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A nonlinear equation for electron surface waves at a plasma boundary

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A nonlinear equation for electrostatic electron surface waves at the boundary between a cold plasma and a dielectric is derived. Our analysis of this equation shows that the nonlinear disturbances at the plasma–dielectric interface may take the form of periodic travelling waves or standing waves with oscillating amplitudes. While the travelling waves are always periodic in space, the standing waves can be both periodic and localized.

1. Introduction

Surface waves in low-temperature plasmas ($<$ 10–20 eV) are of interest because of their importance for plasma-assisted material processing devices, pulse-power switches, microwave generators, etc. (see e.g. Henderson 1986; Claude *et al.* 1987). Surface modes are also useful for diagnostics of low-temperature plasmas, since they provide information on the electron density and the effective collision frequency (Moisan *et al.* 1982; Nonaka and Akao 1983; Zhelyazkov and Atanassov 1995). Unlike high-temperature plasmas that require strong inertial forces or high magnetic fields for confinement, low-temperature plasmas can be confined by rigid dielectric walls. For this reason, a typical experimental and theoretical configuration for studies of low-temperature plasmas is a semi-infinite plasma bounded by a planar dielectric wall. As the characteristic length scales of electron surface waves at the boundary between the plasma and the dielectric are usually much larger than the Debye length, the surface waves are often considered within the cold-plasma approximation.

The properties of electron surface waves localized close to the wall depend strongly on the dielectric characteristics, first of all on the dielectric permittivity ε_d. It is a well-known theoretical result confirmed experimentally (see e.g. Moisan *et al.* 1982; Grozev *et al.* 1991; Vladimirov *et al.* 1994) that small perturbations of the electron density at the plasma–dielectric interface oscillate with a frequency $\omega = (1 + \varepsilon_d)^{-1/2}\omega_p$, where ω_p is the electron plasma frequency. Unlike linear oscillations of small amplitude, the nonlinear dynamics of surface waves is much less well understood. Several exact large-amplitude wave solutions have, however, been found in cylindrical and spherical geometries (Vladimirov *et al.* 1994; Stenflo 1996; Stenflo and Yu 1997). They describe the coupling between bulk electron oscillations affected by the dielectric boundary and electron surface waves. An attempt was recently made to derive a general equation for surface plasma waves of arbitrary amplitudes (Gradov and Stenflo 1998). However, in order to overcome the

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mathematical difficulties of that problem, some ambiguous approximations had to be made. The results obtained were therefore not reliable in all respects.

In the present paper we are going to reconsider the nonlinear problem of electrostatic electron surface waves in a more strict way. We derive a nonlinear equation for surface waves of moderate amplitudes, keeping the nonlinear terms of second order, while neglecting the third-order terms. Although this approach does not allow us to describe nonlinear modulational waves, many nonlinear surface-wave effects are still retained. In particular, an analysis of the derived nonlinear equation will demonstrate the existence of nonlinear travelling waves and oscillating standing waves. We shall show that the travelling waves always have a periodic spatial structure, while the oscillating standing waves may be both periodic and localized in space.

2. Derivation of the nonlinear equation

We consider a two-dimensional configuration with a semi-infinite cold plasma that occupies the space $z > 0$ (the x axis is directed along the plasma boundary). The plasma is bounded by a dielectric of permittivity ε_d. Initially the plasma is uniform, with electron density n_0. Our purpose is to study nonlinear surface waves at the interface between the plasma and the dielectric. The electric field inside the plasma and the dielectric is described by the Poisson equation

$$\boldsymbol{\nabla} \cdot (\varepsilon \boldsymbol{\nabla} \phi) = 4\pi e \, (n - n_0) \, , \tag{1}$$

with $\varepsilon = 1$ in the plasma and $\varepsilon = \varepsilon_d$ in the dielectric (formally $n - n_0 = 0$ in the dielectric). We suppose that the plasma ions are at rest and that the electron motion in the plasma is governed by the equations

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n \mathbf{u}) = 0, \tag{2}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla}) \, \mathbf{u} = \frac{e}{m} \boldsymbol{\nabla} \phi, \tag{3}$$

where $-e$ and m are the electron charge and mass respectively. Since the plasma motion is irrotational, it is convenient to introduce the velocity potential f (with $\mathbf{u} = \boldsymbol{\nabla} f$), for which the equation of motion of the electrons takes the form

$$\frac{\partial f}{\partial t} + \tfrac{1}{2} (\boldsymbol{\nabla} f)^2 - \frac{e}{m} \phi = 0. \tag{4}$$

According to the definition, an arbitrary function of time may be always added to the velocity potential. Assuming that λ is the characteristic length scale of the plasma surface waves, we introduce the scaled dimensionless variables

$$\tau = \omega_p t, \qquad (\eta; \zeta) = \left(\frac{x}{\lambda}; \frac{z}{\lambda} \right), \qquad h = \frac{n}{n_0} - 1,$$

$$F = \frac{f}{\omega_p \lambda^2}, \qquad \Phi = \frac{e \phi}{m \omega_p^2 \lambda^2},$$

where the plasma frequency $\omega_p^2 = 4\pi n_0 e^2 / m$. Then Eqs. (2), (4) and (1) can be written as

$$\frac{\partial h}{\partial \tau} + \nabla^2 F + \boldsymbol{\nabla} \cdot (h \boldsymbol{\nabla} F) = 0, \tag{5}$$

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$$\frac{\partial F}{\partial \tau} + \tfrac{1}{2}(\boldsymbol{\nabla} F)^2 - \Phi = 0, \tag{6}$$

$$\boldsymbol{\nabla} \cdot (\varepsilon \boldsymbol{\nabla} \Phi) - h = 0. \tag{7}$$

In (5)–(7) and in what follows, $\boldsymbol{\nabla}$ denotes the scaled gradient.

In the *linear* approximation one has (from (5) and (6)) $\partial h/\partial \tau = -\nabla^2 F$ and $\partial F/\partial \tau = \Phi$. The Poisson equation can thus be written in the form

$$\boldsymbol{\nabla} \cdot \left(\hat{\varepsilon} \frac{\partial^2}{\partial \tau^2} \boldsymbol{\nabla} \Phi \right) = 0, \tag{8}$$

with $\hat{\varepsilon} = \varepsilon_d$ in the dielectric and $\hat{\varepsilon}\partial^2/\partial\tau^2 = 1 + \partial^2/\partial\tau^2$ in the plasma. It follows from (8) that the potential is continuous at the interface between the plasma and the dielectric, $\Phi|_- = \Phi|_+ = \Phi_0$, while the normal derivative of the potential experiences a jump,

$$\varepsilon_d \frac{\partial^2}{\partial \tau^2} \frac{\partial \Phi}{\partial \zeta}\bigg|_- = \left(1 + \frac{\partial^2}{\partial \tau^2}\right) \frac{\partial \Phi}{\partial \zeta}\bigg|_+ . \tag{9}$$

Both in the plasma and in the dielectric, one has to solve the Laplace equation $\nabla^2 \Phi = 0$, so that for the waves localized at the plasma–dielectric interface, the potential is

$$\Phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_k \exp(ik\eta \pm |k|\,\zeta)\, dk, \tag{10}$$

where Φ_k is the Fourier transform of the potential, and the + and − signs in the exponent in (10) correspond to the dielectric ($z < 0$) and the plasma ($z > 0$) respectively. It follows from (10) that the normal derivative of the potential at both sides of the plasma–dielectric interface is related to the potential by means of a linear operator $\hat{J}$:

$$\frac{\partial \Phi}{\partial \zeta}\bigg|_{\pm} \pm \hat{J}\Phi_0 = 0, \tag{11}$$

where $\hat{J}$ is defined as

$$\hat{J}\Phi_0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |k|\, \Phi_{0k} \exp(ik\eta)\, dk. \tag{12}$$

Substituting the solution for the potential in the plasma and the dielectric into the boundary condition (9), we obtain the well-known linear equation

$$\left[1 + (1 + \varepsilon_d) \frac{\partial^2}{\partial \tau^2}\right] \hat{J}\Phi_0 = 0, \tag{13}$$

which describes harmonic plasma oscillations with the scaled frequency $\boldsymbol{\Omega} = (1 + \varepsilon_d)^{-1/2}$.

In order to find the *nonlinear* equation for the surface waves, it is convenient to write the Poisson equation in the form

$$\boldsymbol{\nabla} \cdot \left(\hat{\varepsilon}_p \frac{\partial}{\partial \tau} \boldsymbol{\nabla} \Phi \right) = 0, \tag{14}$$

where in the dielectric the operator $\hat{\varepsilon}_p$ is equal to the dielectric permittivity ε_d, while in the half-space $z > 0$ it is defined by

$$\boldsymbol{\nabla} \cdot \left(\hat{\varepsilon}_p \frac{\partial}{\partial \tau} \boldsymbol{\nabla} \Phi \right) = \nabla^2 \frac{\partial \Phi}{\partial \tau} - \frac{\partial h}{\partial \tau}. \tag{15}$$

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Taking into account the continuity equation (5) and the equation of motion (6), we then rewrite the Poisson equation in the plasma in terms of the velocity potential F:

$$\boldsymbol{\nabla}\cdot\left(\hat{\varepsilon}_p\frac{\partial}{\partial\tau}\boldsymbol{\nabla}\Phi\right)=\nabla^2\left[\frac{\partial^2F}{\partial\tau^2}+F+\frac{1}{2}\frac{\partial}{\partial\tau}(\boldsymbol{\nabla}F)^2\right]+\boldsymbol{\nabla}\cdot\left[\left(\nabla^2\frac{\partial F}{\partial\tau}\right)\boldsymbol{\nabla}F\right]+\tfrac{1}{2}\boldsymbol{\nabla}\cdot\{[\nabla^2(\boldsymbol{\nabla}F)^2]\boldsymbol{\nabla}F\}. \quad (16)$$

Integrating the Poisson equation (14) across the interface between the plasma and the dielectric from $z=-0$ to $z=+0$ and taking (16) into account, we obtain

$$\varepsilon_d\frac{\partial}{\partial\tau}\left.\frac{\partial\Phi}{\partial\zeta}\right|_-=\frac{\partial}{\partial\zeta}\left[\frac{\partial^2F}{\partial\tau^2}+F+\frac{1}{2}\frac{\partial}{\partial\tau}(\boldsymbol{\nabla}F)^2\right]\Bigg|_+ +\left(\frac{\partial F}{\partial\zeta}\nabla^2\frac{\partial F}{\partial\tau}\right)\Bigg|_+ +\frac{\partial}{\partial\eta}\left(\frac{\partial F}{\partial\eta}\frac{\partial^2F}{\partial\zeta\,\partial\tau}\right)\Bigg|_+ +\frac{1}{2}\left[\frac{\partial F}{\partial\zeta}\nabla^2(\boldsymbol{\nabla}F)^2\right]\Bigg|_+ +\frac{1}{2}\frac{\partial}{\partial\eta}\left[\frac{\partial F}{\partial\eta}\frac{\partial}{\partial\zeta}(\boldsymbol{\nabla}F)^2\right]\Bigg|_+. \quad (17)$$

The continuity of the potential at the interface is written in the form

$$\Phi|_-=\Phi|_+=\left.\frac{\partial F}{\partial\tau}\right|_+ +\tfrac{1}{2}(\boldsymbol{\nabla}F)^2\Big|_+. \quad (18)$$

The solution for the potential in the dielectric obeys the Laplace equation $\nabla^2\Phi=0$ in both the linear and nonlinear regimes, so that the solution for the potential may be presented in the form (10), and the boundary condition at $z=-0$ has the same form as (11). Then, taking (17) and (18) into account, we obtain the boundary condition for the plasma region:

$$\varepsilon_d\left.\frac{\partial^2}{\partial\tau^2}\hat{J}F\right|_+ +\frac{\varepsilon_d}{2}\frac{\partial}{\partial\tau}\hat{J}(\boldsymbol{\nabla}F)^2\Big|_+ =\frac{\partial}{\partial\zeta}\left[\frac{\partial^2F}{\partial\tau^2}+F+\frac{1}{2}\frac{\partial}{\partial\tau}(\boldsymbol{\nabla}F)^2\right]\Bigg|_+ +\left(\frac{\partial F}{\partial\zeta}\nabla^2\frac{\partial F}{\partial\tau}\right)\Bigg|_+ +\frac{\partial}{\partial\eta}\left(\frac{\partial F}{\partial\eta}\frac{\partial^2F}{\partial\zeta\,\partial\tau}\right)\Bigg|_+ +\frac{1}{2}\left[\frac{\partial F}{\partial\zeta}\nabla^2(\boldsymbol{\nabla}F)^2\right]\Bigg|_+ +\frac{1}{2}\frac{\partial}{\partial\eta}\left[\frac{\partial F}{\partial\eta}\frac{\partial}{\partial\zeta}(\boldsymbol{\nabla}F)^2\right]\Bigg|_+, \quad (19)$$

where the operator $\hat{J}$ is defined by (12). The boundary condition (19) for the plasma flow has been obtained with no restrictions on the amplitude of the surface waves, and may be used both in the limits of weak and strong nonlinearity.

Inside the plasma, one has to solve the equation for the velocity potential,

$$\nabla^2\left[\frac{\partial^2F}{\partial\tau^2}+F+\frac{1}{2}\frac{\partial}{\partial\tau}(\boldsymbol{\nabla}F)^2\right]+\boldsymbol{\nabla}\cdot\left[\left(\nabla^2\frac{\partial F}{\partial\tau}\right)\boldsymbol{\nabla}F\right]+\tfrac{1}{2}\boldsymbol{\nabla}\cdot\{[\nabla^2(\boldsymbol{\nabla}F)^2]\boldsymbol{\nabla}F\}=0. \quad (20)$$

This equation will be solved here in the approximation of weak nonlinearity. To first order, one then has $\nabla^2(\partial F/\partial\tau)=0$. The second-order version of (20) is thus

$$\nabla^2\left[\frac{\partial^2F}{\partial\tau^2}+F+\frac{1}{2}\frac{\partial}{\partial\tau}(\boldsymbol{\nabla}F)^2\right]=0. \quad (21)$$

Since (21) is a Laplace equation with respect to the nonlinear function in the square brackets, at the interface between the plasma and the dielectric we therefore have

$$\left(\frac{\partial}{\partial\zeta}+\hat{J}\right)\left[\frac{\partial^2F}{\partial\tau^2}+F+\frac{1}{2}\frac{\partial}{\partial\tau}(\boldsymbol{\nabla}F)^2\right]\Bigg|_+=0. \quad (22)$$

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Substituting (22) into the boundary condition (19), we then obtain the nonlinear equation

$$\left[1+(1+\varepsilon_d)\frac{\partial^2}{\partial\tau^2}\right]\hat{J}F+\frac{1+\varepsilon_d}{2}\frac{\partial}{\partial\tau}\hat{J}\left[\left(\frac{\partial F}{\partial\eta}\right)^2+(\hat{J}F)^2\right]+\frac{\partial}{\partial\eta}\left(\frac{\partial F}{\partial\eta}\hat{J}\frac{\partial F}{\partial\tau}\right)=0, \quad (23)$$

which describes surface waves of moderate amplitude at the interface between the plasma and the dielectric.

Although the above calculations have been performed for the two-dimensional case, they may be generalized in a straightforward way to the three-dimensional case, where the nonlinear equation (23) appears as

$$\left[1+(1+\varepsilon_d)\frac{\partial^2}{\partial\tau^2}\right]\hat{J}F+\frac{1+\varepsilon_d}{2}\frac{\partial}{\partial\tau}\hat{J}[(\boldsymbol{\nabla}_\perp F)^2+(\hat{J}F)^2]+\boldsymbol{\nabla}_\perp\cdot\left(\boldsymbol{\nabla}_\perp F\hat{J}\frac{\partial F}{\partial\tau}\right)=0, \quad (24)$$

where $\boldsymbol{\nabla}_\perp$ is the $\boldsymbol{\nabla}$ operator with respect to the coordinates along the surface between the plasma and the dielectric. The operator $\hat{J}$ in the three-dimensional case is

$$\hat{J}F=\frac{1}{4\pi^2}\int|\mathbf{k}|\,F_k\exp(i\mathbf{k}\cdot\boldsymbol{\eta})\,d\mathbf{k}. \quad (25)$$

3. Solution of the nonlinear equation for plasma surface waves

In this section we shall show that (23) admits solutions in the form of travelling waves and standing waves. For a travelling wave $F(\eta,\tau)=F(\eta/U-\tau)$, introducing the variable $\xi=(1+\varepsilon_d)^{-1/2}(\eta/U-\tau)$ and the scaling $F=(1+\varepsilon_d)^{3/2}U^2\psi$, we reduce (23) to the nonlinear equation

$$\left(1+\frac{d^2}{d\xi^2}\right)\hat{J}\psi-(1+\varepsilon_d)\frac{d}{d\xi}\hat{J}\left[\left(\frac{d\psi}{d\xi}\right)^2+(\hat{J}\psi)^2\right]-2\frac{d}{d\xi}\left(\frac{d\psi}{d\xi}\hat{J}\frac{d\psi}{d\xi}\right)=0. \quad (26)$$

To find a solution of (26), we take into account that for any function

$$\psi=\sum_l\psi_l\cos(l\xi) \quad (27)$$

the operator $\hat{J}$ acts as

$$\hat{J}\psi=\sum_l l\psi_l\cos(l\xi). \quad (28)$$

Therefore, in the first-order approximation for a 2π-periodic function, we have $\hat{J}\psi\approx\psi$, while in the second-order approximation the following estimates hold:

$$\frac{d}{d\xi}\left(\frac{d\psi}{d\xi}\hat{J}\frac{d\psi}{d\xi}\right)\approx\tfrac{1}{2}\hat{J}\frac{d}{d\xi}\left(\frac{d\psi}{d\xi}\right)^2 \quad (29a)$$

$$\frac{d}{d\xi}\left[\left(\frac{d\psi}{d\xi}\right)^2+(\hat{J}\psi)^2\right]\sim\psi^3. \quad (29b)$$

With the help of Eq. (29) and neglecting nonlinear terms of third order, we reduce (26) to

$$\frac{d^2\psi}{d\xi^2}+\psi-\frac{d}{d\xi}\left(\frac{d\psi}{d\xi}\right)^2=0. \quad (30)$$

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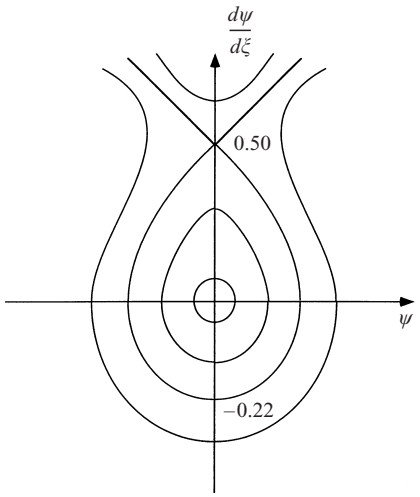


Figure 1. Phase portrait of (31) describing travelling surface waves.

According to (30), the characteristic length scale of a surface wave increases with the wave velocity as $\Delta x \propto (1+\varepsilon_d)^{1/2} U\lambda$. However, since the original problem does not contain any length scale, we can always take $\Delta x \equiv \lambda$ as a definition of λ, so that $(1+\varepsilon_d)^{1/2} U = 1$. The dimensional velocity of the surface wave is $U_{\rm dim} = (1+\varepsilon_d)^{-1/2}\omega_p\lambda$ and the amplitude of the potential of the electric field increases with the wave velocity as

$$\Delta\phi \propto (1+\varepsilon_d)\,\frac{m}{e}U_{\rm dim}^2\,\Delta\,(d\psi/d\xi)\,.$$

Here $\Delta\,(d\psi/d\xi)$ denotes a typical amplitude of $d\psi/d\xi$, which has to be obtained from (30).

The first integral of (30) is

$$\left(\frac{d\psi}{d\xi}\right)^2 + \psi^2 - \frac{4}{3}\left(\frac{d\psi}{d\xi}\right)^3 = C, \tag{31}$$

The phase portrait of (31) is shown in Fig. 1. For small values of the constant $C \ll 1$ the solution of (31) describes linear electron oscillations. For sufficiently large values of the constant ($C > \frac{1}{12}$) the solution of (31) is not regular. The critical value $C = \frac{1}{12}$ corresponds to the separatrix, and the solution for this parameter value looks like a sawtoothed oscillation with a steep increase of the scaled velocity potential ψ and a smoother decrease of the potential. Thus the last nonlinear term in (26) leads to asymmetry of the electron oscillations in time. During the oscillations, the value of $d\psi/d\xi$ corresponding to the scaled electric potential pulsates within the limits $-0.22 < d\psi/d\xi < 0.5$, which gives a typical amplitude of the derivative $\Delta\,(d\psi/d\xi) \approx 0.7$.

Another class of solutions of (23), describing standing surface waves, may be obtained by means of the variable separation $F\,(\tau,\eta) = \Theta\,(\tau)\,A\,(\eta)$. The equations for $\Theta\,(\tau)$ and $A\,(\eta)$ are then

$$(1+\varepsilon_d)\,\frac{d^2\Theta}{d\tau^2} + \Theta - \frac{1}{2}\,\frac{d}{d\tau}\Theta^2 = 0, \tag{32}$$

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$$\hat{J}A + (1+\varepsilon_d)\,\hat{J}\left[\left(\frac{dA}{d\eta}\right)^2 + (\hat{J}A)^2\right] + \frac{d}{d\eta}\left(\frac{dA}{d\eta}\hat{J}A\right) = 0. \tag{33}$$

In general, (32) and (33) should contain a separation constant. However, since there is no fixed length scale in the original problem, the separation constant may always be included in the definition of the chosen length scale λ. In order to solve (32), we introduce the function ϑ from $d\vartheta/d\tau = (1+\varepsilon_d)\,\Theta$ and reduce (32) to

$$(1+\varepsilon_d)\frac{d^2\vartheta}{d\tau^2} + \vartheta - \frac{1+\varepsilon_d}{2}\left(\frac{d\vartheta}{d\tau}\right)^2 = 0. \tag{34}$$

Making the substitution $d\vartheta/d\tau = T(\vartheta)$, we obtain from (34)

$$(1+\varepsilon_d)\frac{d}{d\vartheta}\frac{T^2}{2} - (1+\varepsilon_d)\frac{T^2}{2} + \vartheta = 0. \tag{35}$$

The first integral of (35) may be presented in the form of an equation of energy conservation for a particle of zero initial energy in a potential well:

$$\frac{1+\varepsilon_d}{2}\left(\frac{d\vartheta}{d\tau}\right)^2 = \vartheta + 1 - C_1\exp(\vartheta), \tag{36}$$

with the 'potential energy' given by

$$W_\tau(\vartheta) = C_1\exp(\vartheta) - \vartheta - 1. \tag{37}$$

A typical form of the potential well (37) is shown in Fig. 2. The minimum potential energy of the well may be calculated as

$$\min\, W_\tau\,(\vartheta) = \ln\, C_1 \tag{38}$$

so that an oscillating solution of (32) is possible for the parameter range $0 < C_1 < 1$. For $C_1 > 1$, the solution of (32) is not real, while for $C_1 < 0$, the solution of (32) is not regular and does not correspond to any physical motion. As $C_1 \to 1$, the solution of (32) describes linear electron oscillations with small amplitudes. As $C_1 \to 0$, the amplitudes of the oscillations increase and the oscillations become strongly asymmetric with oscillation amplitude

$$\Delta\Theta = \frac{2}{1+\varepsilon_d}\left[2\ln\left(\frac{1}{C_1}\right)\right]^{1/2}.$$

While it is difficult to find an exact analytical solution of the coordinate-dependent equation for standing surface waves (33), the characteristic features of the solution of (33) may be understood qualitatively from an analysis of the equation

$$\hat{J}a + \left(\frac{da}{d\eta}\right)^2 + (\hat{J}a)^2 = C_2. \tag{39}$$

One can obtain (39) from (33) in the limit of large dielectric permittivity $\varepsilon_d \gg 1$ after introducing the scaled function $a = (1+\varepsilon_d)\,A$ and replacing $\hat{J}a$ by $\hat{J}^2 a$. The latter is possible if the principal term of the solution of (39) is proportional to $\cos\eta$. This is supported by the calculations below. Equation (39) admits a solution in the form of a pole decomposition

$$a = B_0 - \sum_{\alpha=1}^{2M} B_\alpha P_\alpha \tag{40}$$

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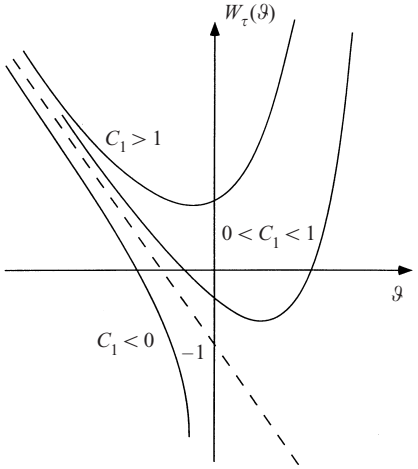


Figure 2. The shape of the 'potential well' (37) for the time-dependent part of standing surface waves.

where the superposition of the terms $P_\alpha = \ln \sin[\frac{1}{2}(\eta - \eta_\alpha)]$ describes some periodic solution of period 2π, B_α are real coefficients, and η_α are poles in the complex plane coming in M conjugate pairs $(\eta_\alpha, \eta_\alpha^*)$ with imaginary parts $\operatorname{Im} \eta_\alpha^* = -\operatorname{Im} \eta_\alpha$. The method of pole decomposition is often useful in nonlinear problems for surface waves involving the operator $\hat{J}$, such as in the problem of the dynamics of curved flame fronts (Thual *et al.* 1985; Joulin 1991; Bychkov *et al.* 1997; Bychkov 1998). The operator $\hat{J}$ acts on the terms P_α as (Joulin 1991)

$$\hat{J}(P_\alpha + P_\alpha^*) = 1 + i \operatorname{sign}(\operatorname{Im} \eta_\alpha) \frac{d}{d\eta}(P_\alpha - P_\alpha^*). \tag{41}$$

Let us consider a solution corresponding to one pair of poles $\eta_1 = ib$, $\eta_2 = -ib$ with $B_1 = B_2 = B$, so that the representation (40) becomes

$$a = -B \ln \left[\sin\left(\frac{\eta - ib}{2} \right) \sin\left(\frac{\eta + ib}{2} \right) \right] + B_0. \tag{42}$$

Substituting (42) into (39), we find an equation relating the amplitude B and the pole position ($\coth b = 1 - (2B)^{-1}$). As a result, we obtain a 2π-periodic solution depending on a single parameter B that determines the amplitude of the solution:

$$a = -B \ln \left[1 - \frac{1}{2B}(4B - 1)^{1/2} \cos \eta \right]. \tag{43}$$

It can be seen from (43) that the parameter B is limited from below ($B > \frac{1}{4}$) and that the solution (43) is close to harmonic for $B - \frac{1}{4} \ll 1$, when $a \approx \frac{1}{2}(4B - 1)^{1/2} \cos \eta$. The solution is also close to the harmonic one in the limit of large parameter values $B \gg 1$, when $a \approx B^{1/2} \cos \eta$. For $B = 2$, the difference in shape of the solution (43) and the linear harmonic is strongest, but even for this parameter value the solution still remains rather close to the harmonic one (see Fig. 3). Thus the obtained solution provides an understanding of the qualitative features of the solution of (33).

Apart from periodic solutions, (39) has non-periodic solutions as well, when the pole decomposition takes the form (40) with $P_\alpha = \ln(\eta - \eta_\alpha)$. In particular, for the case of two poles, such solutions are $a = B \ln(\eta^2 + b^2)$. One may check by direct

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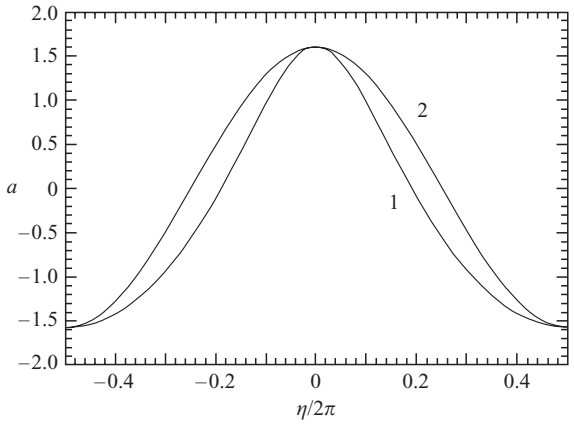


Figure 3. The solution (43) of the coordinate-dependent equation for standing surface waves for the parameter value $B = 2$ (curve 1). A harmonic solution of the same amplitude is shown for comparison (curve 2).

calculation of Fourier transforms and substitution of the result into (39) that in this case

$$\hat{J}a = -\frac{2bB}{\eta^2 + b^2}, \qquad \frac{da}{d\eta} = \frac{2\eta B}{\eta^2 + b^2}, \qquad b = 2B.$$

For the non-periodic solutions, the scaled electric potential diverges at infinity as $\Phi \sim \partial F/\partial t \sim a$; however, the electric field for this solution is localized in space:

$$E_x \sim \frac{\partial \Phi}{\partial \eta} \sim \frac{2B\eta}{\eta^2 + b^2}, \qquad E_z \sim \hat{J}\Phi \sim \frac{2bB}{\eta^2 + b^2}.$$

The latter result indicates that standing electron surface waves may be spatially periodic or localized in space like a solitary wave.

An interesting class of exact analytical solutions of (23) may be found in the limit of large dielectric permittivity, $\varepsilon_d \gg 1$, when the time scale of the solution is small compared with the period of linear oscillations, $\varepsilon_d^{1/2}\partial/\partial\tau \gg 1$. In this case (23) may be reduced to

$$\frac{\partial^2}{\partial\tau^2}\hat{J}F + \frac{1}{2}\frac{\partial}{\partial\tau}\hat{J}\left[\left(\frac{\partial F}{\partial\eta}\right)^2 + \left(\hat{J}F\right)^2\right] = 0, \tag{44}$$

or

$$2\frac{\partial F}{\partial\tau} + \left(\frac{\partial F}{\partial\eta}\right)^2 + \left(\hat{J}F\right)^2 = C_\tau(\tau) + C_\eta(\eta), \tag{45}$$

where $C_\tau(\tau)$ and $C_\eta(\eta)$ are arbitrary functions of time and space respectively. In this case the scaled potential at the plasma surface Φ is related to the scaled velocity potential as

$$(1 + \varepsilon_d)\frac{\partial}{\partial\tau}\hat{J}\Phi = -\hat{J}F - \frac{\partial}{\partial\eta}\left(\frac{\partial F}{\partial\eta}\hat{J}\frac{\partial F}{\partial\tau}\right). \tag{46}$$

Exact analytical solutions of (45) may be obtained in the form of a pole decomposition with pole coordinates depending on time $\eta_\alpha = \eta_\alpha(\tau)$:

$$F = B_0(\tau) - \sum_{\alpha=1}^{2M} B_\alpha \ln\sin\left(\frac{\eta - \eta_\alpha}{2}\right). \tag{47}$$

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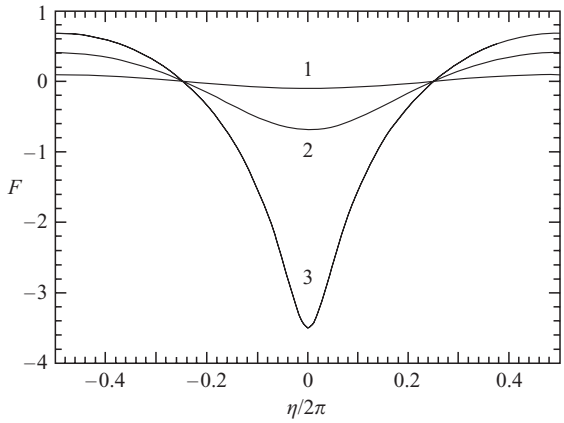


Figure 4. The solution (50) and (51) corresponding to standing surface waves in a pole decomposition form for the parameter values $B = 1$ and $b = 3$ (curve 1), $b = 1.3$ (curve 2) and $b = 0.25$ (curve 3).

Substituting (47) into (45) and taking into account (41) and the trigonometric formula $\cot\gamma\,\cot\chi = -1 + \cot(\gamma-\chi)\,(\cot\chi - \cot\gamma)$, we find the following set of equations describing the motion of poles in the complex plane:

$$2\frac{d\eta_\alpha}{d\tau} + \sum_{\beta,\alpha\neq\beta}^{2M} B_\beta \cot\left(\frac{\eta_\alpha-\eta_\beta}{2}\right)[1-\operatorname{sign}(\operatorname{Im}\eta_\alpha \operatorname{Im}\eta_\beta)] + i\operatorname{sign}(\operatorname{Im}\eta_\alpha)\sum_{\beta}^{2M} B_\beta = 0. \quad (48)$$

In particular, for the case of two poles $\eta_1 = \eta_2^* = ib\,(\tau)$ with $B_1 = B_2 = B$, the set of equations (48) is reduced to one equation

$$\frac{db}{d\tau} = B\,(\coth\,b - 1)\,, \quad (49)$$

so that the pole positions in time are determined by the equation

$$\tfrac{1}{2}\exp(2b) - b = 2B\,(\tau-\tau_0) + \tfrac{1}{2}\exp(2b_0) - b_0. \quad (50)$$

Eliminating the coordinate-independent terms by use of the function $C_\tau\,(\tau)$, we obtain the scaled velocity potential

$$F = B\ln\left(1 - \frac{\cos\eta}{\cosh b}\right). \quad (51)$$

As can be seen from (51), the velocity potential becomes singular at the points $\eta = 2\pi l$, $l = 0, 1, 2, ...$, when the pole position is $b = 0$ and $\cosh b = 1$. If we chose $b_0 = 0$ at the time instant $\tau = 0$ then, after this instant, for $\tau > 0$, the obtained solution (50) and (51) describes a decrease of the perturbation amplitude, while before this instant (for $\tau < 0$) the obtained solution corresponds to perturbation growth. A typical change of the perturbation shape and amplitude related to the change of the pole position in time is shown in Fig. 4. It should be emphasized that the growth in the points $\eta = 2\pi l$ at the time instant $\tau = 0$ does not really imply singularity of any physical value, and the obtained decrease of the perturbation amplitude cannot be complete, since for very large and very small perturbation

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amplitudes (44) is no longer valid. One of the validity conditions of (44),

$$(1+\varepsilon_d)\frac{\partial^2}{\partial\tau^2}\hat{J}F \gg \frac{\partial}{\partial\eta}\left(\frac{\partial F}{\partial\eta}\hat{J}\frac{\partial F}{\partial\tau}\right), \tag{52}$$

imposes restrictions on the perturbation amplitude from above, while the other condition,

$$F \ll (1+\varepsilon_d)\frac{\partial}{\partial\tau}\left(\frac{\partial F}{\partial\eta}\right)^2, \tag{53}$$

provides limitations from below. In the case of large and small amplitudes other terms in (23), which have been neglected in (44), are important. The neglected terms presumably lead to oscillations with a characteristic time scale $(1+\varepsilon_d)^{1/2}$ much larger than the time scale of the obtained solution (51) and (52). Therefore we conclude that the above solution corresponds to strongly nonlinear oscillations, for which the perturbation amplitude changes slowly during a rather long time of the order of $(1+\varepsilon_d)^{1/2}$, and the period of the slow change is followed by a short pulse during a time of about 1 in the scaled units. The obtained solution describes standing surface waves, though these waves differ from the waves obtained by the method of variable separation.

Equation (45) also admits non-periodic solutions in a pole-decomposition form. The electric field is, however, not regular at infinity for these solutions.

Thus our analysis of the nonlinear equation (23) indicates that nonlinear electrostatic electron surface waves at a boundary between a plasma and a dielectric may take the form of travelling waves and standing waves with oscillating amplitudes. The travelling waves are always spatially periodic, whereas the oscillating standing waves may be both periodic and localized in space like a solitary wave. It would be interesting to generalize our results to the case of a magnetized semi-infinite plasma in the presence of incident laser radiation (cf. Aliev and Brodin 1990; Brodin and Lundberg 1991; Ostrikov 1995; Sharma 1997; Parashar *et al.* 1998). This is, however, outside the scope of the present paper.

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