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A Note on Bipartite Graphs Without $2k$-Cycles

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For Béla Bollobás on his 60th birthday

The question of the maximum number $\mathrm{ex}(m,n,C_{2k})$ of edges in an m by n bipartite graph without a cycle of length $2k$ is addressed in this note. For each $k \geqslant 2$, it is shown that

$$\mathrm{ex}(m,n,C_{2k}) \leqslant \begin{cases} (2k-3)\left[(mn)^{\frac{k+1}{2k}} + m + n\right] & \text{if } k \text{ is odd,} \\ (2k-3)\left[m^{\frac{k+2}{2k}}\, n^{\frac{1}{2}} + m + n\right] & \text{if } k \text{ is even.} \end{cases}$$

1. Introduction

In this note, we study the maximum number of edges in an m by n bipartite graph containing no $2k$-cycles. This problem was studied in the papers by Erdős, Sós and Sárközy [3] and by Győri [4], in the context of a number-theoretic problem. It is also related to the size of subsets of points in projective planes, such as arcs and caps. A connection to a geometric problem involving points and lines in Euclidean space is described by de Caen and Székely [1].

Throughout the material to follow, $\gamma(m,n,g,k)$ denotes the maximum number of edges in an m by n bipartite graph of girth at least $2g$ containing no $2k$-cycle. In particular, we write $\gamma(m,n,2,k) = \mathrm{ex}(m,n,C_{2k})$, which is the maximum number of edges in a $2k$-cycle-free m by n bipartite graph. For $d, g \geqslant 2$, let $c(d,g)$ be the largest integer such that every bipartite graph of average degree at least $2d$ and girth at least $2g$ contains a cycle of length at least $c(d,g)$ with at least one chord. Our main result is as follows.

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Theorem 1.1. *Let $k, g \geqslant 2$ be integers. Then, for any d such that $c(d,g) \geqslant 2(k-g+1)$,*

$$\gamma(m,n,g,k) \leqslant \begin{cases} 2d\left[(mn)^{\frac{k+1}{2k}} + m + n\right] & \text{if } k \text{ is odd,} \\ 2d\left[m^{\frac{k+2}{2k}} n^{\frac{1}{2}} + m + n\right] & \text{if } k \text{ is even.} \end{cases}$$

Theorem 1.1 depends explicitly on the parameter $c(d,g)$. Let us make a few remarks about $c(d,g)$. First, a straightforward argument shows that $c(d,g) \geqslant 2d(g-1)+1$. Stronger results were obtained by Erdős, Faudree, Rousseau and Schelp [2]. We deduce from their paper that $c(d,g) \geqslant d^{g/4}$. In particular, by using these two bounds on $c(d,g)$ in Theorem 1.1, we respectively obtain the following two corollaries.

Corollary 1.2.

$$ex(m,n,C_{2k}) \leqslant \begin{cases} (2k-3)\left[(mn)^{\frac{k+1}{2k}} + m + n\right] & \text{if } k \text{ is odd,} \\ (2k-3)\left[m^{\frac{k+2}{2k}} n^{\frac{1}{2}} + m + n\right] & \text{if } k \text{ is even.} \end{cases}$$

Corollary 1.3. *For any number $\delta > 0$, there exists a constant $c(\delta)$ such that*

$$\gamma(m,n,\delta \log k,k) \leqslant \begin{cases} c(\delta)\left[(mn)^{\frac{k+1}{2k}} + m + n\right] & \text{if } k \text{ is odd,} \\ c(\delta)\left[m^{\frac{k+2}{2k}} n^{\frac{1}{2}} + m + n\right] & \text{if } k \text{ is even.} \end{cases}$$

Using the requirement on $c(d,g)$ from Theorem 1.1 and the inequality $c(d,g) \geqslant d^{g/4}$, a short computation shows that we may certainly take $c(\delta) = 4^{4/\delta}$ in Corollary 1.3. Recent results of Hoory [5] and Lam [6] show that $\gamma(m,n,k,k)$ satisfies the bounds given in Corollary 1.3, with $c(\delta) = 1$. In the case $k \in \{2,3,5\}$, the existence of rank two geometries known as generalized polygons (see [7] for constructions) show that the constant $c(\delta) = 1$ is best possible when $m = n$. The strength in Corollary 1.3 is that it shows that excluding cycles of length $O(\log k)$ has, to within an absolute constant factor, the same effect on the upper bounds as excluding all cycles of length at most $2k$. On the other hand, our next result gives an indication that $\gamma(m,n,k,k)$ and $\gamma(m,n,2,k)$ may differ substantially.

Theorem 1.4. *For all integers m, n and $k \geqslant 3$,*

$$\gamma((k-1)m,n,k,k) \geqslant (k-1) \cdot \gamma(m,n,2,k).$$

If we make the assumption that, for each even positive integer k, $\gamma(m,n,2,k)$ is asymptotically $c_1(mn)^{1/2+1/(2k)} + c_2(m+n)$ as $m,n \to \infty$, for some constants c_1, c_2, then Theorem 1.4 gives

$$\liminf_{m,n\to\infty} \frac{\gamma(m,n,k,k)}{\gamma(m,n,2,k)} \geqslant \sqrt{k-1}.$$

Similar observations may be made for odd values of k, namely

$$\liminf_{m,n\to\infty} \frac{\gamma(m,n,k,k)}{\gamma(m,n,2,k)} \geqslant (k-1)^{1/2-1/(2k)}.$$

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One can deduce from these observations and the constructions of generalized quadrangles and hexagons that

$$\liminf_{n\to\infty} \frac{\gamma(n,2n,3,3)}{\gamma(n,2n,2,3)} \geqslant 2^{1/3}, \tag{1.1}$$

$$\liminf_{n\to\infty} \frac{\gamma(n,2n,5,5)}{\gamma(n,2n,2,5)} \geqslant 4^{1/5}. \tag{1.2}$$

Inequality (1.1) comes from [8] and (1.2) is implicit in the work of Lazebnik, Ustimenko and Woldar [7]. The next section is devoted to proving Theorem 1.1, and the construction for Theorem 1.4 is presented in Section 3.

2. Proof of Theorem 1.1

The main tool in our proof will be the first theorem in [9]. Although the next proposition is not the statement of this theorem, it is straightforward to verify, from the proof appearing in [9].

Proposition 2.1. *Let G be a bipartite graph of average degree at least $4d$ and girth $2g$. Then G contains cycles of $\frac{1}{2}c(2d,2g)$ consecutive even lengths, the shortest of which has length at most twice the radius of G.* □

Proof of Theorem 1.1. We proceed by induction on $m+n$. Suppose, for a contradiction, that G is a bipartite graph without cycles of length at most $2g-2$ and containing no $2k$-cycle, with parts A and B of sizes m and n, respectively, and with more edges than the corresponding upper bound in Theorem 1.1. By Proposition 2.1, and using the definition of d in the statement of Theorem 1.1, it is sufficient to show that G contains a subgraph of radius at most k and average degree at least d. Indeed, since G has even girth $2g$, Proposition 2.1 would then imply that there is an integer r such that $g \leqslant r \leqslant k$ and such that G contains the cycles $C_{2r}, C_{2r+2}, \ldots, C_{2r+2k-2g}$. Since $2k \in [2r, 2r+2k-2g]$ whenever $g \leqslant r \leqslant k$, one of these cycles has length $2k$, as required.

Case 1. k is odd.

In this case, we may assume that the minimum degree in A is at least

$$d_A = d\frac{m^{\frac{1}{2}+\frac{1}{2k}}}{n^{\frac{1}{2}-\frac{1}{2k}}} + d.$$

Indeed, if there were a vertex $v \in A$ with degree less than d_A then by deleting it we would arrive at an $m \times (n-1)$ bipartite graph G' with

$$\begin{aligned} e(G') > e(G) - d_A &\geqslant 2dm^{\frac{1}{2}+\frac{1}{2k}}\left(n^{\frac{1}{2}+\frac{1}{2k}} - \tfrac{1}{2}n^{-\frac{1}{2}+\frac{1}{2k}}\right) + 2d(m+n) - d \\ &> 2d\left([m(n-1)]^{\frac{1}{2}+\frac{1}{2k}} + (m+n-1)\right), \end{aligned}$$

so that G' contains a $2k$-gon by the inductive hypothesis. Similarly, we may assume that the minimum degree in B is at least

$$d_B = d\frac{n^{\frac{1}{2}+\frac{1}{2k}}}{m^{\frac{1}{2}-\frac{1}{2k}}} + d.$$

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Choose a vertex $v \in A$ and let H_r be the subgraph of G induced by vertices at distance at most r from v. Let us show that H_r has average degree at least d for some $r \leqslant k$. Suppose this cannot be done. Let D_r denote the set of vertices of H_r at distance exactly r from v. Then the average number of neighbours in D_{r-1} of a vertex in D_r is less than d. It follows that if $D_r \subset A$, then $|D_{r-1}|(d_B - d) < d|D_r|$ and if $D_r \subset B$, then $|D_{r-1}|(d_A - d) < d|D_r|$. Therefore,

$$|D_r| > \begin{cases} \left(\frac{d_A}{d} - 1\right)|D_{r-1}| = \frac{m^{\frac{1}{2}+\frac{1}{2k}}}{n^{\frac{1}{2}-\frac{1}{2k}}}|D_{r-1}| & \text{if } r \text{ is odd,} \\ \left(\frac{d_B}{d} - 1\right)|D_{r-1}| = \frac{n^{\frac{1}{2}+\frac{1}{2k}}}{m^{\frac{1}{2}-\frac{1}{2k}}}|D_{r-1}| & \text{if } r \text{ is even.} \end{cases}$$

Iterating these inequalities for $r = 1, 2, \ldots, k$ we get that since k is odd,

$$|D_k| > \left(\frac{m^{\frac{1}{2}+\frac{1}{2k}}}{n^{\frac{1}{2}-\frac{1}{2k}}}\right)^{\lceil \frac{k}{2} \rceil} \left(\frac{n^{\frac{1}{2}+\frac{1}{2k}}}{m^{\frac{1}{2}-\frac{1}{2k}}}\right)^{\lfloor \frac{k}{2} \rfloor} = m.$$

On the other hand, using the fact that k is odd once more, $D_k \subset B$, so that $|D_k| \leqslant m$, which is a contradiction. The proof of the second part is complete.

Case 2. k is even.

The proof here is similar, so we only indicate the necessary changes to the argument. The inductive hypothesis implies that the minimum degree in A is at least

$$d'_A = d\frac{m^{\frac{1}{2}+\frac{1}{k}}}{\sqrt{n}} + d,$$

and the minimum degree in B is at least

$$d_B = d\frac{\sqrt{n}}{m^{\frac{1}{2}-\frac{1}{k}}} + d.$$

We now start with a vertex $v \in B$, and repeat the above argument. Since k is even, $D_k \subset B$, so that $|D_k| \leqslant m$, but

$$|D_k| > \left(\frac{m^{\frac{1}{2}+\frac{1}{k}}}{\sqrt{n}}\right)^{\frac{k}{2}} \left(\frac{\sqrt{n}}{m^{\frac{1}{2}-\frac{1}{k}}}\right)^{\frac{k}{2}} = m,$$

so we once more arrive at a contradiction. □

3. Proof of Theorem 1.4

Suppose we are given an m by n bipartite graph H, of girth at least $2k + 2$. From H, we construct a $(k-1)m$ by n bipartite graph containing no $2k$-cycles, and with $k - 1$ times as many edges as H. Let A, B be the parts of H, let $A_1, A_2, \ldots, A_{k-1}$ be disjoint sets, and let $\phi : \bigcup_{i=1}^{k-1} A_i \to A$ be defined so that ϕ restricted to A_i is a bijection $A_i \leftrightarrow A$. Define a new graph G with parts $\bigcup_{i=1}^{k-1} A_i$ and B, with edge set

$$E = \{ab : \phi(a)b \in H\}.$$

In words, we are taking $(k-1)$ identical edge-disjoint copies of H which share B as one of their parts. We now show that G has no $2k$-cycles.

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Suppose, for a contradiction, that G contains a $2k$-cycle $C = (a_1, b_1, a_2, a_2, \ldots, a_k, b_k, a_1)$ with $b_i \in B$ for all $i \in \{1, 2, \ldots, k\}$. Then

$$W = (\phi(a_1), b_1, \phi(a_2), b_2, \ldots, \phi(a_k), b_k, \phi(a_1))$$

is a closed walk of length $2k$ in H. As H has girth at least $2k + 2$, W takes place on a tree $T \subset H$ with at most k edges. On the other hand, the tree contains the k vertices in $V(C) \cap B$, and there are at least two vertices $a, a' \in V(C) \cap A_i$ for some $i \in \{1, 2, \ldots, k-1\}$, by the pigeonhole principle. Now $\phi(a)$ and $\phi(a')$ are distinct, since ϕ restricted to A_i is a bijection. Therefore the tree has at least $k + 2$ vertices, a contradiction. Therefore the graph G is $2k$-cycle-free.

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