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Orbital Chromatic and Flow Roots

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The chromatic polynomial $P_\Gamma(x)$ of a graph Γ is a polynomial whose value at the positive integer k is the number of proper k-colourings of Γ. If G is a group of automorphisms of Γ, then there is a polynomial $\mathrm{OP}_{\Gamma,G}(x)$, whose value at the positive integer k is the number of orbits of G on proper k-colourings of Γ.

It is known that real chromatic roots cannot be negative, but they are dense in $[\frac{32}{27}, \infty)$. Here we discuss the location of real orbital chromatic roots. We show, for example, that they are dense in $\mathbb{R}$, but under certain hypotheses, there are zero-free regions.

We also look at orbital flow roots. Here things are more complicated because the orbit count is given by a multivariate polynomial; but it has a natural univariate specialization, and we show that the roots of these polynomials are dense in the negative real axis.

1. Introduction

The chromatic polynomial $P_\Gamma(x)$ of a graph Γ is the polynomial whose value at the positive integer k is the number of proper k-colourings of Γ. (By convention, if Γ contains a loop, then $P_\Gamma(x) = 0$.)

Let G be a group of automorphisms of Γ. For each element $g \in G$, the number of proper k-colourings of Γ fixed by g is equal to $P_{\Gamma/g}(k)$, where Γ/g denotes the graph obtained by shrinking each vertex cycle of the permutation g to a single vertex, putting a loop at that vertex if the cycle contains an edge, and putting an edge between two vertices if there is an edge in Γ between the corresponding cycles of g. (For a colouring is fixed by g if and only if it is constant on the vertex cycles of g, and any such colouring induces a colouring of Γ/g; conversely, any proper colouring of Γ/g lifts to a colouring of Γ fixed by g. If a cycle of g contains an edge, then no proper colouring is fixed by g.)

By the Orbit-Counting Lemma, the number of G-orbits on proper k-colourings is

$$\frac{1}{|G|}\sum_{g\in G} P_{\Gamma/g}(k).$$

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So, if we define the *orbital chromatic polynomial* of Γ and G to be

$$\mathrm{OP}_{\Gamma,G}(x) = \frac{1}{|G|}\sum_{g\in G} P_{\Gamma/g}(x),$$

then substituting k into this polynomial gives the number of G-orbits on proper k-colourings of G.

In this paper, we will investigate whether and to what extent the known results on real zeros of the chromatic polynomial extend to the orbital chromatic polynomial. We begin with two simple observations.

Proposition 1.1.

(a) *A positive integer is a root of* $\mathrm{OP}_{\Gamma,G}(x) = 0$ *if and only if it is a root of* $P_\Gamma(x) = 0$.

(b) *If G is the trivial group, then* $\mathrm{OP}_{\Gamma,G} = P_\Gamma$; *so every chromatic root is an orbital chromatic root*.

Proof. **(a)** A graph has no orbits on proper k-colourings if and only if it has no proper k-colourings!

(b) Clear. □

It is known that there are intervals of $\mathbb{R}$ containing no chromatic roots. We show first that no such result holds for orbital chromatic roots in general, and then that zero-free intervals do exist provided that certain parity conditions on the elements of G are satisfied. We end the section with some open problems.

In the final section of the paper, we look at orbital flow roots, and prove some similar results.

We use the notation $\Gamma + \Delta$ for the graph obtained from the disjoint union of graphs Γ and Δ by adding all possible edges between them.

2. Orbital chromatic roots

2.1. Orbital chromatic roots are dense in $\mathbb{R}$

For information about chromatic roots, we refer to the survey paper by Jackson [3].

The chromatic polynomial of a loopless graph has no negative real roots, and no roots in the interval $(0, 1)$. Jackson [2] showed that there are no roots in the interval $(1, \frac{32}{27}]$, and Thomassen [5] showed that this is best possible: roots of chromatic polynomials are dense in $[\frac{32}{27}, \infty)$. Moreover, Sokal [4] showed that complex chromatic roots are dense in the whole complex plane.

Our first result asserts that there are no zero-free intervals for orbital chromatic roots. We begin by observing that such roots can be negative. Let Γ be the null graph on n vertices, and G the symmetric group S_n. Then orbits of G on k-colourings of Γ correspond to choices of n colours from the set of k, where repetitions are allowed and the order of the choices is not significant. From elementary combinatorics, this number is $\binom{k+n-1}{n}$. So we have

$$\mathrm{OP}_{\Gamma,G}(x) = \frac{1}{n!}x(x+1)\cdots(x+n-1),$$

with roots $0, -1, \ldots, -(n-1)$.

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Theorem 2.1. *Orbital chromatic roots are dense in* $\mathbb{R}$.

Proof. First we consider the graph Γ consisting of m disjoint triangles, where the ith triangle has vertex set x_i, y_i, z_i. The group G will be the group induced on the vertices by the symmetric group S_m permuting the m triangles (preserving the three sets $\{x_1, \dots, x_m\}$, $\{y_1, \dots, y_m\}$, and $\{z_1, \dots, z_m\}$).

The number of colourings of a triangle with k colours is $k(k-1)(k-2) = p(k)$, say. Now a proper colouring of Γ consists simply of a selection of m colourings, one for each triangle (with repetitions allowed); and counting orbits of G means that the order of the selection is not significant. By the elementary combinatorial result cited earlier, the number of colourings is

$$\frac{1}{m!}p(k)(p(k)+1)\cdots(p(k)+m-1) = \binom{p(k)+m-1}{m}.$$

That is, the orbital chromatic polynomial is

$$\mathrm{OP}_{\Gamma,G}(x) = \binom{p(x)+m-1}{m}, \quad \text{where } p(x) = x(x-1)(x-2).$$

This shows that the roots of $f_r(x) = x(x-1)(x-2) + r = 0$ are orbital chromatic roots, for any non-negative integer r. Clearly $f_r(x)$ has a unique negative root α_r for $r > 0$, and $\alpha_r - \alpha_{r+1} \to 0$ as $r \to \infty$. (For $\alpha_r = -m$ when $r = m(m+1)(m+2)$, and so the number of roots between consecutive negative integers $-m$ and $-m-1$ increases with m. Moreover, the second derivative of $f_r(x)$ is negative for negative x, so the gaps between successive α_r decrease as r increases.)

Finally, choose a positive integer s, and let $\Gamma_s = \Gamma + K_s$, where K_s is the complete graph on s vertices. In any colouring of Γ_s, we must use s distinct colours for K_s, leaving $k-s$ colours for the remainder. So

$$\mathrm{OP}_{\Gamma_s,G}(x) = x(x-1)\cdots(x-s+1)\mathrm{OP}_{\Gamma,G}(x-s).$$

So every number of the form $\alpha_r + s$ is an orbital chromatic root. It is clear that these numbers are dense in $\mathbb{R}$. □

2.2. Parity and zero-free intervals

However, under parity conditions, we do get zero-free intervals for orbital chromatic roots. To prove these, we need the following result [3, Theorem 3].

Proposition 2.2. *Let the graph* Γ *have* n *vertices,* c *connected components and* b *blocks. Then, for real* x*, the sign of* $P_\Gamma(x)$ *is* $(-1)^n$ *for* $x < 0$*;* $(-1)^{n-c}$ *for* $0 < x < 1$*; and* $(-1)^{n-c-b}$ *for* $1 < x \leqslant \frac{32}{27}$.

Any automorphism induces permutations not only on the set of vertices of Γ, but also on the set of connected components.

Theorem 2.3. *Let* G *be a group of automorphisms of the loopless graph* Γ*. Let* V *and* C *be the sets of vertices and connected components of* Γ.

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(a) *If all elements of G are even permutations of V, then $\mathrm{OP}_{\Gamma,G}(x)$ has no negative real roots.*
(b) *If all elements of G are even permutations of $V \cup C$, then $\mathrm{OP}_{\Gamma,G}(x)$ has no roots in $(0,1)$.*

Proof. **(a)** The parity of a permutation g on n points is that of $n-k$, where k is the number of cycles of g. So, if every element of G is an even permutation on vertices, then the number of vertices of Γ/g is congruent mod 2 to the number n of vertices of Γ. So the sign of $P_{\Gamma/g}(x)$ is $(-1)^n$ for $x<0$ (unless Γ/g has a loop, in which case it is zero). To find $\mathrm{OP}_{\Gamma,G}(x)$ for negative x, we have to average $P_{\Gamma/g}(x)$ over the group; all these terms have the same sign or are zero, and not all are zero. So the average cannot be zero.

(b) is proved in the same way, using the parity on $V \cup C$ instead of that on V. If Γ/g has n' vertices and c' connected components, then $n-c$ and $n'-c'$ have the same parity; so the sign of $P_{\Gamma/g}(x)$ is the same as that of $P_\Gamma(x)$ for $x \in (0,1)$. □

Continuing the example before Theorem 2.1, let Γ be the null graph on n vertices and G the alternating group A_n. It can be shown that

$$\mathrm{OP}_{\Gamma,G}(x) = \frac{1}{n!}\big(x(x+1)\cdots(x+n-1) + x(x-1)\cdots(x-n+1)\big).$$

It is not hard to see that the exponents of all powers of x in this polynomial have the same parity as n; so the only real root is at the origin, with multiplicity 1 or 2 depending on the parity of n.

It might be thought that, if all elements of G are even permutations of $V \cup C \cup B$, where B is the set of blocks, then $\mathrm{OP}_{\Gamma,G}(x)$ would have no roots in $(1, \frac{32}{27}]$. But the argument fails, since the number of blocks of Γ/g is not always equal to the number of cycles of g on blocks of Γ. For example, let p be an odd prime, let Γ be the complete bipartite graph $K_{2,p}$, and let G be the cyclic group C_p, acting on the second partite set. Then Γ has only one block; but for any non-identity element $g \in G$, Γ/g is a path of length 2, and has two blocks. It can be shown that $\mathrm{OP}_{\Gamma,G}(x) = x(x-1)f_p(x)/p$, where

$$f_p(x) = (x-2)^p + (x-1)^{p-1} + (p-1)(x-1).$$

For $p \geqslant 5$, we have $f_p(1) < 0$ and $f_p(\frac{32}{27}) > 0$, so there is an orbital chromatic root in $(1, \frac{32}{27})$. Indeed, f_5 has a root at approximately 1.12683.

2.3. Open problems

Problem 1. For some pairs (Γ, G), there exists a graph Γ' such that $\mathrm{OP}_{\Gamma,G}(x) = P_{\Gamma'}(x)/|G|$. Of course, for such pairs, restrictions on chromatic roots (including zero-free intervals) apply also to orbital chromatic roots.

For example, it can be shown that if there are disjoint subsets A and B of the vertex set of Γ such that B induces a complete graph, every vertex in A has neighbour set B, and $|A| \leqslant |B|+1$, and G is the symmetric group on A, then such a graph Γ' exists.

The problem is to formulate reasonably general sufficient conditions on (Γ, G) for the existence of Γ'.

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Problem 2. Is it true that the real roots of $\mathrm{OP}_{\Gamma,G}(x)$ are bounded above by the largest real root of $P_\Gamma(x)$? (This would be false if we said instead: 'the real parts of the roots of $\mathrm{OP}_{\Gamma,G}(x)$.')

Problem 3. Is it true that, if Γ is 2-connected and G consists of even permutations of the vertices, then the roots of $\mathrm{OP}_{\Gamma,G}(x)$ are dense in $[1,\infty)$? (Thomassen's result [5] shows that they are dense in $[\frac{32}{27},\infty)$.)

3. Orbital flow roots

3.1. The orbital flow polynomial

Given a graph Γ, an orientation of the edge set of Γ, and an abelian group A, an *A-flow* on Γ is a function from oriented edges to A with the property that the signed sum of its values on the edges incident with any vertex is 0. The flow is *nowhere-zero* if it never takes the value 0. Tutte [6] showed that the number of nowhere-zero A-flows depends only on the order k of A and not on its structure, and moreover that this number is given by a polynomial $F_\Gamma(k)$, the *flow polynomial* of Γ. (The flow in any bridge must be zero, so we assume that the graph Γ is bridgeless.)

In [1], the authors showed that the number of orbits of a group G of automorphisms of Γ on nowhere-zero A-flows may depend on the structure of A. More precisely, this number is given by substituting $x_i = \alpha_i(A)$ into a polynomial $\mathrm{OF}_{\Gamma,G}(x_0, x_1, x_2, \dots)$, the *orbital flow polynomial* of Γ and G, where $\alpha_i(A)$ is the number of solutions of $ia = 0$ in A. (Note that $\alpha_0(A) = |A|$ and $\alpha_1(A) = 1$.)

It is shown in [1] that if the variable x_i occurs in the orbital flow polynomial, for $i > 0$, then G must have an element of order i. Thus, if A is an abelian group of order coprime to $|G|$, then the number of G-orbits on nowhere-zero A-flows is given by $\mathrm{OF}^1_{\Gamma,G}(k)$, where $k = |A|$ and

$$\mathrm{OF}^1_{\Gamma,G}(x) = \mathrm{OF}_{\Gamma,G}(x, 1, 1, \dots).$$

We call OF^1 the *univariate orbital flow polynomial* of Γ and G. (This explains Tutte's observation that the number of nowhere-zero A-flows depends only on the order of A and not on its structure: if G is the trivial group then every finite abelian group has order coprime to $|G|$.)

In this section we allow graphs to have multiple edges. (Duplicating an edge has no effect on the chromatic polynomial but can change the flow polynomial.)

3.2. Negative orbital flow roots

By analogy with the result on zero-free intervals for the chromatic polynomial, Wakelin [7] showed that the flow polynomial of a bridgeless connected graph has no roots in $(-\infty, 1)$ (and has sign $(-1)^{m-n+1}$ there, where n and m are the numbers of vertices and edges), and no roots in $(1, \frac{32}{27}]$ (and has sign $(-1)^{n-m+b+1}$ there, where b is the number of blocks).

Modifying our method for orbital chromatic roots, we show the following.

Theorem 3.1. *The roots of univariate orbital flow polynomials are dense in* $(-\infty, 0]$.

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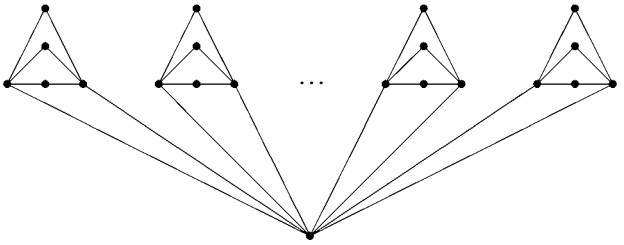


Figure 1. A graph with negative orbital flow roots.

Proof. Let Δ consist of two vertices joined by n parallel edges, where n is even. Then the flow polynomial of Δ is known to be $(x-1)((x-1)^{n-1}+1)/x$. So, if Γ consists of the disjoint union of m copies of Δ, and G is the symmetric group of degree m permuting these copies, the number of G-orbits on flows in Γ is obtained by substituting $x = |A|$ into the polynomial

$$\mathrm{OF}^1_{\Gamma,G}(x) = \binom{q(x)+m-1}{m}, \quad \text{where } q(x) = (x-1)((x-1)^{n-1}+1)/x.$$

Hence every solution of $q(x) = -k$, for positive integers k, is an orbital flow root.

Putting $y = 1-x$, this equation becomes

$$k+1 = \frac{y^n-1}{y-1} = y^{n-1}+\cdots+1.$$

The polynomial on the right is clearly positive, and has its first n derivatives positive, for $y > 0$. So there is a unique real positive root $\alpha(n,k)$, and for fixed n the spacing of the roots decreases as k increases. Clearly $\alpha(n,n-1) = 1$, and $\alpha(n,k+1) - \alpha(n,k) < 1/(n-1)$ for $k \geqslant n$. Hence the values of $\alpha(n,k)$ are dense in $[1,\infty)$, and the values of $1-\alpha(n,k)$ (which are orbital flow roots) are dense on the negative real line. □

We cannot use the second part of the earlier proof to show that the orbital flow roots are dense in $\mathbb{R}$ by 'shifting' them up by integers. Indeed, it has been conjectured that a real flow root cannot be larger than 4.

Note that, for n large compared to k, $\alpha(n,k)$ is approximately $1-1/(k+1)$, so the corresponding orbital flow root is close to $1/(k+1)$. So, although there are infinitely many orbital flow roots in the interval $[0,1]$, the only limit points we obtain from this construction are reciprocals of positive integers.

We can modify this example to produce a simple connected planar graph. First, make the graph simple by subdividing each edge. This does not change the number of flows. Then identify the mid-points of one set of corresponding edges from the graphs Δ. Since the net flow on the edges to this mid-point from each copy of Δ must be zero, again no extra flows are produced. See Figure 1 for $n = 4$.

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