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ON A GENERALIZED Q-URN MODEL

MAY-RU CHEN
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 804, Taiwan, Republic of China
E-mail: mayru@faculty.nsysu.edu.tw

SHOOU-REN HSIAU
Department of Mathematics, National Changhua University of Education, No. 1, Jin-De Road, Changhua 500, Taiwan, Republic of China
E-mail: srhsiau@cc.ncue.edu.tw

Recently, Chen, Hsiau & Yang [1] proposed a new two-urn model with red and white balls and showed that the fractions of red balls in both urns converge almost surely to the same limit. We extend the results for the two-urn model to the q-urn model ($q \geq 3$) with similar dynamics of drawing and adding balls. We use matrix forms and martingale theory to show that the fractions of red balls in all urns converge almost surely to the same limit.

1. INTRODUCTION

The study of urn models has a long history. The Pólya urn model and the Ehrenfest urn model are two examples of famous urn models. The Pólya urn model was introduced by Eggenberger and Pólya [3], and described briefly as follows. There is an urn that initially contains r red and w white balls. A ball is drawn at random and then replaced along with c balls of the same color. Repeat the above procedure *ad infinitum*. It is a well-known fact that the fraction of red balls converges almost surely to a beta distributed random variable with parameters r/c and w/c. The Ehrenfest urn model was originally proposed as a model for dissipation of heat. Briefly, there are m balls numbered 1 to m distributed between two urns, say urn A and urn B. Randomly choose an integer between 1 to m and transfer the ball with this number to the other urn. After n transferences, the probability distribution of the number of balls in urn A is concerned; see Kac [4] and Takács [7]. There are several generalizations and applications of these two models can be found in the literature. For a detailed review about urn models see, for example, Kotz and Balakrishnan [5] as well as Mahmoud [6].

In 2005, Chen and Wei [2] proposed a generalization of Pólya urn, in which more than one ball are drawn at each time step. They used martingale theory to show that the fraction of red balls converges almost surely to an absolutely continuous random variable. In 2012, Chen, Hsiau & Yang [1] proposed a two-urn model with red and white balls whose main character is that we add balls to are the other urn, not the urn from which we drew balls. More precisely, assume there are two urns, say urn A and urn B. At the beginning, urn A contains r_A red and w_A white balls and urn B contains r_B red and w_B white balls. We

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first draw m balls from urn A and note their colors, say i red and $m-i$ white balls. Then return them into urn A and add bi red and $b(m-i)$ white balls into urn B. Next, we draw ℓ balls from urn B and note their colors, say j red and $\ell - j$ white balls. Then return them into urn B and add aj red and $a(\ell - j)$ white balls into urn A. Repeat the above action n times and let X_n and Y_n be the respective fraction of red balls in each urn. They first showed that EX_n and EY_n converge to the same limit and then used martingale theory to show that X_n and Y_n converge to the same limit almost surely.

An interesting line of research would be to extend the results for the two-urn model to a q-urn model ($q \geq 3$). The dynamics of the q-urn model are similar to the two-urn model. For the q-urn model, we also want to get the results similar to those given in the two-urn model. To study the limit of the fraction of red balls in each urn, it needs, like the two-urn model, to construct recurrence relations among the fractions of red balls in q urns. We find it is suitable to express these recurrence relations in matrix form and take texicab norm for derivations. For clarity of expression, we first consider the three-urn model in Section 2. In this section, we first describe the dynamics of the model and introduce the notations. Next, we show that the expectations of the fractions of red balls in every urn converge to the same limit. In the final part of Section 2, we show that the fractions of red balls in every urn converge almost surely to the same limit. The main tool we use is martingale theory. Because we should deal with matrices, the derivation and estimate are more complicated than the case of two-urn model. In Section 3, for $q \geq 4$, the q-urn model is investigated but the proofs are just outline because they are similar to the proofs of three-urn model.

2. THREE-URN MODEL

In this section, we first introduce a three-urn model. Then we show that the respective limits of the expectations of the fraction of red balls in every urn converge to the same limit. Finally, we use martingale theory to show that the respective limits of the fractions of red ball in every urn converge almost surely to the same limit.

Assume that there are three urns, say urn A, urn B and urn C. Suppose at the beginning, there are $x_0 t_{A,0}$ red and $(1-x_0)t_{A,0}$ white balls in urn A; $y_0 t_{B,0}$ red and $(1-y_0)t_{B,0}$ white balls in urn B; $z_0 t_{C,0}$ red and $(1-z_0)t_{C,0}$ white balls in urn C, where $t_{A,0}$, $t_{B,0}$ and $t_{C,0}$ are the respective number of balls in urns A, B and C, and x_0, y_0 and z_0 are the respective fraction of red balls in urns A, B and C. First, draw m_A balls ($m_A \leq t_{A,0}$) from urn A and note their colors, say i red and $m_A - i$ white balls. Then return them into urn A and add bi red and $b(m_A - i)$ white balls into urn B. Next, draw m_B balls ($m_B \leq t_{B,0} + bm_A$) from urn B and note their colors, say j red balls and $m_B - j$ white balls. Then return them into urn B and add cj red and $c(m_B - j)$ white balls into urn C. Next, draw m_C balls ($m_C \leq t_{C,0} + cm_B$) from urn C and note their colors, say k red balls and $m_C - k$ white balls. Then return them into urn C and add ak red and $a(m_C - k)$ white balls into urn A. Repeat the above action *ad infinitum*.

After the nth action, let $R_A(n)$, $R_B(n)$ and $R_C(n)$ be the respective numbers of red balls in urns A, B and C and X_n, Y_n and Z_n the respective fractions of red balls in urns A, B and C. Then $X_n = R_A(n)/t_{A,n}$, $Y_n = R_B(n)/t_{B,n}$ and $Z_n = R_C(n)/t_{C,n}$, where $t_{A,n} = t_{A,0} + am_C n$, $t_{B,n} = t_{B,0} + bm_A n$ and $t_{C,n} = t_{C,0} + cm_B n$ are the respective total number of balls in urns A, B and C. Let $\mathcal{F}_n$ denote the σ-field generated by $X_0, X_1, \ldots, X_n$, $Y_0, Y_1, \ldots, Y_n$ and $Z_0, Z_1, \ldots, Z_n$. Then $(R_A(n), R_B(n), R_C(n))$ is clearly a Markov chain. For clarity, let

$$R_A(n+1) = R_A(n) + am_C U_{n+1};$$

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$$R_B(n+1) = R_B(n) + bm_A V_{n+1};$$
$$R_C(n+1) = R_C(n) + cm_B W_{n+1},$$

where $m_A V_{n+1}$, $m_B W_{n+1}$ and $m_C U_{n+1}$ are the respective numbers of red balls drawn from urns A, B and C at the $(n+1)$th action. Then, we have for any nonnegative integer n,

$$X_{n+1} = \frac{R_A(n+1)}{t_{A,n+1}} = \frac{t_{A,n}X_n + am_C U_{n+1}}{t_{A,n+1}} = X_n + \alpha_{n+1}(U_{n+1} - X_n); \tag{2.1}$$

$$Y_{n+1} = \frac{R_B(n+1)}{t_{B,n+1}} = \frac{t_{B,n}Y_n + bm_A V_{n+1}}{t_{B,n+1}} = Y_n + \beta_{n+1}(V_{n+1} - Y_n); \tag{2.2}$$

$$Z_{n+1} = \frac{R_C(n+1)}{t_{C,n+1}} = \frac{t_{C,n}Z_n + cm_B W_{n+1}}{t_{C,n+1}} = Z_n + \gamma_{n+1}(W_{n+1} - Z_n), \tag{2.3}$$

where $\alpha_{n+1} = am_C/t_{A,n+1}$, $\beta_{n+1} = bm_A/t_{B,n+1}$ and $\gamma_{n+1} = cm_B/t_{C,n+1}$. Note that

$$\alpha_n = \frac{1}{n} + O\left(\frac{1}{n^2}\right); \quad \beta_n = \frac{1}{n} + O\left(\frac{1}{n^2}\right); \quad \gamma_n = \frac{1}{n} + O\left(\frac{1}{n^2}\right).$$

Since $m_A V_{n+1}$ is the number of red balls drawn from urn A, we see that $m_A V_{n+1}$ is (conditionally) hypergeometric random variable with parameters $t_{A,n}$, $t_{A,n}X_n$, m_A, and so

$$E(V_{n+1} \mid \mathcal{F}_n) = \frac{1}{m_A} E(m_A V_{n+1} \mid \mathcal{F}_n) = \frac{1}{m_A}\left(\frac{m_A t_{A,n} X_n}{t_{A,n}}\right) = X_n. \tag{2.4}$$

Similarly,

$$E(W_{n+1} \mid \mathcal{F}_n, V_{n+1}) = \frac{1}{m_B}\left(\frac{m_B t_{B,n+1} Y_{n+1}}{t_{B,n+1}}\right) = Y_{n+1}$$

and

$$E(U_{n+1} \mid \mathcal{F}_n, V_{n+1}, W_{n+1}) = \frac{1}{m_C}\left(\frac{m_C t_{C,n+1} Z_{n+1}}{t_{C,n+1}}\right) = Z_{n+1}.$$

From the above identities, it is easy to derive that

$$E(W_{n+1} \mid \mathcal{F}_n) = E(E(W_{n+1} \mid \mathcal{F}_n, V_{n+1}) \mid \mathcal{F}_n) = E(Y_{n+1} \mid \mathcal{F}_n); \tag{2.5}$$

$$E(U_{n+1} \mid \mathcal{F}_n) = E(E(U_{n+1} \mid \mathcal{F}_n, V_{n+1}, W_{n+1}) \mid \mathcal{F}_n) = E(Z_{n+1} \mid \mathcal{F}_n). \tag{2.6}$$

2.1. The Limits of the Expectations

As in the two-urn model, the expectations of the fractions of red balls in all urns converge to the same limit.

THEOREM 2.1: *In the three-urn model,* $\lim_{n\to\infty} EX_n = \lim_{n\to\infty} EY_n = \lim_{n\to\infty} EZ_n$.

To prove Theorem 2.1, we use the texicab norm on vector space $\mathbb{C}^2$ which is defined by

$$\|v\| = |v_1| + |v_2|, \quad \text{where } v = (v_1, v_2) \in \mathbb{C}^2.$$

With the texicab norm, it is natural to define a norm on 2×2 matrix space $M_2(\mathbb{C})$ by

$$\|M\| = |m_{11}| + |m_{12}| + |m_{21}| + |m_{22}|, \quad \text{where } M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}.$$

Furthermore, it is easy to derive that $\|MN\| \le \|M\|\|N\|$ and $\|Mv\| \le \|M\|\|v\|$ for any $M, N \in M_2(\mathbb{C})$ and $v \in \mathbb{C}^2$.

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Proof of Theorem 2.1: For all nonnegative integer n, let $\Phi_n = X_n - Y_n$ and $\Psi_n = Y_n - Z_n$, and let $\phi_n = E\Phi_n$ and $\psi_n = E\Psi_n$. Using (2.2)–(2.6), we can derive that

$$E(V_{n+1} - Y_n \mid \mathcal{F}_n) = X_n - Y_n = \Phi_n; \tag{2.7}$$

$$E(W_{n+1} - Z_n \mid \mathcal{F}_n) = E(Y_{n+1} - Z_n \mid \mathcal{F}_n) = \Psi_n + \beta_{n+1}\Phi_n; \tag{2.8}$$

$$E(U_{n+1} - X_n \mid \mathcal{F}_n) = E(Z_{n+1} - X_n \mid \mathcal{F}_n) = -(\Phi_n + \Psi_n) + \gamma_{n+1}\Psi_n + \beta_{n+1}\gamma_{n+1}\Phi_n, \tag{2.9}$$

which, combining with (2.1)–(2.3), imply that

$$E(X_{n+1} \mid \mathcal{F}_n) = X_n - \alpha_{n+1}(\Phi_n + \Psi_n - \gamma_{n+1}\Psi_n - \beta_{n+1}\gamma_{n+1}\Phi_n); \tag{2.10}$$

$$E(Y_{n+1} \mid \mathcal{F}_n) = Y_n + \beta_{n+1}\Phi_n; \tag{2.11}$$

$$E(Z_{n+1} \mid \mathcal{F}_n) = Z_n + \gamma_{n+1}(\Psi_n + \beta_{n+1}\Phi_n). \tag{2.12}$$

Taking the expectation on both sides of (2.10)–(2.12), it is not difficult to get a recurrence relation for $\{(\phi_n, \psi_n)\}_{n\geq 1}$,

$$\begin{bmatrix}\phi_{n+1}\\ \psi_{n+1}\end{bmatrix} = \begin{bmatrix}1 - \alpha_{n+1} - \beta_{n+1} + \alpha_{n+1}\beta_{n+1}\gamma_{n+1} & -\alpha_{n+1} + \alpha_{n+1}\gamma_{n+1}\\ \beta_{n+1} - \beta_{n+1}\gamma_{n+1} & 1 - \gamma_{n+1}\end{bmatrix}\begin{bmatrix}\phi_n\\ \psi_n\end{bmatrix}$$

$$= M_{n+1}\begin{bmatrix}\phi_n\\ \psi_n\end{bmatrix}. \tag{2.13}$$

Here we define

$$M_n = \begin{bmatrix}1 - \alpha_n - \beta_n + \alpha_n\beta_n\gamma_n & -\alpha_n + \alpha_n\gamma_n\\ \beta_n - \beta_n\gamma_n & 1 - \gamma_n\end{bmatrix}$$

for $n \geq 1$. Because $\alpha_n = 1/n + O(1/n^2)$, $\beta_n = 1/n + O(1/n^2)$ and $\gamma_n = 1/n + O(1/n^2)$, we can represent M_n as

$$M_n = \begin{bmatrix}1 - \dfrac{2}{n} & -\dfrac{1}{n}\\ \dfrac{1}{n} & 1 - \dfrac{1}{n}\end{bmatrix} + \epsilon_n,$$

where $\epsilon_n = (\epsilon_{ij}(n))_{2\times 2}$ is a 2×2 matrix with entries $\epsilon_{ij}(n) = O(1/n^2)$. Let

$$N_n = \begin{bmatrix}1 - \dfrac{2}{n} & -\dfrac{1}{n}\\ \dfrac{1}{n} & 1 - \dfrac{1}{n}\end{bmatrix}.$$

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Then for $n \in \mathbb{Z}^+$,

$$\begin{aligned}
\begin{bmatrix}\phi_n\\ \psi_n\end{bmatrix} &= M_n \begin{bmatrix}\phi_{n-1}\\ \psi_{n-1}\end{bmatrix}\\
&= (M_n - N_n)\begin{bmatrix}\phi_{n-1}\\ \psi_{n-1}\end{bmatrix} + N_n \begin{bmatrix}\phi_{n-1}\\ \psi_{n-1}\end{bmatrix}\\
&= (M_n - N_n)\begin{bmatrix}\phi_{n-1}\\ \psi_{n-1}\end{bmatrix} + N_n(M_{n-1} - N_{n-1})\begin{bmatrix}\phi_{n-2}\\ \psi_{n-2}\end{bmatrix} + N_n N_{n-1}\begin{bmatrix}\phi_{n-2}\\ \psi_{n-2}\end{bmatrix}\\
&\vdots\\
&= (M_n - N_n)\begin{bmatrix}\phi_{n-1}\\ \psi_{n-1}\end{bmatrix} + N_n(M_{n-1} - N_{n-1})\begin{bmatrix}\phi_{n-2}\\ \psi_{n-2}\end{bmatrix} + \cdots\\
&\quad + \left(\prod_{j=2}^{n} N_j\right)(M_1 - N_1)\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix} + \left(\prod_{j=1}^{n} N_j\right)\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix}. \qquad \textbf{(2.14)}
\end{aligned}$$

By the definitions of ϕ_n and ψ_n, it is easy to see that $|\phi_n| \le 1$ and $|\psi_n| \le 1$. Hence, from (2.14), we have

$$\begin{aligned}
\left\|\begin{bmatrix}\phi_n\\ \psi_n\end{bmatrix}\right\| &\le \|M_n - N_n\|\left\|\begin{bmatrix}\phi_{n-1}\\ \psi_{n-1}\end{bmatrix}\right\| + \|N_n\|\|M_{n-1} - N_{n-1}\|\left\|\begin{bmatrix}\phi_{n-2}\\ \psi_{n-2}\end{bmatrix}\right\| + \cdots\\
&\quad + \left\|\prod_{j=2}^{n} N_j\right\|\|M_1 - N_1\|\left\|\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix}\right\| + \left\|\prod_{j=1}^{n} N_j\right\|\left\|\begin{bmatrix}\phi_0\\ \psi_0\end{bmatrix}\right\|\\
&\le 2\left(\|M_n - N_n\| + \sum_{k=2}^{n}\left\|\prod_{j=k}^{n} N_j\right\|\|M_{k-1} - N_{k-1}\| + \left\|\prod_{j=1}^{n} N_j\right\|\right). \qquad \textbf{(2.15)}
\end{aligned}$$

Later we will show that there exists some constant $\rho > 1$ such that

$$\|M_k - N_k\| \le \frac{\rho}{(k+1)^2} \qquad \textbf{(2.16)}$$

and

$$\left\|\prod_{j=k}^{n} N_j\right\| \le \rho\left(\frac{k}{n+1}\right)^{3/2} \qquad \textbf{(2.17)}$$

hold for all n, k with $n \ge k$. Then, from (2.15),

$$\begin{aligned}
\left\|\begin{bmatrix}\phi_n\\ \psi_n\end{bmatrix}\right\| &\le 2\rho^2\left[\frac{1}{(n+1)^2} + \sum_{k=2}^{n}\left(\frac{k}{n+1}\right)^{3/2}\frac{1}{k^2} + \left(\frac{1}{n+1}\right)^{3/2}\right]\\
&\le \frac{2\rho^2}{(n+1)^{3/2}}\left(\int_1^{n+1}\frac{dx}{\sqrt{x}} + 1\right)\\
&\le \frac{4\rho^2}{n}, \qquad \textbf{(2.18)}
\end{aligned}$$

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which implies that $|\phi_n| \le 4\rho^2/n$ and $|\psi_n| \le \rho^2/n$, and so $\phi_n \to 0$ and $\psi_n \to 0$. Moreover, taking expectations on both sides of (2.11) yields that $EY_{n+1} = EY_n + \beta_{n+1}\phi_n$ and so

$$|EY_{m+n} - EY_n| = \left| \sum_{j=n}^{m+n-1} (EY_{j+1} - EY_j) \right| \le \sum_{j=n}^{m+n-1} \beta_{j+1}|\phi_j| \le \sum_{j=n}^{m+n-1} \frac{|\phi_j|}{j+1}.$$

Here the last inequality holds since $\beta_{j+1} = bm_A/[t_{B,0} + bm_A(j+1)] \le 1/(j+1)$. Using (2.18),

$$\sum_{j=n}^{m+n-1} \frac{|\phi_j|}{j+1} \le \sum_{j=n}^{m+n-1} \frac{4\rho^2}{j(j+1)} = \frac{4\rho^2}{n} - \frac{4\rho^2}{n+m} \le \frac{4\rho^2}{n},$$

which leads to $|EY_{m+n} - EY_n| \le 4\rho^2/n$, and so $\{EY_n\}_{n\ge1}$ is a Cauchy sequence. Consequently, $\lim_{n\to\infty} EY_n$ exists, and so do $\lim_{n\to\infty} EX_n$ and $\lim_{n\to\infty} EZ_n$ since $\lim_{n\to\infty} \phi_n = \lim_{n\to\infty} \psi_n = 0$.

It remains to prove (2.16) and (2.17). Because $M_n - N_n = \epsilon_n$ and $\epsilon_n = (\epsilon_{ij}(n))_{2\times2}$ is a 2×2 matrix with $\epsilon_{ij}(n) = O(1/n^2)$, (2.16) holds. To prove (2.17), let

$$L = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$

Since the eigenvalues of L are $1-\omega$ and $1-\omega^2$ with $\omega = \exp(2\pi i/3)$, there exists $Q \in M_2(\mathbb{C})$ such that $L = Q^{-1}DQ$, where $D = \begin{bmatrix} 1-w & 0 \\ 0 & 1-w^2 \end{bmatrix}$. This leads to

$$N_j = I - \frac{L}{j} = Q^{-1}\left(I - \frac{D}{j}\right)Q = Q^{-1}\begin{bmatrix} \lambda_j^+ & 0 \\ 0 & \lambda_j^- \end{bmatrix} Q,$$

where

$$\lambda_j^+ = 1 - \frac{1}{j} + \frac{w}{j} \quad \text{and} \quad \lambda_j^- = 1 - \frac{1}{j} + \frac{w^2}{j}.$$

Thus,

$$\begin{aligned} \|N_n N_{n-1} \cdots N_k\| &= \left\| Q^{-1} \begin{bmatrix} \prod_{j=k}^n \lambda_j^+ & 0 \\ 0 & \prod_{j=k}^n \lambda_j^- \end{bmatrix} Q \right\| \\ &\le \|Q^{-1}\| \left\| \begin{bmatrix} \prod_{j=k}^n \lambda_j^+ & 0 \\ 0 & \prod_{j=k}^n \lambda_j^- \end{bmatrix} \right\| \|Q\| \\ &= 2\|Q^{-1}\|\|Q\| \prod_{j=k}^n \left(1 - \frac{3}{j} + \frac{3}{j^2}\right)^{1/2}. \end{aligned} \tag{2.19}$$

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Here the last identity holds since $|\lambda_j^+| = |\lambda_j^-| = \lambda_j^+\lambda_j^- = \left(1 - \frac{3}{j} + \frac{3}{j^2}\right)^{1/2}$. Using $1 - x \le e^{-x}$, $x \in \mathbb{R}$, and the fact $\sum_{j=1}^{\infty} j^{-2} = \pi^2/6$, we have

$$\begin{aligned}\prod_{j=k}^{n}\left(1 - \frac{3}{j} + \frac{3}{j^2}\right)^{1/2} &\le \exp\left[-\frac{3}{2}\sum_{j=k}^{n}\left(\frac{1}{j} - \frac{1}{j^2}\right)\right] \\ &\le \exp\left(\frac{3}{2}\sum_{j=1}^{\infty}\frac{1}{j^2}\right)\exp\left(-\frac{3}{2}\int_k^{n+1}\frac{1}{x}\,dx\right) \\ &= e^{\pi^2/4}\left(\frac{k}{n+1}\right)^{3/2}.\end{aligned}$$

This inequality and (2.19) imply that (2.17) holds, and hence the proof is complete. ■

Remark 2.1: In the three-urn model, if $x_0 = y_0 = z_0$, then $\phi_0 = \psi_0 = 0$ and so by repeatedly using (2.13), $\phi_n = \psi_n = 0$ for all $n \in \mathbb{N}$. Hence from the definitions of ϕ_n and ψ_n, we see that $EX_n = EY_n = EZ_n = x_0$.

Remark 2.2: Applying (2.13) repeatedly yields that for a positive integer j,

$$\begin{bmatrix}\phi_j \\ \psi_j\end{bmatrix} = M_j\begin{bmatrix}\phi_{j-1} \\ \psi_{j-1}\end{bmatrix} = \cdots = M_jM_{j-1}\cdots M_1\begin{bmatrix}\phi_0 \\ \psi_0\end{bmatrix}$$

and consequently we get $\phi_j = v(M_jM_{j-1}\cdots M_1)\begin{bmatrix}\phi_0 \\ \psi_0\end{bmatrix}$, where $v = [1 \quad 0]$. Then the expectation of Y_n can be derived by repeatedly applying (2.11),

$$EY_{n+1} = y_0 + \sum_{j=0}^{n}\beta_{j+1}\phi_j = y_0 + \beta_1\phi_0 + \sum_{j=1}^{n}\beta_{j+1}v(M_jM_{j-1}\cdots M_1)\begin{bmatrix}\phi_0 \\ \psi_0\end{bmatrix}, \tag{2.20}$$

and so by Theorem 2.1

$$\lim_{n\to\infty}EX_n = \lim_{n\to\infty}EZ_n = \lim_{n\to\infty}EY_n = y_0 + \beta_1\phi_0 + \sum_{j=1}^{\infty}\beta_{j+1}v(M_jM_{j-1}\cdots M_1)\begin{bmatrix}\phi_0 \\ \psi_0\end{bmatrix}.$$

For example, if $t_{A,0} = t_{B,0} = t_{C,0}$ and $am_C = bm_A = cm_B$, then $t_{A,n} = t_{B,n} = t_{C,n}$ and $\alpha_n = \beta_n = \gamma_n$ for $n \in \mathbb{N}$. These lead to for all positive integer k,

$$M_k = (1 - \alpha_k)\begin{bmatrix}1 - \alpha_k - \alpha_k^2 & -\alpha_k \\ \alpha_k & 1\end{bmatrix} = \left(\frac{t_{B,k-1}}{t_{B,k}}\right)L_k,$$

where $t_{B,-1} = t_{B,0} - bm_A$ and L_k is defined by

$$L_k = \begin{bmatrix}1 - \alpha_k - \alpha_k^2 & -\alpha_k \\ \alpha_k & 1\end{bmatrix}$$

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and consequently

$$M_j M_{j-1} \cdots M_1 = \left(\prod_{k=1}^{j} \frac{t_{B,k-1}}{t_{B,k}} \right) L_j L_{j-1} \cdots L_1 = \left(\frac{t_{B,0}}{t_{B,j}} \right) L_j L_{j-1} \cdots L_1. \tag{2.21}$$

Therefore, if $t_{A,0} = t_{B,0} = t_{C,0}$ and $am_C = bm_A = cm_B$, then from (2.20), (2.21) and the definition of β_k,

$$\begin{aligned} EY_n &= y_0 + \beta_1 \phi_0 + \sum_{j=1}^{n-1} \beta_{j+1} v (M_j M_{j-1} \cdots M_1) \begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix} \\ &= y_0 + \left(\frac{bm_A}{t_{B,1}} \right) \phi_0 + bm_A t_{B,0} \sum_{j=1}^{n-1} \left(\frac{1}{t_{B,j} t_{B,j+1}} \right) v L_j L_{j-1} \cdots L_1 \begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix}, \end{aligned}$$

where $L_j L_{j-1} \cdots L_1$ is defined as the identity matrix if $j = 0$. Thus, by Theorem 2.1

$$\begin{aligned} \lim_{n\to\infty} EX_n = \lim_{n\to\infty} EZ_n = \lim_{n\to\infty} EY_n \\ = y_0 + \left(\frac{bm_A}{t_{B,1}} \right) \phi_0 + bm_A t_{B,0} \sum_{j=1}^{\infty} \left(\frac{1}{t_{B,j} t_{B,j+1}} \right) v L_j L_{j-1} \cdots L_1 \begin{bmatrix} \phi_0 \\ \psi_0 \end{bmatrix}. \end{aligned}$$

Using mathematical software Maple 16, we have the following numerical result for $T_A(0) = T_B(0) = T_C(0) = 2$ and $bm_A = cm_B = am_C = 1$.

x_0	y_0	z_0	EY_{5000}
1	1	0.5	0.8569722334
1	0.5	1	0.8471054099
0.5	1	1	0.7959223560
1	0.5	0.5	0.7040854010
0.5	1	0.5	0.6528886058
0.5	0.5	1	0.6430259961
1	0.5	0	0.5610594071
1	0	0.5	0.5511967977
0.5	1	0	0.5098626099

x_0	y_0	z_0	EY_{5000}
0.5	0	1	0.4901373901
0	1	0.5	0.4488032023
0	0.5	1	0.4389405929
0.5	0.5	0	0.3569740039
0.5	0	0.5	0.3471113942
0	0.5	0.5	0.2959145990
0.5	0	0	0.2040854010
0	0.5	0	0.1528886058
0	0	0.5	0.1430259961

Remark 2.3: In a three-urn model, assume $t_{A,0} = t_{B,0} = t_{C,0}$ and $am_C = bm_A = cm_B = \kappa$. Let w_n be the vector

$$\begin{aligned} w_n^{\mathrm{tr}} = \big[& ER_A(n) \;\; ER_B(n) \;\; ER_C(n) \;\; ER_A^2(n) \;\; ER_B^2(n) \;\; ER_C^2(n) \\ & ER_A(n)R_B(n) \;\; ER_A(n)R_C(n) \;\; ER_B(n)R_C(n) \big], \end{aligned}$$

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where w_n^{tr} is the transport of w_n. Then from the dynamics of three-urn model, it follows that $w_{n+1} = K_{n+1} w_n$, here K_n is defined by

$$K_{n+1} = \begin{bmatrix}
1+\frac{\kappa^3}{t(t+\kappa)^2} & \frac{\kappa^2}{(t+\kappa)^2} & \frac{\kappa}{t+\kappa} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\kappa}{t} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\kappa^2}{t(t+\kappa)} & \frac{\kappa}{t+\kappa} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\kappa^4}{t(t+\kappa)^2} & \frac{\kappa^3}{(t+\kappa)^2} & \frac{\kappa^2}{(t+\kappa)^2} & 1+\frac{2\kappa^3}{t(t+\kappa)^2} & 0 & 0 & \frac{2\kappa^2}{(t+\kappa)^2} & \frac{2\kappa}{t+\kappa} & 0 \\
\frac{\kappa^2}{t} & 0 & 0 & 0 & 1 & 0 & \frac{2\kappa}{t} & 0 & 0 \\
\frac{\kappa^3}{t(t+\kappa)} & \frac{\kappa^2}{t+\kappa} & 0 & 0 & 0 & 1 & 0 & \frac{2\kappa^2}{t(t+\kappa)} & \frac{2\kappa}{t+\kappa} \\
\frac{\kappa^4}{t(t+\kappa)^2} & 0 & 0 & \frac{\kappa}{t} & \frac{\kappa^2}{(t+\kappa)^2} & 0 & 1+\frac{2\kappa^3}{t(t+\kappa)^2} & \frac{\kappa^2}{t(t+\kappa)} & \frac{\kappa}{t+\kappa} \\
\frac{t^4}{t(t+\kappa)^2} & \frac{\kappa^3}{(t+\kappa)^2} & 0 & \frac{\kappa^2}{t(t+\kappa)} & 0 & \frac{\kappa}{t+\kappa} & \frac{\kappa}{t+\kappa} & 1+\frac{2\kappa^3}{t(t+\kappa)^2} & \frac{2\kappa^2}{(t+\kappa)^2} \\
\frac{\kappa^3}{t(t+\kappa)} & 0 & 0 & 0 & \frac{\kappa}{t+\kappa} & 0 & \frac{2\kappa^2}{t(t+\kappa)} & \frac{\kappa}{t} & 1
\end{bmatrix},$$

where $t = t_{A,n}$. Repeatedly using the recurrent relation of w_n, one can get $w_n = \prod_{j=1}^{n} K_j w_0$ and so the variances of X_n, Y_n and Z_n can be derived.

2.2. Strong Convergence

Next, we further study the long-term behavior of the processes $\{X_n\}_{n\geq 1}$, $\{Y_n\}_{n\geq 1}$ and $\{Z_n\}_{n\geq 1}$.

THEOREM 2.2: *In the three-urn model, the fractions of red balls in urns A, B and C converge almost surely to the same limit.*

To prove Theorem 2.2, we first need to prove the following theorem. Recall that $\Phi_n = X_n - Y_n$ and $\Psi_n = Y_n - Z_n$.

THEOREM 2.3: *In the three-urn model,* $\{X_n + Y_n + Z_n - \sum_{k=0}^{n-1} \Lambda_n\}_{n\geq 1}$ *is a bounded martingale and* $X_n + Y_n + Z_n$ *converges almost surely, where* $\Lambda_n = (\beta_{n+1} - \alpha_{n+1} + \alpha_{n+1}\beta_{n+1}\gamma_{n+1} + \beta_{n+1}\gamma_{n+1})\Phi_n + (\gamma_{n+1} - \alpha_{n+1} + \alpha_{n+1}\gamma_{n+1})\Psi_n$.

PROOF: From (2.10)–(2.12), we see that for any positive integer n,

$$\begin{aligned}
&E[X_{n+1} + Y_{n+1} + Z_{n+1} \mid \mathcal{F}_n] \\
&= X_n + Y_n + Z_n - \alpha_{n+1}(\Phi_n + \Psi_n - \gamma_{n+1}\Psi_n - \beta_{n+1}\gamma_{n+1}\Phi_n) + \beta_{n+1}\Phi_n \\
&\quad + \gamma_{n+1}(\Psi_n + \beta_{n+1}\Phi_n) \\
&= X_n + Y_n + Z_n + \Lambda_n,
\end{aligned}$$

where $\Lambda_n = (\beta_{n+1} - \alpha_{n+1} + \alpha_{n+1}\beta_{n+1}\gamma_{n+1} + \beta_{n+1}\gamma_{n+1})\Phi_n + (\gamma_{n+1} - \alpha_{n+1} + \alpha_{n+1}\gamma_{n+1})\Psi_n$. Thus $\{X_n + Y_n + Z_n - \sum_{k=0}^{n-1} \Lambda_k\}_{n\geq 1}$ is a martingale. Furthermore, since $|\Phi_n| \leq 1$,

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$|\Psi_n| \le 1$,

$$\beta_{n+1} - \alpha_{n+1} + \alpha_{n+1}\beta_{n+1}\gamma_{n+1} + \beta_{n+1}\gamma_{n+1} = O\left(\frac{1}{n^2}\right) \quad \text{and}$$

$$\gamma_{n+1} - \alpha_{n+1} + \alpha_{n+1}\gamma_{n+1} = O\left(\frac{1}{n^2}\right),$$

it follows that $\sum_{k=0}^{n} \Lambda_k$ converges almost surely. This implies that the martingale $\{X_n + Y_n + Z_n - \sum_{k=0}^{n} \Lambda_k\}_{n\ge1}$ is bounded and therefore convergent almost surely. Hence $\{X_n + Y_n + Z_n\}_{n\ge1}$ converges almost surely. ■

Proof of Theorem 2.2: In view of Theorem 2.3, we have $X_n + Y_n + Z_n$ converges almost surely. If we can further prove that both $X_n - Y_n$ and $Y_n - Z_n$ converge to 0 almost surely, then the proof is complete since $X_n = [(X_n + Y_n + Z_n) + 2(X_n - Y_n) + (Y_n - Z_n)]/3$. In the following, we prove that both $\Phi_n = X_n - Y_n$ and $\Psi_n = Y_n - Z_n$ converge to 0 almost surely.

From (2.1)–(2.3), we have

$$\Phi_{n+1} = \Phi_n + \alpha_{n+1}(U_{n+1} - X_n) - \beta_{n+1}(V_{n+1} - Y_n) := \Phi_n + \delta_{n+1};$$
$$\Psi_{n+1} = \Psi_n + \beta_{n+1}(V_{n+1} - Y_n) - \gamma_{n+1}(W_{n+1} - Z_n) := \Psi_n + \hat{\delta}_{n+1},$$

where $\delta_{n+1} = \alpha_{n+1}(U_{n+1} - X_n) - \beta_{n+1}(V_{n+1} - Y_n)$ and $\hat{\delta}_{n+1} = \beta_{n+1}(V_{n+1} - Y_n) - \gamma_{n+1}(W_{n+1} - Z_n)$. Note that $|\delta_{n+1}| \le 2/(n+1)$ and $|\hat{\delta}_{n+1}| \le 2/(n+1)$. Using (2.7)–(2.9), we have that

$$\begin{aligned} E[\Phi_{n+1}^2 \mid \mathcal{F}_n] &= \Phi_n^2 + E[\delta_{n+1}^2 \mid \mathcal{F}_n] + 2\Phi_n E[\delta_{n+1} \mid \mathcal{F}_n] = \Phi_n^2 + E[\delta_{n+1}^2 \mid \mathcal{F}_n] \\ &\quad + 2\Phi_n\{\alpha_{n+1}[-(\Phi_n + \Psi_n) + \gamma_{n+1}\Psi_n + \beta_{n+1}\gamma_{n+1}\Phi_n] - \beta_{n+1}\Phi_n\} \\ &= (1 - 2\alpha_{n+1} - 2\beta_{n+1})\Phi_n^2 - 2\alpha_{n+1}\Phi_n\Psi_n + E[\delta_{n+1}^2 \mid \mathcal{F}_n] \\ &\quad + 2\alpha_{n+1}\gamma_{n+1}\Phi_n\Psi_n + 2\alpha_{n+1}\beta_{n+1}\gamma_{n+1}\Phi_n^2 \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} E[\Psi_{n+1}^2 \mid \mathcal{F}_n] &= \Psi_n^2 + E[\hat{\delta}_{n+1}^2 \mid \mathcal{F}_n] + 2\Psi_n E[\hat{\delta}_{n+1} \mid \mathcal{F}_n] \\ &= \Psi_n^2 + E[\hat{\delta}_{n+1}^2 \mid \mathcal{F}_n] + 2\Psi_n[\beta_{n+1}\Phi_n - \gamma_{n+1}(\Psi_n + \beta_{n+1}\Phi_n)] \\ &= (1 - 2\gamma_{n+1})\Psi_n^2 + 2\beta_{n+1}\Phi_n\Psi_n + E[\hat{\delta}_{n+1}^2 \mid \mathcal{F}_n] - 2\beta_{n+1}\gamma_{n+1}\Phi_n\Psi_n. \end{aligned} \tag{2.23}$$

Since $|\Phi_n|, |\Psi_n| \le 1$ and $\alpha_{n+1} = 1/(n+1) + O(1/n^2)$, $\beta_{n+1} = 1/(n+1) + O(1/n^2)$ and $\gamma_{n+1} = 1/(n+1) + O(1/n^2)$, we see that

$$2\alpha_{n+1}\gamma_{n+1}\Psi_n + 2\alpha_{n+1}\beta_{n+1}\gamma_{n+1}\Phi_n^2 \le \frac{2}{(n+1)^2} + O\left(\frac{1}{n^3}\right) \le \frac{3}{(n+1)^2}$$

and

$$-2\beta_{n+1}\gamma_{n+1}\Phi_n\Psi_n \le \frac{2}{(n+1)^2} + O\left(\frac{1}{n^3}\right) \le \frac{3}{(n+1)^2}.$$

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Applying the above inequalities and the facts $|\delta_{n+1}| \le 2/(n+1)$ and $|\hat{\delta}_{n+1}| \le 2/(n+1)$ to (2.22) and (2.23), we get that

$$\begin{aligned}
&E[\Phi_{n+1}^2 + \Psi_{n+1}^2 \mid \mathcal{F}_n] \\
&\le (1 - 2\alpha_{n+1} - 2\beta_{n+1})\Phi_n^2 + (1 - 2\gamma_{n+1})\Psi_n^2 + 2(\beta_{n+1} - \alpha_{n+1})\Phi_n\Psi_n + \frac{14}{(n+1)^2} \\
&\le \tau_{n+1}(\Phi_n^2 + \Psi_n^2) + 2(\beta_{n+1} - \alpha_{n+1})\Phi_n\Psi_n + \frac{14}{(n+1)^2},
\end{aligned}$$

where $\tau_{n+1} = \max\{1 - 2\alpha_{n+1} - 2\beta_{n+1}, 1 - 2\gamma_{n+1}\}$. Since

$$\begin{aligned}
\beta_{n+1} - \alpha_{n+1} &= \frac{bm_A}{t_{B,0} + bm_A(n+1)} - \frac{am_c}{t_{A,0} + am_c(n+1)} \\
&\le \frac{t_{A,0}/am_c - t_{B,0}/bm_A}{(n+1)^2} \\
&\le \frac{|t_{A,0} - t_{B,0}|}{(n+1)^2},
\end{aligned}$$

one can derive that

$$E[\Phi_{n+1}^2 + \Psi_{n+1}^2 \mid \mathcal{F}_n] \le \tau_{n+1}(\Phi_n^2 + \Psi_n^2) + \frac{\lambda}{(n+1)^2}, \tag{2.24}$$

where $\lambda = 14 + 2|t_{A,0} - t_{B,0}|$.

Since $\tau_n < 1$, it follows from (2.24) that

$$E\left[\Phi_{n+1}^2 + \Psi_{n+1}^2 + \frac{\lambda}{n+1} \,\middle|\, \mathcal{F}_n\right] \le \Phi_n^2 + \Psi_n^2 + \frac{\lambda}{(n+1)^2} + \frac{\lambda}{n+1} \le \Phi_n^2 + \Psi_n^2 + \frac{\lambda}{n}.$$

Hence $\{\Phi_n^2 + \Psi_n^2 + \lambda/n\}_{n\ge 0}$ is a nonnegative supermartingale. Thus, by supermartingale convergence theorem, $\{\Phi_n^2 + \Psi_n^2 + \lambda/n\}_{n\ge 0}$ converges almost surely and so does $\{\Phi_n^2 + \Psi_n^2\}_{n\ge 0}$ because $\lim_{n\to\infty} \lambda/n = 0$.

Moreover, observe that

$$\begin{aligned}
0 \le \tau_n &= \max\{1 - 2\alpha_n - 2\beta_n, 1 - 2\gamma_n\} \\
&= 1 - 2\min\{\alpha_n, \beta_n, \gamma_n\} \\
&\le 1 - 2\min\left\{\frac{1}{n + t_{A,0}}, \frac{1}{n + t_{B,0}}, \frac{1}{n + t_{C,0}}\right\} \\
&\le \frac{n + t_0 - 1}{n + t_0},
\end{aligned}$$

where $t_0 = \max\{t_{A,0}, t_{B,0}, t_{C,0}\}$. Then

$$0 \le \lim_{k\to\infty} \prod_{n=1}^{k} \tau_n \le \lim_{k\to\infty} \prod_{n=1}^{k} \frac{n + t_0 - 1}{n + t_0} = \lim_{k\to\infty} \frac{t_0}{k + t_0} = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\lambda}{n^2} < \infty.$$

Hence applying (2.24) and Lemma 2.1 which can be also found in [1] yields that $\lim_{n\to\infty} E[\Phi_n^2 + \Psi_n^2] = 0$. Thus, by dominated convergence theorem, $E[\lim_{n\to\infty}(\Phi_n^2 + \Psi_n^2)] = 0$

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and so $\lim_{n\to\infty}(\Phi_n^2+\Psi_n^2)=0$ almost surely. Hence, we have both $\lim_{n\to\infty}\Phi_n=0$ and $\lim_{n\to\infty}\Psi_n=0$ almost surely.

LEMMA 2.4 (Chen *et al.*): *Suppose* $\{x_n\}_{n\geq 0}$, $\{a_n\}_{n\geq 1}$ *and* $\{b_n\}_{n\geq 1}$ *are nonnegative real sequences satisfying* $x_{n+1}\leq a_{n+1}x_n+b_{n+1}$, *where* $0<a_n<1$ *for* $n\geq 1$. *If* $\prod_{n=1}^{k}a_n\to 0$ *and* $\sum_{n=1}^{\infty}b_n<\infty$, *then* $x_n\to 0$.

3. MORE THAN THREE-URN MODEL

In this section, we study an extension of the three-urn model to a q-urn model. Denote the q urns by $A_1,\ldots,A_q$. The dynamics of the q-urn model are analogous to the three-urn model. After the nth action, let $X_1(n),\ldots,X_q(n)$ be the respective fractions of red balls in urns $A_1,\ldots,A_q$. Denote the total number of balls in the jth urn by $t_j(n)$ and $\alpha_j(n)=1-t_j(n-1)/t_j(n)$. Then for all $j=1,2,\ldots,q$,

$$\alpha_j(n)=\frac{1}{n}+O\left(\frac{1}{n^2}\right).$$

As in the three-urn model, we have the following theorem.

THEOREM 3.1: *In the q-urn model,* $\lim_{n\to\infty}EX_1(n)=\cdots=\lim_{n\to\infty}EX_q(n)$.

PROOF: Concerning the limits of $EX_1(n),\ldots,EX_q(n)$, one can adapt the argumentation of Theorem 2.1, and we here outline the main steps and present the new ingredient of the proof for the q-urn model. Let $\Phi_j(n)=X_j(n)-X_{j+1}(n)$ and let $\phi_j(n)=E\Phi_j(n)$ for all $n\in\mathbb{Z}^+$ and $j=1,2,\ldots,q-1$. As the proof of Theorem 2.1,

$$\begin{bmatrix}\phi_1(n+1)\\ \vdots\\ \phi_{q-1}(n+1)\end{bmatrix}=M_{n+1}\begin{bmatrix}\phi_1(n)\\ \vdots\\ \phi_{q-1}(n)\end{bmatrix}.$$

Since $\alpha_j(n)=1/n+O(1/n^2)$ for all $j=1,2,\ldots,q$, M_n can be decomposed as $M_n=N_n+\epsilon_n$, where $N_n=1-2/n$, if $q=2$ and

$$N_n=\begin{bmatrix}1-\frac{2}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n}\\ \frac{1}{n} & 1-\frac{1}{n} & 0 & \cdots & 0 & 0 & 0\\ 0 & \frac{1}{n} & 1-\frac{1}{n} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{1}{n} & 1-\frac{1}{n} & 0\\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{n} & 1-\frac{1}{n}\end{bmatrix}$$

if $q\geq 3$, and $\epsilon_n=(\epsilon_{ij}(n))_{(q-1)\times(q-1)}$ is a $q-1\times q-1$ matrix with entries $\epsilon_{ij}(n)=O(1/n^2)$. If we can show that there exists some constant $\rho>1$ such that

$$\|M_k-N_k\|\leq\frac{\rho}{(k+1)^2} \tag{3.1}$$

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and

$$\left\| \prod_{j=k}^{n} N_j \right\| \le \rho \left(\frac{k}{n+1} \right)^{\delta} \tag{3.2}$$

hold for all n, k with $n \ge k$, where $\delta = 2\sin^2(\pi/q)$, then as the proof of Theorem 2.1,

$$\begin{aligned}
\left\| \begin{bmatrix} \phi_1(n) \\ \vdots \\ \phi_{q-1}(n) \end{bmatrix} \right\| &\le \|M_n - N_n\| \left\| \begin{bmatrix} \phi_1(n-1) \\ \vdots \\ \phi_{q-1}(n-1) \end{bmatrix} \right\| + \|N_n\| \|M_{n-1} - N_{n-1}\| \left\| \begin{bmatrix} \phi_1(n-2) \\ \vdots \\ \phi_{q-1}(n-2) \end{bmatrix} \right\| \\
&\quad + \cdots + \left\| \prod_{j=2}^{n} N_j \right\| \|M_1 - N_1\| \left\| \begin{bmatrix} \phi_1(0) \\ \vdots \\ \phi_{q-1}(0) \end{bmatrix} \right\| + \left\| \prod_{j=1}^{n} N_j \right\| \left\| \begin{bmatrix} \phi_1(0) \\ \vdots \\ \phi_{q-1}(0) \end{bmatrix} \right\| \\
&\le (q-1) \left(\|M_n - N_n\| + \sum_{k=2}^{n} \left\| \prod_{j=k}^{n} N_j \right\| \|M_{k-1} - N_{k-1}\| + \left\| \prod_{j=1}^{n} N_j \right\| \right) \\
&\le \rho^2 (q-1) \left[\frac{1}{(n+1)^2} + \sum_{k=2}^{n} \left(\frac{k}{n+1} \right)^{\delta} \frac{1}{k^2} + \left(\frac{1}{n+1} \right)^{\delta} \right] \\
&\le \frac{\rho^2 (q-1)}{(n+1)^{\delta}} \left(\int_1^{n+1} \frac{dx}{x^{2-\delta}} + 1 \right). \qquad (3.3)
\end{aligned}$$

Note that $\delta = 1$ if $q = 4$ and $\delta \ne 1$ otherwise. Thus (3.3) becomes

$$\left\| \begin{bmatrix} \phi_1(n) \\ \vdots \\ \phi_{q-1}(n) \end{bmatrix} \right\| \le \begin{cases} \dfrac{\rho^2 (q-1) \ln(n+1)}{n+1} & \text{if} \quad q = 4, \\ \dfrac{\rho^2 (q-1)}{\delta - 1} \left[\dfrac{1}{n+1} + \dfrac{\delta - 2}{(n+1)^{\delta}} \right] & \text{if} \quad q \ne 4, \end{cases}$$

which implies that for all $j = 1, 2, \ldots, q-1$,

$$|\phi_j(n)| \le \begin{cases} \dfrac{\rho^2 (q-1) \ln(n+1)}{n+1} & \text{if} \quad q = 4, \\ \dfrac{\rho^2 (q-1)}{\delta - 1} \left[\dfrac{1}{n+1} + \dfrac{\delta - 2}{(n+1)^{\delta}} \right] & \text{if} \quad q \ne 4, \end{cases} \tag{3.4}$$

and so $\lim_{n\to\infty} \phi_j(n) = 0$ for all $j = 1, 2, \ldots, q-1$ and $q \ge 2$. Moreover, as in the three-urn model, $EX_2(n+1) = EX_2(n) + \alpha_2(n+1)\phi_1(n)$ and so

$$|EX_2(m+n) - EX_2(n)| = \left| \sum_{k=n}^{m+n-1} \alpha_2(k+1)\phi_1(k) \right| \le \sum_{k=n}^{m+n-1} \frac{|\phi_1(k)|}{k+1}$$

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since $\alpha_2(k+1) = 1 - t_2(k)/t_2(k+1) \le 1/(k+1)$. Furthermore, using (3.4), we see that for any $q \ge 4$,

$$\begin{aligned}\sum_{k=n}^{m+n-1} \frac{|\phi_1(k)|}{k+1} &\le \rho^2(q-1) \sum_{k=n}^{m+n-1} \left[\frac{\ln(k+1)}{(k+1)^2} + \frac{1}{(1-\delta)(k+1)^{\delta+1}}\right] \\ &\le \rho^2(q-1)\left[\int_n^{m+n} \frac{\ln x}{x^2}\,dx + \frac{1}{(1-\delta)}\int_n^{m+n} \frac{dx}{x^{\delta+1}}\right] \\ &= \rho^2(q-1)\left[\frac{1+\ln n}{n} - \frac{1+\ln(m+n)}{m+n} + \frac{1}{\delta(1-\delta)}\left(\frac{1}{n^\delta} - \frac{1}{(m+n)^\delta}\right)\right] \\ &\le \rho^2(q-1)\left[\frac{1+\ln n}{n} + \frac{1}{\delta(1-\delta)n^\delta}\right].\end{aligned}$$

Then, as the proof of Theorem 2.1, one can derive that $\lim_{n\to\infty} EX_j(n)$ exists for all $j = 1, \ldots, q$.

Note that (3.1) holds since the entries $\epsilon_{ij}(n)$ of ϵ_n satisfy $\epsilon_{ij}(n) = O(1/n^2)$. To prove (3.2), let

$$L = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

Then

$$N_j = I - \frac{L}{j}$$

and its eigenvalues are $\lambda_1(j), \ldots, \lambda_{q-1}(j)$, where

$$\lambda_k(j) = 1 - \frac{1}{j} + \frac{\exp\{2k\pi i/q\}}{j}, \quad k = 1, \ldots, q-1, \text{and} i = \sqrt{-1}.$$

Note that for all $k = 1, \ldots, q-1$,

$$|\lambda_k(j)| = \left[1 - \left(\frac{4}{j} - \frac{4}{j^2}\right)\sin^2\left(\frac{\pi}{q}\right)\right]^{1/2},$$

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and then, as the proof of Theorem 2.1, there exists a positive constant η independent on k and n such that

$$\begin{aligned}\|N_n N_{n-1}\cdots N_k\| &\le \eta \prod_{j=k}^{n}\left[1-\left(\frac{4}{j}-\frac{4}{j^2}\right)\sin^2\left(\frac{\pi}{q}\right)\right]^{1/2}\\ &\le \eta\ \exp\left[-2\sin^2\left(\frac{\pi}{q}\right)\sum_{j=k}^{n}\left(\frac{1}{j}-\frac{1}{j^2}\right)\right]\\ &\le \eta\ \exp\left(2\sin^2\left(\frac{\pi}{q}\right)\sum_{j=k}^{\infty}\frac{1}{j^2}\right) exp\left(-2\sin^2\left(\frac{\pi}{q}\right)\int_{j=k}^{n+1}\frac{dx}{x}\right)\\ &= \eta \exp\left(\frac{\pi^2}{3}\sin^2\left(\frac{\pi}{q}\right)\right)\left(\frac{k}{n+1}\right)^{\delta},\end{aligned}$$

where $\delta = 2\sin^2(\pi/q)$. Hence (3.2) holds. ■

By calculating $E[\sum_{j=1}^{q} X_j(n+1)|\mathcal{F}_n]$ and then adapting the argumentation of the proof of Theorem 2.3, we have the following theorem and skip its proof here.

THEOREM 3.2: *In the q-urn model, $\sum_{j=1}^{q} X_j(n)$ converges almost surely.*

Next, we further study the long-term behavior of the processes $\{X_j(n)\}_{n\ge1}$, $j = 1, 2, \ldots, q$.

THEOREM 3.3: *In the q-urn model, the fraction of red balls in urns $A_1, \ldots, A_q$ converge almost surely to the same limit, that is, $\lim_{n\to\infty} X_1(n) = \cdots = \lim_{n\to\infty} X_q(n)$ almost surely.*

PROOF: Since the proof of this theorem is similar to that of Theorem 2.1, and we here outline the main steps and present the new ingredient of the proof for the q-urn model. Recall that $\Phi_j(n) = X_j(n) - X_{j+1}(n)$, $n \in \mathbb{Z}^+$ and $j = 1, 2, \ldots, q-1$. From the dynamics of the q-urn model, we have that for all $j = 1, \ldots, q-1$

$$\Phi_j(n+1) = \Phi_j(n) + \delta_j(n+1),$$

where

$$E[\delta_1(n+1)\mid\mathcal{F}_n] = -\sum_{i=1}^{q}\alpha_i(n+1)\Phi_1(n) - \alpha_1(n+1)\sum_{j=2}^{q-1}\Phi_j(n) + O\left(\frac{1}{n^2}\right)$$

and for all $j = 2, \ldots, q-1$,

$$E[\delta_j(n+1)\mid\mathcal{F}_n] = \alpha_j(n+1)\Phi_{j-1}(n) - \alpha_{j+1}(n+1)\Phi_j(n) + O\left(\frac{1}{n^2}\right).$$

Note that $|\delta_j(n+1)| = |\Phi_j(n+1) - \Phi_j(n)| \le 2/(n+1)$ for all $j = 1, \ldots, q-1$. Moreover,

$$\begin{aligned}E[\Phi_1(n+1)^2\mid\mathcal{F}_n] &= \Phi_1(n)^2 + 2\Phi_1(n)E[\delta_1(n+1)|\mathcal{F}_n] + E[(\delta_1(n+1))^2|\mathcal{F}_n]\\ &= \left(1-2\sum_{i=1}^{q}\alpha_i(n+1)\right)\Phi_1^2(n) - 2\alpha_1(n+1)\Phi_1(n)\sum_{j=2}^{q-1}\Phi_j(n) + O\left(\frac{1}{n^2}\right)\end{aligned}$$

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and for $j = 1, \ldots, q-2$,

$$E[\Phi_j(n+1)^2 \mid \mathcal{F}_n] = \Phi_j(n)^2 + 2\Phi_j(n)E[\delta_j(n+1)|\mathcal{F}_n] + E[(\delta_j(n+1))^2|\mathcal{F}_n]$$
$$= (1 - 2\alpha_{j+1}(n+1))\Phi_j^2(n) + 2\alpha_j(n+1)\Phi_{j-1}(n)\Phi_j(n) + O\left(\frac{1}{n^2}\right),$$

and so there are two sequences of positive constants $\{\tau_n\}$ and $\{\lambda_n\}$ such that

$$\begin{aligned}
&\sum_{j=1}^{q-1} E[\Phi_{n+1}^2(j) \mid \mathcal{F}_n] \\
&= \left(1 - 2\sum_{i=1}^{q} \alpha_i(n+1)\right)\Phi_1^2(n) + \sum_{j=2}^{q-1}(1 - 2\alpha_{j+1}(n+1))\Phi_j^2(n) \\
&\quad - 2\sum_{j=2}^{q-1} \Phi_j(n)[\alpha_1(n+1)\Phi_1(n) - \alpha_j(n+1)\Phi_{j-1}(n)] + O\left(\frac{1}{n^2}\right) \\
&\le \tau_{n+1}\left(\sum_{j=1}^{q-1} \Phi_n^2(j)\right) + \frac{\lambda}{(n+1)^2}.
\end{aligned}$$

Here

$$\begin{aligned}
\tau_{n+1} &= \max\left\{1 - 2\sum_{i=1}^{q} \alpha_i(n+1), 1 - 2\alpha_3(n+1), \ldots, 1 - 2\alpha_q(n+1)\right\} \\
&\le \max\{1 - 2\alpha_1(n+1), 1 - 2\alpha_2(n+1), \ldots, 1 - 2\alpha_q(n+1)\} \\
&\le \frac{n + t_0}{n + 1 + t_0},
\end{aligned}$$

where $t_0 = \max\{t_1(0), t_2(0), \ldots, t_q(0)\}$. Next, adapting the argumentation of Theorem 2.2, one can derive that $\lim_{n\to\infty} \Phi_j(n) = 0$ almost surely for all $j = 1, 2, \ldots, q-1$ and so, combining with Theorem 3.2, the fractions of red balls in all urns converge almost surely to the same limit. ■

Remark 3.1: For the q-urn model, as in the three-urn model, we have $\phi_1(n) = \cdots = \phi_q(n) = 0$ if $x_1(0) = \cdots = x_q(0)$, and so $EX_1(n) = \cdots = EX_q(n) = x_1(0)$ for all $n \in \mathbb{N}$. We also have that

$$\lim_{n\to\infty} EX_1(n) = \cdots = \lim_{n\to\infty} EX_q(n)$$
$$= x_2(0) + \alpha_2(1)\phi_1(0) + \sum_{j=1}^{\infty} \alpha_2(j+1) v M_j M_{j-1} \cdots M_1 \begin{bmatrix} \phi_1(0) \\ \vdots \\ \phi_q(0) \end{bmatrix},$$

where $v = [1 \quad 0 \quad \cdots \quad 0]$.

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References

1. Chen, M.-R. Hsiau, S.-R. & Yang, T.-H. (2014). A new two-urn model. *Journal of Applied Probability*, 51: 590–597.
2. Chen, M.-R. & Wei, C.-Z. (2005). A new urn model. *Journal of Applied Probability*, 42: 964–976.
3. Eggenberger, F. & Pólya, G. (1923). Über die statistik verketetter vorgänge. *Zeitschrift für Angewandte Mathematik und Mechanik.*, 1: 279–289.
4. Kac, M. (1947). Random walk and the theory of Brownian motion, *American Mathematical Monthly*, 54: 369–391.
5. Kotz, S. & Balakrishnan, N. (1997). Advances in urn models during the past two decades. In *Advances in Combinatorial Methods and Applications to Probability and Statistics*. Birkhäuser Boston, Cambridge, MA, U.S.A., pp. 203–257.
6. Mahmoud, H.M. (2009). *Pólya Urn Models*, Boca Raton, FL: CRC Press.
7. Takács, L. (1979). On an urn problem of Pal and Tatiana Ehrenfest, *Mathematical Proceedings of the Cambridge Philosophical Society*, 86: 127–130.

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