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Gravity waves on shear flows

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The eigenvalue problem for gravity waves on a shear flow of depth h and non-inflected velocity profile $U(y)$ (typically parabolic) is revisited, following Burns (1953) and Yih (1972). Complementary variational formulations that provide upper and lower bounds to the Froude number F as a function of the wave speed c and wavenumber k are constructed. These formulations are used to improve Burns's long-wave approximation and to determine Yih's critical wavenumber k_*, for which the wave is stationary ($c = 0$) and to which k must be inferior for the existence of an upstream running wave.

1. Introduction

Straight-crested, linear gravity waves of wavenumber $k > 0$ and wave speed c on the surface of a shear flow of ambient depth h and velocity $U(y)$ are governed by the Rayleigh equation

$$(U - c)(\phi'' - k^2\phi) - U''\phi = 0 \quad (0 < y < h, \quad ' \equiv \mathrm{d}/\mathrm{d}y) \tag{1.1}$$

and the bottom and free-surface boundary conditions

$$\phi = 0 \quad (y = 0), \quad (U - c)^2\phi' = g\phi \quad (y = h), \tag{1.2a,b}$$

where $\phi(y)\exp[ik(x - ct)]$ is a complex stream function. Following Burns (1953) and Yih (1972), I consider this eigenvalue problem for a velocity profile for which

$$U(0) = 0, \quad U(h) \equiv U_1 > 0, \quad U'(h) = 0, \quad U''(y) < 0. \tag{1.3a–d}$$

The simplest solution of (1.3) is the parabolic profile

$$U(y) = U_1 y(2h - y)/h^2, \tag{1.4}$$

which is realized for a nearly inviscid flow down a slightly inclined plane.

The basic problem is to determine the characteristic relation $f(c, k, F) = 0$ or, as proves more convenient, $G = G(c, k)$, among the dimensionless parameters

$$c = c/U_1, \quad k = kh, \quad F = U_1/(gh)^{1/2}, \quad G = gh/U_1^2 \equiv 1/F^2. \tag{1.5a–d}$$

The still-water wave speed and drift speed are given by

$$C \equiv C/U_1 = [(G/k)\tanh k]^{1/2} \tag{1.6}$$

and

$$D \equiv D/U_1 = c \mp C \begin{pmatrix} c > 1 \\ c < 0 \end{pmatrix} \tag{1.7}$$

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for waves moving to the right/left (down/upstream). The dispersion relation $c = c(k)$ is implicitly determined by $G = G(c,k)$, and the corresponding group velocity is given by

$$c_g = \frac{\mathrm{d}}{\mathrm{d}k}[kc(k)] = c - k\left(\frac{\partial G/\partial k}{\partial G/\partial c}\right). \tag{1.8}$$

Burns (1953) solves (1.1)–(1.4) in the long-wave limit $k \downarrow 0$. Yih (1972) shows that the eigenvalue problem for prescribed k and F admits one solution with $c > 1$ for all $k > 0$ and a second solution with $c < 0$ if and only if $0 \leqslant k < k_*$, where k_* is a critical value of k for which the wave is stationary. There are no other solutions; accordingly, the singular point at $U = c$ lies outside the physical domain, and the admissible running waves are stable. The stationary ($c = 0$) wave, for which the singular point $U = 0$ lies on the lower boundary, is exceptional; however, the singular solution of the Rayleigh equation then may be excluded (see § 4).

In the present investigation, I establish complementary variational formulations that provide upper and lower bounds to $G = G(c,k)$. As a first, brief example, I improve, and provide a measure of the truncation error in, Burns's long-wave ($k \ll 1$) approximation. As a more detailed example, I consider the stationary wave and derive variational approximations to the critical wavenumber k_* for the parabolic profile (1.4). These last results are relevant to the earlier controversy over the existence of upstream waves for large Froude numbers (see Benjamin 1962; Velthuizen & Wijngaarden 1969; Yih 1972; and Yih & Schultz 1999). In particular, the limit $F \uparrow \infty$ in (4.2) yields the asymptote

$$k_* h \sim (gh/\langle U^2\rangle)^{1/2} \equiv 1/\langle F\rangle, \tag{1.9}$$

where $\langle F\rangle$ is the Froude number based on the r.m.s. flow speed $\langle U^2\rangle^{1/2}$.

2. Variational formulations

Introducing the normalized streamline inclination θ and the dimensionless perturbation pressure $\tilde{\omega}$ through the transformations (Miles 1962)

$$\phi(y)/U_1 h = (U-c)\theta(y) = (U-c)^{-1}\tilde{\omega}'(y), \quad k^2\tilde{\omega}(y) = Q\theta'(y), \tag{2.1a,b}$$

where

$$y = y/h, \quad U(y) = U(y)/U_1, \quad Q = (U-c)^2, \tag{2.2a–c}$$

we transform (1.1) and (1.2a,b) to the complementary Sturm–Liouville systems

$$(Q\theta')' - k^2 Q\theta = 0 \quad (0 < y < 1, \quad ' \equiv \mathrm{d}/\mathrm{d}y), \tag{2.3}$$

$$(U-c)\theta = 0 \quad (y = 0), \quad Q\theta' = G\theta \quad (y = 1), \tag{2.4a,b}$$

and

$$(Q^{-1}\tilde{\omega}')' - k^2 Q^{-1}\tilde{\omega} = 0, \tag{2.5}$$

$$(U-c)^{-1}\tilde{\omega}' = 0 \quad (y = 0), \quad G\omega' = k^2 Q\tilde{\omega} \quad (y = 1), \tag{2.6a,b}$$

where c, k, and G are defined by (1.5), and either $c < 0$ or $c > 1$.

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Multiplying (2.3) by θ, integrating by parts over $0 < y < 1$, invoking (2.4*a*, *b*), and dividing by $\theta_1^2 \equiv \theta^2(1)$, we obtain the variational integral

$$G = \frac{1}{\theta_1^2}\int_0^1 (\theta'^2 + k^2\theta^2)Q\,\mathrm{d}y, \tag{2.7}$$

which is stationary with respect to variations of θ about the true solution of (2.3) and (2.4), is invariant under a scale transformation of θ (so that we may choose $\theta_1 = 1$), and provides an upper bound to the true value of G. Similarly,

$$\frac{1}{G} = \frac{1}{k^2\tilde{\omega}_1^2}\int_0^1 \frac{(\tilde{\omega}'^2 + k^2\tilde{\omega}^2)}{Q}\mathrm{d}y \tag{2.8}$$

provides a lower bound to the true value of G.

3. Long-wave approximation for running waves

Burns's (1953) solution of (2.3) and (2.4) for $k = 0$ is given by

$$\theta = \theta_1\frac{R(y)}{R_1}, \quad R(y) = \int_0^y \frac{\mathrm{d}y}{Q}, \quad R_1 \equiv R(1). \tag{3.1a–c}$$

Adopting (3.1*a*) as a trial function in (2.7), we obtain

$$G = G_0(c) + k^2G_1(c), \quad G_0 = \frac{1}{R_1}, \quad G_1 = \frac{1}{R_1^2}\int_0^1 QR^2\,\mathrm{d}y. \tag{3.2a–c}$$

The error in (3.1*a*) is $O(k^2)$, whence that in the variational approximation (3.2*a*) is $O(k^4)$. We remark that (3.2) remains valid for $c \uparrow 0$, in which limit it reduces to the dominant term in (4.3).

Combining (l.6), (1.7) and (3.2*a*), we obtain

$$D = D_0(c) + k^2D_1(c), \quad D_0 = c \mp G_0^{1/2}, \quad D_1 = \mp\tfrac{1}{2}(G_0^{-1/2}G_1 - \tfrac{1}{3}G_0^{1/2})\begin{pmatrix} c > 1 \\ c < 0 \end{pmatrix}. \tag{3.3a–c}$$

It follows from (3.3*b*) and (3.2*b*) that

$$0 < D_0 < \langle U\rangle \quad \text{for} \quad 0 < -c < \infty \tag{3.4a}$$

and

$$1 > D_0 > \langle U\rangle \quad \text{for} \quad 1 < c < \infty, \tag{3.4b}$$

where $\langle U\rangle$ is the dimensionless, depth-averaged flow speed.

The results (3.2*b*, *c*) and (3.3*b*, *c*) are plotted in figures 1 and 2 for the parabolic profile (1.4), for which

$$U = 2y - y^2. \tag{3.5}$$

4. Stationary wave

The stationary wave ($c = 0$) is distinguished by the presence of the Rayleigh-equation singularity of exponents 0 and 1 at the lower boundary. The boundary condition (2.4*a*) then requires that the former solution be rejected, and hence that $\theta(y)$ be regular at $y = 0$.

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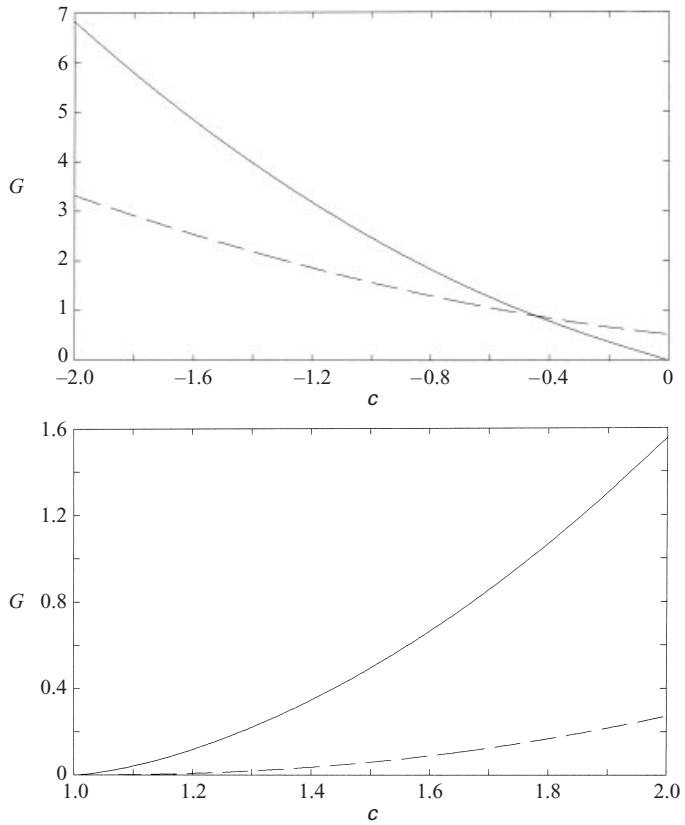


FIGURE 1. $G_0(c)$ (——) and $G_1(c)$ (– – –), as determined by (3.2b, c) for the parabolic profile (1.4).

Considering first the long-wave regime, we expand the solution of (2.3) and (2.4), with $Q = U^2$ therein, in powers of k^2 to obtain the trial function

$$\theta = 1 - k^2 \int_y^1 \frac{P}{U^2} \mathrm{d}y + O(k^4), \quad P \equiv \int_0^y U^2 \, \mathrm{d}y. \tag{4.1a, b}$$

Substituting (4.1) into (2.7) and integrating by parts, we obtain the upper bound

$$G = k^2 \left[P_1 - k^2 \int_0^1 (P/U)^2 \, \mathrm{d}y + k^4 \int_0^1 U^2 \left(\int_y^1 (P/U)^2 \, \mathrm{d}y \right)^2 \mathrm{d}y \right] + O(k^8), \tag{4.2}$$

in which $P_1 = \langle U^2 \rangle$ and the error is of the order of the square of that in the trial function. The limit $k \downarrow 0$ ($F \uparrow \infty$) of (4.2) yields (1.9).

For the parabolic profile (3.5), (4.2) reduces to

$$G = \tfrac{8}{15} k^2 - 0.06036 k^4 + 0.00194 k^6 + O(k^8), \tag{4.3}$$

the inversion of which yields (see figure 3)

$$k_*^2 = \tfrac{15}{8} G + 0.3180 G^2 + 0.1450 G^3 + O(G^4). \tag{4.4}$$

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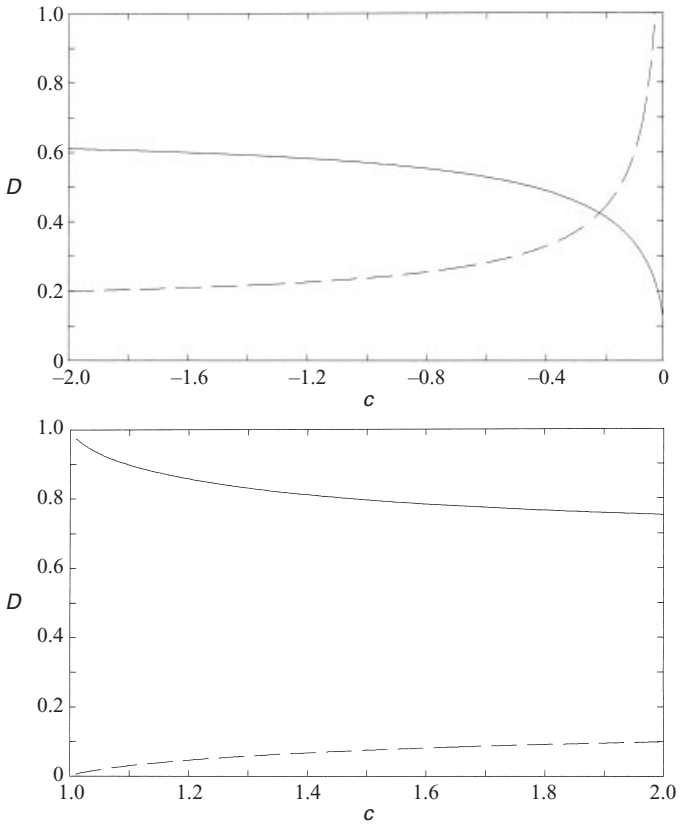


FIGURE 2. $D_0(c)$ (——) and $D_1(c)$ (– – –), as determined by (3.3*b*, *c*) for the parabolic profile (1.4).

The asymptotic solution of (1.1) and (1.2*a*) for $k \uparrow \infty$ (which is equivalent to that for uniform flow), $\phi \sim \sinh ky / \sinh k$, yields the short-wave trial function

$$\theta(y) = \frac{\sinh ky}{U(y)\sinh k}. \tag{4.5}$$

Substituting (4.5) into (2.7), integrating by parts, and invoking $U_1' = 0$, we obtain

$$G = \frac{1}{\sinh^2 k}\int_0^1 \left[k^2\cosh 2ky - k\frac{U'}{U}\sinh 2ky + \left(\frac{U'\sinh ky}{U}\right)^2\right]\mathrm{d}y \tag{4.6a}$$

$$= k\coth k\,[1 - k^{-2}I(-U''/U)], \tag{4.6b}$$

where

$$I[f(y)] = \frac{2k}{\sinh 2k}\int_0^1 f(y)\sinh^2 ky\,\mathrm{d}y \tag{4.7a}$$

$$\sim \frac{1}{2}\sum_{n=0}^{\infty}(-)^n(2k)^{-n}[(\mathrm{d}/\mathrm{d}y)^n f(y)]_{y=1} \quad (k \uparrow \infty). \tag{4.7b}$$

Turning to the complementary variational approximation, we substitute (4.5) into

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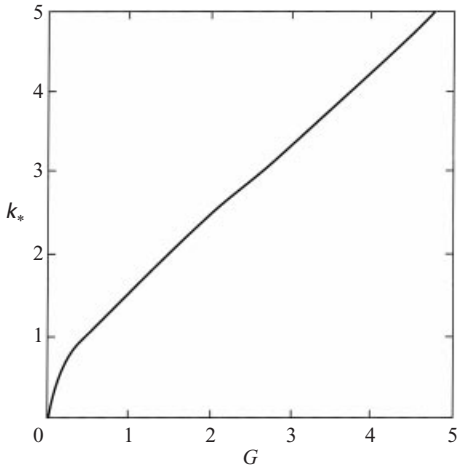


FIGURE 3. $k_*(G)$, as approximated by (4.4) for $G < 2.1$ and (4.12) for $G > 2.1$.

(2.1*b*) to obtain the trial function

$$\tilde{\omega} = \frac{U\cosh ky - k^{-1}U'\sinh ky}{\cosh k}. \tag{4.8}$$

Substituting (4.8) into (2.8) and proceeding as in (4.6), we obtain

$$\frac{1}{G} = \operatorname{sech}^2 k \int_0^1 \left[\cosh 2ky - \frac{U''}{U}\frac{\sinh^2 ky}{k^2} + \left(\frac{U''}{U}\right)^2 \frac{\sinh^2 ky}{k^4}\right] dy \tag{4.9a}$$

$$= k^{-1}\tanh k\{1 + k^{-2}I(-U''/U) + k^{-4}I[(U''/U)^2]\}. \tag{4.9b}$$

For the parabolic profile (3.5), equations (4.6*b*), (4.7*b*) and the inverse of (4.9*b*) yield the lower and upper bounds (in each of which the first two terms are exact)

$$G = k - k^{-1} - \tfrac{1}{2}k^{-3} - k^{-5} + O(k^{-7}) \quad \text{and} \quad G \sim k - k^{-1} - \tfrac{3}{2}k^{-3} + k^{-5} + O(k^{-7}). \tag{4.10a,b}$$

Empirical evidence (Miles 1962) suggests that the average of these bounds,

$$G = k - k^{-1} - k^{-3} + O(k^{-7}), \tag{4.11}$$

is superior to either of them. The inverse of (4.11)

$$k_* = G + G^{-1} + O(G^{-5}), \tag{4.12}$$

which intersects (4.4) at $G = 2.1$ and differs therefrom by less than 3% for $1.8 < G < 2.4$, is plotted in figure 3.

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