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Discrete subsemigroups of hypercyclic C_0-semigroups are hypercyclic

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Abstract. If $S(t)$ is a C_0-semigroup on a Banach space X with a hypercyclic vector x, then x is a hypercyclic vector for each discrete semigroup $\{S(kT) \mid k \in \mathbb{N}\}$ with $T > 0$ fixed.

1. *Introduction*

Definition 1.1. Let J be an additive subsemigroup of $[0, \infty)$. Let X be a Banach space and let $\{S(t) \mid t \in J\}$ be a family of bounded linear operators on X such that $S(0) = \mathrm{id}$ and $S(t)S(s) = S(t+s)$ for $s, t \in J$.

A vector $x \in X$ is said to be hypercyclic with respect to $\{S(t) \mid t \in J\}$ if the orbit $\{S(t)x \mid t \in J\}$ is a dense subset of X.

We call the semigroup $\{S(t) \mid t \in J\}$ a hypercyclic semigroup if and only if it admits a hypercyclic vector.

Hypercyclicity has been thoroughly investigated for discrete semigroups as well as C_0-semigroups of linear operators. It is well known that hypercyclicity is equivalent to topological transitivity: any two non-empty open sets $U, V \subset X$ can be connected by a trajectory of $S(t)$, i.e. there exists some $t \in J$ such that $S(t)U \cap V \neq \emptyset$. For a comprehensive survey of the theory of discrete hypercyclic semigroups, we refer the reader to [**8, 9**]. Hypercyclic C_0-semigroups are treated, among others, in [**2, 3, 6**]. Hypercyclicity is a very unstable property, which can be destroyed by small perturbations. Recently, researchers have paid much attention to the question of whether semigroups constructed from hypercyclic semigroups, such as products or subsemigroups, inherit the property of hypercyclicity. The answers for products are non-trivial and one is lead to additional conditions, such as the hypercyclicity criterion ([**4, 8**] for discrete semigroups, [**5, 7**] for C_0-semigroups). The question of whether hypercyclicity is inherited by subsemigroups, i.e. for discrete samples of strongly continuous semiflows, was posed in [**3**, Open Problem 1] and attributed to Peris. In this paper we give the answer.

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THEOREM 1.2. *Let $S(t)$ be a C_0-semigroup on a Banach space X, and let x be a hypercyclic vector with respect to $\{S(t) \mid t \geq 0\}$. Let $T > 0$. Then x is hypercyclic with respect to $\{S(kT) \mid k \in \mathbb{N}\}$.*

Theorem 1.2 has a discrete counterpart which was proved in [**1**].

THEOREM 1.3. [**1**] *Let T be a bounded linear operator on a Banach space X, and let x be a hypercyclic vector for $\{T^k \mid k \in \mathbb{N}\}$. Then x is also hypercyclic for each subsemigroup $\{T^{nk} \mid k \in \mathbb{N}\}$ (with fixed $n \in \mathbb{N}$).*

While the assertion of this theorem is straightforward, its proof is highly non-trivial.

As early as 1941, Oxtoby and Ulam [**10**, Theorem 6] had proved by a short, but striking, category argument the following theorem.

THEOREM 1.4. *Let $\{S(t) \mid t \in \mathbb{R}\}$ be a topologically transitive continuous flow on a complete metric space X which admits no isolated trajectory. Then $\{S(kT) \mid k \in \mathbb{Z}\}$ is topologically transitive for all periods T except a set of first Baire category.*

We also remark that it is very easy to show that the hypercyclicity criterion is inherited by subsemigroups. Therefore the sophisticated proofs of Theorems 1.2 and 1.3 could be replaced by a short argument, if the following open question would be settled by an affirmative answer: Is the hypercyclicity criterion equivalent to hypercyclicity?

To prove Theorem 1.2, we will use a completely different approach. Since the proof is very technical, we present a short outline in the following section. The technical details are given in §3. We mention that in our proof the linearity of the operators is required, so that it cannot be transferred immediately to semiflows on metric spaces.

2. *Idea of the proof*

Throughout the whole proof, $x \in X$ will be a fixed vector which is hypercyclic for $S(t)$.

Step 1: We start with two basic definitions.

Definition 2.1. For $h > 0$ we define G_h as the additive group of all classes $[t]_h = \{t + kh \mid k \in \mathbb{Z}\}$, with $t \geq 0$, topologized by the distance $d([t]_h, [s]_h) = \inf\{|\tau - \sigma| \mid \tau \in [t]_h, \sigma \in [s]_h\}$.

Clearly, $[t_k + h]_h = [t]_h$. With this notation, G_h is isomorphic to the torus group as a topological Abelian group.

Definition 2.2. Let $h > 0$. For each $y \in X$ we define $\phi_h(y) \subset G_h$ by

$$[t]_h \in \phi_h(y) \quad \text{if and only if} \quad (\exists t_n \to \infty)\ [t_n]_h \to [t]_h \text{ and } S(t_n)x \to y.$$

Since the orbit of x is dense and G_h is compact, $\phi_h(y)$ is not empty. The set $\phi_h(y)$ may contain more than one class.

Step 2: We start with $h = T$. Suppose first that $\phi_T(x) = G_T$. Then for each $t > 0$ we will construct a sequence $t_n \to \infty$ such that $[t_n]_T = 0$ and $S(t_n)x \to S(t)x$. Since $\{S(t)x \mid t \geq 0\}$ is dense, this implies that $\{S(t)x \mid [t]_T = 0\}$ is also dense. Therefore we may assume that $\phi_T(x) \neq G_T$ and we aim at a contradiction.

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Step 3: Now assume that $\phi_T(x) \neq G_T$. We will show that $\phi_T(x)$ is a closed additive subgroup of G_T, and thus there is a divisor h of T such that $\phi_T(x) = \{[kh]_T \mid k \in \mathbb{N}\}$. Now we consider ϕ_h with values in G_h. We will show that $\phi_h(S(t)x) = \{[t]_h\}$ for all $t \geq 0$.

Step 4: We do not know if for each $y \in X$ the set $\phi_h(y)$ contains only one element. However, if U is a bounded linear operator with dense range, commuting with each $S(t)$, then $\phi_h(Ux)$ is single-valued. In particular, this holds for $U = (1-\lambda)\,\mathrm{id} + \lambda S(t)$ for $\lambda \in [0,1]$, $t \in [0,\delta]$.

Step 5: If ϕ_h is single-valued on a subset $K \subset X$, then ϕ_h is continuous. Therefore, the map

$$\begin{cases} [0,1] \times [0,h] \to G_h \\ (\lambda, t) \qquad\quad \mapsto \phi_h((1-\lambda)x + \lambda S(t)x) \end{cases}$$

describes a homotopy between the maps $t \mapsto [0]_h$ and $t \mapsto [t]_h$. This is impossible, since G_h is not simply connected.

3. *Proof of Theorem 1.2*

Throughout this proof we assume that x is hypercyclic for $\{S(t) \mid t \geq 0\}$.

3.1. *Step 1.* We define G_h and ϕ_h by Definitions 2.1 and 2.2. We collect some basic properties.

LEMMA 3.1. *Let $h > 0$ and $y \in X$. Then:*

(1) *$\phi_h(y) \neq \emptyset$;*

(2) *if $y_n \to y$ in X and the $[s_n]_h \in \phi_h(y_n)$ converge to some $[s]_h \in G_h$, then $[s]_h \in \phi_h(y)$;*

(3) *$\phi_h(y)$ is closed;*

(4) *if $K \subset X$ is such that $\phi_h(y)$ is single-valued for all $y \in K$, then ϕ_h is continuous from K into G_h.*

Proof. (1) There exists a sequence $t_n \to \infty$ such that $S(t_n)x \to y$. By compactness of G_h there exists a subsequence such that $[t_{n_k}]_h$ has a limit in G_h.

(2) Let $[s_n]_h \in \phi_h(y_n)$, thus there exist sequences t_n^m such that $\lim_{m\to\infty}[t_n^m]_h \to [s_n]_h$ and $\lim_{m\to\infty} S(t_n^m)x = y_n$. Now choose a diagonal sequence $q_k = t_{n_k}^{m_k}$ with $S(q_k)x \to y$ and $[q_k]_h \to [s]_h$.

(3) This is an immediate consequence of (2) with $y_n = y$.

(4) Let $y_n \in K$, $\{[s_n]_h\} = \phi_h(y_n)$, $\{[s]_h\} = \phi_h(y)$. By compactness of G_h we need only show that each convergent subsequence of $([s_n]_h)$ converges to $[s]_h$. This follows from (3) and the fact that $\phi_h(y)$ is single-valued. □

LEMMA 3.2. *Let $h > 0$, $t \geq 0$. Let $y \in X$ and $[s]_h \in \phi_h(y)$. Then:*

(1) *$[t]_h + [s]_h \in \phi_h(S(t)y)$;*

(2) *let U be a bounded linear operator commuting with each $S(\tau)$, such that $[t]_h \in \phi_h(Ux)$. Then $[t]_h + [s]_h \in \phi_h(Uy)$.*

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Proof. (1) Let $S(s_n)x \to y$ with $[s_n]_h \to [s]_h$. Then $S(s_n + t)x \to S(t)y$ and $[s_n + t]_h \to [s]_h + [t]_h$.

(2) Let $S(s_m)x \to y$ with $[s_m]_h \to [s]_h$, and let $S(t_n)x \to Ux$ with $[t_n]_h \to [t]_h$. Given $\epsilon > 0$, pick s_m such that $d([s_m]_h, [s]_h) < \epsilon/2$ and $\|y - S(s_m)x\| \cdot \|U\| < \epsilon/2$. Next choose t_n such that $d([t_n]_h, [t]_h) < \epsilon/2$ and $\|Ux - S(t_n)x\| \cdot \|S(s_m)\| < \epsilon/2$. Then $d([s_m + t_n]_h, [s + t]_h) < \epsilon$, and

$$\|Uy - S(t_n + s_m)x\| \leq \|Uy - US(s_m)x\| + \|S(s_m)Ux - S(s_m)S(t_n)x\| < \epsilon.$$

With $\epsilon \to 0$ we conclude that $[t + s]_h \in \phi_h(Uy)$. □

3.2. *Step 2.*

LEMMA 3.3. *Let $\phi_T(x) = G_T$. Then x is a hypercyclic vector of $\{S(kT) \mid k \in \mathbb{N}\}$.*

Proof. Take any $t > 0$. All we need to show is that there exists a sequence $k_n \in \mathbb{N}$ such that $S(k_nT)x \to S(t)x$. Since the orbit of $S(t)x$ is dense, this implies that x is hypercyclic for $\{S(kT)\}$.

Given $n \in \mathbb{N}$, fix some $\delta \in (0, 1/n)$ such that $\|S(t)x - S(s)x\| < 1/n$ if $|s - t| < 2\delta$. Let $M = \sup_{q \in [0,2\delta]} \|S(q)\|$. Since $\phi_T(x) = G_T$ we have $\phi_T(S(t)x) = G_T$ by Lemma 3.2(1). In particular, $[-\delta]_T \in \phi_T(S(t)x)$, so that there exists some $s_n > n$ with $\|S(s_n)x - S(t)x\| < 1/(Mn)$ and $d([s_n]_T, [-\delta]_T) < \delta$. Let $q_n \in [0, 2\delta)$ be such that $s_n + q_n = k_nT$ with $k_n \in \mathbb{N}$. Then we have

$$\|S(s_n + q_n)x - S(t)x\| \leq \|S(q_n)\| \cdot \|S(s_n)x - S(t)x\| + \|S(t + q_n)x - S(t)x\| < 2/n. \quad \square$$

3.3. *Step 3.*

LEMMA 3.4. *The set $\phi_T(x)$ is a closed subgroup of G_T. In particular, if x is not hypercyclic with respect to $\{S(kT)\}$, then there exists a minimal $h > 0$ such that h is a divisor of T and $\phi_T(x) = \{[kh]_T \mid k \in \mathbb{N}\}$.*

Proof. Let $[t]_T$ and $[s]_T$ be in $\phi_T(x)$. By Lemma 3.2(2) with $y = x$ and $U = 1$, we obtain that $[s]_T + [t]_T \in \phi_T(x)$. By Lemma 3.1(3), $\phi_T(x)$ is closed. Thus $\phi_T(x)$ is a closed subsemigroup of G_T. If x is not hypercyclic for $\{S(kT)\}$, then $\phi_T(x) \neq G_T$ by Lemma 3.3. The only closed proper subsemigroups of G_T are the finite subgroups $\{[kT/m]_T \mid k \in \mathbb{N}\}$ with $m \in \mathbb{N}$. □

LEMMA 3.5. *Suppose x is not hypercyclic with respect to $\{S(kT)\}$ and let h be as in Lemma 3.4. Then $\phi_h(S(t)x) = \{[t]_h\}$.*

Proof. We start with $t = 0$. Let $[s]_h \in \phi_h(x)$. There exists a sequence $s_n \to \infty$ such that $[s_n]_h \to [s]_h$ and $S(s_n)x \to x$. Consequently, $\min\{d([s_n]_T, [s + hm]_T) \mid m = 0, 1, \ldots, T/h\} \to 0$. At least for one integer m there exists a subsequence such that $d([s_{n_k}]_T, [s + hm]_T) \to 0$. Consequently, $[s + hm]_T \in \phi_T(x)$, and thus $[s + hm]_T$ is a multiple of $[h]_T$. We infer that $[s]_h = 0$. Hence $\phi_h(x) = [0]_h$.

From Lemma 3.2(1) we infer that $[t]_h \in \phi_h(S(t)x)$. Now let $[s]_h \in \phi(S(t)x)$. Let $u_n \to \infty$ with $S(u_n)x \to x$ and $v_m \to \infty$ with $S(v_m)x \to S(t)x$, $[v_m]_h \to [s]_h$.

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We can construct subsequences such that $S(u_n - t + v_{m_n})x \to x$. Since $\phi(x) = \{[0]_h\}$, we infer that $[u_n]_h \to [0]_h$. Similarly $[u_n - t + v_{m_n}]_h \to [0]_h$, so that $[0 - t + s]_h = 0$. Thus $[s]_h = [t]_h$. □

3.4. *Step 4.*

LEMMA 3.6. *Suppose x is not hypercyclic with respect to $\{S(kT)\}$ and let h be as in Lemma 3.4. Let $\epsilon > 0$. Then there exists an open set $Y \subset X$ such that for all $y \in Y$ and all $[s]_h, [t]_h \in \phi_h(y)$ we have $d([s]_h, [t]_h) < \epsilon$.*

Proof. Let $\delta < \epsilon/4$ be such that $\delta m = h$ for some $m \in \mathbb{N}$. For $k = 0, \dots, m-1$ we define $M_k = \{y \in X \mid (\exists [t]_h \in \phi_h(y))\ d([t]_h, [k\delta]_h) \le \delta\}$. Since the δ-balls with centers $[k\delta]_h$ cover the whole group G_h, the sets M_k are a covering of X. Moreover, from Lemma 3.1(2) and the compactness of G_h we obtain that each M_k is closed. Let $J \subset \{0, \dots, m-1\}$ be a minimal subset such that $\bigcup_{j \in J} M_j$ contains a non-empty open set Y. If J contains more than one element, say $i \ne j \in J$, then both sets $Y - M_i$, $Y - M_j$ are open. At least one of them is not empty, in contradiction to the minimality of J. Thus there exists a non-empty, open set contained in a single set M_k. Now let $y \in Y$ and assume that $[t]_h \in \phi_h(y)$ and $[s]_h \in \phi_h(y)$ with $d([t]_h, [s]_h) \ge \epsilon > 4\delta$. There exist $s_n, t_n \to \infty$ such that $S(t_n)x \to y$, $S(s_n)x \to y$, $[s_n]_h \to [s]_h$, and $[t_n]_h \to [t]_h$. For sufficiently large n we have $S(s_n)x \in Y$ and $S(t_n)x \in Y$, and $d([t_n]_h, [s_n]_h) > 2\delta$. Since $S(s_n)x \in Y \subset M_k$, and $\phi_h(S(s_n)x) = \{[s_n]_h\}$ by Lemma 3.5, we infer that $d([s_n]_h, [k\delta]_h) \le \delta$. Similarly, $d([t_n]_h, [k\delta]_h) \le \delta$ with the same k. We arrive at a contradiction. □

LEMMA 3.7. *Suppose x is not hypercyclic with respect to $\{S(kT)\}$ and let h be as in Lemma 3.4. Let U be a continuous linear operator, commuting with each $S(t)$ and with dense range. Then $\phi_h(Ux)$ is single-valued.*

Proof. Assume that $[s]_h, [t]_h \in \phi_h(Ux)$ and $\epsilon > 0$. Choose an open set Y according to Lemma 3.6. Since the range of U is dense, there exists $z \in X$ such that $Uz \in Y$. Let $[v]_h \in \phi_h(z)$. By Lemma 3.2(2) we infer that both, $[v]_h + [t]_h$ and $[v]_h + [s]_h$ are contained in $\phi_h(Uz)$. This implies that $d([t]_h, [s]_h) = d([v]_h + [t]_h, [v]_h + [s]_h) < \epsilon$. As ϵ was arbitrary, we infer $[t]_h = [s]_h$. □

LEMMA 3.8. *For all fixed $\lambda \in \mathbb{R}$ and $t > 0$ the operator $(1 - \lambda)\,\mathrm{id} + \lambda S(t)$ has dense range in X.*

Proof. Suppose that the range of $(1 - \lambda)\,\mathrm{id} + \lambda S(t)$ is not dense. Clearly $\lambda \ne 0$. Put $\mu = (\lambda - 1)/\lambda$. Let $f^* \in X^* \setminus 0$ such that $\mu f^* - S^*(t) f^* = 0$. Since x is hypercyclic, we can fix some $s \ge 0$ such that $\langle f^*, S(s)x\rangle \ne 0$. Clearly, for all $m \in \mathbb{N}$ we have $\langle f^*, S(s + mt)x\rangle = \mu^m \langle f^*, S(s)x\rangle$. Since arbitrarily small vectors lie on the orbit $\{S(\sigma)x \mid \sigma \ge 0\}$ and $\|S(\sigma)\|$ is uniformly bounded for $\sigma \in [0, t]$, there is a sequence $k_n \in \mathbb{N}$ such that $\|S(k_n t + s)x\| \to 0$. This implies that $|\mu| < 1$. Consequently, for all $\sigma \in [0, t]$, $k \in \mathbb{N}$ we have an estimate $|\langle f^*, S(\sigma + kt)x\rangle| \le |\langle f^*, S(\sigma)x\rangle|$. Hence the set $\{\langle f^*, S(\sigma)x\rangle \mid \sigma \ge 0\}$ is bounded in contradiction to the fact that x is a hypercyclic vector for $S(t)$. □

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3.5. *Step 5.*

LEMMA 3.9. *Suppose x is not hypercyclic with respect to $\{S(kT)\}$ and let h be as in Lemma 3.4. Then the map*

$$\Psi : \begin{cases} [0,1] \times [0,h] & \to G_h \\ (\lambda, t) & \mapsto [s]_h \in \phi_h((1-\lambda)x + \lambda S(t)x) \end{cases}$$

is uniquely determined and continuous. Moreover, $\Psi(0,t) = [0]_h$ and $\Psi(1,t) = [t]_h$.

Proof. Ψ is well defined by Lemmas 3.7 and 3.8. By Lemma 3.1(4) and the strong continuity of $S(t)x$ in t, Ψ is continuous. $\Psi(0,t) = [0]_h$ and $\Psi(1,t) = [t]_h$ follow from Lemma 3.5. □

3.6. *Conclusion.* With Lemma 3.9 we have constructed a homotopy between the closed curves

$$\begin{cases} [0,h] & \to G_h \\ t & \to [0]_h \end{cases} \quad \text{and} \quad \begin{cases} [0,h] & \to G_h \\ t & \to [t]_h. \end{cases}$$

Since these curves are not homotopic in G_h, we have a contradiction and Theorem 1.2 is proved. □

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