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The generalized function $(x+i0)^\lambda$

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The generalized function $(x+i0)^\lambda$ is defined by

$$(x+i0)^\lambda = x_+^\lambda + e^{i\lambda\pi} x_-^\lambda$$

for $\lambda \neq -1, -2, \ldots$ and $(x+i0)^{-n}$ is defined by

$$\begin{aligned}(x+i0)^{-n} &= \lim_{\lambda \to -n} (x_+^\lambda + e^{i\lambda\pi} x_-^\lambda) \\ &= x^{-n} + \frac{i\pi(-1)^n}{(n-1)!}\delta^{(n-1)}(x)\end{aligned}$$

for $n = 1, 2, \ldots$, see (1).

The generalized function $(x+i0)^\lambda \ln^r (x+i0)$ is defined by

$$(x+i0)^\lambda \ln^r (x+i0) = \frac{\partial^r}{\partial \lambda^r}(x+i0)^\lambda$$

for $\lambda \neq -1, -2, \ldots$ and $r = 1, 2, \ldots$ and $(x+i0)^{-n} \ln^r (x+i0)$ is defined by

$$(x+i0)^{-n} \ln^r (x+i0) = \lim_{\lambda \to -n} \frac{\partial^r}{\partial \lambda^r}(x_+^\lambda + e^{i\lambda\pi} x_-^\lambda)$$

for $n, r = 1, 2, \ldots$.

In particular we have

$$\begin{aligned}\ln (x+i0) &= \ln x_+ + \ln x_- + i\pi H(-x) \\ &= \ln |x| + i\pi H(-x),\end{aligned}$$

where $H(x)$ denotes Heaviside's function. It follows that $\ln (x+i0)$ and all positive integer powers $\{\ln (x+i0)\}^r$ exist as locally summable functions.

We further have

$$\begin{aligned}\ln^r (x+i0) &= \lim_{\lambda \to 0} \frac{\partial^r}{\partial \lambda^r}(x_+^\lambda + e^{i\lambda\pi} x_-^\lambda) \\ &= \lim_{\lambda \to 0} \left\{ x_+^\lambda \ln^r x_+ + \sum_{s=0}^{r} {}_rC_s (i\pi)^{r-s} e^{i\lambda\pi} x_-^\lambda \ln^s x_- \right\} \\ &= \ln^r x_+ + \sum_{s=0}^{r} {}_rC_s (i\pi)^{r-s} \ln^s x_- \\ &= \{\ln |x| + i\pi H(-x)\}^r.\end{aligned}$$

We thus have the result that

$$\ln^r (x+i0) = \{\ln |x| + i\pi H(-x)\}^r.$$

Thus the product of $\ln^r (x+i0)$ and $\ln^s (x+i0)$ exists and

$$\{\ln^r (x+i0)\}.\{\ln^s (x+i0)\} = \ln^{r+s} (x+i0)$$

for $r, s = 1, 2, \ldots$.

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We can now prove

THEOREM 1. *The product of the generalized functions* $(x+i0)^\lambda$ *and* $(x+i0)^\mu$ *exists and*

$$(x+i0)^\lambda . (x+i0)^\mu = (x+i0)^{\lambda+\mu}$$

for all λ, μ.

Proof. $(x+i0)^\lambda$ is an entire function of λ, see (1), and so can be expanded in a Taylor's series

$$(x+i0)^\lambda = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \ln^r (x+i0),$$

the series converging for all λ. Likewise we have

$$(x+i0)^\mu = \sum_{s=0}^{\infty} \frac{\mu^s}{s!} \ln^s (x+i0).$$

The product of $\sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \ln^r (x+i0)$ and $\sum_{s=0}^{\infty} \frac{\mu^s}{s!} \ln^s (x+i0)$

exists and

$$\begin{aligned}\left\{\sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \ln^r (x+i0)\right\} \times \left\{\sum_{s=0}^{\infty} \frac{\mu^s}{s!} \ln^s (x+i0)\right\} &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\lambda^r \mu^s}{r!\, s!} \ln^{r+s} (x+i0) \\ &= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{\lambda^r \mu^{n-r}}{r!\,(n-r)!} \ln^n (x+i0) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda+\mu)^n}{n!} \ln^n (x+i0) \\ &= (x+i0)^{\lambda+\mu}.\end{aligned}$$

It follows that the product $(x+i0)^\lambda . (x+i0)^\mu$ exists and

$$(x+i0)^\lambda . (x+i0)^\mu = (x+i0)^{\lambda+\mu}.$$

An immediate extension of this theorem is

THEOREM 2. *The product of the generalized functions* $(x+i0)^\lambda \ln^r (x+i0)$ *and* $(x+i0)^\mu \ln^s (x+i0)$ *exists and*

$$\{(x+i0)^\lambda \ln^r (x+i0)\} . \{(x+i0)^\mu \ln^s (x+i0)\} = (x+i0)^{\lambda+\mu} \ln^{r+s} (x+i0)$$

for all λ, μ *and* $r, s = 0, 1, 2, \ldots$.

Proof. Differentiating the equation

$$(x+i0)^\lambda . (x+i0)^\mu = (x+i0)^{\lambda+\mu}$$

r times partially with respect to λ and s times partially with respect to μ gives the result.

REFERENCE

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