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Groups with two projective characters

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All representations and characters studied in this paper are taken over the field of complex numbers, and all groups considered are finite. The reader unfamiliar with projective representations is referred to [**8**] for basic definitions and terminology.

If G is a group and α is a cocycle of G, we denote the set of irreducible projective characters of G with cocycle α by $\mathrm{Proj}(G, \alpha)$. In 1964 Iwahori and Matsumoto [**7**] conjectured that a group G with a cocycle α for which $|\mathrm{Proj}(G, \alpha)| = 1$ is solvable (we shall say that such a group is of α-central type). This conjecture was verified by Howlett and Isaacs [**6**] in 1982. We generalize this work by investigating a group G with a cocycle α for which $|\mathrm{Proj}(G, \alpha)| = 2$; for the sake of brevity we say that such a group is of 2α-central type. An example of a group with this property is the dihedral group of order 8.

We shall establish two major results concerning groups of 2α-central type; the first describes the degrees of the projective characters in such a group.

Theorem A. *Let* $\mathrm{Proj}(G, \alpha) = \{\xi_1, \xi_2\}$; *then* $\xi_1(1) = \xi_2(1)$.

This theorem enables us to utilize the techniques established in [**6**] and [**9**] for dealing with groups of α-central type, and as a consequence we obtain

Theorem B. *Let* G *be of* 2α*-central type; then* G *is solvable.*

1. *Preliminaries*

In order to minimize repetition we employ the following notation for the rest of this paper. Let G denote a group of 2α-central type, and $\mathrm{Proj}(G, \alpha) = \{\xi_1, \xi_2\}$ with degrees a and b respectively. We remark here that since degrees are unaffected under projective equivalence we can assume without loss of generality that α is special. In particular, then, the projective character table of G is as follows:

$\lvert\mathscr{C}_i\rvert$	1	m
c_i	1	x
ξ_1	a	$\xi_1(x)$
ξ_2	b	$\frac{-a}{b}\xi_1(x)$

where $c_1 = 1$ and $c_2 = x$ are representative elements respectively for the trivial and non-trivial α-regular conjugacy classes $\mathscr{C}_1$ and $\mathscr{C}_2$ of G.

We start by considering the situation when α is trivial.

Lemma 1·1. *Let* $|\mathrm{Irr}(G)| = 2$. *Then* $|G| = 2$.

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Proof. Let ξ_1 denote the trivial character of G. Then $1+b^2 = |G|$, so that $b|(|G|-1)$; however $b||G|$ as well, and so $b = 1$. ∎

As a consequence of 1·1, we can assume henceforward where necessary that $o([\alpha]) > 1$ in $M(G)$, the Schur multiplier of G. We now quote the following well known result:

LEMMA 1·2. *Let H be a group and β be a cocycle of H with $o([\beta]) = e$ in $M(H)$. Then* (i) *$e|\xi(1)$ for all $\xi \in \mathrm{Proj}(H, \beta)$, and* (ii) *if p is a prime number such that $p|\xi(1)$ for all $\xi \in \mathrm{Proj}(H, \beta)$, then $p|e$.*

The next result describes an important special case for which Theorem A is true.

LEMMA 1·3. *Suppose that G possesses a non-trivial cyclic Sylow p-subgroup S for some prime p. Then* (i) *$a = b$, and* (ii) *$G = SN$ where $N \lhd G$ of index 2, and $N \cap S = 1$.*

Proof. Since S is cyclic we have that $p \nmid o([\alpha])$, and it follows that every element of p-power order in G is α-regular. Consequently G has only one α-regular p-regular class, and therefore the decomposition matrix of the only possible 'projective' p-block must be $(1\,1)^T$, where T stands for transpose. Thus $a = b$. Now $|G| = 2a^2$, so $p = 2$ and $2 \nmid a$ from 1·2(ii); in particular G is solvable and hence has a normal Hall $2'$-subgroup N. ∎

We note in the situation of 1·3 that it is easy to show that $\xi_2 = \lambda\xi_1$ where λ is the non-trivial linear character of G/N, and that N is of α_N-central type. This observation motivates the following lemma.

LEMMA 1·4. *Suppose $x \notin G'$. Then* (i) *G has a normal subgroup N of index 2 and N is of α_N-central type, and* (ii) *$\xi_2 = \lambda\xi_1$, where λ is the non-trivial linear character of G/N.*

Proof. The group A of linear characters of G acts on Proj (G, α) by multiplication; also $\xi_1(x) \neq 0$ since x is α-regular. Let $A(\xi_1)$ denote the stabilizer of ξ_1 in A. Then $A(\xi_1) = \{\mu \in A : \mu(x) = 1\}$, and $[A : A(\xi_1)] = 2$ since $x \notin G'$. Thus by the normal duality between A and G/G' there exists $N \lhd G$ with $x \notin N$ and $[G:N] = 2$, and (ii) is then obvious. Now since $\xi_1(x) \neq 0$ we have that $(\xi_i)_N$ for $i = 1, 2$ are irreducible and equal, but $a^2 = |N|$ so that N is of α_N-central type. ∎

The last of the special cases when Theorem A can easily be shown to be true is when G is a p-group.

LEMMA 1·5. *Suppose G is a p-group for some prime p. Then $a = b$ and $p = 2$.*

Proof. We may assume that $a \leqslant b$. Now $|G| = a^2 + b^2 = a^2(1 + b^2/a^2)$, so $1 + b^2/a^2$ is a power of p. However, $1 + b^2/a^2 \equiv 1 \pmod{p}$ unless $a = b$ and $p = 2$. ∎

We now recall that if H is a group, β is a cocycle of H, and $M \trianglelefteq H$, then H acts on Proj (M, α_M) via the action

$$\zeta^h(y) = f_\beta(y, h^{-1})\,\zeta(hyh^{-1})$$

for $h \in H$, and all $y \in M$, where $f_\beta(y, h^{-1}) = \beta(h, yh^{-1})\,\beta(y, h^{-1})(\beta(h, h^{-1}))^{-1}$. It is well known that Clifford's theorem as stated for ordinary characters holds for projective characters with this action, which is called conjugation. We now use a strong version of this theorem to show that the property of G possessing two irreducible projective

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characters can be inherited by certain groups involved in G. We remark for future reference that the first paragraph of the proof of 1·6 holds for any group G with cocycle α.

PROPOSITION 1·6. *Let $N \trianglelefteq G$, $\zeta \in \mathrm{Proj}(N, \alpha_N)$, and let I denote the inertia subgroup $I_G(\zeta)$ of ζ in G.*

(i) *If $x \notin N$, then G acts transitively on* $\mathrm{Proj}(N, \alpha_N)$, *and there is a cocycle β of I/N such that I/N is of 2β-central type. Moreover, $a = b$ if and only if the degrees of the two elements of* $\mathrm{Proj}(I/N, \beta)$ *are equal.*

(ii) *If $x \in N$, then G has two orbits on* $\mathrm{Proj}(N, \alpha_N)$, *and there exists a cocycle β of I/N such that I/N is of β-central type.*

Proof. Let $\zeta \in \mathrm{Proj}(N, \alpha_N)$, I be its inertia subgroup in G, and $[\,,\,]$ stand for the usual inner product of characters. Then by Clifford's theorem there is a cocycle β of I/N and a bijection from $\mathrm{Proj}(I/N, \beta)$ to $\mathscr{A} = \{\gamma \in \mathrm{Proj}(I, \alpha_I) : [\gamma_N, \zeta] \neq 0\}$ defined by $\tau \mapsto \tau\kappa$, where $\kappa_N = \zeta$ and $\kappa \in \mathrm{Proj}(I, \beta^{-1}\alpha_I)$. Moreover, by Clifford's theorem again, there exists a bijection from $\mathscr{A}$ to $\mathscr{B} = \{\xi \in \mathrm{Proj}(G, \alpha) : [\xi_N, \zeta] \neq 0\}$ defined by $\gamma \mapsto \gamma^G$.

Now if $x \notin N$, then

$$[(\xi_1)_N, \zeta] = \frac{a\zeta(1)}{|N|} \neq 0,$$

and it follows that G acts transitively on $\mathrm{Proj}(N, \alpha_N)$. Thus $\mathscr{B} = \mathrm{Proj}(G, \alpha)$ and so $|\mathrm{Proj}(I/N, \beta)| = 2$, with elements τ_1, τ_2 say. Thus by suitable labelling we have $[G:I]\tau_1(1)\kappa(1) = a$ and $[G:I]\tau_2(1)\kappa(1) = b$, and so $\tau_1(1) = \tau_2(1)$ if and only if $a = b$.

Now suppose $x \in N$. Then $[(\xi_1)_N, (\xi_2)_N] = 0$ and so G must have exactly two orbits on $\mathrm{Proj}(N, \alpha_N)$. Thus $|\mathscr{B}| = 1$ and so $|\mathrm{Proj}(I/N, \beta)| = 1$. ∎

2. *Behaviour of groups of 2α-central type*

In this section we shall demonstrate that provided Theorem A is true, groups of 2α-central type behave exactly like groups of α-central type. This will mean that we shall eventually be able to employ the large body of known results about groups of α-central type in our situation.

If G is a group of 2α-central type we shall continue to use the notation introduced in §1, so that in particular a and b stand for the degrees of the two elements of $\mathrm{Proj}(G, \alpha)$. We begin by recalling that we may assume α is special, so that if ξ lies in $\mathrm{Proj}(G, \alpha)$ then $Z(\xi) = \{g \in G : |\xi(g)| = \xi(1)\}$ is a normal subgroup of G. It is obvious that if G is of 2α-central type with $a = b$, then $Z(\xi_1) = Z(\xi_2)$.

LEMMA 2·1. *Let G be of 2α-central type with $a = b$, x be a non-trivial α-regular element of G, and $V = \langle x \rangle$. Suppose $Z(\xi_1) \neq \{1\}$. Then* (i) *x is unique and $x \in Z(G)$,* (ii) *$Z(\xi_1) = V$ and $|V| = 2$, and* (iii) *G/V is of β-central type for some cocycle β.*

Proof. Since $Z(\xi_1) \neq \{1\}$ we must have $Z(\xi_1) = \{1\} \cup \mathscr{C}_2$, but then $2a^2 = a^2 + |\mathscr{C}_2|a^2$ so that $|\mathscr{C}_2| = 1$. Thus $x \in Z(G)$ is the unique non-trivial α-regular element in G and $|V| = 2$. Now $(\xi_1)_V = a\lambda$ for $\lambda \in \mathrm{Proj}(V, \alpha_V)$ so that G/V is of β-central type from 1·6(ii). ∎

Our next result shows that if G is of 2α-central type satisfying Theorem A, then a Sylow 2-subgroup S of G is of $2\alpha_S$-central type.

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PROPOSITION 2·2. *Let G be a group and α be a cocycle of G. Then G is of 2α-central type with $a = b$ if and only if a Sylow 2-subgroup S of G is of $2\alpha_S$-central type, and for each odd prime p, a Sylow p-subgroup S_p of G is of α_{S_p}-central type.*

Proof. Let p be any prime, $S_p \in \mathrm{Syl}_p(G)$, and α_p denote α_{S_p}. Suppose G is of 2α-central type with $a = b$. Then from proposition 1 of [5], $\min\{\zeta(1) : \zeta \in \mathrm{Proj}\,(S_p, \alpha_p)\}$ is the pth part a_p of a. Now if $p \neq 2$ we have that $a_p^2 = |S_p|$, whereas if $p = 2$ we have that $a_2^2 = \frac{1}{2}|S_2|$, and so S_2 must be of $2\alpha_2$-central type.

Conversely, suppose S_2 is of $2\alpha_2$-central type and S_p is of α_p-central type for all primes $p \neq 2$. Then from 1·5 and proposition 1 of [5], g.c.d. $\{(\xi(1))^2 : \xi \in \mathrm{Proj}\,(G, \alpha)\}$ is equal to $\frac{1}{2}|G|$, and G is forced to be of 2α-central type with $a = b$. ∎

Using 2·2, we can now describe the α-regular elements in a group of 2α-central type satisfying Theorem A.

COROLLARY 2·3. *Let G be of 2α-central type with $a = b$, and $S \in \mathrm{Syl}_2(G)$. Then the α_S-regular elements of S are α-regular in G.*

Proof. Suppose that a non-trivial α_S-regular element of $S \in \mathrm{Syl}_2(G)$ is not α-regular. Then from 2·2

$$[(\xi_1)_S, (\xi_1)_S] = \frac{a^2}{|S|} = \tfrac{1}{2}[G:S] \notin \mathbb{Z},$$

a contradiction. ∎

Let G be of 2α-central type satisfying Theorem A and $N \lhd G$; then it is not necessarily true that the elements of $\mathrm{Proj}\,(N, \alpha_N)$ have the same degree. However, we now aim to show that this is the case for certain nilpotent normal subgroups of G.

LEMMA 2·4. *Let G be a 2-group of 2α-central type, x be a non-trivial α-regular element of G, and $N \trianglelefteq G$. Define*

$$K = \begin{cases} N & \text{if } x \notin N, \\ \langle y : y \in \mathscr{C}_2 \rangle & \text{if } x \in N. \end{cases}$$

Then the elements of $\mathrm{Proj}\,(K, \alpha_K)$ all have the same degree.

Proof. If $x \notin N$ the required result is immediate from 1·6 (i). So we now assume that $x \in N$, and let $M \lhd G$ with $[K:M] = 2$. Now let λ be the non-trivial linear character of K/M, so that $\lambda(y) = -1$ for all $y \in \mathscr{C}_2$. Then we have $(\xi_2)_K = \lambda(\xi_1)_K$, so if ζ is an irreducible constituent of $(\xi_1)_K$ we obtain that $\lambda\zeta$ is an irreducible constituent of $(\xi_2)_K$, and the desired result is yielded by 1·6 (ii). ∎

PROPOSITION 2·5. *Let $\mathrm{Proj}\,(G, \alpha) = \{\xi_1, \xi_2\}$ with $a = b$, x be a non-trivial α-regular element of G, and N be a nilpotent normal subgroup of G. Define*

$$K = \begin{cases} N & \text{if } x \notin N, \\ \langle y : y \in \mathscr{C}_2 \rangle \times H & \text{if } x \in N, \end{cases}$$

where H is the Hall $2'$-subgroup of N. Now for $i = 1, 2$ let ζ_i be an irreducible constituent of $(\xi_i)_K$ with $[(\xi_i)_K, \zeta_i] = e_i$ and $[G : I_G(\zeta_i)] = s_i$. Then $\zeta_1(1) = \zeta_2(1)$, $e_1 = e_2$ and $s_1 = s_2$, so that in particular the elements of $\mathrm{Proj}\,(K, \alpha_K)$ all have the same degree.

Proof. If $x \notin N$, then the result is immediate from 1·6 (i) and does not rely on the fact that N is nilpotent.

We now assume $x \in N$, so that $P_1 = \langle y : y \in \mathscr{C}_2 \rangle$ is a 2-group from 2·2 and 2·3. Let

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$p_2, \ldots, p_n$ be the odd prime divisors of $|N|$, and $S_i \in \mathrm{Syl}_{p_i}(G)$ for $2 \leqslant i \leqslant n$. Then for $2 \leqslant i \leqslant n$, $P_i = S_i \cap N \in \mathrm{Syl}_{p_i}(N)$ and $P_i \trianglelefteq S_i$. Since degrees are unaffected under projective equivalence, we can assume in what follows that $\mathrm{Proj}\,(K, \alpha_K)$ is equal to $\{\gamma_1 \times \cdots \times \gamma_n : \gamma_i \in \mathrm{Proj}\,(P_i, \alpha_i)\}$, where α_i denotes α_{P_i}. Now the elements of $\mathrm{Proj}\,(P_1, \alpha_1)$ all have the same degree from 2·2 and 2·4, whereas for $2 \leqslant i \leqslant n$ we have from 2·2 that the elements of $\mathrm{Proj}\,(P_i, \alpha_i)$ are all conjugate under S_i and therefore all have the same degree as well. Finally $[(\xi_i)_K, (\xi_i)_K] = [G:K]$ for $i = 1, 2$ and so $e_1^2 s_1 = e_2^2 s_2$; but we also have that $a = e_1 s_1 \zeta_1(1) = e_2 s_2 \zeta_2(1) = b$ and so $e_1 s_1 = e_2 s_2$. We conclude that $e_1 = e_2$ and $s_1 = s_2$. ∎

3. *Two properties of simple groups*

In this section we shall establish two properties of non-Abelian simple groups which will be used in the proof of Theorem A. We remark before commencing 3·1 that the restrictions on n and q which we shall impose can certainly be improved upon; however, they are sufficient for our purposes.

LEMMA 3·1. *Let p be a prime number and $q = p^m$. Then the number of conjugacy classes in $H = PSL(n, q)$ or $PSU(n, q)$ for $n \geqslant 6$ and $q \geqslant 3$ is less than the order of a Sylow p-subgroup of H.*

Proof. Let $c(L)$ denote the number of conjugacy classes in the group L. The following inequalities can easily be established from [**10**]: $c(PSL(n, q)) \leqslant c(PSU(n, q)) \leqslant c(SU(n, q)) \leqslant (q+1)\, c(GU(n, q))$. Thus it will certainly suffice if we show that $(q+1)\, c(GU(n, q)) < q^{\frac{1}{2}[n(n-1)]}$.

Now in [**10**], §6, a function f is defined from the conjugacy classes of $GU(n, q)$ onto the partitions of n. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of n with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0$; then if i occurs n_i times in λ there are

$$c(\lambda) = \prod_{n_i > 0} (q^{n_i} + q^{n_i - 1})$$

conjugacy classes of $GU(n, q)$ which map under f to λ. Let Q denote the set of $n \times n$ upper-triangular matrices over $GF(q)$ which have 1 on the main diagonal, so that $|Q| = q^{\frac{1}{2}[n(n-1)]}$. Now let $Q(\lambda)$ denote the set of elements (a_{ij}) of Q such that (1) $a_{i, i+1} = 0$ for all i with $0 < i < n$ and $i = \lambda_1,\ \lambda_1 + \lambda_2,\ \lambda_1 + \lambda_2 + \lambda_3, \ldots$, and (2) $a_{i, i+1} \neq 0$ otherwise. We note that if $\lambda' \neq \lambda$ then $Q(\lambda) \cap Q(\lambda') = \varnothing$, since the positions of the zeros on the second diagonal determine λ. Now $|Q(\lambda)| \geqslant q^{\frac{1}{2}[(n-1)(n-2)]}$, whereas $(q+1)\, c(\lambda) \leqslant (q+1)^{n+1} < q^{\frac{4}{3}(n+1)}$ since $q + 1 < q^{\frac{4}{3}}$ for $q \geqslant 3$. Finally $q^{\frac{4}{3}(n+1)} < q^{\frac{1}{2}[(n-1)(n-2)]}$ for $n \geqslant 6$. ∎

Our second and third results use the fact that the Schur multipliers of all non-Abelian simple groups are listed in [**3**].

THEOREM 3·2. *Let H be a non-Abelian simple group and β be a cocycle of H. Then the degrees of the elements of* $\mathrm{Proj}\,(H, \beta)$ *are not all equal.*

Proof. Suppose the elements of $\mathrm{Proj}\,(H, \beta)$ have common degree m, and let t denote the number of α-regular conjugacy classes of H, so that $tm^2 = |H|$. Now from 1·2 if $\Pi = \{p : p \text{ is prime and } p \mid m\}$, then Π is also the set of prime divisors of $o([\beta])$ and $t_{\Pi'} = |H|_{\Pi'}$. In particular, if $o([\beta]) = p^d$ for p prime and $d \geqslant 0$, then $m = p^e$ for $e \geqslant 0$ and so H is solvable from [**4**], a contradiction. Thus $M(H)$ is not of prime-power

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order. This means that either $H = PSL(n,q)$ or $PSU(n,q)$ for suitable n, q such that $n \geqslant 6$ and $q \geqslant 5$, or H is one of a finite number of groups. For all of these groups we obtain a contradiction, since either the number of conjugacy classes is too small from 3·1 and [1], or the projective character tables of H are known [1] and there is always more than one degree. ∎

Now we note from (3·1) of [9] (see also p. 556 of [6]) that if H is a non-Abelian simple group then there is a prime p such that H has a non-trivial cyclic Sylow p-subgroup and $p \nmid |\mathrm{Out}(H)|$. This implies from 1·3 that a group of 2α-central type is not a non-Abelian simple group, and it also enables us to prove the third result.

LEMMA 3·3. *Let H be a non-Abelian simple group with $H \ncong A_6$. Then there are at least two distinct prime numbers p, q, such that $p \mid |H|$ and $q \mid |H|$ but $p \nmid |M(H)|$ and $q \nmid |M(H)|$.*

Proof. From above we have that H has a non-trivial cyclic Sylow p-subgroup for some prime p, and so $p \nmid |M(H)|$. So suppose for all prime numbers q with $q \mid |H|$ and $q \neq p$ that $q \mid |M(H)|$. Then by Burnside's $p^a q^b$-theorem we have that $M(H)$ is not of prime-power order, and so we are in the same situation as in the proof of 3·2. Now if $H = PSL(n, q^a)$ or $PSU(n, q^a)$ for $n \geqslant 6$ and $q^a \geqslant 5$ where q is prime, then $q \nmid |M(H)|$, but a Sylow q-subgroup of H is not cyclic, a contradiction. For the remaining finite number of groups to be considered we also obtain a contradiction by inspection of $|H|$ and $|M(H)|$. ∎

4. *Degree equality*

Before embarking on the proof of Theorem A, we need the following result which necessitates the use of an α-representation group; see p. 99 of [8] for details. We shall continue to use all the notation of §1, so that in particular $\mathscr{C}_2$ is once again the non-trivial α-regular class of G.

LEMMA 4·1. *Let G be a group of 2α-central type, and $\hat{G}$ be an α-representation group of G. Suppose that $K = \langle y : y \in \mathscr{C}_2 \rangle$ is a p-group for some prime p, and let q be a prime with $q \neq p$. Then* (i) *a Sylow q-subgroup Q of $\hat{G}$ is of central type and $Z(Q) = Q \cap Z(\hat{G})$;* (ii) *if $N \trianglelefteq \hat{G}$ and $q \mid |N|$, then $q \mid |N \cap Z(\hat{G})|$.*

Proof. Since $\hat{G}$ is an α-representation group for G, there exists a central subgroup A of $\hat{G}$ and an isomorphism $\Pi : \hat{G}/A \to G$. Now by (3·5) of [12], $\Pi(Z(\hat{G})/A)$ is the group of α-regular elements of G contained in $Z(G)$. Thus if $A \neq Z(\hat{G})$, we must actually have $|K| = 2$, and $[Z(\hat{G}) : A] = 2$.

For $i = 1, 2$ let $\gamma_i \in \mathrm{Proj}\,(K, \alpha_K)$ be such that $[(\xi_i)_K, \gamma_i] = e_i > 0$, and let $s_i = [G : I_G(\gamma_i)]$. Then $[(\xi_i)_K, (\xi_i)_K] = [G:K] = e_i^2 s_i$, and so $[G:K] \mid \mathrm{g.c.d.}\{a^2, b^2\}$. Let $q \neq p$ be a prime and $S \in \mathrm{Syl}_q(G)$. Then as in the proof of 2·2 we have

$$|S| = ([G:K])_q \mid \min\{\zeta(1)^2 : \zeta \in \mathrm{Proj}\,(S, \alpha_S)\},$$

and so S must be of α_S-central type. Now let $R/A = \Pi^{-1}(S)$. Then by the normal relationship between $\mathrm{Irr}\,(R)$ and $\mathrm{Proj}\,(S, \alpha_S)$ there exists $\chi \in \mathrm{Irr}\,(R)$ with $\chi(1)^2 = |S| = [R:A]$. Let $Q \in \mathrm{Syl}_q(R)$ and $A_{q'}$ be the Hall q'-subgroup of A, so that $Q \in \mathrm{Syl}_q(\hat{G})$ and $R = Q \times A_{q'}$. Then we have $\chi(1)^2 = [Q : Q \cap A] = [Q : Q \cap Z(\hat{G})]$, and χ_Q is irreducible since $A_{q'} \leqslant Z(\hat{G})$. However if $\psi \in \mathrm{Irr}\,(Q)$ then $\psi(1)^2 \leqslant [Q : Z(Q)]$, and so we must have that $Z(Q) = Q \cap Z(\hat{G})$. This completes the proof of (i); the proof of (ii) is now exactly the same as that of (4·5) of [6]. ∎

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Proof of Theorem A. Let G be a group of minimal order such that G has a cocycle α for which $|\mathrm{Proj}\,(G,\alpha)| = 2$ but $a \neq b$.

Choose if possible a proper normal subgroup N of G such that $x \notin N$. Then if I is as in 1·6, we obtain from the minimality of $|G|$ and 1·6(i) that $a = b$, a contradiction. Thus we must now assume that $K = \langle y : y \in \mathscr{C}_2 \rangle$ is the unique minimal normal subgroup of G.

Case 1. $K = H_1 \times \cdots \times H_n$, with $H_i \cong H$ a non-Abelian simple group.

In this case $M(K) \cong M(H_1) \times \cdots \times M(H_n)$, and so since degrees are unaffected under projective equivalence we can assume that $\mathrm{Proj}\,(K,\alpha_K) = \{\zeta_1 \times \cdots \times \zeta_n : \zeta_i \in \mathrm{Proj}(H_i, \alpha_{H_i})\}$ in what follows. Now from 1·6 the elements of $\mathrm{Proj}\,(K,\alpha_K)$ have at most two distinct degrees, and it follows from 3·2 that $K \cong H$ and there are exactly two distinct degrees. However, $K \not\leqslant C_G(K) \trianglelefteq G$, so that $C_G(K) = 1$, and hence $K < G \leqslant \mathrm{Aut}\,(K)$. But $K \cong \mathrm{Inn}\,(K)$, so G has a cyclic Sylow p-subgroup for some prime p, which is clearly impossible from 1·3.

Case 2. $K = C_p \times \cdots \times C_p$ for some prime p.

In this case we combine the arguments of [**6**] and [**9**] to show that Theorem A is true. We first emphasize that $K = \langle y : y \in \mathscr{C}_2 \rangle$. Consequently, (a) $[\alpha_K] = [1]$, (b) if δ lies in $\mathrm{Proj}\,(K,\alpha_K)$ then $\delta(1) = 1$, and (c) $C_G(K) \leqslant I_G(\delta)$ for all $\delta \in \mathrm{Proj}\,(K,\alpha_K)$. In particular, then, we note that $C = C_G(K)$ is solvable from 1·6(ii). Now for $i = 1, 2$ let $\delta_i \in \mathrm{Proj}\,(K,\alpha_K)$ such that $[(\xi_i)_K, \delta_i] = e_i > 0$, and let $[G : I_G(\delta_i)] = s_i$, so that $s_1 + s_2 = |K|$. It is useful to observe that $(s_1)_p = (s_2)_p$ and $\mathrm{g.c.d.}\{s_1, s_2\} = (s_1)_p$.

Stage 1. $I_G(\delta) < G$ for all $\delta \in \mathrm{Proj}\,(K,\alpha_K)$.

Proof. Suppose $I_G(\delta) = G$ for some $\delta \in \mathrm{Proj}\,(K,\alpha_K)$; then without loss of generality we may assume that $s_1 = 1$ and $s_2 = |K| - 1$. Now $[(\xi_i)_K, (\xi_i)_K] = [G:K] = e_1^2 = e_2^2 s_2$ for $i = 1, 2$; however, we also have $a^2 = e_1^2$ and $b^2 = e_2^2 s_2^2$. Hence $b^2/a^2 = s_2$, and s_2 is a square. Thus from p. 301 of [**11**] we must have $|K| = p$, and $s_2 = p - 1$. Now we have $C = C_G(y)$ for $y \in \mathscr{C}_2$ and $[G:C] = p - 1$, so that $I_G(\delta_2) = C$. By 1·6(ii) there exist cocycles β_1 of $\bar{G} = G/K$ and β_2 of $\bar{C} = C/K$ such that $\bar{G}, \bar{C}$ are of β_1-central, respectively β_2-central, type. Thus in particular G is solvable. Let H be a Hall p'-subgroup of G, so that $L = H \cap C$ is a Hall p'-subgroup of C, and $[H:L] = p - 1$. Then it follows from theorem 2 of [**2**] that $\bar{H} = HK/K$ and $\bar{L} = LK/K$ are of $(\beta_1)_{\bar{H}}$-central, respectively $(\beta_2)_{\bar{L}}$-central, type. Thus from the bijections in the proof of 1·6 we obtain two elements χ_1, χ_2 of $\mathrm{Proj}\,(HK, \alpha_{HK})$ with $\chi_1(1)^2 = |\bar{H}|$ and $\chi_2(1)^2 = |\bar{L}|(p-1)^2$. Therefore $\chi_1(1)^2 + \chi_2(1)^2 = |HK|$, and HK is of $2\alpha_{HK}$-central type. Now by 1·3 we have $p = 2$, and so $a = b$, a contradiction.

Stage 2. If $C < R \trianglelefteq G$ and $\bar{R} = R/C$, then $p \mid |\bar{R}|$.

Proof. We start by noting that if $\delta \in \mathrm{Proj}\,(K,\alpha_K)$ and $V = \mathrm{Irr}\,(K)$, then $\mathrm{Proj}\,(K, \alpha_K) = \{\delta v : v \in V\}$. Now suppose there exists $R \trianglelefteq G$ with $C < R$ such that $p \nmid |\bar{R}|$. Then we may use the arguments in the proofs of (2·5) and (2·4) of [**9**] (in that order) in our situation to show that some $\delta \in \mathrm{Proj}\,(K,\alpha_K)$ is G-invariant, and Stage 1 gives a contradiction.

Now suppose $\bar{R} = R/C$ is an Abelian chief factor of G. Then $\bar{R}$ acts faithfully on V and $C_V(\bar{R})$ is G-invariant. Thus since K is the minimal normal subgroup of G we must have that $C_V(\bar{R})$ is trivial, and so $\bar{R}$ has order coprime to p, contrary to Stage 2. Thus

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we assume henceforward that C is the maximal solvable normal subgroup of G. We now have $K \leqslant F(G) = O_p(G) \leqslant C$. Let $F = F(G)$, $\chi \in \mathrm{Proj}(F, \alpha_F)$, and $\delta \in \mathrm{Proj}(K, \alpha_K)$ such that $[\chi_K, \delta] \neq 0$.

Stage 3. $[I_G(\delta) : I_G(\chi)] = p^c$ for some $c \geqslant 0$.

Proof. We start by noting that $\chi_K = l\delta$ for some $l \geqslant 1$, and so $I_G(\chi) \leqslant I_G(\delta)$. Now by 1·6(ii) there exists a cocycle β of $I_G(\delta)/K$ such that $I_G(\delta)/K$ is of β-central type; however, $\bar{F} = F/K \trianglelefteq I_G(\delta)/K$, and so the elements of $\mathrm{Proj}(\bar{F}, \beta_{\bar{F}})$ all have the same degree. As in the proof of 1·6, there exists a bijection from $\mathrm{Proj}(\bar{F}, \beta_{\bar{F}})$ onto $\mathscr{A} = \{\chi' \in \mathrm{Proj}(F, \alpha_F) : [\chi'_K, \delta] \neq 0\}$, and thus $|\mathscr{A}| = |\mathrm{Proj}(\bar{F}, \beta_{\bar{F}})| = p^d$ for some $d \geqslant 0$. Now we may assume without loss of generality that $[(\xi_1)_K, \delta] \neq 0$, so that $[(\xi_2)_K, \delta] = 0$. Thus we must have $[(\xi_1)_F, \chi'] \neq 0$ for all $\chi' \in \mathscr{A}$, and hence the elements of $\mathscr{A}$ are all G-conjugate. It now follows easily that $[G : I_G(\chi)] = p^a s_1$ with the assumption we have made, and so $[I_G(\delta) : I_G(\chi)] = p^a$ as claimed.

Stage 4. $F = C$.

Proof. Suppose $F < C$ and let M/F be a chief factor of G with $M \leqslant C$. Then M/F is an elementary Abelian q-group for some prime $q \neq p$. Let $Q \in \mathrm{Syl}_q(M)$, and $H = N_G(Q)$. Then $H < G$, since $FQ = M$ and M is not nilpotent. Now by Stage 3 we have $O_q(C/F) \leqslant I_G(\chi)/F$ for all $\chi \in \mathrm{Proj}(F, \alpha_F)$, and hence $Q \leqslant I_G(\chi)$ for all $\chi \in \mathrm{Proj}(F, \alpha_F)$. Thus writing $\mathrm{Proj}_Q(F, \alpha_F)$ for the set of Q-invariant elements of $\mathrm{Proj}(F, \alpha_F)$, we have that $\mathrm{Proj}_Q(F, \alpha_F) = \mathrm{Proj}(F, \alpha_F)$. Let $L = H \cap F$, so that the conditions of theorem (3·7) of [**6**] are satisfied (we use the theorem as it applies to projective characters). By (3·7)(*a*) of [**6**], the mapping $\alpha_Q(F : L)$, which we denote by f, defines a bijection from $\mathrm{Proj}(F, \alpha_F)$ onto $\mathrm{Proj}_Q(L, \alpha_L)$. Now since L and Q have coprime orders, it follows firstly that $Q \leqslant C_G(L)$, and secondly that $\alpha(y, g) = \alpha(g, y)$ for all $y \in L$ and all $g \in Q$. Thus we also have $\mathrm{Proj}_Q(L, \alpha_L) = \mathrm{Proj}(L, \alpha_L)$.

Let $\phi_1, \ldots, \phi_r$ be representatives for the orbits of H on $\mathrm{Proj}(L, \alpha_L)$, and $\chi_i = f^{-1}(\phi_i)$ for $1 \leqslant i \leqslant r$, so that $\chi_i \in \mathrm{Proj}(F, \alpha_F)$. If $r > 1$, it is clear that we may choose the ϕ_i so that by suitable labelling χ_1 and χ_2 are representatives for the two orbits of G on $\mathrm{Proj}(F, \alpha_F)$. Now let $T_i = I_G(\chi_i)$, $U_i = I_H(\phi_i)$, and $t_i = [G : I_G(\chi_i)]$. Then by (3·7)(*b*) of [**6**] we have $U_i = T_i \cap H$. However, since $G = FH$ we have $G = T_i H$, and hence $[H : U_i] = [G : T_i] = t_i$. Now $|\mathrm{Proj}(F, \alpha_F)| = t_1 + t_2 = |\mathrm{Proj}(L, \alpha_L)|$, so certainly $r > 1$, and H has exactly two orbits on $\mathrm{Proj}(L, \alpha_L)$ of lengths t_1 and t_2.

Let $i = 1$ or 2. From the proof of 1·6 there exist unique ψ_1, ψ_2 such that ψ_i lies in $\mathrm{Proj}(T_i, \alpha_{T_i})$ and $[(\psi_i)_F, \chi_i] \neq 0$, where we may assume without loss of generality that $\psi_i^G = \xi_i$. By (3·7)(*c*) of [**6**], there exist unique η_1, η_2 such that $\eta_i \in \mathrm{Proj}(U_i, \alpha_{U_i})$ and $[(\eta_i)_L, \phi_i] \neq 0$. Now $\eta_i^H = \tau_i \in \mathrm{Proj}(H, \alpha_H)$, and by the proof of 1·6 we have that τ_1, τ_2 are the unique elements of $\mathrm{Proj}(H, \alpha_H)$ such that $[(\tau_i)_L, \phi_i] \neq 0$. However if $\tau \in \mathrm{Proj}(H, \alpha_H)$, then either $[\tau_L, \phi_1] \neq 0$ or $[\tau_L, \phi_2] \neq 0$, and so we must have $\mathrm{Proj}(H, \alpha_H) = \{\tau_1, \tau_2\}$. Thus by the minimality of $|G|$ we obtain $\tau_1(1) = \tau_2(1)$. Now from 1·6 we have that L contains the non-trivial α_H-regular elements of H, and so from 2·2 and 2·3 we have $p = 2$.

To finish the proof, we have $e_1^2 s_1 = e_2^2 s_2 = [G : K]$, and so $\mathrm{g.c.d.}\{a^2, b^2\} = [G : K](s_1)_2 = [G : K](s_2)_2$. Thus both $(s_1)_{2'}$ and $(s_2)_{2'}$ are squares and their sum is a power of 2, which is impossible mod 4 unless $(s_1)_{2'} = (s_2)_{2'} = 1$; but then $a = b$, a contradiction.

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Since $C < G$ by Stage 1, we let $\bar{R} = R/C$ be a chief factor of G. Then $\bar{R}$ is a direct product of copies of some non-Abelian simple group H, and we must also assume that $p \mid |H|$ from Stage 2.

Stage 5: $H \cong A_6$ and $p = 5$.

Proof. Choose if possible a prime $q \neq p$ such that $q \mid |H|$ but $q \nmid |M(H)|$. Then by working in an α-representation group for G and using 4·1 we obtain a contradiction as in Step 3, p. 565 of [**6**]. Thus by 3·3 we have $H \cong A_6$, and since $|M(A_6)| = 6$ we also have that $p = 5$.

Stage 6. Final contradiction.

Proof. Let $\bar{X} = X/C$ be a composition factor of $\bar{R}$, so that $\bar{X} \cong A_6$. Let $i = 1$ or 2. Define $J_i = X \cap I_G(\delta_i)$; then from 1·6(ii) we have that $I_G(\delta_i)$, and hence also J_i, is solvable. Moreover, since X is subnormal in G, $[X:J_i]$ divides $[G:I_G(\delta_i)] = s_i$, and so g.c.d.$\{[X:J_1], [X:J_2]\} = 5^c$ for some $c \geqslant 0$. Thus from the known subgroups of A_6 we have without loss of generality $10 \mid [X:J_1] \mid 40$ and $15 \mid [X:J_2] \mid 45$. In particular J_1/C contains a Sylow 3-subgroup $\overline{S(1)} = S(1)/C$ of $\bar{X}$, and J_2/C contains a Sylow 2-subgroup $\overline{S(2)} = S(2)/C$ of $\bar{X}$. Now, since all Sylow 2-subgroups, respectively 3-subgroups, of $\bar{X}$ are conjugate in $\bar{X}$, it follows that $S(i)$ fixes at least one element of each X-orbit on the G-orbit of δ_i. Let P_i denote the set of $S(i)$-invariant elements of Proj (K, α_K); we then certainly have

$$|\text{Proj}\,(K, \alpha_K)| \leqslant 40|P_1| + 45|P_2|.$$

However, if $\delta \in P_i$, then as in Stage 2 δ can be written in the form $\delta_i v$ such that $v \in V = \text{Irr}\,(K)$. In this case, since δ_i is $S(i)$-invariant, we also have that v is $S(i)$-invariant (where of course we have different conjugation actions), and so $|P_i| \leqslant |C_V(S(i))|$. Thus letting P and S be respectively P_1 and $S(1)$ or P_2 and $S(2)$ according to whether or not $|P_1| \geqslant |P_2|$, we obtain

$$|V| \leqslant 85|P| \leqslant 85|C_V(S)|.$$

Thus $|V/C_V(S)| \leqslant 5^2$, since $V/C_V(S)$ is a 5-group. It is now easy to check that there exists $gC \in \bar{X}$ such that $\langle \bar{S}, \bar{S}^{gC} \rangle = \bar{X}$, and hence

$$[V:C_V(X)] = [V:C_V(S) \cap C_V(S^g)] \leqslant 5^4.$$

We have thus shown that $\bar{X}$ acts faithfully on $V/C_V(X)$, and $|V/C_V(X)| \leqslant 5^4$, so that we have a monomorphism of $\bar{X} \cong A_6$ into $SL(4, 5)$. However, the degree of a faithful representation of A_6 over $GF(5)$ is at least five, a contradiction. This contradiction completes the proof. ∎

One immediate consequence of Theorem A is that the conclusions of 1·3 now hold under the weaker hypothesis that $p \nmid o([\alpha])$.

We can now also prove Theorem B. However, most of the details of this proof are exactly as given in [**6**], [**9**], or the proof of Theorem A. For this reason, and for the sake of brevity, we merely sketch the various stages in the proof and leave the reader to verify that our assertions are indeed valid.

Proof of Theorem B. Let G be a group of 2α-central type. Then any minimal normal subgroup of G is solvable, the proof being exactly as in (2·1) of [**9**] because of 2·2 and the fact that no non-Abelian simple group has a cyclic Sylow 2-subgroup.

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Now let G be a non-solvable group of 2α-central type of minimal order, and K be a fixed but arbitrary minimal normal subgroup of G. Then K is an elementary Abelian p-group for some prime p, and there are two cases to consider, according to whether $K = \langle y : y \in \mathscr{C}_2 \rangle$ and $p = 2$ or $K \cap \mathscr{C}_2 = \varnothing$. We next note that if $\delta \in \mathrm{Proj}(K, \alpha_K)$, then $I_G(\delta)/K$ is solvable from 1·6. Thus in both cases, mimicing the techniques of [9] and using 2·5 as necessary yields first that $C = C_G(K)$ is the maximal solvable normal subgroup of G, and second that if R is a non-solvable subnormal subgroup of G containing C, then R/C has a proper solvable subgroup of p-power index.

Now of course $K \leqslant F(G) \leqslant C$. Suppose $F = F(G)$ is not a p-group. Then by the argument just given, we have that if $\bar{X} = X/C$ is a composition factor of G, then $\bar{X}$ has two proper solvable subgroups of p-power and q-power index for two distinct primes p and q, which is impossible by (3·2) of [9]. Thus we must assume $F = O_p(G)$.

Again there are now two cases to consider according to whether (i) $\mathscr{C}_2 \subset F$ and $p = 2$, or (ii) $F \cap \mathscr{C}_2 = \varnothing$. Now the difficult case is of course (i); however by 2·2, 2·3 and 1·6(ii), if $S \in \mathrm{Syl}_2(G)$ then both S and G have two orbits on $\mathrm{Proj}(F, \alpha_F)$, which must therefore be the same, and so both these orbits have 2-power length. Thus in either case (i) or (ii) we are essentially now at Stage 4 of the proof of Theorem A (having omitted Stage 3). The argument given there in case (i) ultimately gives that H is solvable, and hence since $G = FH$ that G is solvable, a contradiction. A similar (and in fact easier) argument in case (ii) also gives a contradiction. Thus in either case we must assume that $F = C$.

Now Stage 5 of the proof of Theorem A remains the same, and a final contradiction is obtained easily since A_6 has no solvable subgroup of 5-power index.

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