, H\}$ which contains X. Clearly $X^H = \{X, [H, X]\}$ and $X^H = X^{\{X,H\}} = \{X\}^H$.

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

If x is an element of a group and x has finite order, we will use $|x|$ to denote this order. Let Π be any set of primes: x is said to be a Π-*element* if all the prime divisors of $|x|$ belong to Π and a group G is said to be a Π-*group* if every element of G is a Π-element. For any group G, $\varpi(G)$ is to be the set of all primes p such that G has an element of order p.

Let G be a nilpotent group: then for any set of primes Π the set of all Π-elements of G forms a fully invariant subgroup of G which we denote by G_Π. Hence $\varpi(G)$ is the set of all primes p such that $G_p \neq 1$ and the torsion-subgroup of G is the direct product of the subgroups G_p where p runs through $\varpi(G)$. If Π is a subset of $\varpi(G)$, let Π' be the complementary set in $\varpi(G)$, i.e. the set $\varpi(G) - \Pi$. Thus the torsion-subgroup of G is the direct product of G_Π and $G_{\Pi'}$.

2. SOME ELEMENTARY PROPERTIES OF T-GROUPS

2·1. *T-groups.* By definition a group is a T-group if and only if every subnormal subgroup is normal. Thus if X is any non-empty subset of a T-group G and $X_1 = X^G$, then $X_1 = X^{X_1}$. For $X^{X_1} \lhd X_1 \lhd G$ and so $X^{X_1} \lhd G$, showing that $X^{X_1} = X_1$. It is interesting that T-groups can be characterized in terms of the normal closures of their cyclic subgroups.

LEMMA 2·1·1. *A group G is a T-group if and only if, for each $x \in G$, $X = x^X$ where $X = x^G$. Also if G is a T-group, then $x^G = x^{[G',G]}$ for all $x \in G$.*

Proof. We have just seen that the condition is necessary. Suppose that it is satisfied in the group G. To show that G is a T-group it will be enough if we prove that $H \lhd G$ always follows from $H \lhd K \lhd G$. Let $x \in H$ and $X = x^G$. Then $X \leqslant K$ and so $X = x^X \leqslant H$: hence $H \lhd G$.

Next let x be an element of the T-group G. $x^{[G',G]}$ is normal in $\{x, [G', G]\}$ and the latter is subnormal in G (since every subgroup of a nilpotent group is subnormal). Hence $x^{[G',G]} \lhd G$, from which it follows that $x^{[G',G]} = x^G$.

COROLLARY 1. *Let G be a T-group, $L = [G', G]$, $x \in G$ and $X_0 = \{x\} \cap G'$. Then $x^G \cap G' = X_0[L, x]$.*

For $x^G = x^L = \{x, [L, x]\} = \{x\}[L, x]$, since $[L, x] \lhd \{x, [L, x]\}$. Hence $x^G \cap G' = X_0[L, x]$.

COROLLARY 2. *A group G is a T-group if and only if every finite subset of G is contained in a T-subgroup of G. Hence T is a local property in the sense of McLain* (9).

Suppose that every finite subset of G is contained in a T-subgroup. Let $x \in G$ and let $X = x^G$. If $g \in G$, there is a T-subgroup H containing both x and g; let $X_1 = x^H$. Then $x^g \in x^H = x^{X_1} \leqslant x^X$. Since X can be generated by all the conjugates of x in G, $X \leqslant x^X$ and so $X = x^X$. By Lemma 2·1·1 G is a T-group.

It is an immediate consequence of the definition that normal subgroups and homomorphic images of T-groups are also T-groups. However, in general the same cannot be said of subgroups of T-groups. For example, every simple group is a T-group but not every subgroup of a simple group. (The alternating group of degree 5 contains a subgroup isomorphic with the alternating group of degree 4, which is not a T-group.)

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

Nor is it true that subgroups of soluble T-groups are T-groups—in this respect see Corollary 3 to Theorem 3·1·1 and also § 6·1.

2·2. *Nilpotent T-groups.* Since every subgroup of a nilpotent group is subnormal, a nilpotent T-group has each of its subgroups normal. Following Gaschütz we shall call such a group a *Dedekind group*. There is a well-known structure theorem for such groups according to which a Dedekind group is either Abelian or Hamiltonian, i.e. the direct product of a quaternion group of order 8 and a periodic Abelian group without elements of order 4. (See, for example, Zassenhaus (13).)

LEMMA 2·2·1. *Let G be a T-group.*

(i) *The lower central series of G terminates with $L = [G', G]$, i.e. $[L, G] = L$; also $|G' : L| = 1$ or 2.*

(ii) $L = \prod_{x \in G} X(x)'$ *where* $X(x) = x^G$.

Proof. Let K be a normal subgroup of G such that G/K is nilpotent; then G/K is a Dedekind group and its nilpotent class does not exceed 2. Hence $K \geqslant [G', G] = L$. This shows that $[L, G] = L$. If G/L is not Abelian, it is Hamiltonian and hence its derived group G'/L has order 2.

Let x be any element of G; then $X(x) \leqslant \{x, G'\}$ and hence $X(x)' \leqslant [G', G] = L$. Let $H = \prod_{x \in G} X(x)'$; then $H \leqslant L$. Also $H \lhd G$ and G/H has the property that each element commutes with all its conjugates. By a theorem of Levi (8), G/H is nilpotent and so $L \leqslant H$ by (i).

LEMMA 2·2·2. *Let G be a T-group and let $L = [G', G]$. Then $C = C_G(G')$ is the unique maximal nilpotent normal subgroup of G. Also $C = C_G(L)$ and C is a Dedekind group.*

Proof. First of all, in any group G if $K \lhd G$ and G/K is nilpotent, then $C_G(K)$ is nilpotent and normal in G. Now let G be a T-group, let N be a nilpotent normal subgroup of G and let $x \in N$. Then $\{x\} \lhd G$ and, since the automorphisms of a cyclic group commute, $C_G(x)$ contains G'. Hence $x \in C = C_G(G')$ and $N \leqslant C$. By the first remark of the proof, C is nilpotent and $C \lhd G$. Therefore C is the unique maximal nilpotent normal subgroup of G. Also $C_G(L)$ is nilpotent and normal in G, so $C_G(L) \leqslant C$. But $C \leqslant C_G(L)$, since $L \leqslant G'$, so we get the result stated.

2·3. *Soluble T-groups.* Most of our results are about soluble T-groups or *ST-groups*, as we shall call them. The next theorem is fundamental in the theory of ST-groups.

THEOREM 2·3·1. *Every ST-group is metabelian.*

Proof. Suppose that the theorem is false. Then, since a normal subgroup of a T-group is a T-group, there exists an ST-group G such that G'' is a non-trivial Abelian group. By Lemma 2·2·2 $[G', G''] = 1$ and so G' is nilpotent. By the same result $[G', G'] = 1$, contradicting our hypothesis that $G'' \neq 1$.

LEMMA 2·3·2. *Let G be a T-group, let $R = G''$ and let $S = C_G(R)$.*

(i) *The derived series of G terminates with R, i.e. $R = R'$.*

(ii) *S is the unique maximal soluble normal subgroup of G. Also S is metabelian and $[S, G', G'] = 1$.*

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

Proof. The first part follows at once from Theorem 2·3·1. Let N be any soluble normal subgroup of G. Then N' is Abelian and normal in G; from this it follows via Lemma 2·2·2 that $[N', G'] = 1$ and $[N, G'] \leqslant N'$. Hence $[N, G', G'] = 1$ which implies that

$$[N, G''] = 1 \quad \text{or} \quad N \leqslant S = C_G(R).$$

(See Zassenhaus (13), p. 84, Theorem 14.) On the other hand S is soluble and normal in G, so S is the unique maximal soluble normal subgroup of G.

2·4. Let G be an ST-group, let $L = [G', G]$ and $C = C_G(G')$. The subgroups L, G' and C play an important part in the theory of ST-groups. We have already seen that $C = C_G(L)$ and that C is a Dedekind group. Some further useful facts about these subgroups are given in the next lemma.

LEMMA 2·4·1. *Let G be an ST-group, $L = [G', G]$ and $C = C_G(G')$.*

(i) $L \leqslant G' \leqslant C$.

(ii) *If $G' = L$, C is Abelian.*

(iii) *L_2 is a divisible Abelian 2-group.*

Proof. (i) Since G is soluble, G' is Abelian and hence $G' \leqslant C$.

(ii) Since G has the property T, each subgroup of L/L^2 is normal in G/L^2 and hence L/L^2 is a central factor of G. Therefore $L = L^2$ by Lemma 2·2·1(i), from which it follows that $L_2 = L_2^2$ and L_2 is divisible.

(iii) Let $G' = L$ and suppose if possible that C is a Hamiltonian group. Then C' has order 2 and C_2/C' has exponent 2. Now $1 < C' \leqslant (G')_2 = L_2 \leqslant C_2$, so C_2 and therefore L_2 has finite exponent. But L_2 is divisible by (iii), so $L_2 = 1$ which is impossible. Hence C is Abelian.

A group G is said to be *supersoluble* if there exist normal subgroups G_i of G such that $1 = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_n = G$, each factor G_i/G_{i-1} is cyclic and n is finite. In his paper (4), Gaschütz pointed out that finite ST-groups are supersoluble. Since supersoluble groups are finitely generated, there is no hope of extending this result to infinite ST-groups. However, it is not difficult to show that any ST-group is necessarily locally supersoluble (i.e. every finite subset is contained in a supersoluble subgroup or, equivalently, every finitely generated subgroup is supersoluble).

LEMMA 2·4·2. *Every ST-group is locally supersoluble.*

Proof. Let H be a subgroup of the ST-group G and suppose that H has a finite set of generators $x_1, x_2, \ldots, x_n$. Let $c_{ij} = [x_i, x_j]$; since G' is Abelian, $\{c_{ij}\} \lhd G$ and H' can be generated by all the c_{ij}'s with $i < j = 1, 2, \ldots, n$. In the series

$$1 \leqslant \{c_{12}\} \leqslant \{c_{12}, c_{23}\} \leqslant \{c_{12}, c_{23}, c_{13}\} \leqslant \ldots \leqslant \{c_{ij} : i < j = 1, 2, \ldots, n\} = H'$$

each term is normal in G (so *a fortiori* in H) and each factor is cyclic. H/H' is finitely generated and Abelian, so the series can be continued from H' to H with terms each of which is normal in H and with cyclic factors. The length of the resulting series is finite and hence H is supersoluble.

COROLLARY. *In any ST-group the set of all elements of odd order forms a fully invariant subgroup of G.*

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

In (12) Zappa has shown that in a supersoluble group all the elements of odd order form a subgroup. It follows at once that this is true for locally supersoluble groups and so for ST-groups.

LEMMA 2·4·3. *Let G be a T-group and suppose that G has a normal subgroup N such that G/N is a non-periodic Abelian group.*

(i) *If N is a finite nilpotent group, then it is Abelian.*

(ii) *If N is a direct product of cyclic groups, then G is Abelian.*

Proof. (i) Since N is finite, there is an element x of infinite order which centralizes N. $\{x, N\}$ is nilpotent and also normal in G, because G/N is Abelian. Hence $\{x, N\}$ is Abelian, as a Dedekind group with elements of infinite order.

(ii) Suppose for the moment that N is cyclic, generated by an element a, say. By hypothesis there is an element of G/N which has infinite order, say xN. Since N is cyclic, there is a positive integer m such that x^m commutes with a; hence $\{x^m, a\}$ is an Abelian normal subgroup of G. By the T-property $\{x^m a\} \lhd G$ and so there exist integers l and n such that $a^x = a^l$ and $(x^m a)^x = (x^m a)^n$. These equations imply that $x^{m(n-1)} = a^{l-n} \in N$. Since xN has infinite order, this can only mean that $n = 1$ and $a^x = a$. G/N is a non-periodic Abelian group and hence it can be generated by elements of infinite order. It follows that N is central in G, G is nilpotent and hence G is Abelian in view of the presence of elements of infinite order. Now let N be a direct product of cyclic groups and let N_1 be any cyclic direct factor of N, $N = N_1 \times M$ say. Since G is a T-group, $N_1 \lhd G$ and $M \lhd G$. By the case which we have just settled, G/M is Abelian and so $[N_1, G] = 1$. Therefore $[N, G] = 1$, G is nilpotent and hence Abelian.

2·5. *Classifying ST-groups.* The presence of elements of infinite order in an ST-group may greatly affect the structure of the group, as we shall see. For this reason it is natural to treat separately periodic and non-periodic (non-Abelian) ST-groups. We divide the class of ST-groups into the following four mutually disjoint subclasses:

(1) the class of Abelian groups;

(2) the class of non-Abelian periodic ST-groups;

(3) the class of ST-groups of *type* 1.

An ST-group G is of type 1 if G is non-Abelian and $C = C_G(G')$ is non-periodic;

(4) the class of ST-groups of *type* 2.

An ST-group G is of type 2 if G is non-Abelian and non-periodic, and $C = C_G(G')$ is periodic.

3. ST-GROUPS OF TYPE 1

3·1. Let G be an ST-group of type 1 and let $C = C_G(G')$. By definition C contains an element x of infinite order and this means that C is Abelian, since it is a Dedekind group. Let y be any other element of C and let z be an element of G. Since $\{x\} \lhd G$ and $\{y\} \lhd G$, $x^z = x^\epsilon$ and $y^z = y^\delta$ for some integers δ, ϵ. If $\{x\} \cap \{y\} \neq 1$, $x^\eta = y^\zeta$ where the integers η and ζ are non-zero; hence $x^{\eta\epsilon} = y^{\zeta\delta} = x^{\eta\delta}$. Since x has infinite order, $\delta = \epsilon$ and $y^z = y^\epsilon$. Suppose that $\{x\} \cap \{y\} = 1$; $\{xy\} \lhd G$ and hence, for some integer θ, $(xy)^z = (xy)^\theta$. Therefore $x^\epsilon y^\delta = x^\theta y^\theta$ (since C is Abelian), and $x^\epsilon = x^\theta$, $y^\delta = y^\theta$. It follows that $\epsilon = \theta$ and $y^z = y^\epsilon$. Thus we have shown that $y^z = y^\epsilon$ for all elements y in C. Now G is non-Abelian by definition, so we can choose z to be outside C. z does

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

not centralize C, for if it did, it would centralize G' and so belong to $C = C_G(G')$. Hence $\epsilon \neq 1$: now x has infinite order, so $\epsilon = -1$ and $y^z = y^{-1}$ for all $y \in C$. Since $G' \leqslant C$, $|G:C| = 2$ and $G = \{z, C\}$. z^2 belongs to C and is transformed by z into its inverse: hence $z^4 = 1$. Also $[C, z] = C^2$ and $[C^2, z] = C^4$ from which it follows that $z^G = \{z, C^2\}$ and $z^{\{z, C^2\}} = \{z, C^4\}$, giving $\{z, C^2\} = \{z, C^4\}$ by the T-property. Taking intersections with C we get

$$\{z^2, C^2\} = \{z^2, C^4\}.$$

THEOREM 3·1·1. *Let the group G have an Abelian subgroup C and an element $z \notin C$ such that*

(i) $|G:C| = 2$,

(ii) *for all $c \in C$, $c^z = c^{-1}$,*

(iii) $\{z^2, C^2\} = \{z^2, C^4\}$.

Then G is an ST-group; if C has an element of infinite order, G is an ST-group of type 1. *Conversely every ST-group of type* 1 *has this structure.*

Proof. By the preceding discussion every ST-group of type 1 has a subgroup C satisfying the conditions of the theorem. Let G be a group with an Abelian subgroup C satisfying (i), (ii) and (iii). We shall show that G is a T-group. Let $H \lhd K \lhd G$: every subgroup of C is normal in G, so we may assume that H contains an element $y = zc$, $c \in C$. Then K contains $[C, zc] = [C, z] = C^2$ and H contains $[C^2, zc] = C^4$, by (ii). Hence H contains $\{(zc)^2, C^4\}$: but $(zc)^2 = zczc = z^2$, by (ii), and so $H \geqslant \{z^2, C^2\}$ by (iii). Now $G' = [C, x] = C^2$, so $H \lhd G$. It is of course clear that G is soluble: also $C = C_G(G')$, showing that G is an ST-group of type 1, if C is not periodic.

COROLLARY 1. *Torsion-free ST-groups are Abelian.*

COROLLARY 2. *In an ST-group of type* 1 *the set of all elements of finite order is not a subgroup.*

Indeed by Theorem 3·1·1 an ST-group of type 1 can be generated by elements of orders 2 or 4.

COROLLARY 3. *Each ST-group of type* 1 *has a subgroup which is not a T-group.*

Let $G = \{z, C_G(G')\}$ be an ST-group of type 1 and let c be an element of infinite order in $C = C_G(G')$. The subgroup $H = \{z, c\}$ is not a T-group, since

$$\{z, c^4\} \lhd \{z, c^2\} = \{z, c^4\}^H \lhd \{z, c\} = H \quad \text{but} \quad \{z, c^4\} \neq \{z, c^2\}.$$

3·2. *Constructing ST-groups of type* 1. Let $C_\infty = \{c_1, c_2, \ldots\}$ be a group of the type C_{2^∞} with the usual relations $c_i^2 = c_{i-1}$ $(i > 0)$ and $c_0 = 1$. Let $C_i = \{c_i\}$. We define D_i, for $i = 0, 1, 2, \infty$, to be the group generated by C_i and an element z such that $c^z = c^{-1}$ for all $c \in C_i$ and $z^2 = 1, c_1, c_1, c_1$ respectively for $i = 0, 1, 2, \infty$. Thus D_0 and D_1 are cyclic groups of orders 2 and 4 while D_2 is a quaternion group of order 8. Let A be an Abelian group such that $A = A^2$ and A is non-trivial if $i = 0, 1$ or 2. Let E_i be the split extension of A by D_i defined by $[C_i, A] = 1$ and $a^z = a^{-1}$ for all $a \in A$. Finally, let B be a group of exponent 2 at most and dimension r (which may be infinite). The group $G = G(i, A, r)$ is defined by

$$G = G(i, A, r) = E_i \times B. \tag{I}$$

From the definition it follows easily that $C = C_G(G') = C_i \times A \times B$, $G = \{z, C\}$ and

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

$|G:C| = 2$. Also for all $c \in C$, $c^z = c^{-1}$ and $\{z^2, C^2\} = \{z^2, C^4\}$. By Theorem 3·1·1 G is an ST-group and, if A is non-periodic, G is of type 1. G cannot be nilpotent by our convention that $A \neq 1$ if $i = 0$, 1 or 2.

THEOREM 3·2·1. (i) *If $G = G(i, A, r)$ is a group of the type defined in* (I), *then G is a non-nilpotent ST-group. If A is not periodic, G is an ST-group of type* 1.

(ii) *Every ST-group of type* 1 *is isomorphic with some $G(i, A, r)$.*

(iii) *$G(i_1, A_1, r_1) \cong G(i_2, A_2, r_2)$ if and only if $i_1 = i_2$, $A_1 \cong A_2$ and $r_1 = r_2$. Thus (i, A, r) is a complete set of invariants for the group $G(i, A, r)$ determining it to within isomorphism.*

Proof. It remains only to establish (ii) and (iii).

Let G be any ST-group of type 1 and let $G = \{z, C\}$, $C = C_G(G')$. Then by Theorem 3·1·1

$$\{z^2, C^2\} = \{z^2, C^4\}. \tag{II}$$

At this point we need the following simple result on Abelian groups.

LEMMA 3·2·2. *Let A be an Abelian group such that $A^2 = A^4$. Then $A = B \times D$ where $B^2 = B$ and $D^2 = 1$. If $A_i = B_i \times D_i$ $(i = 1, 2)$, are two groups of this type, $A_1 \cong A_2$ if and only if $B_1 \cong B_2$ and $D_1 \cong D_2$.*

Proof. Let $a \in A$; then there is an element $b \in A^2$ such that $a^2 = b^2$. $(ab^{-1})^2 = 1$ and so $A = A^2 . A(2)$ where $A(2)$ is the subgroup of all elements x of A such that $x^2 = 1$. Let $B = A^2$ and let $A(2) = (B \cap A(2)) \times D$. Then $A = B \times D$, $B^2 = B$ and $D^2 = 1$. Let $A_1 \cong A_2$ where $A_i = B_i \times D_i$. Since $B_i = A_i^2$, $B_1 \cong B_2$ and $D_1 \cong D_2$.

We now return to the proof of Theorem 3·2·1. Suppose that $z^2 = 1$; then (II) becomes $C^2 = C^4$ and so we can write $C = A \times B$ by Lemma 3·2·2, where $A^2 = A$ and $B^2 = 1$. $B \lhd G$ and $\{z\} \cap C = 1$; hence $G = \{z, A\} \times B \cong G(0, A, r)$, where $r = \dim B$. Suppose next that $z^2 \neq 1$ and z^2 belongs to some subgroup C_∞ of C, of the type $C_{2\infty}$. Then we can write $C = C_\infty \times C^*$, since C_∞ is divisible, and deduce from (II) that $(C^*)^2 = (C^*)^4$. By Lemma 3·2·2, $C^* = A \times B$ where $A^2 = A$ and $B^2 = 1$. It follows that $C = C_\infty \times A \times B$ and $G = \{z, C_\infty \times A\} \times B \cong G(\infty, A, r)$ where $r = \dim B$. Assume that z^2 is not contained in any subgroup of C of the type $C_{2\infty}$. By (II) and Lemma 3·2·2 we can write

$$C/\{z^2\} = \bar{A}/\{z^2\} \times \bar{B}/\{z^2\},$$

where the first factor is divisible by 2 and the second has exponent at most 2. Let $A = \bar{A}^2$; now $\bar{A} = \bar{A}^2\{z^2\}$ and $z^4 = 1$, giving $A = A^2$. Hence z^2 does not belong to A and $\bar{A} = \{z^2\} \times A$. If $\bar{B}$ has exponent 2, $\bar{B} = \{z^2\} \times B$, $C = C_1 \times A \times B$, where $C_1 = \{z^2\}$, and $G = \{z, C_1 \times A\} \times B \cong G(1, A, r)$, $(r = \dim B)$. Finally, if $\bar{B}$ does not have exponent 2, it contains an element c_2 of order 4 such that $c_2^2 = z^2$. In this case $\bar{B} = C_2 \times B$ and $C = C_2 \times A \times B$ where $C_2 = \{c_2\}$ and $B^2 = 1$. $G = \{z, C_2 \times A\} \times B \cong G(2, A, r)$, $(r = \dim B)$.

Let θ be an isomorphism of $G_1 = G(i_1, A_1, r_1)$ onto $G_2 = G(i_2, A_2, r_2)$. Let

$$G_\alpha = \{z_\alpha, C(\alpha)\} \quad \text{and} \quad C(\alpha) = C_{G_\alpha}(G'_\alpha) = C_{i_\alpha} \times A_\alpha \times B_\alpha, \quad \alpha = 1, 2;$$

then θ maps $C(1)$ onto $C(2)$. Also θ maps z_1^2 onto z_2^2; for if,

$$z_1 = z_2 c_2, \quad c_2 \in C(2), \quad (z_1^2)^\theta = (z_2 c_2)^2 = z_2^2.$$

It follows at once that $i_1 = i_2$ and, by an easy examination of the cases 0, 1, 2, ∞, that $A_1 \cong A_2$ and $r_1 = r_2$.

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

3·3. *Finitely generated ST-groups.* We have obtained sufficient information about ST-groups to survey the class of finitely generated ST-groups.

THEOREM 3·3·1. *A finitely generated ST-group is finite or Abelian.*

Proof. Let G be a finitely generated ST-group and suppose that G is not Abelian. By Lemma 2·4·2 G is supersoluble and hence every subgroup of G is finitely generated. Let $C = C_G(G')$ and suppose that C is not periodic; then G is an ST-group of type 1. Let $G = \{z, C\}$ and let T be the torsion subgroup of C. Then $T \neq C$ and C/T is a free Abelian group. Now $z^2 \in T$, since $z^4 = 1$, and hence by Theorem 3·1·1 $(C/T)^2 = (C/T)^4$, which is not consistent with C/T being free Abelian of rank > 0. Hence C must be periodic and therefore finite, as a finitely generated periodic nilpotent group. But $G' \leqslant C = C_G(G')$, showing that $|G:C|$ is finite. Hence G is finite.

4. POWER AUTOMORPHISMS OF ABELIAN GROUPS

4·1. If A is an Abelian normal subgroup of a T-group G, each element of G induces by transformation an automorphism in A which leaves invariant every subgroup of A. Such a subgroup-preserving automorphism of an Abelian group will be called a *power automorphism*, the motivation for this terminology being

LEMMA 4·1·1. *Let A be an Abelian group and let σ be a power automorphism of A.*

(i) *If A is non-periodic, σ is either the identity automorphism or the automorphism in which every element is mapped onto its inverse.*

(ii) *If A is periodic, then for each positive integer r and each prime $p \in \pi(A)$ there exists an integer $m_{p,r}$ such that $0 < m_{p,r} < p^r$, $(m_{p,r}, p) = 1$, $m_{p,r} \equiv m_{p,r-1} \bmod p^{r-1}$ and for all elements a of A_p with order p^r at most, $a^\sigma = a^{m_{p,r}}$.*

Conversely any mapping σ of A into itself which satisfies (i) *or* (ii) *is a power automorphism of A.*

Proof. If A is non-periodic, the assertion is true by the arguments used at the beginning of §3·1. Let A be periodic, let $p \in \pi(A)$ and let $r > 0$. We show that there is an integer $m_{p,r}$ with the properties given in (ii). To do this let a be an element of A_p with order p^r and let a' be another element of A_p with order not exceeding p^r. Let $a^\sigma = a^m$ where the integer $m = m_{p,r}$ satisfies $0 < m < p^r$ and $(m, p) = 1$. Now

$$\{a, a'\} = \{a\} \times \{b\}$$

for some element b of order at most p^r. If $b^\sigma = b^l$ and $(ab)^\sigma = (ab)^n$, then $a^{n-m} = 1 = b^{l-n}$, since $\{a\} \cap \{b\} = 1$, so $b^\sigma = b^l = b^n = b^m$, since $|b|$ divides p^r. But $a' = a^\alpha b^\beta$ for some integers α, β and $(a')^\sigma = a^{\alpha m} b^{\beta m} = (a')^m$ as required. Also a^p has order p^{r-1}, so

$$(a^p)^\sigma = a^{pm_{p,r}} = a^{pm_{p,r-1}},$$

giving $m_{p,r} \equiv m_{p,r-1} \bmod p^{r-1}$, $(r > 1)$.

The converse of the result just proved is immediate, for the given conditions imply that σ is an automorphism which leaves invariant each cyclic subgroup of A. This concludes the proof.

Let $\Pi(A)$ denote the set of all power automorphisms of the Abelian group A. It is easy to show that $\Pi(A)$ is a subgroup of the centre of the group Aut (A) of all

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

automorphisms of A; thus in particular $\Pi(A)$ is Abelian. If A is non-periodic, $\Pi(A)$ has order 2, and if A is periodic, $\Pi(A)$ is isomorphic with the Cartesian product of the groups $\Pi(A_p)$ where p runs through the set $\varpi(A)$.

Let A be an Abelian p-group and let $\sigma \in \Pi(A)$. Then if $a \in A$ and $|a| \leqslant p^r$, $a^\sigma = a^{m_{p,r}}$ where $(m_{p,r}, p) = 1$, $0 < m_{p,r} < p^r$ and $m_{p,r} \equiv m_{p,r-1} \bmod p^{r-1}$. These conditions imply that for $r = 1, 2, \ldots$

$$m_{p,r} = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots + \alpha_{r-1} p^{r-1},$$

where $0 < \alpha_0 < p$ and $0 \leqslant \alpha_i < p$, the α's being integral. Hence to σ there corresponds a p-adic unit

$$\alpha = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots.$$

Conversely given a p-adic unit α, there is a unique power automorphism σ of A determined by α; this is defined by $a^\sigma = a^\alpha \equiv a^{\alpha_0 + \alpha_1 p + \ldots + \alpha_{r-1} p^{r-1}}$ where $a \in A$ and $|a| = p^r$. α is uniquely determined by σ if A has infinite exponent and, if A has exponent p^r, α is determined modulo p^r by σ. Hence the following result has been established.

LEMMA 4·1·2. *Let A be an Abelian group.*

(i) *If A is not periodic, $\Pi(A)$ has order 2 and is generated by the automorphism $a \to a^{-1}$.*

(ii) *If A is periodic, $\Pi(A)$ is isomorphic with the Cartesian product of the groups $\Pi(A_p)$ where p runs over $\varpi(A)$. If A_p has infinite exponent, $\Pi(A_p)$ is isomorphic with the multiplicative group $X_{p,\infty}$ of all p-adic units. If A_p has finite exponent p^n, $\Pi(A_p)$ is isomorphic with $X_{p,n}$ the multiplicative group of all prime residue classes modulo p^n.*

4·1·3. We now recall some properties of the groups $X_{p,n}$ and $X_{p,\infty}$. Details may be found in Hasse (5).

(*a*) If p is odd, $X_{p,n}$ is cyclic of order $(p-1)p^{n-1}$, and $X_{p,\infty}$ is the direct product of a cyclic group of order $p-1$ and a group $F(p)$, consisting of all $\alpha \equiv 1 \bmod p$, which is isomorphic with the additive group of all p-adic integers.

(*b*) $X_{2,1}$ is the unit group: if $n > 1$, $X_{2,n}$ is the direct product of a group of order 2 (generated by -1) and a cyclic group of order 2^{n-2} (consisting of all $\alpha \equiv 1 \bmod 4$).

$X_{2,\infty}$ is the direct product of a group of order 2 (generated by $-1 = 1 + 2 + 2^2 + \ldots$) and a group $F(2)$ which is isomorphic with the additive group of 2-adic integers.

For all p, $F(p)$ is torsion-free, so the only available periodic power automorphisms of an Abelian p-group of infinite exponent are those from the cyclic group of order $p-1$, if p is odd, and order 2, if $p = 2$.

4·1·4. We now consider some restrictions on cardinality in ST-groups. Let G be an ST-group, let $L = [G', G]$ and $C = C_G(G')$. If C is non-periodic, then $|G:C| = 1$ or 2. Let C be periodic; then since $C = C_G(L)$ (Lemma 2·2·2), G/C is isomorphic with a group of power automorphisms of L. By Lemma 4·1·2(ii)

$$|G:C| \leqslant (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}.$$

If G is periodic, $G/C(p)$ is finite, where $C(p) = C_G(L_p)$ by 4·1·3. If also $\varpi(L)$ is finite, then $|G:C|$ is finite. However, in general for an ST-group G, G/C need not be countable, even if G is periodic.

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

For example, let $A = A_3 \times A_5 \times \ldots \times A_p \times \ldots$, where p runs over all odd primes and A_p is a group of order p. For each set Π of odd primes we define an automorphism $x(\Pi)$ of A by the rules $a^{x(\Pi)} = a^{-1}$ if $p \in \Pi$ and $a^{x(\Pi)} = a$ otherwise, where $a \in A_p$. Let X be the set of all $x(\Pi)$. For any two sets of odd primes Π_1 and Π_2,

$$x(\Pi_1)\, x(\Pi_2) = x(\Pi_3)$$

where $\Pi_3 = \Pi_1 \cup \Pi_2 - \Pi_1 \cap \Pi_2$, showing that X is an elementary Abelian 2-group. Clearly $|X| = 2^{\aleph_0}$. Now define G to be the split extension of A by X. G is a periodic soluble group, $G' = A^2 = A$ and $C = C_G(G') = A$. Thus $|G:C| = 2^{\aleph_0}$. Also G is a T-group: this can easily be proved directly, but it is also an immediate consequence of Lemma 5·2·3 below, since $\varpi(A) \cap \varpi(G/A)$ is empty.

4·2. *Periodic ST-groups.* We begin by noting

LEMMA 4·2·1. *Let G be a soluble p-group with the property T.*

(i) *If p is odd, G is Abelian.*

(ii) *If $p = 2$, G is either a Dedekind 2-group or isomorphic with a group of the type $G(i, A, r)$ defined by* (I) *in* §3·2.

Proof. Let G have exponent p^n. Any subgroup of G' which has exponent p lies in the centre of G, since all its subgroups are normal in G and G is a p-group. Hence the $(n+2)$th term of the lower central series of G is trivial and G is nilpotent. If p is odd, G is Abelian and if $p = 2$, G is a Dedekind 2-group.

Let G' have infinite exponent. If p is odd, then $|G:C_G(G')|$ divides $p-1$, by §4·1·3, and hence $G = C_G(G')$, since G is a p-group; thus G is nilpotent and hence G is Abelian. Suppose that $p = 2$. $C = C_G(G')$ is Abelian, because it is a Dedekind 2-group of infinite exponent. If G is not Abelian, then by §4·1·3 $c^z = c^{-1}$ for all $c \in C$ and $z \in G - C$. Hence in this case $G = \{z, C\}$, $z^4 = 1$ and $\{z^2, C^2\} = \{z^2, C^4\}$. It is now clear that $G \cong G(i, A, r)$ where $i = 0$, 1, 2 or ∞, r is some cardinal and A is a divisible Abelian 2-group, non-trivial if $i = 0$, 1 or 2. Conversely every $G(i, A, r)$ of this description is a soluble 2-group with the property T.

We come now to our main result on periodic ST-groups.

THEOREM 4·2·2. *Let G be a periodic ST-group, let $L = [G', G]$ and let $C(p) = C_G(L_p)$ where $p \in \varpi(L)$.*

(i) *Let p be an odd prime in the set $\varpi(L)$. Then p does not belong to the set $\varpi(G/L)$, $G/C(p)$ is a non-trivial cyclic group whose order divides $p-1$ and each element of G outside $C(p)$ commutes with no elements of L_p other than the identity.*

(ii) *If 2 belongs to $\varpi(L)$, then 2 also belongs to $\varpi(G/L)$. $G/C(2)$ has order 2, in this case, and each element of G outside $C(2)$ transforms every element of L_2 into its inverse.*

Proof. Let p be an odd prime in $\varpi(L)$ and let $M(p)/L$ be the p-component of G/L. By Lemma 4·2·1 $M(p)/L_{p'}$ is Abelian and this implies that $M(p) \leqslant C(p)$. By results in §4·1·3 $G/C(p)$ is a cyclic group of order dividing $p-1$. Also $C(p)/L_{p'}$ is nilpotent and $L_{p'} < L$, so $C(p) < G$. Let x be an element in the set $G - C(p)$. Then x induces in L_p a power automorphism $a \to a^x = a^w$ of order dividing $p-1$, where w is a p-adic unit $\neq 1$. From the structure of the groups $X_{p,n}$ and $X_{p,\infty}$ (see §4·1·3), it follows that $w \not\equiv 1 \bmod p$ and x commutes with no element of L_p other than the identity. Hence

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

the map $a \to [a, x]$ is an automorphism of $M_p/L_{p'}$ and $[M_p, x]L_{p'} = M_p$. But M_p/L lies in the centre of G/L, so $M_p = L$ and $p \notin \varpi(G/L)$ as required.

Let 2 belong to $\varpi(L)$; then L_2 is a direct product of groups of the type C_{2^∞}, by Lemma 2·4·1 and $|G:C(2)| = 2$, by results in §4·1·3. Since $L \leqslant C(2)$, 2 belongs to $\varpi(G/L)$. Also each element in $G - C(2)$ induces in L_2 the automorphism $a \to a^{-1}$.

COROLLARY. If L has finite exponent, $\varpi(L) \cap \varpi(G/L)$ is empty. In particular this is true if G is finite (when Theorem 4·2·2 reduces to Gaschütz's Satz 1, (4)).

4·3. *ST-groups of type* 2. We recall that an ST-group G is of type 2 if it contains elements of infinite order and $C_G(G')$ is periodic. It follows at once from the definition that in an ST-group of type 2 the set of all elements of finite order is a subgroup.

THEOREM 4·3·1. *Let G be an ST-group of type 2, let $C = C_G(G')$ and let P be the subgroup of all elements of finite order.*

(i) *$[G', G] = G'$ and C is Abelian: also $[P', P] = P'$.*

(ii) *G' is a divisible Abelian group and $C = G' \times B$ where B is contained in the centre of G. P' is a divisible Abelian group and $G' = P' \times D$ where D is contained in the centre of P.*

(iii) *No element of infinite order centralizes any non-trivial primary component of G'.*

(iv) *If $p \in \varpi(G')$, B_p has finite exponent $p^{n(p)}$. If $x \in G$, x induces in C_p a power automorphism $c \to c^\alpha$ where the p-adic unit α satisfies $\alpha \equiv 1 \bmod p^{n(p)}$.*

Proof. (i) Let $L = [G', G]$. Since $L \leqslant C \leqslant P < G$, G/L is a Dedekind group containing elements of infinite order. Hence G/L is Abelian and $G' = L$: by Lemma 2·4·1 C is Abelian. Suppose that $[p', P] \neq P'$. Then, since P has the property T, there is a normal subgroup N of G contained in P such that P/N is a quaternion group. But G/P is Abelian and non-periodic and it follows via Lemma 2·4·3 (i) that P/N is Abelian. By this contradiction $P' = [P', P]$.

(ii) Suppose that G' is not divisible and there is a prime $p \in \varpi(G')$ such that G' has a subgroup H of index p. G' is Abelian, so $H \lhd G$. If x is any element of infinite order, Lemma 2·4·3 (ii) shows that $\{x, G'\}/H$ is Abelian and x centralizes G'/H. Now the set of all elements of finite order in G is a subgroup and hence G can be generated by elements of infinite order. The factor G'/H must be central in G and hence G/H is nilpotent; therefore $[G', G] \leqslant H < G'$, which contradicts (i). Hence G' is divisible and we may write $C = G' \times B$. Now $B \lhd G$, so $[B, G] \leqslant B \cap G' = 1$ and B lies in the centre of G.

P is a periodic ST-group and, by (i), $P' = [P', P]$. Theorem 4·2·2 shows that the set $\varpi((P')_{2'}) \cap \varpi(P/(P')_{2'})$ is empty. Since $P' \leqslant G' \leqslant P$, this implies that $(P')_{2'}$ is a direct factor of the divisible group G' and hence that $(P')_{2'}$ is divisible. On the other hand $(P')_2$ is known to be divisible, from Lemma 2·4·1, so $P' = (P')_{2'} \times (P')_2$ is divisible. Let $G' = P' \times D$: then $D \lhd P$, $[D, P] \leqslant D \cap P' = 1$ and D lies in the centre of P.

(iii) Let x be an element of infinite order and let $[(G')_p, x] = 1$ for some $p \in \varpi(G')$; then $G/(G')_{p'}$ contains an element of infinite order which centralizes its derived group $G'/(G')_{p'}$. Now an ST-group which has this property is either Abelian or of type 1, and $G/(G')_{p'}$ cannot be an ST-group of type 1, since the set of all its elements of finite order

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

is a subgroup and no ST-group of type 1 has this property, by Corollary 2 to Theorem 3·1·1. We conclude that $G/(G')_{p'}$ is Abelian and $(G')_p = 1$ which is a contradiction.

(iv) Let $p \in \varpi(G')$ and let y be an element of G with infinite order. Suppose that y induces in C_p the power automorphism $a \to a^\alpha$. Since B lies in the centre of G, the p-adic unit α has to satisfy $\alpha \equiv 1 \bmod |b|$ for every b in B_p. By (iii) $\alpha \neq 1$, so B_p cannot have infinite exponent. Let B_p have exponent $p^{n(p)}$ and let x be any element of G. Then if x induces in C_p the power automorphism $a \to a_w$, $w \equiv 1 \bmod p^{n(p)}$.

5. Splitting properties of periodic ST-groups

5·1. Let G be a periodic ST-group and let $L = [G', G]$. If G is finite, the set $\varpi(L) \cap \varpi(G/L)$ is empty (by Theorem 4·2·2), and hence by a well-known theorem associated with the names of Schur and Zassenhaus (13), G splits over L. This fact has been pointed out by Gaschütz. However, if G is infinite, G need not split over L, as can easily be seen from the example of the group $G(\infty, 1, 0)$. A less obvious question is whether G splits over $L_{2'}$, the odd component of L. A periodic ST-group which splits over the odd component of the third term of its lower central series will be said to have the *splitting property*.

To make much progress in the problem of deciding which periodic ST-groups have the splitting property, we are going to need some sort of analogue of the Schur–Zassenhaus theorem for periodic soluble groups.

LEMMA 5·1·1. *Let N be a normal subgroup of the locally finite group G and suppose that*

(i) *$\varpi(N) \cap \varpi(G/N)$ is empty, and*

(ii) *G/N is countable.*

Then G splits over N.

The proof of the lemma depends on work of Schur and Zassenhaus ((13), p. 162) and on the recent result of Feit and Thompson on the solubility of groups of odd order (2). The information needed is summarized as follows.

(S) *If G is a finite group and N is a normal subgroup of G whose order is prime to its index, then G splits over N, all complements of N in G are conjugate and every $\varpi(G/N)$-subgroup of G is contained in some complement of N.*

Proof of Lemma 5·1·1. By condition (ii) there is a countable set of elements $x_1, x_2, \ldots$ such that $G = \{x_1, x_2, \ldots, N\}$. Let X_n be the finite subgroup generated by $x_1, x_2, \ldots, x_n$. Then by (S) X_n splits over $X_n \cap N$ and all complements of $X_n \cap N$ in X_n are conjugate. By (S) again, it is possible to choose complements Y_n of $X_n \cap N$ in X_n such that $Y_1 \leqslant Y_2 \leqslant \ldots$. Let $Y = \bigcup_{i=1}^{\infty} Y_i$: then Y is the desired complement of N in G.

Remark 1. The lemma has been stated in greater generality than we need: as far as we are concerned the restrictive condition is (ii). However, in its present form the lemma may be false if (ii) is omitted—see the paper of Kovacs, Neumann and de Vries (6), Theorem 4·4, for an example. (I wish to thank the referee for drawing my attention to this example.)

Remark 2. The conditions of the lemma do not imply that all complements of N in G are conjugate. For example, let G be the direct product of a countably infinite

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

number of dihedral groups of order 6 and let N be the unique 3-component of G. G is locally finite and $\varpi(N) \cap \varpi(G/N)$ is empty, but N has $2^{\aleph_0}$ distinct complements in G and these cannot all be conjugate, since G is countable.

We come now to our first result on the splitting of periodic ST-groups.

THEOREM 5·1·2. *Let G be a periodic ST-group and let $C = C_G(G')$. Then if G/C is countable, G has the splitting property.*

Proof. In the first place we may assume that $L = [G', G]$ has trivial 2-component. For if this case has been settled, since L/L_2 is the limit of the lower central series of G/L_2 and

$$|G/L_2 : C_{G/L_2}(L/L_2)| \leqslant |G/L_2 : C/L_2| = |G:C| \leqslant \aleph_0,$$

we can conclude that G/L_2 splits over L/L_2. From this it follows that G splits over $L_{2'}$. Let $L_2 = 1$: then $\varpi(L) \cap \varpi(G/L)$ is empty, by Theorem 4·2·2. Also we can suppose that $C = L$. For again suppose that this case has been settled. C is nilpotent and so $C = L \times M$ for some $M \lhd G$. C/M is the limit of the lower central series of G/M: also since $C = C_G(L)$, C/M coincides with its centralizer in G/M. Hence we can conclude that G/M splits over C/M and G splits over L.

Let $L_{2'} = L = C$. A periodic soluble group is locally finite so Lemma 5·1·1 is directly applicable and G splits over L.

COROLLARY. *If G is a periodic ST-group, $L = [G', G]$ and $\varpi(L)$ is finite, then G has the splitting property. In particular an ST-group of finite exponent has the splitting property.*

For by § 4·1·4, $\varpi(L)$ finite implies that $G/C_G(G')$ is finite.

Further results on the splitting of periodic ST-groups may be obtained from

LEMMA 5·1·3. *Let G be an ST-group and let $L = [G', G]$. If x is any element of G, let $X_0 = \{x\} \cap G'$ and let $C(x) = C_G(x \bmod X_0)$ (i.e. the set of all $g \in G$ such that $[x, g] \in X_0$).*

(i) $G = C(x)\, L$.

(ii) *If G is periodic and $\varpi(L)$ does not contain the prime 2, then $G = C(x)\, L_\Pi$ and $C(x) \cap L_\Pi = 1$, where Π is the set of all $p \in \varpi(L)$ such that $[L_p, x] \neq 1$.*

Proof. Since G is an ST-group, G' is Abelian and $X_0 \lhd G$; hence $C(x)$ is a subgroup. By Corollary 1 to Lemma 2·1·1, $x^G \cap G' = X_0[L, x]$. Hence for each $y \in G$ there is an element $z \in L$ such that $[y, x] \equiv [z, x] \bmod X_0$, since G' is Abelian. It follows that $y \equiv z \bmod C(x)$ and $G = C(x)\, L$.

Now let G be periodic and $2 \notin \varpi(L)$. Π is the set of all p such that $[L_p, x] \neq 1$. Then if $p \in \Pi$, $C_G(x) \cap L_p = 1$, for otherwise x commutes with some element of L_p different from the identity and this means that $[L_p, x] = 1$ by Theorem 4·2·2. Now let z be an element in $C(x) \cap L_p$ where $p \in \Pi$. Then $[x, z] \in X_0 \cap L_p \leqslant C_G(x) \cap L_p = 1$ and hence $z \in C_G(x) \cap L_p = 1$. It follows that $C(x) \cap L_\Pi = 1$; also $L_{\Pi'} \leqslant C(x)$ by definition of Π. By (i) $G = C(x)\, L = C(x)\, L_\Pi$ and we have just proved that $C(x) \cap L_\Pi = 1$.

THEOREM 5·1·4. *Let G be a periodic ST-group and let $L = [G', G]$. If G contains an element x which centralizes only a finite number of non-trivial primary components of L, then G has the splitting property.*

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

Proof. As in the proof of Theorem 5·1·2 we can suppose that $L_2 = 1$. Let Π be the set of all $p \in \varpi(L)$ such that $[L_p, x] \neq 1$. By Lemma 5·1·3 G splits over L_Π and we can write $G = XL_\Pi$ where $X \cap L_\Pi = 1$ and $L_{\Pi'} \leqslant X$. By hypothesis Π' is finite, so G/L_Π splits over L/L_Π by the corollary to Theorem 5·1·2. Hence G splits over $L_{\Pi'}$ and so $G = YL_{\Pi'}$ where $Y \cap L_{\Pi'} = 1$. Since $L_{\Pi'} \leqslant X$, it follows that $G = XL_\Pi \cap YL_{\Pi'} = (X \cap Y)L$. $X \cap Y \cap L = 1$ and so G splits over L.

5·1·5. We now consider the structure of periodic ST-groups with the splitting property. Let G be a periodic ST-group, let $L = [G', G]$ and let $A = L_{2'}$. Suppose that G has the splitting property: then $G = EA$ where $E \cap A = 1$. $E \cong G/A$, so E is a periodic ST-group and $F = [E', E]$ is a divisible Abelian 2-group. If M/F is the subgroup of all elements of odd order in E/F, F is central in M, by Theorem 4·2·2(ii), and therefore M is a Dedekind group. Let $M = B \times F$ where B is an Abelian group all of whose elements have odd order. M/F is a direct factor of E/F; hence E splits over B. Now it is clear that the set of all 2-elements in E forms a subgroup, D say. Therefore $E = B \times D$; D is a soluble 2-group with the property T and the set $\varpi(A) \cap \varpi(B)$ is empty. By Theorem 4·2·2(i), for each prime p in $\varpi(A)$ we can find an element e_p in E such that e_p commutes with no element of A_p except the identity.

5·2. *Constructing periodic ST-groups with the splitting property.* As we have seen, periodic ST-groups with the splitting property have a rather simple structure. In this section we outline a method of constructing all such groups and then we solve the isomorphism problem for the constructed groups. By Theorem 5·1·2 these results will apply to countable periodic ST-groups.

Let A and B be two Abelian groups such that all elements of A or B have odd order and $\varpi(A) \cap \varpi(B)$ is empty. Let D be a soluble 2-group with the property T; by Lemma 4·2·1 D is either a Dedekind group or isomorphic with one of the groups $G(*, *, *)$ of §3·2.

Let $E = B \times D$ and let θ be homomorphism of E into the torsion-subgroup of $\Pi(A)$, the group of power automorphisms of A, such that for each $p \in \varpi(A)$ there is an element e_p in E with the property that $(e_p)^\theta$ leaves invariant no element of A_p other than the identity. We now define the group $G = G(A, B, D, \theta)$ to be the split extension of A by E determined by θ. We shall prove the following result.

THEOREM 5·2·1. *Let $G(A, B, D, \theta)$ be the group constructed in* 5·2.

(i) *$G(A, B, D, \theta)$ is a periodic ST-group with the splitting property.*

(ii) *Every periodic ST-group with the splitting property is isomorphic with some $G(A, B, D, \theta)$.*

(iii) *$G(A_1, B_1, D_1, \theta_1)$ is isomorphic with $G(A_2, B_2, D_2, \theta_2)$ if and only if there exist isomorphisms ψ_{A_1} and ψ_{E_1} of A_1 onto A_2 and E_1 onto E_2 (where $E_i = B_i \times D_i$) such that $\theta_1 \Psi_{A_1} = \psi_{E_1} \theta_2$, Ψ_{A_1} being the isomorphism of $\Pi(A_1)$ onto $\Pi(A_2)$ induced by ψ_{A_1}.*

(iv) *$G(A, B, D, \theta_1)$ is isomorphic with $G(A, B, D, \theta_2)$ if and only if $\theta_1 = \chi\theta_2$ for some automorphism χ of $E = B \times D$.*

Proof. In the first place by the discussion in §5·1·5 every periodic ST-group with the splitting property is isomorphic with some $G(A, B, D, \theta)$. To establish (i) we need a criterion for a group to have the property T.

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

LEMMA 5·2·2. *Let G have a periodic normal subgroup N such that*

(i) *each periodic subnormal subgroup of G/N is normal in G/N,*

(ii) *the set $\varpi(N) \cap \varpi(G/N)$ is empty,*

(iii) *each subnormal subgroup of N is normal in G.*

Then every periodic subnormal subgroup of G is normal in G: if G is periodic, it is a T-group.

Proof. If M is any subnormal subgroup of G, then M inherits from G the properties (i), (ii) and (iii), with $M \cap N$ in place of N. Hence it is enough to show that if H is periodic, $H \lhd K \lhd G$ implies that $H \lhd G$. By (iii) $H \cap N \lhd G$, so we can pass to quotient groups and assume that $H \cap N = 1$. By (i) $HN \lhd G$; hence $H(K \cap N) = HN \cap K \lhd G$. $H \lhd H(K \cap N)$ and $K \lhd H(K \cap N)$, so $H(K \cap N) = H \times (K \cap N)$. By (ii) $\varpi(H) \cap \varpi(K \cap N)$ is empty, so H is characteristic in $H(K \cap N)$ and hence $H \lhd G$.

Proof of Theorem 5·2·1 (continued). Let $G \equiv G(A, B, D, \theta) = EA$ where $E = B \times D$ and $E \cap A = 1$. D is a 2-group and every element of B has odd order; also both D and B are T-groups, so E is a T-group by Lemma 5·2·2. By construction the set $\varpi(A) \cap \varpi(E)$ is empty and every subgroup of A is normal in G. Hence G is a T-group by Lemma 5·2·2. Of course it is clear that G is periodic and soluble.

Let $p \in \varpi(A)$: by hypothesis there is an element $e_p \in E$ which induces a power automorphism in A_p and commutes with no element of A_p except the identity. Hence $[A_p, e_p] = A_p$ and $[A, E] = A$; $\Pi(A)$ is an Abelian group, so $[A, E'] = 1$. It follows that $G' = A \times D'$ and $[G', G] = A \times [D', D]$, showing that G has the splitting property. Hence (i) is established.

Let ψ be an isomorphism of $G_1 = G(A_1, B_1, D_1, \theta_1)$ onto $G_2 = G(A_2, B_2, D_2, \theta_2)$. Now $[G_i', G_i] = A_i \times [D_i', D_i]$, so that ψ maps A_1 onto A_2. Also we may suppose that ψ maps B_1 onto B_2 and D_1 onto D_2: let the induced isomorphisms of A_1 onto A_2 and E_1 onto E_2 be ψ_{A_1} and ψ_{E_1} respectively. ψ_{A_1} induces an isomorphism Ψ_{A_1} of $\Pi(A_1)$ onto $\Pi(A_2)$ defined by $(a^\psi)^{\alpha\Psi_{A_1}} = (a^\alpha)^\psi$ $(a \in A_1, \alpha \in \Pi(A_1))$. From the fact that ψ is a homomorphic mapping it follows by a standard argument that

$$\theta_1 \Psi_{A_1} = \psi_{E_1} \theta_2. \tag{III}$$

(See Gaschütz (4) and Taunt (10).)

Conversely if we are given isomorphisms ψ_{A_1} and ψ_{E_1} of A_1 onto A_2 and E_1 onto E_2, respectively such that the relation (III) is satisfied, then there is an isomorphism of G_1 onto G_2. (This may be proved by reversing the argument which leads to (III).) Finally if $A_1 = A_2$ and $E_1 = E_2$ then Ψ_{A_1} is the trivial automorphism of $\Pi(A_1)$ and we can write equation (III) as

$$\theta_1 = \chi \theta_2 \quad \text{where} \quad \chi = \psi_{E_1}.$$

6. ST-GROUPS WITH SPECIAL PROPERTIES

6·1. *Subgroups of ST-groups—the property $\overline{T}$.* A narrower class of groups than the class of T-groups is the class of $\overline{\text{T}}$-groups which is defined as follows. A group G has the *property $\overline{T}$* or is a *$\overline{T}$-group* if and only if given subgroups H, K, L of G such that

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

$H \lhd K \lhd L$, we can always conclude that $H \lhd L$. Hence the class of $\overline{\mathrm{T}}$-groups is the largest subgroup-closed class of groups which is contained in the class of T-groups. Gaschütz has shown that for finite soluble groups the properties T and $\overline{\mathrm{T}}$ coincide ((4), Satz 4). This is not true for infinite soluble groups; indeed according to Corollary 3 of Theorem 3·1·1 no ST-group of type 1 is a $\overline{\mathrm{T}}$-group. We now proceed to determine which ST-groups are S$\overline{\mathrm{T}}$-groups, i.e. which ST-groups have the property $\overline{\mathrm{T}}$.

THEOREM 6·1·1. *Let G be an S$\overline{T}$-group. Then G is either periodic or Abelian. If G is periodic and $L = [G', G]$, then $L_2 = 1$ and $\varpi(L) \cap \varpi(G/L)$ is empty.*

Conversely an ST-group with this structure is an S$\overline{T}$-group.

Proof. Let G be a non-Abelian S$\overline{\mathrm{T}}$-group. G cannot be of type 1 and hence $C = C_G(G')$ is periodic. Suppose that G contains an element x with infinite order and let c be any element of C. $\{c\} \lhd G$ and by hypothesis $\{x, c\}$ is a T-group. Hence $\{x, c\}$ is Abelian by Lemma 2·4·3 (ii) and $x \in C_G(C) \leqslant C$, contradicting the periodicity of C. Therefore G is periodic. Now assume that $\varpi(L)$ contains the prime 2 and L_2 is the direct product of groups of the type C_{2^∞}. Theorem 4·2·2 (ii) shows that G has a factor A/B which is a group of the type $G(0, A_1, 0)$ where $A_1 \cong C_{2^\infty}$. Thus we are led to conclude that every subgroup of $G(0, A_1, 0)$ is a T-group, which is impossible, since $G(0, A_1, 0)$ contains nilpotent subgroups of class greater than 2. By this contradiction $L_2 = 1$ and $\varpi(L) \cap \varpi(G/L)$ is empty (by Theorem 4·2·2).

Conversely let G be a periodic ST-group such that $L_2 = 1$. Then if H is any subgroup of G, $\varpi(H \cap L) \cap \varpi(H/H \cap L)$ is empty and H is a T-group by Lemma 5·2·3.

COROLLARY 1. *For any prime p, a soluble p-group is Abelian if it has the property $\overline{T}$.*

COROLLARY 2. *For soluble groups of finite exponent the properties T and $\overline{T}$ coincide.*

COROLLARY 3. *For periodic soluble groups without elements of even order the properties T and $\overline{T}$ coincide.*

For $\overline{\mathrm{T}}$-groups certain of the classes of generalized soluble groups coincide. Let G be a locally soluble group with the property $\overline{\mathrm{T}}$. Then each finitely generated subgroup of G is an ST-group and therefore is metabelian. Hence G is metabelian and each locally soluble $\overline{\mathrm{T}}$-group is soluble.

From this it follows that any $\overline{\mathrm{T}}$-group which has an ascending or descending series with locally soluble factors is soluble. For let G have an ascending series

$$1 = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_\gamma = G,$$

where each factor $G_{\alpha+1}/G_\alpha$ is locally soluble, and let G be a $\overline{\mathrm{T}}$-group. If G is not soluble, there is a smallest ordinal α such that G_α is not soluble. If α is a limit number, $G = \bigcup_{\beta < \alpha} G_\beta$ and each G_β is soluble ($\beta < \alpha$), and hence metabelian. Hence G_α is metabelian. On the other hand, if α is not a limit number $G_\alpha/G_{\alpha-1}$ is soluble, as a locally soluble $\overline{\mathrm{T}}$-group, and hence G_α is soluble. The proof in the case where G has a descending series with locally soluble factors is similar. In particular every SN*-group with the property T is soluble; (an SN*-group is a group with an ascending series with Abelian factors, see (7)).

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

6·2. *ST-groups which coincide with the normal closure of a cyclic subgroup.* Let G be a T-group, let $x \in G$ and let $X = x^G$: then $x^X = X$. Hence G is a product of normal T-subgroups each of which coincides with the normal closure (with respect to itself) of a cyclic subgroup. Another result which links the structure of a T-group G with its subgroups of the form x^G is $[G', G] = \prod_{x \in G} (x^G)'$ (Lemma 2·2·1 (ii)). In view of these facts it is natural to look at the structure of ST-groups which coincide with the normal closure of one of their cyclic subgroups.

Let G be an ST-group and suppose that there exists an element $x \in G$ such that $x^G = G$. Then $G = \{x\}G'$ and so $G = \{x\}[G', G]$: this shows that $G' = [G', G]$ and that C is Abelian.

(*a*) If G is a Dedekind group, $\{x\} \lhd G$ and $G = \{x\}$.

(*b*) Let $G \cong G(i, A, r)$, in the notation of §3·2. Let $G = \{z, C\}$ where $C = C_G(G')$. Now x must lie outside C, since G is non-Abelian. Let $x = zc$, $c \in C$: then

$$(zc)^G = \{zc, C^2\} = \{z, C\} = G.$$

Intersecting with C, we get $\{z^2, C^2\} = C$; conversely this relation implies that $z^G = G$. It follows that $G \cong G(i, A, r)$ coincides with the normal closure of a cyclic subgroup if and only if $r = 0$ and $i = 0$, 1 or ∞, by an easy examination of the relation $\{z^2, C^2\} = C$.

(*c*) Let G be a group of type 2. By Theorem 4·3·1 G' is divisible and non-trivial: also x has infinite order and therefore induces a power automorphism of infinite order in each non-trivial primary component of G'. Any ST-group of type 2 can be generated by elements of infinite order and hence it is a product of normal ST-subgroups of type 2 each of which coincides with the normal closure in itself of a cyclic subgroup.

Conversely let G be the split extension of a non-trivial periodic divisible Abelian group A by an infinite cyclic group $\{x\}$, such that x induces in each non-trivial primary component of A a power automorphism of infinite order. Then G is an ST-group of type 2 and $x^G = G$.

For let $p \in \pi(A)$ and let m be any positive integer. A_p is divisible and by hypothesis $[A_p, x^m] \neq 1$, so $[A_p, x^m] = A_p$ and $[A, x^m] = A$. Therefore $x^G = \{x, [G, x]\} = G$, $G' = [A, x] = A$ and $C_G(G') = A$. Now let $H \lhd K \lhd G$. If $H \leqslant A$, $H \lhd G$; if $H \not\leqslant A$, H contains an element of the form ax^m, $m > 0, a \in A$. Then $K \geqslant [A, ax^m] = [A, x^m] = A$ and so $H \geqslant [A, ax^m] = A$. It follows that $H \lhd G$ and G is a T-group. Clearly G is an ST-group of type 2.

(*d*) Let G be a periodic ST-group. Since G/G' is cyclic, G splits over A, the odd component of G', by Theorem 5·1·2. We can write $G = EA$ where $E \cap A = 1$ and $E = B \times D$. Here B is Abelian and D is a soluble 2-group. Clearly each quotient group of G coincides with the normal closure of one of its cyclic subgroups. Hence, by (*a*) and (*b*), B is cyclic and D is either cyclic or isomorphic with $G(i, A_1, 0)$ (for $i = 0$, 1 or ∞, and some A_1).

Conversely let $G = G(A, B, D, \theta)$ where $B = \{b\}$ and $D = \{d\}$ or $D \cong G(i, A_1, 0), i = 0, 1,$ or ∞. Then G contains an element x such that $x^G = G$. If D is cyclic, let $x = bd$; if D is not cyclic and $D = \{z, C^*\}$, $C^* = C_D(D')$, let $x = bz$. Let $E = B \times D$. Then in either case $x^E = E$, since B has all its elements of odd order and D is a 2-group. By construc-

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press

tion, for each $p \in \varpi(A)$ there is an element in E which commutes with no element of A_p except the identity. Since $x^E = E$, x commutes with no element of A_p except the identity. Hence $[A_p, x] = A_p$ and $[A, x] = A$. It follows that

$$x^G = (x^E)^A = E^A = EA = G.$$

I would like to thank Prof. P. Hall of King's College, Cambridge, for his generous help and valuable criticism during the preparation of this paper. Also I wish to thank the Carnegie Trust for the Universities of Scotland for financial support.

REFERENCES

(1) BEST, E. and TAUSSKY, O. A class of groups. *Proc. Roy. Irish. Acad. Sect. A*, **47** (1942), 55–62.
(2) FEIT, W. and THOMPSON, J. A solvability criterion for finite groups and some consequences. *Proc. Nat. Acad. Sci. U.S.A.* **48** (1962), 968–970.
(3) FUCHS, L. *Abelian groups* (Pergamon Press, 1960).
(4) GASCHÜTZ, W. Gruppen in denen das Normalteilersein transitiv ist. *J. Reine Angew. Math.* **198** (1957), 87–92.
(5) HASSE, H. *Zahlentheorie* (Akademie, 1949).
(6) KOVACS, L. G., NEUMANN, B. H. and DE VRIES, H. Some Sylow subgroups. *Proc. Roy. Soc. London, Ser. A*, **260** (1961), 304–316.
(7) KUROŠ, A. G. *The theory of groups* (Chelsea, 1956).
(8) LEVI, F. W. Groups in which the commutator operation satisfies certain algebraic conditions. *J. Indian Math. Soc.* **6** (1942), 87–97.
(9) MCLAIN, D. H. Local theorems in universal algebras. *J. London Math. Soc.* **34** (1959), 177–184.
(10) TAUNT, D. R. Remarks on the isomorphism problem in theories of construction of finite groups. *Proc. Cambridge Philos. Soc.* **51** (1955), 16–24.
(11) ZACHER, G. Caratterizzazione dei t-gruppi finiti risolubili. *Ricerche Mat.* **1** (1952), 287–294.
(12) ZAPPA, G. Sui gruppi di Hirsch supersolubili I. *Rend. Sem. Math. Univ. Padova*, **12** (1941), 1–11.
(13) ZASSENHAUS, H. *The theory of groups* (Chelsea; 2nd ed., 1958).

https://doi.org/10.1017/S0305004100037403 Published online by Cambridge University Press