THE EIGENVALUES OF ELECTROMAGNETIC ANGULAR MOMENTUM

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Introduction

It has been generally recognized that the eigenvalues of the angular momentum of a radiation field are integral multiples of $\hbar$, and that half-integral multiples cannot occur, but no proof of this seems ever to have been given. Born and Infeld, in a paper on their new field theory†, have in fact tentatively assumed that half-integral values are possible. It is therefore of interest to clear this matter up.

It might be thought that the presence of charged particles in the field could in some way cause the occurrence of half-integral values. The problem is therefore worked out here for the general case of a field containing charged particles. The pure radiation field is a particular case of this, with the number of particles equal to zero. The result is that *the angular momentum can take only integral values*, in units of $\hbar$.

Although they interpret their results as applying to material systems, Born and Infeld give no analysis of (point) charges in the field, and in fact, by the use of the equation $\operatorname{div} \mathbf{D} = 0$, exclude the possibility of their occurrence. Their results must therefore be considered to apply to pure radiation fields only. It is possible, however, to generalize the Born-Infeld quantum electrodynamics to take in point charges‡. The theory is formally identical with the current (Heisenberg-Pauli) theory, with the mass of the particles zero and the energy density function replaced by the one given by the Born-Infeld theory. The reasoning of this paper is independent of these and will therefore apply to the Born-Infeld theory as well as to the usual electrodynamics.

The total angular momentum of the composite system is the sum of two independent parts. The first is the sum of the orbital angular momentum of the particles and the electromagnetic angular momentum, which do not separate completely. The second is the spin angular momentum, $\pm \frac{1}{2}\hbar$ for each particle. It is not of electromagnetic origin, but arises from the α's of Dirac's relativistic theory of the electron; it is quite different from the "inner angular momentum"

† M. Born and L. Infeld, "On the quantization of the new field theory. II", *Proc. Roy. Soc.* A, 150 (1935), 141.

‡ M. H. L. Pryce, "On the new field theory. II", *Proc. Roy. Soc.* A (in the Press); quoted as A.

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of Born and Infeld, which they have attempted to identify with spin, but which really belongs to the first part.

I. The angular momentum and the commutation rules

The angular momentum (apart from spin) is

$$\mathbf{m} = \int \mathbf{r} \times \mathbf{S} dv + \sum_s \boldsymbol{\xi}_s \times \boldsymbol{\Pi}_s. \tag{1·1}$$

Here $\mathbf{S}$ is the Poynting vector $\mathbf{S} = \mathbf{D} \times \mathbf{B}$†;

$\mathbf{r}$ is the position vector $(x, y, z) = (x_1, x_2, x_3)$; its components are c-numbers;

dv is the element of volume $dx\,dy\,dz$;

$\boldsymbol{\xi}_s$ is the coordinate vector $(\xi_s, \eta_s, \zeta_s) = (\xi_{s1}, \xi_{s2}, \xi_{s3})$ of the sth particle;

$\boldsymbol{\Pi}_s$ is the gauge-invariant vector $\mathbf{p}_s - e/c\mathbf{A}_s$, which plays an important role in quantum electrodynamics‡; it also occurs in the Born-Infeld theory.

$\mathbf{D}$ and $\mathbf{B}$ are expressed in reduced units (i.e. divided by $\sqrt{(4\pi)}$). They satisfy the divergence relations

$$\operatorname{div} \mathbf{B} = 0, \tag{1·2}$$

$$\operatorname{div} \mathbf{D} = \Delta = \sum_s \Delta_s, \tag{1·3}$$

where

$$\Delta_s = \sqrt{(4\pi)}\, e_s \delta(x - \xi_s)\, \delta(y - \eta_s)\, \delta(z - \zeta_s)$$

and e_s is the charge on the sth particle.

The velocity of light is chosen to be unity.

The commutation rules for the quantities involved are summarized in the following equations:

$$[D_\alpha(x, y, z), B_\beta(x', y', z')] = -\epsilon_{\alpha\beta\gamma} \frac{\partial}{\partial x_\gamma} \delta(x - x')\, \delta(y - y')\, \delta(z - z'), \tag{1·4}$$

$$[\xi_{s\alpha}, \Pi_{s\beta}] = \delta_{\alpha\beta}, \tag{1·5}$$

$$[\Pi_{s\alpha}, \Pi_{s\beta}] = -\epsilon_{\alpha\beta\gamma} \int \Delta_s B_\gamma dv \S, \tag{1·6}$$

$$[\Pi_{s\alpha}, D_\beta] = \Delta_s \delta_{\alpha\beta}. \tag{1·7}$$

All other brackets vanish.

In these equations the symbols have the following meanings:

The bracket of two quantities A and B is defined by

$$AB - BA = i\hbar\,[A, B].$$

The indices α, β, γ label the components of a vector; they take on the values 1, 2, 3.

† The field strengths $\mathbf{D}$ and $\mathbf{B}$ at the same point commute. This happens because the commutation brackets for different points contain the *derivative* of a delta function, which vanishes at the point zero. The order of the factors is therefore immaterial. That this is so may be verified at every stage of the calculation.

‡ See Pauli, *Handbuch der Physik*, 24 (1), (1933), 2nd edition, p. 233, equation 77 and p. 261, equation 261.

§ This can also be written symbolically: $-\epsilon_{\alpha\beta\gamma}\sqrt{(4\pi)}\, e_s B_{s\gamma}$.

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$\epsilon_{\alpha\beta\gamma}$ is the completely antisymmetrical array of third rank.

$\delta(x-x')$, etc. are Dirac delta functions.

The index s labels the particles.

From these equations it is easy to verify (see A) that $\mathbf{m}$ satisfies the usual commutation rule for angular momentum:

$$\mathbf{m} \times \mathbf{m} = i\hbar\mathbf{m}.$$

In the above scheme the potentials and momenta play no independent part. It is well known that the introduction of potentials A_α with the commutation rules

$$[D_\alpha(x,y,z), A_\beta(x',y',z')] = \delta_{\alpha\beta}\delta(x-x')\delta(y-y')\delta(z-z'), \qquad (1\cdot8)$$

and momenta which commute with $\mathbf{D}$, leads to an incompatibility with equation (1·3), and that if one wants to do this, one must replace (1·3) by a subsidiary condition acting on allowable wave-functions:

$$\{\operatorname{div}\mathbf{D} - \Delta\}\psi = 0.$$

In the present discussion we shall not introduce potentials satisfying (1·8), and (1·3) will therefore be valid. We shall, however, construct quantities with some of the properties of potentials, which do not satisfy all the equations (1·8), and are consistent with (1·3).

II. LEMMA

Before going any further we shall prove the following lemma, which will be used later:

If two (non-hermitian) quantum operators α and β, and their hermitian conjugates α^* and β^*, commute with all other independent operators, and among themselves satisfy the following commutation rules

$$\left.\begin{array}{ll} [\alpha, \alpha^*] = 0, & [\beta, \beta^*] = 0, \\ [\alpha, \beta] = 0, & [\alpha^*, \beta^*] = 0, \\ [\alpha, \beta^*] = 1, & [\alpha^*, \beta] = 1, \end{array}\right\} \qquad (2\cdot1)$$

then the operator

$$\lambda = -i(\alpha\beta^* - \alpha^*\beta) \qquad (2\cdot2)$$

has the eigenvalues $\lambda' = N\hbar$, where N is any integer, positive, zero or negative.

We define the following *hermitian* operators

$$x = i(\alpha^* - \alpha)/\sqrt{2}, \qquad y = (\alpha^* + \alpha)/\sqrt{2},$$
$$u = i(\beta^* - \beta)/\sqrt{2}, \qquad v = (\beta^* + \beta)/\sqrt{2}.$$

They then satisfy the commutation rules

$$\left.\begin{array}{ll} [x, y] = 0, & [u, v] = 0, \\ [x, v] = 0, & [y, u] = 0, \\ [x, u] = 1, & [y, v] = 1, \end{array}\right\} \qquad (2\cdot3)$$

and in terms of them λ is

$$\lambda = xv - yu. \qquad (2\cdot4)$$

From the theory of the angular momentum of a particle we know that (2·3) and (2·4) imply that λ has the eigenvalues $\lambda' = N\hbar$. This is so familiar that no proof of it need be given here.

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III. The auxiliary variables

We shall consider the angular momentum about the z-axis, and for this it is convenient to use cylindrical polar coordinates r, z, θ, related to the cartesian coordinates by $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$

We denote the radial, axial and transverse components of a vector by the suffixes r, z, θ. Then in these coordinates the divergence relations (1·2) and (1·3) run

$$\frac{\partial}{\partial r}(rB_r) + \frac{\partial}{\partial z}(rB_z) + \frac{\partial B_\theta}{\partial \theta} = 0, \tag{3·1}$$

$$\frac{\partial}{\partial r}(rD_r) + \frac{\partial}{\partial z}(rD_z) + \frac{\partial D_\theta}{\partial \theta} = r\Delta. \tag{3·2}$$

We may therefore express B_z and D_z in terms of the other components by integrating these equations along a path parallel to the z-axis. We have

$$B_z(r,z,\theta) = -\frac{1}{r}\int_{-\infty}^{z}\left[\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_\theta}{\partial \theta}\right]_{r,z',\theta} dz', \tag{3·3}$$

$$D_z(r,z,\theta) = -\frac{1}{r}\int_{-\infty}^{z}\left[\frac{\partial}{\partial r}(rD_r) + \frac{\partial D_\theta}{\partial \theta} - r\Delta\right]_{r,z',\theta} dz'. \tag{3·4}$$

The r and θ components are independent of one another and may be chosen as the variables to describe the field. It is more convenient, however, to replace B_r and B_θ by two auxiliary quantities $\tilde{A}_r$ and $\tilde{A}_\theta$ defined as follows:

$$\tilde{A}_r(r,z,\theta) = -\int_{-\infty}^{z} B_\theta(r,z',\theta)\,dz', \tag{3·5}$$

$$\tilde{A}_\theta(r,z,\theta) = \int_{-\infty}^{z} B_r(r,z',\theta)\,dz'. \tag{3·6}$$

They have the property that

$$\frac{\partial \tilde{A}_\theta}{\partial z} = B_r,$$

$$-\frac{\partial \tilde{A}_r}{\partial r} = B_\theta,$$

$$\frac{1}{r}\left[\frac{\partial \tilde{A}_r}{\partial \theta} - \frac{\partial}{\partial r}(r\tilde{A}_\theta)\right] = -\frac{1}{r}\int_{-\infty}^{z}\left[\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_\theta}{\partial \theta}\right]_{r,z',\theta} dz'$$

$$= B_z. \tag{3·7}$$

The last equation follows from (3·3).

The vector $\tilde{\mathbf{A}} = (\tilde{A}_r, 0, \tilde{A}_\theta)$ therefore behaves as a vector potential, in that

$$\operatorname{curl}\tilde{\mathbf{A}} = \mathbf{B}. \tag{3·8}$$

We can now introduce pseudo-momenta $\tilde{\mathbf{p}}$, defined by

$$\tilde{\mathbf{p}}_s = \mathbf{\Pi}_s + \int \Delta_s \tilde{\mathbf{A}}\,dv. \tag{3·9}$$

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It is easy to prove that the auxiliary quantities satisfy the following commutation rules:

$$[\tilde{p}_{s\alpha}, \tilde{p}_{s\beta}] = 0, \qquad [\tilde{p}_{s\alpha}, \xi_{s\beta}] = \delta_{\alpha\beta}, \tag{3·10}$$

$$\left.\begin{aligned} &[D_r(r,z,\theta), \tilde{A}_r(r',z',\theta')] = \frac{1}{r}\delta(r-r')\,\delta(z-z')\,\delta(\theta-\theta')\dagger, \\ &[D_r(r,z,\theta), \tilde{A}_\theta(r',z',\theta')] = 0, \\ &[D_\theta(r,z,\theta), \tilde{A}_r(r',z',\theta')] = 0, \\ &[D_\theta(r,z,\theta), \tilde{A}_\theta(r',z',\theta')] = \frac{1}{r}\delta(r-r')\,\delta(z-z')\,\delta(\theta-\theta'), \end{aligned}\right\} \tag{3·11}$$

$$\left.\begin{aligned} &[\tilde{\mathfrak{p}}_s, D_r] = 0, \qquad [\tilde{\mathfrak{p}}_s, D_\theta] = 0, \\ &[\tilde{\mathfrak{p}}_s, \tilde{A}_r] = 0, \qquad [\tilde{\mathfrak{p}}_s, \tilde{A}_\theta] = 0. \end{aligned}\right\} \tag{3·12}$$

The quantities ξ_s, $D_r(r,z,\theta)$, $D_\theta(r,z,\theta)$ and $\tilde{\mathfrak{p}}_s$, $\tilde{A}_r(r,z,\theta)$, $\tilde{A}_\theta(r,z,\theta)$ are therefore a set of canonical variables describing the system.

IV. The angular momentum in terms of the auxiliary variables

The angular momentum about the z-axis is

$$m_z = \int r S_\theta dv + \sum_s \{\xi_s \Pi_{sy} - \eta_s \Pi_{sx}\}. \tag{4·1}$$

The first term in this is

$$\begin{aligned} \int r S_\theta dv &= \int r\,(D_r B_z - D_z B_r)\,dv \\ &= \int \left\{ D_r \frac{\partial \tilde{A}_r}{\partial \theta} - D_r \frac{\partial}{\partial r}(r\tilde{A}_\theta) - r D_z \frac{\partial \tilde{A}_\theta}{\partial z} \right\} dv \\ &= \int \left\{ D_r \frac{\partial \tilde{A}_r}{\partial \theta} + D_\theta \frac{\partial \tilde{A}_\theta}{\partial \theta} \right\} dv - \iiint \left\{ D_r \frac{\partial}{\partial r}(r\tilde{A}_\theta) + D_z \frac{\partial}{\partial z}(r\tilde{A}_\theta) + D_\theta \frac{\partial \tilde{A}_\theta}{\partial \theta} \right\} r\,dr\,dz\,d\theta. \end{aligned}$$

The second term here is transformed by partial integration‡:

$$\begin{aligned} \int r S_\theta dv &= \int \left\{ D_r \frac{\partial \tilde{A}_r}{\partial \theta} + D_\theta \frac{\partial \tilde{A}_\theta}{\partial \theta} \right\} dv + \iiint \left\{ \frac{\partial}{\partial r}(rD_r) + \frac{\partial}{\partial z}(rD_z) + \frac{\partial D_\theta}{\partial \theta} \right\} \tilde{A}_\theta\, r\,dr\,dz\,d\theta \\ &= \int \left\{ D_r \frac{\partial \tilde{A}_r}{\partial \theta} + D_\theta \frac{\partial \tilde{A}_\theta}{\partial \theta} \right\} dv + \int r \Delta \tilde{A}_\theta dv. \end{aligned} \tag{4·2}$$

The second term in (4·1) can be written

$$\sum_s \{\xi_s \Pi_{sy} - \eta_s \Pi_{sx}\} = \sum_s \{\xi_s \tilde{p}_{sy} - \eta_s \tilde{p}_{sx}\} - \int \sum_s \Delta_s (\xi_s \tilde{A}_y - \eta_s \tilde{A}_x)\, dv.$$

† $\frac{1}{r}\delta(r-r')\,\delta(z-z')\,\delta(\theta-\theta')$ is the same as $\delta(x-x')\,\delta(y-y')\,\delta(z-z')$.

‡ The integrals over the boundary at infinity are supposed to vanish. If this assumption were not made, the integral in the angular momentum would not be convergent. It can be justified by restricting ourselves to states in which mean field tends to zero at infinity sufficiently rapidly, or by introducing a periodicity condition, or equivalently a closed space.

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Since Δ_s is a delta function which vanishes everywhere except at $\mathbf{r}=\boldsymbol{\xi}_s$, we may replace the ξ_s and η_s in the integral by x and y; this gives us a term $x\tilde{A}_y - y\tilde{A}_x = r\tilde{A}_\theta$. Thus

$$\sum_s \{\xi_s \Pi_{sy} - \eta_s \Pi_{sx}\} = \sum_s (\xi_s \tilde{p}_{sy} - \eta_s \tilde{p}_{sx}\} - \int r \sum_s \Delta_s \tilde{A}_\theta dv$$

$$= \sum_s \{\xi_s \tilde{p}_{sy} - \eta_s \tilde{p}_{sx}\} - \int r\Delta\tilde{A}_\theta dv. \tag{4·3}$$

Combining (4·2) and (4·3), we have:

$$\boxed{m_z = \int D_r \frac{\partial \tilde{A}_r}{\partial\theta} dv + \int D_\theta \frac{\partial \tilde{A}_\theta}{\partial\theta} dv + \sum_s \{\xi_s \tilde{p}_{sy} - \eta_s \tilde{p}_{sx}\}.} \tag{4·4}$$

If we write

$$\lambda_1 = \int D_r \frac{\partial \tilde{A}_r}{\partial\theta} dv, \tag{4·5}$$

$$\lambda_2 = \int D_\theta \frac{\partial \tilde{A}_\theta}{\partial\theta} dv, \tag{4·6}$$

$$\mu_s = \xi_s \tilde{p}_{sy} - \eta_s \tilde{p}_{sx}, \tag{4·7}$$

this is

$$m_z = \lambda_1 + \lambda_2 + \sum_s \mu_s. \tag{4·8}$$

The angular momentum is therefore separated out into the sum of a part depending only on the field and a part depending only on the particles.

V. The eigenvalues

Looking back at the commutation rules (3·10), (3·11), (3·12), we see that $\lambda_1, \lambda_2; \mu_1, \mu_2, \mu_3, \ldots$, commute with one another, and are independent of one another, in the sense that all variables occurring in the definition of one of them commute with all variables occurring in the definition of any other. We can therefore set up wave equations separately for each. The eigenvalues m_z' of m_z will be the sums of eigenvalues of $\lambda_1', \lambda_2'; \mu_1', \mu_2', \mu_3', \ldots$, and the corresponding eigenfunctions will be linear combinations of products of the partial eigenfunctions belonging to the eigenvalues $\lambda_1', \lambda_2'; \mu_1', \mu_2', \mu_3', \ldots$, leading to the same sum m_z'.

We shall now prove that the eigenvalues of each of the terms $\lambda_1, \lambda_2; \mu_1, \mu_2, \ldots$ are integral multiples of $\hbar$, and therefore that the eigenvalues of m_z are integral multiples of $\hbar$.

This is obvious for the terms $\mu_1, \mu_2, \mu_3, \ldots$, in view of the commutation rules (3·10) and the known theory of orbital angular momentum. Of the remaining two terms it is only necessary to analyse one, let us say λ_1, in detail, since the other is of the same structure, and any argument applying to the one will apply to the other.

The analysis of λ_1 proceeds in two stages. First we expand D_r and A_r as a Fourier series in terms of $e^{in\theta}$, the coefficients depending on r and z. Then, in the

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second stage, we expand these coefficients in terms of some system of normalized orthogonal functions defined in the region $0<r<\infty$, $-\infty<z<+\infty$ with weight r. The orthogonal system may be different for different values of n (compare the expansion in spherical harmonics).

Let us therefore write

$$\left.\begin{aligned} D_r &= (2\pi)^{-\frac{1}{2}} \sum_n D_n e^{in\theta}, \\ A_r &= (2\pi)^{-\frac{1}{2}} \sum_n A_n e^{in\theta}. \end{aligned}\right\} \tag{5·1}$$

Owing to the hermitian character of D_r and A_r, we have the symmetry relation

$$D_{-n}=D_n^*, \quad A_{-n}=A_n^*. \tag{5·2}$$

Putting this expansion into (4·5), we get

$$\lambda_1 = \sum_{n=-\infty}^{\infty} n \iint -iD_n A_{-n} r\,dr\,dz,$$

which we can write with the help of (5·2) as

$$\lambda_1 = \sum_{n=1}^{\infty} n \iint -i(D_n A_n^* - D_n^* A_n) r\,dr\,dz. \tag{5·3}$$

This completes the first stage.

We now expand D_n, D_n^*, A_n, A_n^* in terms of a system of real normalized orthogonal functions u_{n1}, u_{n2}, u_{n3}, ... The condition of normality and orthogonality is expressed by

$$\int_0^\infty r\,dr \int_{-\infty}^{+\infty} dz\, u_{nk} u_{nl} = \delta_{kl}.$$

We write the expansion in the form

$$D_n = \sum_{k=1}^{\infty} \alpha_{nk} u_{nk}, \quad A_n = \sum_{k=1}^{\infty} \beta_{nk} u_{nk}. \tag{5·4}$$

Then

$$\iint (D_n A_n^* - D_n^* A_n) r\,dr\,dz = \sum_{k=1}^{\infty} \{\alpha_{nk}\beta_{nk}^* - \alpha_{nk}^*\beta_{nk}\}. \tag{5·5}$$

The final expression for λ_1 is therefore

$$\begin{aligned} \lambda_1 &= \sum_{n=1}^{\infty} n \sum_{k=1}^{\infty} -i\{\alpha_{nk}\beta_{nk}^* - \alpha_{nk}^*\beta_{nk}\} \\ &= \sum_{n=1}^{\infty} n \sum_{k=1}^{\infty} \lambda_{nk}. \end{aligned} \tag{5·6}$$

The commutation rules (3·11) for D_r and $\tilde{A}_r$ lead to the following commutation rules for α_{nk}, α_{nk}^*, β_{nk}, β_{nk}^*:

(1) All α_{nk}, α_{nk}^*, β_{nk}, β_{nk}^* with different pairs of indices n, k commute.

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(2) For the same n, k they satisfy the rules:

$$\left.\begin{aligned} &[\alpha_{nk}, \alpha^*_{nk}] = 0, && [\beta_{nk}, \beta^*_{nk}] = 0, \\ &[\alpha_{nk}, \beta_{nk}] = 0, && [\alpha^*_{nk}, \beta^*_{nk}] = 0, \\ &[\alpha_{nk}, \beta^*_{nk}] = 1, && [\alpha^*_{nk}, \beta_{nk}] = 1. \end{aligned}\right\} \tag{5·7}$$

λ_1 is therefore the sum of independent (in the sense previously defined) terms $n\lambda_{nk}$:

$$\lambda_{nk} = -i\{\alpha_{nk}\beta^*_{nk} - \alpha^*_{nk}\beta_{nk}\}.$$

We can apply our lemma of section 2 separately to each pair of values of n and k. It states that the eigenvalues of λ_{nk} are integral multiples of $\hbar$. *A fortiori* the eigenvalues of $n\lambda_{nk}$ are integral multiples of $\hbar$, and therefore the eigenvalues of λ_1 are integral multiples of $\hbar$. The same argument applies to λ_2. This completes the proof.

Summary

It is shown that the eigenvalues of the angular momentum of the electromagnetic field containing a number of charged particles, apart from spin angular momentum, are only the *integral* multiples of $\hbar$. This is shown by using cylindrical polar variables, and taking a particular choice of the vector potential in which the radial component is zero, defined explicitly in terms of the magnetic field strengths. By expanding in terms of the Fourier functions $e^{in\theta}$, the angular momentum is separated out into terms independent of one another, each taking on only integral values in units of $\hbar$.

The arguments all apply equally well to a modified field theory such as that of Born and Infeld.

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