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An eigenvalue optimization problem for the p-Laplacian

Anisa M. H. Chorwadwala
Indian Institute of Science Education and Research, Pune, India
(anisa@iiserpune.ac.in)

Rajesh Mahadevan*
Departamento de Matemática, Universidad de Concepción, Concepción, Chile (rmahadevan@udec.cl)

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It has been shown by Kesavan (*Proc. R. Soc. Edinb.* A **133** (2003), 617–624) that the first eigenvalue for the Dirichlet Laplacian in a punctured ball, with the puncture having the shape of a ball, is maximum if and only if the balls are concentric. Recently, Emamizadeh and Zivari-Rezapour (*Proc. Am. Math. Soc.* **136** (2007), 1325–1331) have tried to generalize this result to the case of the p-Laplacian but could succeed only in proving a domain monotonicity result for a weighted eigenvalue problem in which the weights need to satisfy some artificial conditions. In this paper we generalize the result of Kesavan to the case of the p-Laplacian ($1 < p < \infty$) without any artificial restrictions, and in the process we simplify greatly the proof, even in the case of the Laplacian. The uniqueness of the maximizing domain in the nonlinear case is still an open question.

Keywords: shape optimization; Dirichlet p-Laplacian; shape derivative analysis; moving plane method; comparison principles

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1. Introduction

The p-Laplacian Δ_p ($1 < p < \infty$) is the nonlinear operator defined as $\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f)$. Let B_1 be an open ball in $\mathbb{R}^N$. Let B_0 be another open ball, whose closure is contained in B_1, that is free to move inside B_1. Let $\Omega = B_1 \setminus \overline{B_0}$. We consider the following domain optimization problem: given the eigenvalue problem

$$\left.\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2}u \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

whose principal eigenvalue is

$$\lambda_1(\Omega) := \inf\left\{ \frac{\|\nabla\varphi\|^p_{L^p(\Omega)}}{\|\varphi\|^p_{L^p(\Omega)}} \;\middle|\; \varphi \in W^{1,p}_0(\Omega) \setminus \{0\} \right\}, \tag{1.2}$$

we are interested in maximizing $\lambda_1(\Omega)$ with respect to the position of B_0.

*Corresponding author.

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In the linear case, i.e. for $p = 2$, it has been shown by Ashbaugh and Chatelain (personal communication, 2012), Harrell *et al.* [9], Kesavan [10], and Ramm and Shivakumar [14] that the first eigenvalue λ_1 of problem (1.2) attains its maximum if and only if the balls are concentric.

Analogous results for manifolds, in the linear case, were obtained by Anisa and Aithal [3] in the setting of space-forms (complete simply connected Riemannian manifolds of constant sectional curvature) and by Anisa and Vemuri [4] in the setting of rank 1 symmetric spaces of non-compact type. We extend these results, in a different direction, to the setting of the p-Laplacian operator, which is a nonlinear elliptic operator. Our main result is theorem 3.1.

The idea of the proof in [9, 10] basically consists of obtaining the sign of the *shape derivative* of the eigenvalue functional. For determining the sign of the shape derivative at a given position of the hole, the normal derivatives of the eigenfunction on opposite sides of the hole with respect to a given plane of symmetry of the hole need to be compared. This is achieved, following Serrin [16], by the use of the *moving plane method*, which goes back to Aleksandrov [1].

In the linear case, for the eigenvalue problem, this comparison analysis is done via some intricate arguments in Kesavan [10]. We propose a simpler analysis in which we directly apply a weak comparison principle proved recently in Chorwadwala *et al.* [5]. Thus, in addition to partly extending the result to the case of the p-Laplacian, we greatly simplify the analysis, even in the linear case.

A domain monotonicity result for a p-Laplacian eigenvalue problem has been proved by Emamizadeh and Zivari-Rezapour [7] as we do here. But the authors have to consider a weighted eigenvalue problem of the form (1.1) with a sign-changing weight f on the right-hand side. The conditions imposed on f therein do not permit the analysis of the weight $f = 1$. In fact, the main difficulty with their approach is that their weights need to be roughly negative in order to adapt to the classical weak comparison principle of Tolksdorff [19]. So, strictly speaking, their result does not directly extend the result of Ramm and Shivakumar [14], which they cite. The weak comparison principle we have used here from Chorwadwala [5] applies to positive weights, and thus extends the previously known result from the linear case. Like us, they are also unable to show the strict monotonicity.

For the sake of those interested, we would like to point out that in [7] the development of the shape derivative for the eigenvalue functional is purely formal since they do not show the existence of the shape derivative u', and several regularity issues are ignored.

In § 2 we establish some notation and introduce some definitions and technical preliminaries. We recall the formula for the shape derivative of the first eigenvalue of the Dirichlet p-Laplacian defined through (1.2). Finally, in § 3 we prove the main result by analysing the sign of the shape derivative.

2. Preliminaries

In this section we introduce some notions and results that will be used later on.

Properties related to the first eigenvalue of the Dirichlet p-Laplacian. Given a bounded domain Ω in $\mathbb{R}^n$ with Lipschitz boundary, let $\lambda_1(\Omega)$ denote the first eigenvalue of the Dirichlet p-Laplacian defined through (1.2). Then $\lambda_1(\Omega)$ is invariant

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under orthogonal transformations of the domain. For a bounded connected domain Ω, it is known that the first eigenvalue is simple. It is also known that any eigenfunction for λ_1 has a strict sign in Ω and is the only eigenfunction with this property. We refer the reader to Lindqvist [12] for these results. Hereafter, by first eigenfunction we shall mean a positive eigenfunction whose L^p-norm is 1. In a domain Ω with $C^{2,\gamma}$ boundary, the normal derivative of the first eigenfunction is strictly negative on $\partial\Omega$ (see [15, lemma A.3]). Furthermore, if Ω has a connected $C^{2,\gamma}$ boundary, then the first eigenfunction is known to belong to $C^{1,\alpha}(\bar{\Omega}) \cap C^{2,\gamma}(\overline{\Omega_\varepsilon})$ for some $\varepsilon > 0$, where $\Omega_\varepsilon = \{x \in \Omega\colon d(x, \partial\Omega) < \varepsilon\}$ (see [2, theorem 1.3]).

Shape derivative. Given a shape functional J that depends on the domain Ω (usually, a smooth open set in $\mathbb{R}^N$) and given a variation of the domain Ω by a fairly smooth perturbative vector field V that has its support in a neighbourhood of $\partial\Omega$, the infinitesimal variation of J in the direction V is defined as

$$J'(\Omega; V) = \lim_{t\to 0} \frac{J(\Omega_t) - J(\Omega)}{t}, \tag{2.1}$$

where, for t sufficiently small, Ω_t is the diffeomorphic image $\Phi_t(\Omega)$ of Ω under the smooth perturbation of the identity map, namely, $\Phi_t(x) = (I + tV)(x)$. This shape derivative is a frequently used tool in the analysis of shape optimization problems for understanding the variations of shape functionals (see [17, 18]).

We now recall the shape derivative for the eigenvalue functional $\lambda_1(\Omega)$ defined through (1.2). We have

$$\lambda_1'(\Omega; V) = -(p-1) \int_{\partial\Omega} \left|\frac{\partial y_1}{\partial n}\right|^p \langle V, n\rangle \, \mathrm{d}S, \tag{2.2}$$

where y_1 is the first eigenfunction and n is the vector field of outward unit normals to Ω on $\partial\Omega$. This result has been proved in [8, 11].

Weak comparison principle for the p-Laplacian. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $\beta\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be continuous and assume that there exists $\kappa \in [0, 1]$ and $M > 0$ such that β satisfies the growth assumption

$$\beta(x, s) \leqslant M(\kappa + |s|^{p-2})|s| \quad \text{for all } x \in \Omega, \text{ for all } s \text{ in } \mathbb{R}. \tag{2.3}$$

The following weak comparison principle, which generalizes [6, proposition 2.3], has been proved in [5].

THEOREM 2.1. *Let u, v in $W^{1,p}(\Omega)$ be non-negative weak solutions of*

$$\left.\begin{array}{ccc} -\Delta_p u = \beta(x,u) + f(x), & -\Delta_p v = \beta(x,v) + g(x) & \text{in } \Omega, \\ u = f', & v = g' & \text{on } \partial\Omega, \end{array}\right\} \tag{2.4}$$

where Ω is just a Lipschitz domain and we assume that

(*A*1) *$\beta(x, s)$ is non-decreasing in s for $(x, s) \in \Omega \times \mathbb{R}$, and $\beta(x, 0) \geqslant 0$ for all $x \in \Omega$;*

(*A*2′) *the problem*

$$\begin{array}{cc} -\Delta_p w = \beta(x, w) + f & \text{in } \Omega, \\ w = f' & \text{on } \partial\Omega, \end{array}$$

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given $f \in W^{-1,p/(p-1)}(\Omega)$, $f \geqslant 0$, in Ω, and $f' \in W^{1-1/p,p}$, $f' \geqslant 0$, on $\partial\Omega$, admits a unique non-negative solution $w \in W^{1,p}(\Omega)$;

(A3′) *$f, g \in W^{-1,p/(p-1)}(\Omega)$, $0 \leqslant g \leqslant f$, on Ω and $f', g' \in W^{1-1/p,p}(\partial\Omega)$, $0 \leqslant g' \leqslant f'$, on $\partial\Omega$.*

We then conclude that $u \geqslant v$ almost everywhere on Ω, that is, the weak comparison principle holds.

3. Main result

Let $0 < r_0 < r_1$ and let B_1 be a fixed ball $B(0, r_1)$. Consider the family $\mathcal{F} = \{B_1 \setminus \overline{B(y, r_0)} : \overline{B(y, r_0)} \subset B_1\}$ of domains in $\mathbb{R}^N$. We study the extrema of the functional $\lambda_1(\Omega)$ over $\mathcal{F}$, associated with problem (1.1).

Set $\Omega_0 = B(0, r_1) \setminus \overline{B(0, r_0)}$.

THEOREM 3.1. *The first Dirichlet eigenvalue $\lambda_1(\Omega)$ attains its maximum value on $\mathcal{F}$ when $\Omega = \Omega_0$, i.e. when the balls are concentric.*

Proof of theorem 3.1. The proof will be accomplished at the end of the following steps.

STEP 1. In this step we reduce the problem to analysing the sign of the derivative of a function defined on the real line. For this we observe that the functional to be optimized is invariant under the isometries of $\mathbb{R}^N$. Therefore, it is enough to study the optimization problem for the class of domains $\Omega(s) := B_1 \setminus \overline{B(se_1, r_0)}$ for $0 \leqslant s < r_1 - r_0$, where e_1 is the unit vector in the direction of the first coordinate axis. So, if we define $j_1 : (r_0 - r_1, r_1 - r_0) \to \mathbb{R}$ as

$$j_1(s) = \lambda_1(\Omega(s)),$$

then the problem of maximizing λ_1 in the class $\mathcal{F}$ corresponds to studying the maximum of j_1 on the interval $(r_0 - r_1, r_1 - r_0)$. We note, furthermore, that j_1 is an even function and so the optimization problem needs to be studied only on the interval $[0, r_1 - r_0)$.

STEP 2. Now we relate the derivative of the function j_1 to the shape derivative of the functional λ_1.

Let $\rho : \mathbb{R}^N \to [0, 1]$ be a smooth function with compact support in B_1 such that $\rho \equiv 1$ on a neighbourhood of $B(se_1, r_0)$ that we denote by B_0. Let V, defined by $V(x) := \rho(x)e_1$ for all $x \in \mathbb{R}^N$, be the vector field that has the effect of translating the hole $B(se_1, r_0)$ in the direction e_1. Then, given $s \in [0, r_1 - r_0)$, for t sufficiently small, $\Phi_t = I + tV$ is a diffeomorphism between $\Omega(s)$ and $\Phi_t(\Omega(s)) = \Omega(s + t)$. Therefore, the derivative of j_1 at s is exactly the shape derivative of λ_1 at $\Omega(s)$ in the direction V given by (2.2). We obtain that

$$j_1'(s) = -(p-1) \int_{x \in \partial B_0} \left| \frac{\partial y_1}{\partial n}(x) \right|^p n_1(x) \, \mathrm{d}S \tag{3.1}$$

since V is zero on ∂B_1 and since, for all $x \in \partial B_0$, $\langle V, n \rangle(x) = \rho e_1 \cdot n(x) = n_1(x)$, the first component of the normal vector.

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STEP 3. In this step we will write $j_1'(s)$ in a suitable form.

We shall adopt the following notation. Given s in $[0, r_1 - r_0)$, we simply denote $\Omega(s)$ by Ω and let H denote the hyperplane $H := \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \mid x_1 = s\}$. Let r_H be the reflection function about H. We define $\mathcal{O}$ to be the subdomain $\mathcal{O} := \{x \in \Omega \mid x_1 > s\}$ in Ω. We then see that the reflection of $\mathcal{O}$ about H, namely, $\mathcal{O}' = r_H(\mathcal{O})$, is contained in B_1, whereas B_0 is symmetric with respect to H. Thus, $\mathcal{O}' \subset \Omega$. For $x \in \mathcal{O}$, let x' denote the reflection of x about H, namely, the point $r_H(x)$. We now consider the subdomain $\mathcal{O}$ and let us define $\tilde{y}_1$ on $\mathcal{O}$ by $\tilde{y}_1(x) := y_1(x')$, the value of y_1 at the reflection x' of x about H. Let us note that $(\partial\tilde{y}_1/\partial n)(x) = (\partial y_1/\partial n)(x')$ and that $n_1(x') = -n_1(x)$ for all $x \in \partial B_0$. Now, we may rewrite the expression (3.1) as follows:

$$\begin{aligned} j_1'(s) &= -(p-1)\int_{x\in\partial B_0\cap\partial\mathcal{O}} \left|\frac{\partial y_1}{\partial n}(x)\right|^p n_1(x)\,\mathrm{d}S \\ &\quad -(p-1)\int_{x'\in\partial B_0\cap\partial\mathcal{O}'} \left|\frac{\partial y_1}{\partial n}(x')\right|^p n_1(x')\,\mathrm{d}S \\ &= -(p-1)\int_{x\in\partial B_0\cap\partial\mathcal{O}} \left\{\left|\frac{\partial y_1}{\partial n}(x)\right|^p - \left|\frac{\partial\tilde{y}_1}{\partial n}(x)\right|^p\right\} n_1(x)\,\mathrm{d}S. \end{aligned} \tag{3.2}$$

STEP 4. Since j_1 is a differentiable and even function, we have $j_1'(0) = 0$. To show that j_1 attains its maximum at 0, it is enough to show that $j_1'(s) \leqslant 0$ for all $s \in (0, r_1 - r_0)$.

On $\mathcal{O}$, the function y_1 satisfies

$$\begin{aligned} -\Delta_p y_1 &= \lambda_1 y_1^{p-1} && \text{in } \mathcal{O}, \\ y_1 &= 0 && \text{on } \partial\mathcal{O}\cap\partial B_0, \\ y_1 &= y_1^* && \text{on } \partial\mathcal{O}\cap H, \\ y_1 &= 0 && \text{on } \partial\mathcal{O}\cap\partial B_1, \end{aligned}$$

whereas $\tilde{y}_1$ satisfies

$$\begin{aligned} -\Delta_p \tilde{y}_1 &= \lambda_1 \tilde{y}_1^{p-1} && \text{in } \mathcal{O}, \\ \tilde{y}_1 &= 0 && \text{on } \partial\mathcal{O}\cap\partial B_0, \\ \tilde{y}_1 &= y_1^* && \text{on } \partial\mathcal{O}\cap H, \\ \tilde{y}_1 &> 0 && \text{on } \partial\mathcal{O}\cap\partial B_1, \end{aligned}$$

where y_1^* denotes the common value of y_1 and $\tilde{y}_1$ on H. So, we obtain that $\tilde{y}_1 \geqslant y_1$ on $\partial\mathcal{O}$. Hence, by theorem 2.1 applied with $\beta(s) = \lambda_1 s^{p-1}$, we obtain $\tilde{y}_1 \geqslant y_1$ in $\mathcal{O}$. Since $\tilde{y}_1 \equiv 0 \equiv y_1$ on $\partial B_0 \cap \partial\mathcal{O}$, we conclude that

$$\frac{\partial\tilde{y}_1}{\partial n}(x) \leqslant \frac{\partial y_1}{\partial n}(x) \quad \text{for all } x \in \partial B_0 \cap \partial\mathcal{O}. \tag{3.3}$$

We have $(\partial y_1/\partial n)(x) < 0$ on $\partial B_0 \cup \partial B_1$ by the properties listed for the first eigenfunction in § 2 and $n_1 < 0$ on $\partial\mathcal{O}\cap\partial B_0$. So, from (3.3) and (3.2), we conclude that $j_1'(s) \leqslant 0$, which means that the first eigenvalue is a maximum when the hole B_0 is concentric with the ball B_1.

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STEP 5. We show that when $p = 2$, $j_1'(s) < 0$ for all $s \in (0, r_1 - r_0)$. This has been proved in [10] but now we can give a very short proof of our own.

First we deduce that $y_1 < \tilde{y_1}$ in $\mathcal{O}$. To see this, let $w := \tilde{y_1} - y_1$. Then, by the previous step, $w \geqslant 0$ on Ω with $w = 0$ on $\partial\mathcal{O} \cap (H \cup \partial B_0)$ and $w > 0$ on $\partial\mathcal{O} \cap \partial B_1$. Taking the difference of the two equations for $\tilde{y_1}$ and y_1, we have

$$-\Delta w \geqslant 0 \quad \text{in } \mathcal{O} \quad \text{and} \quad w \geqslant 0 \quad \text{on } \mathcal{O} \tag{3.4}$$

in the weak sense. By the strong maximum principle [13, ch. 2, § 3, theorem 5], we need to have either $w > 0$ in $\mathcal{O}$ or $w \equiv 0$ in $\mathcal{O}$. However, since $w > 0$ on $\partial\mathcal{O} \cap \partial B_1$, we conclude that only $w > 0$ can hold in $\mathcal{O}$. Then, since $w = 0$ on $\partial\mathcal{O} \cap \partial B_0$, it follows from [13, ch. 2, § 3, theorem 7] that $\partial w/\partial n < 0$ on $\partial\mathcal{O} \cap \partial B_0$. That is, for all $x \in \partial\mathcal{O} \cap \partial B_0$ we have the following:

$$\frac{\partial \tilde{y_1}}{\partial n}(x) < \frac{\partial y_1}{\partial n}(x) < 0.$$

This leads to the conclusion that $j_1'(s) < 0$. □

REMARK 3.2. The uniqueness of the maximizing domain for $p \neq 1$ can be proved provided it can be shown that $\tilde{y_1} > y_1$ in a neighbourhood of a point on $\partial\mathcal{O} \cap \partial B_0$. We believe the result to be true but we have a technical impediment in the form of a strong comparison principle, which is not available so far for this case. All strong comparison results available so far are applicable for non-negative solutions that are zero on the boundary. Unfortunately, the functions y_1 and $\tilde{y_1}$ are not zero on the entire boundary of $\mathcal{O}$, but only on parts of it.

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