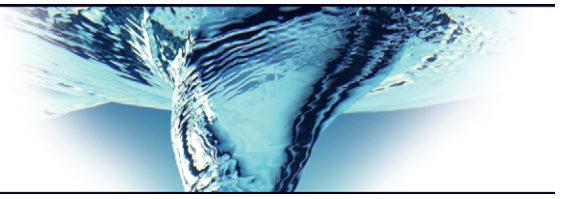
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Available potential energy in Rayleigh–Bénard convection

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The mechanical energy budget for thermally equilibrated Rayleigh–Bénard convection is developed theoretically, with explicit consideration of the role of available potential energy, this being the form in which all the mechanical energy for the flow is supplied. The analysis allows derivation for the first time of a closed analytical expression relating the rate of mixing in symmetric fully developed convection to the rate at which available potential energy is supplied by the thermal forcing. Only about half this supplied energy is dissipated viscously. The remainder is consumed by mixing acting to homogenize the density field. This finding is expected to apply over a wide range of Rayleigh and Prandtl numbers for which the Nusselt number is significantly greater than unity. Thus convection at large Rayleigh number involves energetically efficient mixing of density variations. In contrast to conventional approaches to Rayleigh–Bénard convection, the dissipation of temperature or density variance is shown not to be of direct relevance to the mechanical energy budget. Thus, explicit recognition of available potential energy as the source of mechanical energy for convection, and of both mixing and viscous dissipation as the sinks of this energy, could be of further use in understanding the physics.

Key words: Bénard convection, stratified turbulence, turbulent convection

1. Introduction

The convective flow that arises in a volume of fluid as a result of uniform heating from below and cooling from above, known as Rayleigh–Bénard convection, has been studied extensively for the complexity of its fundamental physics (e.g. see reviews Siggia 1994; Ahlers, Grossmann & Lohse 2009; Lohse & Xia 2010). One of the most fundamental aspects of the flow is that the thermal forcing maintains statically unstable fluid adjacent to each horizontal boundary, a process that corresponds to the continual generation of available potential energy. Fluid motion arises upon conversion of the available potential energy to kinetic energy by buoyancy fluxes.

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The energy budget of Rayleigh–Bénard convection is the primary tool for studies of the dynamics at large Rayleigh number, yet current analyses consider only the kinetic and thermal energy forms (e.g. Shraiman & Siggia 1990; Siggia 1994; Grossmann & Lohse 2000; Ahlers *et al.* 2009). As discussed by Lohse & Xia (2010), the details of the energy budget, specifically the pathways for energy conversion through differing length scales of motion, the relative roles of the boundary layers and interior, the variation of these with Rayleigh and Prandtl number, and the behaviour at extremely large Rayleigh number, all remain as major unresolved questions. Potential energy has not been considered explicitly, despite being the only source of mechanical energy to sustain the flow. Once potential energy is included in the accounting, an accurate representation of the budget requires separate consideration of the available and background components of gravitational potential energy, as has been shown for the case of horizontal convection (Hughes, Hogg & Griffiths 2009; Gayen *et al.* 2013*a*). The available component represents the energy that can be released to drive motion (Lorenz 1955), whereas the background component cannot drive motion but can exchange energy with other forms. The recent studies of the mechanical energy budget of horizontal convection indicate that two mechanisms of exchange between background and available potential energy are particularly important. First, boundary buoyancy fluxes associated with the thermal forcing convert background potential energy to available potential energy. Secondly, irreversible mixing, which tends to homogenize the density field, corresponds to a conversion in the opposite direction. In horizontal convection, at large Rayleigh number, the mixing is a much larger sink of mechanical energy than is the viscous dissipation (Scotti & White 2011; Gayen *et al.* 2013*a*), an aspect overlooked in analyses that focus only on kinetic energy and viscous dissipation.

In the case of statically stable stratified flows, accounting for the energy consumed by turbulence offers useful insights. For instance, the proportion of supplied energy used for irreversible mixing is closely related to the 'turbulent mixing efficiency' of the flow (e.g. Linden 1979; Peltier & Caulfield 2003). In stratified shear flows, much of the turbulent kinetic energy is viscously dissipated and only a small percentage is consumed by mixing (Linden 1979; Peltier & Caulfield 2003; Ivey, Winters & Koseff 2008; Prastowo *et al.* 2009). This contrasts with the relatively small viscous dissipation and much higher mixing efficiencies that characterize horizontal convection: the efficiency approaches 1 at very large Rayleigh number (Scotti & White 2011; Gayen *et al.* 2013*a*). However, these principles have not been applied to Rayleigh–Bénard convection.

The aim of this paper is to suggest a suitable theoretical framework to describe the mechanical energy budget of Rayleigh–Bénard convection, recognizing that the energy for the flow is supplied as available potential energy generated by the buoyancy forcing at the horizontal boundaries. We examine additional insights into the flow obtained from the full mechanical energy budget, which includes the decomposition of gravitational potential energy into its available and background components. A framework such as that given here is shown to be essential in properly relating the overall heat and mechanical energy fluxes. We concentrate in this paper on domain-integrated results that are amenable to analytical predictions, while noting that the framework has also proven useful for interpreting numerical simulations of Rayleigh–Bénard convection in terms of the dissipation and mixing, the boundary layers and the interior, the large and small scales of motion, and the Nusselt number–Rayleigh number relation (Gayen, Hughes & Griffiths 2013*b*).

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2. Theory

We develop a mechanical energy budget for convection in a fluid volume V due to an arbitrary distribution of buoyancy forcing over its boundaries. We then apply this in § 3 to a layer of fluid with destabilizing buoyancy at both horizontal boundaries (i.e. Rayleigh–Bénard convection, with heating from below and cooling from above). For simplicity the volume is assumed to be a closed rectangular box, and the fluid is assumed to be incompressible and Boussinesq with a linear equation of state,

$$\frac{\rho - \rho_0}{\rho_0} = \alpha(T - T_0), \tag{2.1}$$

where $\rho - \rho_0$ and $T - T_0$ are departures of the local fluid density and concentration of the stratifying scalar (e.g. temperature) from reference values (denoted by subscript 0), respectively, and $\alpha = \partial\rho/\partial T(\rho_0)$ is the coefficient of expansion. The continuity, momentum and density equations for an incompressible Boussinesq fluid are written as

$$\frac{\partial u_i}{\partial x_i} = 0, \quad \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \frac{\rho g\delta_{i3}}{\rho_0} + \nu\frac{\partial^2 u_i}{\partial x_j^2}, \quad \frac{\partial \rho}{\partial t} + u_i\frac{\partial \rho}{\partial x_i} = \kappa\frac{\partial^2 \rho}{\partial x_i^2}, \tag{2.2}$$

respectively, where u_i is the component of velocity in the x_i direction (i.e. u, v and w in the x, y and z directions, respectively), p is pressure, g is the gravitational acceleration (assumed to be in the $-z$ direction), δ is the Kronecker delta, ν is the kinematic viscosity, κ is the coefficient of molecular diffusion for the stratifying scalar and t is time. From (2.2) we derive (in the standard manner; see Winters *et al.* 1995; Hughes *et al.* 2009) evolution equations for the specific kinetic energy $E_k = (1/2)\langle {u_i}^2\rangle_V$ and specific potential energy $E_p = (g/\rho_0)\langle \rho z\rangle_V$ in the volume of interest:

$$\frac{\mathrm{d}E_k}{\mathrm{d}t} = -\Phi_z + \Phi_\tau - \epsilon, \tag{2.3}$$

and

$$\frac{\mathrm{d}E_p}{\mathrm{d}t} = \Phi_z + \Phi_{b1} + \Phi_i, \tag{2.4}$$

where each term is a volume-integrated energy conversion rate normalized by the reference mass $\rho_0 V$ in the volume. Similarly, the energies E_k and E_p are volume-integrated quantities, normalized by the reference mass $\rho_0 V$ in the volume. The operator $\langle\cdot\rangle_V$ is used to denote a volume average (or, alternatively, a horizontal average with a subscript A). The individual conversions expand to

$$\left.\begin{aligned} \Phi_z &= \frac{g}{\rho_0}\langle \rho w\rangle_V, \quad \Phi_\tau = \frac{\nu}{V}\oint u_i\frac{\partial u_i}{\partial x_j}n_j\,\mathrm{d}S, \quad \epsilon = \nu\langle(\partial u_i/\partial x_j)^2\rangle_V, \\ \Phi_{b1} &= \frac{\kappa g}{\rho_0 V}\oint z\frac{\partial \rho}{\partial x_j}n_j\,\mathrm{d}S, \quad \Phi_i = -\frac{\kappa g}{\rho_0}\frac{\Delta\langle\rho\rangle_A}{\Delta z}, \end{aligned}\right\} \tag{2.5}$$

and represent the buoyancy flux (Φ_z), the rate of working by boundary stresses (Φ_τ), the rate of viscous dissipation (ϵ), the rate of change of potential energy by net buoyancy flux through any level (Φ_{b1}) and the rate at which the centre of mass of the volume would be raised or lowered as a result of molecular diffusion of the stratifying scalar if there were no fluid motion (Φ_i). Here S is the surface with local outward normal n_j that encloses the volume V and $\Delta z = z_t - z_b$ is the height of that volume, where z_t and z_b are the heights of the top and bottom boundaries, respectively,

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and $\Delta\langle\rho\rangle_A = \langle\rho_t\rangle_A - \langle\rho_b\rangle_A$ is the top-to-bottom difference in horizontally averaged density. The dissipation ϵ is positive definite and represents a sink of kinetic energy. In stably stratified flows Φ_i is positive, but this does not have to be the case for convection. For instance, Rayleigh–Bénard convection has $\langle\rho_t\rangle_A > \langle\rho_b\rangle_A$, hence Φ_i is expected to be negative and Φ_{b1} is positive.

We decompose the potential energy (following Winters *et al.* 1995; Hughes *et al.* 2009) into components of background potential energy $E_b = (g/\rho_0)\langle\rho z_*\rangle_V$ and available potential energy $E_a = E_p - E_b$, where z_* is the height at which a fluid parcel of density ρ would reside if the density field in the volume at any given time were adiabatically rearranged to a state of static equilibrium. Thus z_* is related to ρ by a unique one-to-one mapping, $z_* = z_*(\rho)$ (for a more detailed definition see Winters *et al.* 1995), which, in its most general form, could vary with time. The values of E_b and E_a are however unimportant, as only the rates of energy conversion are significant in the dynamics.

The corresponding evolution equations for the potential energy components can be written (Winters *et al.* 1995; Hughes *et al.* 2009)

$$\frac{\mathrm{d}E_b}{\mathrm{d}t} = -\Phi_{b2} + \Phi_d \tag{2.6}$$

and

$$\frac{\mathrm{d}E_a}{\mathrm{d}t} = \Phi_z + \Phi_{b1} + \Phi_{b2} - \Phi_d + \Phi_i, \tag{2.7}$$

where conversions between background potential energy and available potential energy are achieved by two separate physical mechanisms:

$$\Phi_{b2} = -\frac{\kappa g}{\rho_0 V}\oint z_*\frac{\partial\rho}{\partial x_j}n_j\,\mathrm{d}S \quad \text{and} \quad \Phi_d = -\frac{\kappa g}{\rho_0}\langle(\mathrm{d}z_*/\mathrm{d}\rho)(\partial\rho/\partial x_i)^2\rangle_V. \tag{2.8}$$

The first of these is associated with the buoyancy forcing (that is required to maintain a density field away from its adiabatically relaxed state) and the second is the rate of irreversible mixing of the density field. Note that because $\mathrm{d}z_*/\mathrm{d}\rho$ is negative (by definition), contributions to the mixing rate Φ_d are everywhere positive definite, and Φ_d is a sink of available potential energy. The overall rate of working by the buoyancy forcing is $\Phi_{b1} + \Phi_{b2}$, which is the rate of generation of available potential energy that maintains the convective flow.

3. Application to Rayleigh–Bénard convection

We now restrict attention to the thermally equilibrated flow, for which there is no net heating of the fluid volume. The thermal boundary conditions are taken to be imposed temperature on the horizontal boundaries, and adiabatic elsewhere. The boundaries are assumed to be either no-slip or stress-free, for which Φ_τ (equation (2.5)) is identically zero. The mechanical energetics framework in (2.3), (2.6) and (2.7) is summarized schematically in figure 1 for Rayleigh–Bénard convection: arrows indicate the direction in which each net energy conversion occurs (more generally, the direction of some conversions is a function of the type of flow), with a sign change added on some symbols to ensure consistency between these directions and the definitions in (2.5) and (2.8). Note that although diffusive fluxes of the stratifying scalar (i.e. Φ_{b1}, Φ_{b2}, Φ_d and Φ_i) and hence internal energy are involved, we have chosen to simplify the energetics framework here by considering an incompressible

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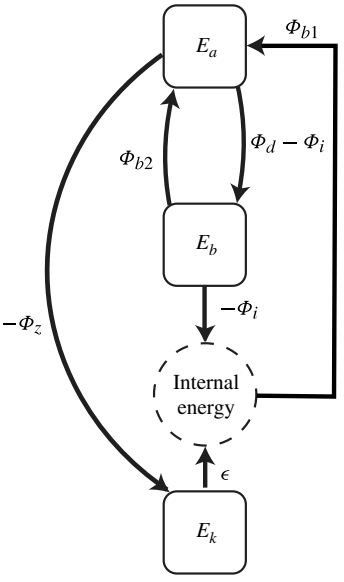


FIGURE 1. Schematic summary of the mechanical energetics framework for Rayleigh–Bénard convection in a volume of incompressible Boussinesq fluid with a linear equation of state (modified from Hughes *et al.* 2009). E_k, E_a and E_b are the specific kinetic energy, the specific available potential energy and the specific background potential energy, respectively, in the volume. The rates of energy conversion are the buoyancy flux (Φ_z), viscous dissipation (ϵ), irreversible mixing of the density field (Φ_d), background molecular diffusion (Φ_i) and available potential energy generation by buoyancy forcing (Φ_{b1} and Φ_{b2}). Arrows indicate the direction in which each net energy conversion occurs; the sign change on some symbols is for consistency between these directions and their definitions in (2.5) and (2.8)).

Boussinesq fluid with a linear equation of state. The energetics governing the convection can then be represented in terms of changes in density and buoyancy forces arising from these fluxes and the consequent rates of mechanical working, without explicit treatment of the internal energy budget. For this reason, the internal energy reservoir is denoted by a dashed outline, and this framework does not relate directly to any thermodynamic quantities (the reader is instead referred to Tailleux 2009, for a treatment that links the mechanical and internal energy budgets).

The amount of mechanical energy present in a given form, and the rates of conversion between forms, may fluctuate but they do not evolve systematically in time. Thus (2.3), (2.6) and (2.7) continue to hold if E_k, E_a, E_b and the conversions defined in (2.5) and (2.8) are instead time-averaged (such quantities are henceforth denoted by an overbar). Hence $\mathrm{d}\overline{E_k}/\mathrm{d}t = 0$ requires that the mean rate of viscous dissipation balances the conversion from potential to kinetic energy,

$$\overline{\epsilon} = -\overline{\Phi_z}, \tag{3.1}$$

where we have set Φ_τ to zero. Equation (3.1), together with $\mathrm{d}\overline{E_p}/\mathrm{d}t = 0$, requires that

$$\overline{\epsilon} = \overline{\Phi_{b1}} + \overline{\Phi_i}, \tag{3.2}$$

i.e. $\overline{\Phi_{b1}}$ is the component of the buoyancy forcing that maintains the flow (and viscous dissipation).

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The heat flux passing through any level is uniform and equal to that through the horizontal boundaries, $\overline{q} = (\rho_0 c_p \kappa/|\alpha|)(\partial\langle\overline{\rho}\rangle_A/\partial z)|_{S'}$, where c_p is the specific heat and S' denotes either the top ($z = z_t$) or bottom ($z = z_b$) boundary, at which it is assumed that there is no normal component of velocity. It follows from (2.5) that

$$\overline{\Phi_{b1}} = \frac{\kappa g}{\rho_0}\frac{\partial\langle\overline{\rho}\rangle_A}{\partial z}\bigg|_{S'} = -\overline{\Phi_i}Nu, \tag{3.3}$$

where the Nusselt number can be written

$$Nu = \frac{(\partial\langle\overline{\rho}\rangle_A/\partial z)|_{S'}}{\Delta\langle\overline{\rho}\rangle_A/\Delta z}. \tag{3.4}$$

Combining (3.2) and (3.3) shows that

$$\overline{\epsilon} = -\overline{\Phi_i}(Nu - 1), \tag{3.5}$$

which is equivalent to previous results (in terms of average rate of viscous dissipation per unit mass, Shraiman & Siggia 1990; Grossmann & Lohse 2000).

In thermally equilibrated convection, (2.6) requires that $\overline{\Phi_d} = \overline{\Phi_{b2}}$, from which it is apparent that $\overline{\Phi_{b2}}$ is positive and represents a supply of energy from the buoyancy forcing. In physical terms, the tendency of irreversible mixing to homogenize the range of densities in the flow must be exactly offset by the action of buoyancy forcing to maintain that range. Equation (2.8) can be used to evaluate $\overline{\Phi_{b2}}$ by noting that the lowest temperature (highest density) and highest temperature (lowest density) fluid will be in contact with the upper and lower horizontal boundaries, respectively. Thus fluid at the upper and lower boundaries has z_* equal to the heights z_b and z_t, respectively. As the mean temperature gradient at both boundaries is the same (for the heat flux to be independent of height; see discussion above), (2.8) shows that

$$\overline{\Phi_{b2}}(=\overline{\Phi_d}) = \frac{\kappa g}{\rho_0}\frac{\partial\langle\overline{\rho}\rangle_A}{\partial z}\bigg|_{S'}, \tag{3.6}$$

and hence, by (3.3), $\overline{\Phi_{b2}}(=\overline{\Phi_d}) = \overline{\Phi_{b1}}$. While these equalities are exact for imposed temperature boundary conditions, we anticipate them to be very closely approached with imposed flux boundary conditions because horizontal variations in plate temperature are expected to be much less than that across each boundary layer. Thus z_* for the fluid at the upper and lower boundaries is expected to lie within a boundary layer thickness of the heights z_b and z_t, respectively.

Available potential energy is generated (per unit mass in the volume V) by the buoyancy forcing in thermally equilibrated Rayleigh–Bénard convection at a rate $\overline{\Phi_{b1}} + \overline{\Phi_{b2}} = (2\kappa g\Delta\langle\overline{\rho}\rangle_A/\rho_0\Delta z)Nu$, determining the rate of release to kinetic energy. Viscous dissipation of kinetic energy occurs at up to half this rate, the remainder of the energy supply being consumed by molecular diffusion acting to homogenize the density field (the irreversible mixing). Rayleigh–Bénard convection can be characterized by the mixing efficiency (e.g. see Peltier & Caulfield 2003),

$$\eta = \frac{\overline{\Phi_d} - \overline{\Phi_i}}{\overline{\Phi_d} - \overline{\Phi_i} + \overline{\epsilon}} = \frac{1}{2}\left(1 + \frac{1}{Nu}\right), \tag{3.7}$$

which measures the proportion of the kinetic energy supply that is used for irreversible mixing by the flow. The limiting value $\eta = 1/2$ is likely to be approached over a wide range of conditions for which $Nu \gg 1$; the results of Ahlers & Xu (2001) suggest that this will hold for Rayleigh numbers $Ra = g\alpha(T - T_0)\Delta z^3/\nu\kappa \gtrsim O(10^7)$,

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for Prandtl numbers $Pr = \nu/\kappa$ in the range $O(1–100)$. Equation (3.4) shows that the Nusselt number increases from unity when the temperature difference across the flow depth starts to be concentrated in boundary layers. This boundary layer structure is expected to be established when the convection becomes turbulent and the Reynolds (and Rayleigh) number characterizing the flow is sufficiently high (see Ahlers *et al.* 2009). Thus the quantity defined above can properly be referred to as the turbulent mixing efficiency for large Rayleigh numbers and small or moderate Prandtl numbers.

The energy analysis above is not specific to any particular constraints on Ra or Pr and (3.7) is predicted to be a universal result. The influence of Prandtl number on convection is not well understood. However, existing experimental data and theoretical scalings all give the Nusselt number dependence on Pr as a weaker (negative) power than the dependence on Ra, which indicates that increasing either viscosity or diffusivity will decrease the Nusselt number and further increase the mixing efficiency away from the asymptotic value for large Nu, in turn indicating a greater relative importance of mixing compared to viscous dissipation. Larger Prandtl number is accompanied by a smaller ratio of Batchelor to Kolmogorov scale, leading to more efficient mixing within the 'thermal turbulence' field. The length scales of density variance must adjust until the rate of mixing Φ_d is equal to half the total rate of generation of available potential energy by the boundary buoyancy fluxes. This continues to hold in the absence of inertial turbulence at $Pr \gg Ra$. Note that the value of κ determines the energy supplied to the flow (equations (3.3) and (3.6)), and therefore that consumed by $\overline{\epsilon}$ and $\overline{\Phi}_d$. Hence, the expression for $\overline{\epsilon}$ in (3.5) does not depend explicitly upon ν. It is also interesting to note that as the Nusselt number decreases towards unity (as a consequence of smaller Ra or larger Pr) (3.7) predicts a growing contribution to the overall heat transport from conduction through the depth (Φ_i), and the remaining advective component of the transport is carried by a weaker flow with relatively less viscous dissipation. For this reason it is perhaps not physically sensible to characterize mixing in the flow itself near the conductive limit $Nu \to 1$ by an efficiency $\eta \to 1$ as given by (3.7).

4. Concluding remarks

The results obtained here show why dissipation of temperature (density) variance (e.g. see Shraiman & Siggia 1990; Siggia 1994; Grossmann & Lohse 2000; Ahlers *et al.* 2009) is not directly related to the dissipation of available potential energy (i.e. irreversible mixing Φ_d) in Rayleigh–Bénard convection. The linkage between the two types of dissipation applies to turbulence in stably stratified flows (e.g. see Gargett & Holloway 1984) upon assuming that the amount of available potential energy generated locally by displacing a fluid parcel from its equilibrium level is

$$\frac{1}{2} g z'^2 \frac{\mathrm{d}\langle\overline{\rho}\rangle_A}{\mathrm{d}z} = \frac{1}{2} \frac{g\rho'^2}{\mathrm{d}\langle\overline{\rho}\rangle_A/\mathrm{d}z}, \tag{4.1}$$

where $\mathrm{d}\langle\overline{\rho}\rangle_A/\mathrm{d}z$ is the mean stratification and ρ' is the density perturbation induced by its vertical displacement z'. However, as the mean density profile $\langle\overline{\rho}\rangle_A$ is unstably stratified in Rayleigh–Bénard convection, the displacements of fluid parcels z' are limited by the height of the layer rather than by the ambient stratification. Thus the rate of change of the volume-averaged integral of (4.1), which is proportional to the dissipation of density variance, cannot be expected to yield a consistent estimate for dissipation of available potential energy Φ_d. Indeed, (3.6) and its equality with (3.3),

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along with exact closure of the full mechanical energy budget, have been confirmed in direct numerical simulations of Rayleigh–Bénard convection (Gayen *et al.* 2013*b*).

Many processes that lead to turbulent mixing in a statically stable stratified fluid lose more than 80 % of their supplied energy to viscous dissipation, and are therefore characterized by mixing efficiencies $\eta \lesssim 20\,\%$ (e.g. Linden 1979; Peltier & Caulfield 2003; Ivey *et al.* 2008; Prastowo *et al.* 2009). In contrast, the theoretical energetics framework developed here shows that Rayleigh–Bénard convection is a flow involving highly efficient mixing (in a domain-integrated sense). Values of $\eta > 1/2$ are predicted because available potential energy is generated directly by the thermal forcing, which maintains statically unstable fluid in the boundary layers and leads readily to motion and density gradients on small length scales, where diffusion acts to mix and homogenize the density or temperature. The high-Rayleigh-number turbulent regime, where $\eta \to 1/2$, is of particular interest. This result, together with recent evidence for convection forced by heat fluxes through a single horizontal boundary (where $\eta \to 1$ at large Ra: Scotti & White 2011; Gayen *et al.* 2013*a*), suggests that high mixing efficiencies are likely to be a general characteristic of convective flows. For large Rayleigh numbers such efficiencies imply that irreversible mixing is largely confined to the (statically unstable) boundary layers, an insight not available from previous analyses of Rayleigh–Bénard convection that focus instead on the supplementary energy sink, viscous dissipation, which plays a dominant role in the interior of the flow (e.g. for $Pr \sim O(1\text{–}100)$ and $Ra \gtrsim O(10^7)$, see Ahlers *et al.* 2009).

We conclude that a full understanding of the energy budget of Rayleigh–Bénard convection requires an account of the available potential energy supplied to the convection and its consumption through both irreversible mixing and viscous dissipation. Although Rayleigh–Bénard convection has not been viewed previously in these terms, the current results clarify the link between the small-scale processes, which are responsible for the mixing, and the large-scale forcing. This linkage might be useful in the development of relatively simple parameterizations for convection. Further work including numerical simulations is needed to extend the analytical, domain-integrated results given here to evaluate the relative roles of the various energy conversions in the interior and boundary layers, and of large and small scales of motion, in the energy budget.

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