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PORTFOLIO OPTIMIZATION IN A DEFAULT MODEL UNDER FULL/PARTIAL INFORMATION

THOMAS LIM
Laboratoire de Mathématiques et,
Modélisation d'Evry, CNRS, UMR 8071, ENSIIE
E-mail: lim@ensiie.fr

MARIE-CLAIRE QUENEZ
Laboratoire de Probabilités et,
Modèles Aléatoires.CNRS, UMR 7599
Université Paris 7
E-mail: quenez@math.jussieu.fr

In this paper, we consider a financial market with an asset exposed to a risk inducing a jump in the asset price, and which can still be traded after the default time. We use a default-intensity modeling approach, and address in this incomplete market context the problem of maximization of expected utility from terminal wealth for logarithmic, power and exponential utility functions. We study this problem as a stochastic control problem both under full and partial information. Our contribution consists of showing that the optimal strategy can be obtained by a direct approach for the logarithmic utility function, and the value function for the power (resp. exponential) utility function can be determined as the minimal (resp. maximal) solution of a backward stochastic differential equation. For the partial information case, we show how the problem can be divided into two problems: a filtering problem and an optimization problem. We also study the indifference pricing approach to evaluate the price of a contingent claim in an incomplete market and the information price for an agent with insider information.

1. INTRODUCTION

One of the important problems in mathematical finance is the portfolio optimization problem when the investor wants to maximize the expected utility from terminal wealth. In this paper, we study this problem for the classical utility functions by considering a small investor on an incomplete financial market who can trade in a finite time interval $[0, T]$ by investing in risky stocks and a riskless bond. We assume that there exists a default time on the market, and this one generates a jump of stock price. We assume that in this market there are two kinds of agents: the classical agents (they only observe the asset prices and the default time) and the insider agents (the agents with insider information). These situations are referred as partial information and full information. We will be interested not only in

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describing the investor's optimal utility, but also in describing the strategies which she may follow to reach her goal.

The utility maximization problem with full information has been largely studied in the literature. In the framework of a continuous-time model the problem was studied for the first time by Merton [23]. Using the methods of stochastic optimal control, the author derives a non-linear partial equation for the value function of the optimization problem. Some papers study this problem by using the dual problem, we can quote, for instance, Karatzas, Lehoczky, and Shreve [14] for the case of complete financial models, and Karatzas *et al.* [15] and Kramkov and Schachermayer [18] for the case of incomplete financial models, they find the solution of the original problem by convex duality. These papers are useful to prove the existence of an optimal strategy in the general case, but in practice it is difficult to find the optimal strategy with the dual method. Some others study the problem by using the dynamic programming principle, we can quote Jeanblanc and Pontier [9] for a complete model with discontinuous prices, Bellamy [1] in the case of a filtration generated by a Brownian motion and a Poisson measure, Hu, Imkeller, and Muller [8] for an incomplete model in the case of a Brownian filtration, and Jiao and Pham [13] in the case with a default, in which the authors study the case before the default and the case after the default.

Models with partial information are essentially studied in the literature in a complete market framework. Detemple [4], Dothan and Feldman [5], Gennotte [7] use dynamic programming methods in a linear Gaussian filtering. Lakner [19,20] solves the optimization problem via a martingale approach and works out the special case of linear Gaussian model. We mention that Frey and Runggaldier [6], and Lasry and Lions [21] study hedging problems in finance under restricted information. Pham and Quenez [26] consider the case of an incomplete stochastic volatility model. Callegaro, Di Masi, and Runggaldier [2] and Roland [28] study the case of a market model with jumps.

We first study the case of full information. For the logarithmic utility function, we use a direct approach, which allows us to give an expression of the optimal strategy depending only on the coefficients of the model satisfied by the stocks. For the power utility function, we look for a necessary condition characterizing the value function which is solution of the maximization problem. We show that this one is the smallest solution of a backward stochastic differential equation (in short BSDE). We also give an approximation of this one by a sequence of solutions of BSDEs. These solutions are the value functions of the maximization problem restricted to some bounded subsets of strategies. For the exponential utility function, we refer to the companion paper Lim and Quenez [22].

In order to solve the partial information problem, the common way is to use the filtering theory, so as to reduce the stochastic control problem with partial information to one with full information as in Pham and Quenez [26] or Roland [28]. Then we can apply the results of the full information problem.

The outline of this paper is organized as follows. In Section 2, we describe the model and formulate the optimization problem. In Section 3, we solve the maximization problem for the logarithmic utility function with a direct approach. In Section 4, we consider the power utility function by giving a characterization of the value function by a BSDE thanks to the dynamic programming principle, then we approximate this one by a sequence of solutions of Lipschitz BSDEs. In Section 5, we use results from filtering theory to reduce the stochastic control problem with partial information to one with full information, then we apply the results of the full information problem to the partial information problem. Finally, we study the indifference price for a contingent claim and the information price linked to the insider information.

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2. THE MODEL

2.1. The Probability Space

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We assume that this space is equipped with a one-dimensional standard Brownian motion B and we denote by $\mathbb{F}^B := (\mathcal{F}_t^B)_{t\geq 0}$ the right continuous complete filtration generated by B. We also consider on this space a random time τ, which represents for example a default time in credit risk or in counterparty risk. The random time τ is not assumed to be an $\mathbb{F}^B$-stopping time. We therefore use in the sequel the standard approach of filtration enlargement by considering $\mathbb{F}$ the smallest right continuous extension of $\mathbb{F}^B$ that turns τ into an $\mathbb{F}$-stopping time (see, e.g., Jeulin [11] and Jeulin and Yor [12]). More precisely $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is defined by

$$\mathcal{F}_t := \bigcap_{\varepsilon>0} \tilde{\mathcal{F}}_{t+\varepsilon},$$

for all $t \geq 0$, where $\tilde{\mathcal{F}}_s := \mathcal{F}_s^B \vee \sigma(\mathbb{1}_{\tau\leq u}, u \in [0,s])$, for all $s \geq 0$.

In the sequel, we denote by N the process $\mathbb{1}_{\tau\leq \cdot}$ and we suppose that this one admits an $\mathbb{F}$-compensator of the form $\int_0^\cdot \lambda_t dt$, that is, $N - \int_0^\cdot \lambda_t dt$ is an $\mathbb{F}$-martingale. We also assume that the process λ is uniformly bounded. M denotes the $\mathbb{F}$-martingale defined by

$$M_t = N_t - \int_0^t \lambda_s \, ds, \tag{2.1}$$

for all $t \geq 0$. We note that $\lambda_t = 0$ on $\{\tau \leq t\}$ almost surely and therefore we get $\langle M \rangle = \int_0^\cdot \lambda_s ds$.

We also suppose the classical assumption in filtration enlargement which is called H'-hypothesis:

Assumption 2.1: *Any $\mathbb{F}^B$-semimartingale is an $\mathbb{F}$-semimartingale.*

In particular under this assumption, there exists an $\mathbb{F}$-adapted process ν such that the process $W := B + \int_0^\cdot \nu_s ds$ is an $\mathbb{F}$-Brownian motion.

2.2. The Financial Market

We consider a financial market, defined on the finite-time interval $[0,T]$, consisting of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at each date, and one risky asset with a price process S evolving according to the following diffusion:

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dN_t), \quad t \in [0,T].$$

We shall make the following standing assumptions about the previous diffusion.

Assumption 2.2:

(i) *The coefficients μ and σ are $\mathbb{R}$-valued uniformly bounded $\mathbb{F}$-adapted processes, satisfying $\sigma_t > 0$ and $\theta_t = \mu_t/\sigma_t$ uniformly bounded.*

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(ii) *β is an $\mathbb{R}$-valued uniformly bounded $\mathbb{F}$-predictable process, satisfying $\beta_t > -1$ for any $t \in [0, \tau]$.*

The hypothesis (ii) of Assumption 2.2 implies that the process S is positive.

For any $x > 0$ and any $\mathbb{F}$-predictable process $\pi = (\pi_t)_{0 \le t \le T}$, we define the process $X^{x,\pi}$ by

$$X_t^{x,\pi} = x \exp\left(\int_0^t \left(\pi_s \mu_s - \frac{|\pi_s \sigma_s|^2}{2} \right) ds + \int_0^t \pi_s \sigma_s dW_s \right) (1 + \pi_\tau \beta_\tau N_t), \tag{2.2}$$

if this one is well defined. The process $X^{x,\pi}$ is the wealth process associated with the self-financing trading strategy π and the initial capital x. The process π describes the part of the wealth invested in the risky asset. The number of shares of the risky asset at time t is then given by $((\pi_t X_t^{x,\pi})/S_t)$. $\pi = (\pi_t)_{0 \le t \le T}$ is called trading strategy if π is $\mathbb{F}$-predictable and $\int_0^{\cdot}((\pi_t X_t^{x,\pi}/S_t) dS_t$ is well defined. For a given initial time t and an initial capital x, the associated wealth process is denoted by $X^{t,x,\pi}$.

Let $U : \mathbb{R} \to \mathbb{R}$ be a utility function. The optimization problem consists of maximizing the expected utility from terminal wealth over the class $\mathcal{A}_{\mathbb{F}}(x)$ of admissible strategies, which will be defined in the sequel. More precisely, we want to characterize the value function of this problem, which is defined by

$$V(x) := \sup_{\pi \in \mathcal{A}_{\mathbb{F}}(x)} \mathbb{E}\big[U(X_T^{x,\pi})\big],$$

and we also want to give the optimal strategy when this one exists. We begin by the simple case when U is the logarithmic utility function, then we study the power and exponential utility functions.

3. LOGARITHMIC UTILITY FUNCTION

In this section, we specify the meaning of optimality for trading strategies by stipulating that the agent wants to maximize her expected utility from her terminal wealth $X_T^{x,\pi}$ with respect to the logarithmic utility function $U(x) = \log(x)$ for any $x > 0$. Our goal is to solve the following optimization problem:

$$V(x) = \sup_{\pi \in \mathcal{A}_{\mathbb{F}}(x)} \mathbb{E}\big[\log(X_T^{x,\pi})\big], \tag{3.1}$$

where $\mathcal{A}_{\mathbb{F}}(x)$ is the set of admissible strategies defined by:

Definition 3.1: The set of admissible trading strategies $\mathcal{A}_{\mathbb{F}}(x)$ consists of $\mathbb{F}$-predictable processes π satisfying $\mathbb{E}[\int_0^T \left(|\pi_t \sigma_t|^2 + \lambda_t |\log(1 + \pi_t \beta_t)| \right) dt] < \infty$, and such that $\pi_t \beta_t > -1$ a.s. for any $t \in [0, \tau]$.

As in Kramkov and Schachermayer [18], we assume that $\sup_{\pi \in \mathcal{A}_{\mathbb{F}}(x)} \mathbb{E}[\log(X_T^{x,\pi})] < \infty$.

We add one of the following assumptions on the coefficients of the model to be able to solve the optimization problem (3.1) directly.

Assumption 3.1: *The process β^{-1} is uniformly bounded.*

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ASSUMPTION 3.2: *The process* $\hat{\pi}$ *defined by*

$$\hat{\pi}_t := \begin{cases} \dfrac{\mu_t}{2\sigma_t^2} - \dfrac{1}{2\beta_t} + \dfrac{\sqrt{(\mu_t\beta_t + \sigma_t^2)^2 + 4\lambda_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2} & \text{if } t \le \tau \text{ and } \beta_t \neq 0, \\ \dfrac{\mu_t}{\sigma_t^2} \quad \text{if } t \le \tau \text{ and } \beta_t = 0 \text{ or } t > \tau, \end{cases} \tag{3.2}$$

satisfied the following inequality

$$\mathbb{E}\left[\int_0^T \left(|\hat{\pi}_t\sigma_t|^2 + \lambda_t|\log(1+\hat{\pi}_t\beta_t)|\right) dt\right] < \infty.$$

If Assumption 3.1 or 3.2 holds, then we easily get the value function $V(x)$ and the optimal strategy associated with this one.

THEOREM 3.1: *The solution of the optimization problem* (3.1) *is given by*

$$V(x) = \log(x) + \mathbb{E}\left[\int_0^T \left(\hat{\pi}_t\mu_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \lambda_t\log(1+\hat{\pi}_t\beta_t)\right) dt\right],$$

where $\hat{\pi}$ *is the optimal trading strategy defined by* (3.2).

PROOF: We first remark $V(x) = \log(x) + V(1)$. By (2.2) we obtain

$$\log(X_t^{1,\pi}) = \int_0^t \left(\pi_s\mu_s - \frac{|\pi_s\sigma_s|^2}{2}\right) ds + \int_0^t \pi_s\sigma_s dW_s + \int_0^t \log(1+\pi_s\beta_s)(dM_s + \lambda_s\, ds).$$

Hence, we get the following expression for $V(1)$ thanks to Definition 3.1

$$V(1) = \sup_{\pi\in\mathcal{A}_{\mathbb{F}}(1)} \mathbb{E}\left[\int_0^T \left(\pi_t\mu_t - \frac{|\pi_t\sigma_t|^2}{2} + \lambda_t\log(1+\pi_t\beta_t)\right) dt\right],$$

which implies that

$$V(1) \le \mathbb{E}\left[\int_0^T \operatorname*{ess\,sup}_{\pi_t\beta_t > -1} \left\{\pi_t\mu_t - \frac{|\pi_t\sigma_t|^2}{2} + \lambda_t\log(1+\pi_t\beta_t)\right\} dt\right]. \tag{3.3}$$

For any $t \in [0,T]$ we remark

$$\operatorname*{ess\,sup}_{\pi_t\beta_t > -1} \left\{\pi_t\mu_t - \frac{|\pi_t\sigma_t|^2}{2} + \lambda_t\log(1+\pi_t\beta_t)\right\} = \hat{\pi}_t\mu_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \lambda_t\log(1+\hat{\pi}_t\beta_t) \quad a.s.,$$

where $\hat{\pi}$ is defined by (3.2). Then from the inequality (3.3), we can see that

$$V(1) \le \mathbb{E}\left[\int_0^T \left(\hat{\pi}_t\mu_t - \frac{|\hat{\pi}_t\sigma_t|^2}{2} + \lambda_t\log(1+\hat{\pi}_t\beta_t)\right) dt\right]. \tag{3.4}$$

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So it is sufficient to show that the strategy $\hat{\pi}$ is admissible to prove that this one is the optimal strategy and that the inequality (3.4) is an equality.

- It is clearly the case if Assumption 3.1 holds.
- If Assumption 3.2 holds we must prove that $\hat{\pi}_t\beta_t > -1$ for any $t \in [0,T]$. It is obvious if $t > \tau$ or if $\beta_t = 0$. If $t \leq \tau$ and $\beta_t \neq 0$ we remark that

$$\begin{aligned}\hat{\pi}_t\beta_t &= \frac{\mu_t\beta_t}{2\sigma_t^2} - \frac{1}{2} + \frac{\sqrt{(\mu_t\beta_t + \sigma_t^2)^2 + 4\lambda_t\beta_t^2\sigma_t^2}}{2\sigma_t^2} \\ &> \frac{\mu_t\beta_t + |\mu_t\beta_t + \sigma_t^2|}{2\sigma_t^2} - \frac{1}{2} \\ &> \frac{\mu_t\beta_t - (\mu_t\beta_t + \sigma_t^2)}{2\sigma_t^2} - \frac{1}{2} \\ &> -1.\end{aligned}$$

Hence, $\hat{\pi}$ is an admissible strategy and the inequality (3.4) is an equality. Thus the strategy $\hat{\pi}$ is optimal. ■

Remark 3.1: Recall in the case without default, the optimal strategy is given by $\tilde{\pi}_t = \mu_t/\sigma_t^2$. Thus, in the case with default, the optimal strategy can be written under the form

$$\hat{\pi}_t = \tilde{\pi}_t - \varepsilon_t,$$

where ε_t is an additional term defined by

$$\varepsilon_t = \begin{cases} \dfrac{\mu_t}{2\sigma_t^2} + \dfrac{1}{2\beta_t} - \dfrac{\sqrt{(\mu_t\beta_t + \sigma_t^2)^2 + 4\lambda_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2}, & \text{if } t \leq \tau \text{ and } \beta_t \neq 0, \\ 0 \quad \text{if } t \leq \tau \text{ and } \beta_t = 0 \text{ or } t > \tau. \end{cases}$$

Note if we assume β is negative (resp. positive), that is, the asset price S has a negative (resp. positive) jump at default time τ, ε is positive (resp. negative), that is, the agent has to invest less (resp. more) in the risky asset than in the case of a market without default.

4. POWER UTILITY

In this section, we keep the notation introduced in Section 3 and we study the case of the power utility function defined by $U(x) = x^\gamma/\gamma$, where $x \geq 0$ and $\gamma \in (0,1)$. In order to formulate the optimization problem we first define the set of admissible trading strategies.

Definition 4.1: The set of admissible trading strategies $\mathcal{A}_{\mathbb{F}}(x)$ consists of $\mathbb{F}$-predictable processes π such that $\int_0^T |\pi_t\sigma_t|^2 dt + \int_0^T \lambda_t|\pi_t\beta_t|dt < \infty$ a.s. and such that $\pi_\tau\beta_\tau \geq -1$ a.s.

As in Kramkov and Schachermayer [18], we assume that $\sup_{\pi\in\mathcal{A}_{\mathbb{F}}(x)} \mathbb{E}[U(X_T^{x,\pi})] < \infty$.

Remark 4.1: From the expression (2.2), it is obvious that the condition $\pi_\tau\beta_\tau \geq -1$ a.s. is equivalent to $X_t^{x,\pi} \geq 0$ a.s. for any $t \in [0,T]$.

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The portfolio optimization problem consists of determining an admissible strategy π which attains the optimal value

$$V(x) = \sup_{\pi \in \mathcal{A}_{\mathbb{F}}(x)} \mathbb{E}\big[U(X_T^{x,\pi})\big]. \tag{4.1}$$

The problem (4.1) can be clearly written as $V(x) = x^\gamma V(1)$. Therefore, it is sufficient to study the case $x = 1$. To solve this optimization problem, we give a dynamic extension of the initial problem. That allows us to use the dynamic programming principle to solve the initial problem. For any initial time $t \in [0, T]$, we define the value function $J(t)$ by the following random variable

$$J(t) := \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{\mathbb{F},t}(1)} \mathbb{E}\big[U\big(X_T^{t,1,\pi}\big)\big|\mathcal{F}_t\big],$$

with $\mathcal{A}_{\mathbb{F},t}(1)$ the set of $\mathbb{F}$-predictable processes $\pi = (\pi_s)_{t \le s \le T}$ such that $\int_t^T |\pi_s \sigma_s|^2 ds + \int_t^T \lambda_s |\pi_s \beta_s| ds < \infty$ a.s. and such that $\pi_\tau \beta_\tau \ge -1$ a.s.

For the sake of brevity, we shall denote X_s^π (resp. $X_s^{t,\pi}$) instead of $X_s^{1,\pi}$ (resp. $X_s^{t,1,\pi}$) and $\mathcal{A}_{\mathbb{F}}$ (resp. $\mathcal{A}_{\mathbb{F},t}$) instead of $\mathcal{A}_{\mathbb{F}}(1)$ (resp. $\mathcal{A}_{\mathbb{F},t}(1)$).
We introduce some sets used in the sequel:

- $\mathcal{S}^2_{\mathbb{F}}$ is the set of $\mathbb{F}$-adapted càd-làg processes Y on $[0, T]$ such that

$$E\left[\sup_{t \in [0,T]} |Y_t|^2\right] < \infty.$$

- $L^2(W)$ (resp. $L^2_{\mathrm{loc}}(W)$) is the set of $\mathbb{F}$-predictable processes Z on $[0, T]$ such that

$$\mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < \infty \left(\text{resp.} \int_0^T |Z_t|^2 dt < \infty \ a.s.\right).$$

- $L^2(M)$ (resp. $L^1_{\mathrm{loc}}(M)$) is the set of $\mathbb{F}$-predictable processes U on $[0, T]$ such that

$$\mathbb{E}\left[\int_0^T \lambda_t |U_t|^2 dt\right] < \infty \left(\text{resp.} \int_0^T \lambda_t |U_t| dt < \infty \ a.s.\right).$$

In the sequel, we assume that the martingale representation property holds.

Assumption 4.1: *Any $(\mathbb{P}, \mathbb{F})$-local martingale has the representation*

$$m_t = m_0 + \int_0^t a_s dW_s + \int_0^t b_s dM_s, \quad \forall t \in [0, T], \ a.s., \tag{4.2}$$

where $a \in L^2_{\mathrm{loc}}(W)$ and $b \in L^1_{\mathrm{loc}}(M)$. If m is a square integrable martingale, each term on the right-hand side of the representation (4.2) is square integrable.

In particular, this result holds if we assume that any $\mathbb{F}^B$-martingale is an $\mathbb{F}$-martingale, this assumption is called immersion property in the literature. But this result also holds under other assumptions which are weaker that the immersion property (see Jeanblanc and Song [10]).

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4.1. Optimization over Bounded Strategies

Before studying the value function $J(t)$, we study the value functions $(J^k(t))_{k\in\mathbb{N}}$ defined by

$$J^k(t) := \operatorname*{ess\,sup}_{\pi\in\mathcal{A}^k_{\mathbb{F},t}} \mathbb{E}\left[U(X_T^{t,\pi})\middle|\mathcal{F}_t\right], \quad \forall t\in[0,T],\ a.s.,$$

where $\mathcal{A}^k_{\mathbb{F},t}$ is the subset of strategies of $\mathcal{A}_{\mathbb{F},t}$ which are uniformly bounded by k. That means that the part of the wealth invested in the risky asset has to be uniformly bounded by a constant k, which makes sense in finance, because the ratio of the amount of money invested or borrowed to the wealth must be bounded according to the financial legislation.

Let us fix $k\in\mathbb{N}$. We want to characterize the value function $J^k(t)$ by a BSDE. For that we introduce for any $\pi\in\mathcal{A}^k_{\mathbb{F}}$ the càd-làg process J^π defined for any $t\in[0,T]$ by

$$J^\pi_t := \mathbb{E}\big[U(X_T^{t,\pi})\big|\mathcal{F}_t\big].$$

Lemma 4.1: *The processes $\{J^\pi, \pi\in\mathcal{A}^k_{\mathbb{F}}\}$ are uniformly bounded by a constant independent of π.*

Proof: Fix $\pi\in\mathcal{A}^k_{\mathbb{F}}$ and $t\in[0,T]$. We have

$$J^\pi_t = \frac{1}{\gamma}\mathbb{E}\left[\exp\left(\gamma\int_t^T\left(\mu_s\pi_s - \frac{|\sigma_s\pi_s|^2}{2}\right)ds + \int_t^T\gamma\sigma_s\pi_s dW_s\right)(1+\pi_\tau\beta_\tau\mathbb{1}_{t<\tau\le T})^\gamma\middle|\mathcal{F}_t\right].$$

Since the coefficients μ, σ and β are supposed to be uniformly bounded, we can see

$$J^\pi_t \le \frac{(1+k|\beta|_\infty)^\gamma}{\gamma}\exp\left(\left(\gamma k|\mu|_\infty + \gamma^2\frac{(k|\sigma|_\infty)^2}{2}\right)T\right),$$

with $|X|_\infty = \sup_{t\in[0,T]}|X_t|$. ■

Classically, for any $\pi\in\mathcal{A}^k_{\mathbb{F}}$, the process J^π can be shown to be the solution of a linear BSDE. More precisely, there exist $Z^\pi\in L^2(W)$ and $U^\pi\in L^2(M)$, such that (J^π, Z^π, U^π) is the solution in $\mathcal{S}^2_{\mathbb{F}}\times L^2(W)\times L^2(M)$ of the linear BSDE with bounded coefficients

$$\begin{cases} -dJ^\pi_t = \left\{\gamma\pi_t(\mu_t J^\pi_t + \sigma_t Z^\pi_t) + \frac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 J^\pi_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(J^\pi_t + U^\pi_t)\right\}dt \\ \qquad\quad -Z^\pi_t dW_t - U^\pi_t dM_t, \\ J^\pi_T = \frac{1}{\gamma}. \end{cases} \tag{4.3}$$

Using the fact $J^k(t) = \operatorname{ess\,sup}_{\pi\in\mathcal{A}^k_{\mathbb{F},t}} J^\pi_t$ for any $t\in[0,T]$, we derive that $J^k(.)$ corresponds to the solution of a BSDE, whose generator is the essential supremum over π of the generators of BSDEs associated with $\{J^\pi, \pi\in\mathcal{A}^k_{\mathbb{F}}\}$. More precisely,

Proposition 4.1:

– *Let (Y,Z,U) be the solution in $\mathcal{S}^2_{\mathbb{F}}\times L^2(W)\times L^2(M)$ of the following Lipschitz BSDE*

$$\begin{cases} -dY_t = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}^k_{\mathbb{F}}}\left\{\gamma\pi_t(\mu_t Y_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2 Y_t\right. \\ \qquad\qquad \left. + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(Y_t + U_t)\right\}dt - Z_t dW_t - U_t dM_t, \\ Y_T = \frac{1}{\gamma}. \end{cases} \tag{4.4}$$

Then, $J^k(t) = Y_t$ a.s. for any $t\in[0,T]$.

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– *There exists a unique optimal strategy for* $J^k(0) = \sup_{\pi \in \mathcal{A}^k_{\mathbb{F}}} \mathbb{E}[U(X^\pi_T)]$.
– *A strategy* $\hat{\pi} \in \mathcal{A}^k_{\mathbb{F}}$ *is optimal for* $J^k(0)$ *if and only if it attains the essential supremum of the generator in* (4.4) $dt \otimes d\mathbb{P} - a.e.$

PROOF: Recall that for any $\pi \in \mathcal{A}^k_{\mathbb{F}}$ there exist $Z^\pi \in L^2(W)$ and $U^\pi \in L^2(M)$ such that (J^π, Z^π, U^π) is the solution of the BSDE

$$-dJ^\pi_t = f(t, \pi_t, J^\pi_t, Z^\pi_t, U^\pi_t)dt - Z^\pi_t dW_t - U^\pi_t dM_t; \quad J^\pi_T = \frac{1}{\gamma},$$

with $f(t, \pi, y, z, u) = (\gamma(\gamma-1)/2)\pi^2_t\sigma^2_t y + \gamma\pi_t(\mu_t y + \sigma_t z) + \lambda_t\left((1+\pi_t\beta_t)^\gamma - 1\right)(y+u)$. Let us introduce the generator f which satisfies $dt \otimes d\mathbb{P} - a.e.$

$$f(t, y, z, u) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}^k_{\mathbb{F}}} f(t, \pi, y, z, u).$$

The supremum of affine functions, whose coefficients are bounded by a positive constant c, is Lipschitz with Lipschitz constant c. Hence, f is Lipschitz and the BSDE

$$-dY_t = f(y, Y_t, Z_t, U_t)dt - Z_t dW_t - U_t dM_t; \quad Y_T = \frac{1}{\gamma},$$

admits a unique solution in $\mathcal{S}^2_{\mathbb{F}} \times L^2(W) \times L^2(M)$ denoted by (Y, Z, U).

By the comparison theorem in case of jumps for BSDEs we get $Y_t \geq J^\pi_t$, $\forall t \in [0,T]$ a.s. As this inequality holds for any $\pi \in \mathcal{A}^k_{\mathbb{F}}$, it is obvious that $Y_t \geq \operatorname{ess\,sup}_{\pi \in \mathcal{A}^k_{\mathbb{F},t}} J^\pi_t$ a.s. Also, by applying a measurable selection theorem, one can easily show there exists $\hat{\pi} \in \mathcal{A}^k_{\mathbb{F}}$ such that for any $t \in [0,T]$, we have

$$\operatorname*{ess\,sup}_{\pi \in \mathcal{A}^k_{\mathbb{F}}} \left\{ \gamma\pi_t(\mu_t Y_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2}\pi^2_t\sigma^2_t Y_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(Y_t + U_t) \right\}$$

$$= \gamma\hat{\pi}_t(\mu_t Y_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2}\hat{\pi}^2_t\sigma^2_t Y_t + \lambda_t((1+\hat{\pi}_t\beta_t)^\gamma - 1)(Y_t + U_t).$$

Thus (Y, Z, U) is a solution of the BSDE (4.3) associated with $\hat{\pi}$. Therefore by uniqueness of the solution of the BSDE (4.3), we have $Y_t = J^{\hat{\pi}}_t$ and thus $Y_t = \operatorname{ess\,sup}_{\pi \in \mathcal{A}^k_{\mathbb{F},t}} J^\pi_t = J^{\hat{\pi}}_t$, $\forall t \in [0,T]$ a.s.
The uniqueness of the optimal strategy is due to the strict concavity of the function $x \mapsto U(x)$. ■

4.2. General Case

In this part, we characterize the value function $J(.)$ by a BSDE, but the general case is more complex than the case with bounded strategies and it needs more technical tools. Note that the random variable $J(t)$ is defined uniquely only up to $\mathbb{P}$-almost sure equivalent and that the process $J(.)$ is adapted but not necessarily progressive. Using dynamic control technics, we derive the following characterization of the value function.

PROPOSITION 4.2: *$J(.)$ is the smallest $\mathbb{F}$-adapted process such that $(X^\pi)^\gamma J(.)$ is a supermartingale for any $\pi \in \mathcal{A}_{\mathbb{F}}$ with $J(T) = 1/\gamma$. More precisely, if $\bar{J}$ is an $\mathbb{F}$-adapted process*

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such that $(X^\pi)^\gamma \bar{J}$ *is a supermartingale for any* $\pi \in \mathcal{A}_\mathbb{F}$ *with* $\bar{J}_T = 1/\gamma$, *then for any* $t \in [0,T]$, *we have* $J(t) \leq \bar{J}_t$ *a.s.*

From the dynamic programming principle, we have the following optimality criterion.

PROPOSITION 4.3: *The following assertions are equivalent:*

(i) $\hat{\pi}$ *is an optimal strategy, that is* $\mathbb{E}[U(X_T^{\hat{\pi}})] = \sup_{\pi \in \mathcal{A}_\mathbb{F}} \mathbb{E}[U(X_T^\pi)]$.

(ii) *The process* $(X^{\hat{\pi}})^\gamma J(.)$ *is a martingale.*

The proof of these propositions is given in Appendix 5.3.

PROPOSITION 4.4: *There exists a càd-làg modification of* $J(.)$ *which is denoted by* J.

PROOF: From Kramkov and Schachermayer [18], there exists an optimal strategy $\hat{\pi} \in \mathcal{A}_\mathbb{F}$ such that $J(0) = \mathbb{E}[U(X_T^{\hat{\pi}})]$. Moreover, using Proposition 4.3, we get that $J(t) = \mathbb{E}[U(X_T^{\hat{\pi}})|\mathcal{F}_t]/(X_t^{\hat{\pi}})^\gamma$ a.s. Which implies the desired result. ■

This càd-làg process is characterized by a BSDE.

DEFINITION 4.2: (Y,Z,U) *is called the minimal solution in* $L^1_\mathbb{F} \times L^2_{\text{loc}}(W) \times L^1_{\text{loc}}(M)$ *to a BSDE if for any solution* (Y',Z',U') *in* $L^1_\mathbb{F} \times L^2_{\text{loc}}(W) \times L^1_{\text{loc}}(M)$ *to this BSDE we have* $Y_t \leq Y'_t \ \forall t \in [0,T]$ *a.s.*

More precisely, we have:

THEOREM 4.1: *There exist* $Z \in L^2_{\text{loc}}(W)$ *and* $U \in L^1_{\text{loc}}(M)$ *such that* (J,Z,U) *is the minimal solution in* $L^1_\mathbb{F} \times L^2_{\text{loc}}(W) \times L^1_{\text{loc}}(M)$ *satisfying* $J_t \geq 0$ *for any* $t \in [0,T]$ *of the following BSDE:*

$$\begin{cases} -dJ_t = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_\mathbb{F}} \left\{ \gamma \pi_t(\mu_t J_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2} \pi_t^2 \sigma_t^2 J_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(J_t + U_t) \right\} dt \\ \qquad\quad -Z_t dW_t - U_t dM_t, \\ J_T = 1/\gamma. \end{cases} \tag{4.5}$$

There exists a unique optimal strategy such that $J(0) = \mathbb{E}[U(X_T^{\hat{\pi}})]$. *Moreover,* $\hat{\pi} \in \mathcal{A}_\mathbb{F}$ *is optimal if and only if it attains the essential supremum of the generator in* (4.5) $dt \otimes d\mathbb{P} - a.e.$

The proof of this theorem is postponed in Appendix 5.3.

There exists another characterization of the value function J as the limit of processes $(J^k)_{k\in\mathbb{N}}$ as k goes to $+\infty$, where J^k is the value function in the case where the strategies are bounded by k.

THEOREM 4.2: *For any* $t \in [0,T]$, *we have*

$$J_t = \lim_{k\to\infty} \uparrow J^k(t) \ a.s.$$

The proof of this theorem is given in Appendix 5.3.

This allows us to approximate the value function J by numerical computation, since the value functions $(J^k)_{k\in\mathbb{N}}$ are the solution of Lipschitz BSDEs and the results of Kharroubi and Lim [17] can be applied.

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5. THE PARTIAL INFORMATION CASE

We consider the financial market introduced in Section 2 and we assume that

$$\sigma_t = \begin{cases} \sigma_1(t, S_t) & \text{if } t \leq \tau, \\ \sigma_2(t, S_t, \tau, \beta_\tau) & \text{if } t > \tau, \end{cases}$$

and

$$\beta_t = \beta(t, S_{t^-}) \quad \text{if } t \leq \tau.$$

The known functions $\sigma_1(t,x)$ and $\beta(t,x)$ are measurable mappings from $[0,T] \times \mathbb{R}$ into $\mathbb{R}$, and the function $\sigma_2(t,x,\theta,b)$ is a measurable mapping from $[0,T] \times \mathbb{R} \times \mathbb{R}_+ \times (-1,\infty)$ into $\mathbb{R}$. We assume that Assumption 2.2 holds and we add the following assumption.

ASSUMPTION 5.1: *The functions $x\sigma_1(t,x)$, $x\sigma_2(t,x,\theta,b)$ and $x\beta_1(t,x)$ are Lipschitz in $x \in \mathbb{R}$, uniformly in $t \in [0,T]$, $\theta \in \mathbb{R}_+$ and $b \in (-1,\infty)$.*

We now consider an agent in this market who can observe neither the Brownian motion W nor the drift μ and the intensity λ, but only the asset price process S and the default time τ. We shall denote by $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ the filtration generated by the price process S and the default process N. The trading strategies are defined as in Section 2, but we add the condition that they are $\mathbb{G}$-predictable. We now want to solve the problem of maximization of expected utility from terminal wealth. It is not possible to use directly the results of the full information case because we do not know the Brownian motion, the drift and the intensity (e.g., the optimal strategy obtained under the full information case is not admissible since it is not $\mathbb{G}$-predictable). As in Pham and Quenez [26], we begin by an operation of filtering.

5.1. Filtering

In this part, we write the diffusion of the price process S with some processes which are $\mathbb{G}$-adapted and we get a martingale representation property in the filtration $\mathbb{G}$ by assuming Assumption 4.1 holds. For that we must introduce a second probability measure.

Recall that we have assumed $\theta = \mu/\sigma$ is uniformly bounded. Consider the positive martingale defined by $L_0 = 1$ and $dL_t = -L_t\theta_t dW_t$. The process L is explicitly given by

$$L_t = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds\right).$$

One can define a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F})$ characterized by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = L_t, \quad t \in [0,T].$$

By Girsanov's theorem, the process defined by

$$\tilde{W}_t = W_t + \int_0^t \theta_s \, ds \tag{5.1}$$

is a $(\mathbb{Q}, \mathbb{F})$-Brownian motion and the compensated martingale M is still a $(\mathbb{Q}, \mathbb{F})$-martingale. The dynamic of S under $\mathbb{Q}$ is given by

$$dS_t = S_{t^-}(\sigma_t d\tilde{W}_t + \beta_t dN_t). \tag{5.2}$$

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Since the processes θ and λ are not $\mathbb{G}$-predictable, it is natural to introduce the $\mathbb{G}$-conditional law of these random variables, say

$$\tilde{\lambda}_t = \mathbb{E}\big[\lambda_t \big| \mathcal{G}_t\big] \text{ and } \tilde{\theta}_t = \mathbb{E}\big[\theta_t \big| \mathcal{G}_t\big].$$

We begin by proving a proposition which will be of paramount importance in the sequel:

Proposition 5.1: *Under Assumption 2.2, the filtration $\mathbb{G}$ is the filtration generated by $(\tilde{W}, N)$.*

Proof: Let $\mathbb{F}^{\tilde{W},N}$ be the filtration generated by $(\tilde{W}, N)$. From (5.2), we have

$$\tilde{W}_t = \int_0^t \sigma_s^{-1} S_{s^-}^{-1} dS_s - \int_0^t \sigma_s^{-1} \beta_s dN_s,$$

for any $t \in [0,T]$, which implies that $\tilde{W}$ is $\mathbb{G}$-adapted and $\mathbb{F}^{\tilde{W},N} \subset \mathbb{G}$. Conversely, under the assumptions on the coefficients, by a classical result of stochastic differential equation (see Protter [27], Theorem V 3.7), the unique solution of (5.2), on $t \in [0,\tau)$, is $\mathbb{F}^{\tilde{W}}$-adapted, and by using a Picard sequence and an iteration we prove that the unique solution of (5.2) is $\mathbb{F}^{\tilde{W},N}$-adapted on $\tau \leq t$. Hence, $\mathbb{G} \subset \mathbb{F}^{\tilde{W},N}$ and finally $\mathbb{G} = \mathbb{F}^{\tilde{W},N}$. ■

Consider the couple of processes $(\bar{W}, \bar{M})$ defined by

$$\begin{cases} \bar{W}_t = \tilde{W}_t - \int_0^t \tilde{\theta}_s \, ds, \\ \bar{M}_t = N_t - \int_0^t \tilde{\lambda}_s \, ds. \end{cases} \tag{5.3}$$

These are the so-called innovation processes of filtering theory. By classical results in filtering theory (see, e.g., Proposition 2.27 in Pardoux [25]), we have:

Proposition 5.2: *The process $\bar{M}$ is a $(\mathbb{Q}, \mathbb{G})$-martingale.*

Proof: Since the processes N and $\tilde{\lambda}$ are $\mathbb{G}$-adapted, we get that the process $\bar{M}$ is $\mathbb{G}$-adapted. We can write from (2.1)

$$\bar{M}_t = M_t + \int_0^t (\lambda_s - \tilde{\lambda}_s) ds.$$

By the law of iterated conditional expectation, it is easy to check that $\bar{M}$ is a $(\mathbb{Q}, \mathbb{G})$-martingale. ■

Remark 5.1: From Proposition 5.1 and the definition (5.3), the filtration $\mathbb{G}$ is equal to the augmented filtration of $(\tilde{W}, \bar{M})$, since

$$[\bar{M}]_t = N_t. \tag{5.4}$$

We have also the following property about the process $\bar{W}$:

Proposition 5.3: *The process $\bar{W}$ is a $(\mathbb{P}, \mathbb{G})$-Brownian motion.*

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PROOF: We can write with (5.1)

$$\bar{W}_t = W_t + \int_0^t \sigma_s^{-1}(\mu_s - \tilde{\mu}_s)ds, \tag{5.5}$$

where $\tilde{\mu}_t = \mathbb{E}\big[\mu_t \big| \mathcal{G}_t\big]$. By Proposition 5.1 and (5.3), $\bar{W}$ is $\mathbb{G}$-adapted. Moreover, we have $[\bar{W}]_t = t$ for any $t \in [0,T]$. By the law of iterated conditional expectation, it is easy to check that $\bar{W}$ is a $\mathbb{G}$-martingale. We then conclude by Levy's characterization theorem on Brownian motion (see, e.g., Theorem 3.3.16 in Karatzas and Shreve [16]). ■

Denote by Λ the $(\mathbb{Q}, \mathbb{F})$-martingale given by $\Lambda_t = 1/L_t$. We then have

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \Lambda_t, \quad t \in [0,T].$$

Let $\tilde{\Lambda}$ be the $(\mathbb{Q}, \mathbb{G})$-martingale given by $\tilde{\Lambda}_t = \mathbb{E}_{\mathbb{Q}}[\Lambda_t \big| \mathcal{G}_t]$. Recall the classical proposition (see, e.g., Lakner [20] or Pham and Quenez [26]), which gives the expression of $\tilde{\Lambda}$.

LEMMA 5.1: *We have*

$$\tilde{\Lambda}_t = \exp\left(\int_0^t \tilde{\theta}_s d\tilde{W}_s - \frac{1}{2}\int_0^t |\tilde{\theta}_s|^2 ds\right).$$

PROPOSITION 5.4: *The process $\bar{M}$ is a $(\mathbb{P}, \mathbb{G})$-martingale.*

PROOF: Since $(d\mathbb{P}/d\mathbb{Q})\big|_{\mathcal{G}_t} = \tilde{\Lambda}_t$, we can apply Girsanov's theorem and we get that the process $\bar{M}$ is a $(\mathbb{P}, \mathbb{G})$-martingale. ■

By means of innovation processes, we can describe from (5.5) the dynamics of the partially observed default model within a framework of full information model

$$\begin{cases} dS_t = S_{t^-}(\tilde{\mu}_t dt + \sigma_t d\bar{W}_t + \beta_t dN_t), \\ d\bar{M}_t = dN_t - \tilde{\lambda}_t dt, \end{cases} \tag{5.6}$$

where $\tilde{\mu}$ is $\mathbb{G}$-adapted and $\tilde{\lambda}$ is $\mathbb{G}$-predictable.

Hence, the operations of filtering and control can be put in sequence and thus separated. To apply the results in the case of full information to solve the optimization problem we need in particular of a martingale representation property for the filtration $\mathbb{G}$ with respect to $\bar{W}$ and $\bar{M}$. Notice it cannot be directly derived from the usual martingale representation theorem since $\mathbb{G}$ is not equal to the filtration generated by $\bar{W}$ and $\bar{M}$. By assuming that the martingale representation property holds for the filtration $\mathbb{F}$ we get

LEMMA 5.2: *Any $(\mathbb{P}, \mathbb{G})$-local martingale has the representation*

$$m_t = m_0 + \int_0^t a_s d\bar{W}_s + \int_0^t b_s d\bar{M}_s, \quad \forall t \in [0,T] \ a.s.,$$

where $a \in L^2_{\mathrm{loc}}(\bar{W})$ and $b \in L^1_{\mathrm{loc}}(\bar{M})$. If m is a square integrable martingale, each term on the right-hand side of the previous equality is square integrable.

The proof of this lemma is postponed in Appendix 5.3.

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5.2. Optimization Problem for the Classical Utilities

It is now possible to apply the previous results because the price process evolves according to the equation

$$\begin{cases} dS_t = S_{t^-}(\tilde{\mu}_t dt + \sigma_t d\bar{W}_t + \beta_t dN_t), \\ d\bar{M}_t = dN_t - \tilde{\lambda}_t dt, \end{cases}$$

where $\tilde{\mu}$ and σ are $\mathbb{G}$-adapted, and β and $\tilde{\lambda}$ are $\mathbb{G}$-predictable, and there exists a martingale representation theorem for $(\mathbb{P}, \mathbb{G})$-martingales.

Logarithmic utility function: we assume that Assumption 3.1 or the strategy $\tilde{\pi}$, defined in Therorem 5.1, is admissible. The set of admissible strategies is defined by

Definition 5.1: *The set of admissible trading strategies $\mathcal{A}_{\mathbb{G}}(x)$ consists of all $\mathbb{G}$-predictable processes π satisfying $\mathbb{E}[\int_0^T (|\pi_t\sigma_t|^2 + \tilde{\lambda}_t|\log(1+\pi_t\beta_t)|)dt] < \infty$, and such that $\pi_t\beta_t > -1$ a.s. for any $t \in [0,\tau]$.*

Theorem 5.1: *The solution of the optimization problem for the logarithmic utility function is given by*

$$\sup_{\pi\in\mathcal{A}_{\mathbb{G}}(x)} \mathbb{E}[\log(X_T^{x,\pi})] = \log(x) + \mathbb{E}\left[\int_0^T \left(\tilde{\pi}_t\tilde{\mu}_t - \frac{|\tilde{\pi}_t\sigma_t|^2}{2} + \tilde{\lambda}_t \log(1+\tilde{\pi}_t\beta_t)\right) dt\right],$$

with $\tilde{\pi}$ the optimal trading strategy defined by

$$\tilde{\pi}_t = \begin{cases} \dfrac{\tilde{\mu}_t}{2\sigma_t^2} - \dfrac{1}{2\beta_t} + \dfrac{\sqrt{(\tilde{\mu}_t\beta_t + \sigma_t^2)^2 + 4\tilde{\lambda}_t\beta_t^2\sigma_t^2}}{2\beta_t\sigma_t^2}, & \text{if } t \le \tau \text{ and } \beta_t \neq 0, \\ \dfrac{\tilde{\mu}_t}{\sigma_t^2} \quad \text{if } t \le \tau \text{ and } \beta_t = 0 \text{ or } t > \tau. \end{cases}$$

Therefore, we can see that the optimal strategy in the case of partial information can be formally derived from the full information case by replacing the unobservable coefficients μ_t and λ_t by their estimates $\tilde{\mu}_t$ and $\tilde{\lambda}_t$.

Remark 5.2: We note that $\tilde{\pi}$ is $\mathbb{F}$-predictable and satisfied $\mathbb{E}[\int_0^T (|\tilde{\pi}_t\sigma_t|^2 + \lambda_t|\log(1+\tilde{\pi}_t\beta_t)|)dt] < \infty$. Hence, it is an admissible strategy for the problem with full information. Thus we get that the value function in the case of partial information is smaller than the value function in the case of full information.

Power utility function: we have

Theorem 5.2:

- *Let $(\bar{Y}, \bar{Z}, \bar{U})$ be the minimal solution in $L^1_{\mathbb{G}} \times L^2_{\text{loc}}(\bar{W}) \times L^1_{\text{loc}}(\bar{M})$ satisfying $\bar{Y}_t \ge 0$ of the BSDE (4.5) with (W, M, μ, λ) replaced by $(\bar{W}, \bar{M}, \tilde{\mu}, \tilde{\lambda})$, then*

$$\bar{Y}_t = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{\mathbb{G},t}} \mathbb{E}\big[U(X_T^{t,\pi})\big|\mathcal{G}_t\big] \ a.s.$$

- *If a strategy $\hat{\pi} \in \mathcal{A}_{\mathbb{G}}$ is optimal for $J_0 = \sup_{\pi\in\mathcal{A}_{\mathbb{G}}} \mathbb{E}[U(X_T^{\pi})]$ then $\hat{\pi}$ attains the essential supremum in the generator of the BSDE (4.5) $dt \otimes d\mathbb{P}$ a.s. with (μ, λ) replaced by $(\tilde{\mu}, \tilde{\lambda})$.*

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– *Moreover the process* $\bar{Y}$ *is the non-decreasing limit of the processes* $(\bar{Y}^k)_{k\in\mathbb{N}}$, *where* $(\bar{Y}^k, \bar{Z}^k, \bar{U}^k)$ *is the solution in* $\mathcal{S}^2_{\mathbb{G}} \times L^2(\bar{W}) \times L^2(\bar{M})$ *of the BSDE* (4.4) *with* (W, M, μ, λ) *replaced by* $(\bar{W}, \bar{M}, \tilde{\mu}, \tilde{\lambda})$.
– *We have* $\bar{Y}_t \leq Y_t$ *for any* $t \in [0, T]$.

The last property of the previous theorem is due to if a strategy is admissible for the partial information problem then it is admissible for the full information problem. Which implies that the value function in the case of partial information is smaller than the value function in the case of full information.

5.3. Optimization Problem for the Exponential Utility Function and Indifference Pricing

We can also apply the results of Lim and Quenez [22] for the exponential utility function. In this case, we assume that the agent buys a contingent claim, which is modeled by a non-negative $\mathcal{G}_T$-measurable random variable ξ (note that all the results still hold under the assumption that ξ is only lower bounded). Without loss of generality we can use a somewhat different notion of trading strategy: ϕ_t corresponds to the amount of money invested in the risky asset at time t. The number of shares is ϕ_t/S_t. With this notation, under the assumption that the trading strategy is self-financing, the wealth process $X^{x,\phi}$, if this one is well defined, associated with a trading strategy ϕ and an initial capital x is equal to

$$X_t^{x,\phi} = x + \int_0^t \phi_s \tilde{\mu}_s \, ds + \int_0^t \phi_s \sigma_s dW_s + \int_0^t \phi_s \beta_s dN_s.$$

Our goal is to solve the following optimization problem for an agent who buys the contingent claim ξ

$$V(x, \xi) = \sup_{\phi \in \mathcal{A}_{\mathbb{G}}(x)} \mathbb{E}\big[-\exp\big(-\gamma\big(X_T^{x,\phi} + \xi\big)\big)\big] = \exp(-\gamma x) V(0, \xi),$$

where $\mathcal{A}_{\mathbb{G}}(x)$ is defined by:

Definition 5.2: *The set of admissible trading strategies* $\mathcal{A}_{\mathbb{G}}(x)$ *consists of* $\mathbb{G}$*-predictable processes* $\phi = (\phi_t)_{0\leq t\leq T}$, *which satisfy* $\int_0^T |\phi_t \sigma_t|^2 ds + \int_0^T \tilde{\lambda}_t |\phi_t \beta_t|^2 dt < \infty$ *a.s. and such that for any* ϕ *fixed and any* $t \in [0, T]$, *there exists a constant* $K_{t,\pi}$ *such that for any* $s \in [t, T]$, *we have* $X_s^{t,\pi} \geq K_{t,\pi}$ *a.s.*

To solve this problem, it is sufficient to study the case $x = 0$. For that we give a dynamic extension of the initial problem as in Section 4. For any initial time $t \in [0, T]$, we define the value function $\bar{J}^{\xi}(t)$ by the following random variable:

$$\bar{J}^{\xi}(t) := \operatorname*{ess\,inf}_{\phi \in \mathcal{A}_{\mathbb{G},t}} \mathbb{E}\big[\exp\Big(-\gamma\Big(X_T^{t,\phi} + \xi\Big)\Big) \, |\mathcal{G}_t\big],$$

where $\mathcal{A}_{\mathbb{G},t}$ is the admissible strategies set defined by:

Definition 5.3: *The set of admissible trading strategies* $\mathcal{A}_{\mathbb{G},t}$ *consists of* $\mathbb{G}$*-predictable processes* $\phi = (\phi_s)_{t\leq s\leq T}$, *which satisfy* $\int_t^T |\phi_s \sigma_s|^2 ds + \int_t^T \tilde{\lambda}_s |\phi_s \beta_s|^2 ds < \infty$ *a.s. and such that for any* ϕ *fixed and any* $s \in [t, T]$, *there exists a constant* $K_{s,\pi}$ *such that for any* $u \in [s, T]$, *we have* $X_u^{s,\pi} \geq K_{s,\pi}$ *a.s.*

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We introduce the two following sets:

- $\mathcal{S}_{\mathbb{G}}^{+,\infty}$ is the set of positive $\mathbb{G}$-adapted $\mathbb{P}$-essentially bounded càd-làg processes on $[0,T]$.
- $\mathcal{A}_{\mathbb{G}}^2$ is the set of the increasing $\mathbb{G}$-adapted càd-làg processes K such that $K_0 = 0$ and $\mathbb{E}|K_T|^2 < \infty$.

Definition 5.4: *(Y,Z,U,K) is called the maximal supersolution of a BSDE in $\mathcal{S}_{\mathbb{G}}^{+,\infty} \times L^2(\bar{W}) \times L^2(\bar{M}) \times \mathcal{A}_{\mathbb{G}}^2$ if for any super solution (Y',Z',U',K') in $\mathcal{S}_{\mathbb{G}}^{+,\infty} \times L^2(\bar{W}) \times L^2(\bar{M}) \times \mathcal{A}_{\mathbb{G}}^2$ to this BSDE we have $Y_t \leq Y'_t,\ \forall t \in [0,T],\ \mathbb{P}-a.s.$*

By applying the results of the companion paper Lim and Quenez [22], we get the following characterizations of the value function:

Theorem 5.3:

- *Let $(\bar{Y},\bar{Z},\bar{U},\bar{K})$ be the maximal supersolution in $\mathcal{S}_{\mathbb{G}}^{+,\infty} \times L^2(\bar{W}) \times L^2(\bar{M}) \times \mathcal{A}_{\mathbb{G}}^2$ of*

$$\begin{cases} -d\bar{Y}_t = -\bar{Z}_t d\bar{W}_t - \bar{U}_t d\bar{M}_t - d\bar{K}_t + \operatorname*{ess\,inf}_{\phi\in\mathcal{A}_{\mathbb{G}}} \Big\{ \frac{\gamma^2}{2}|\phi_t\sigma_t|^2\bar{Y}_t - \gamma\phi_t(\tilde{\mu}_t\bar{Y}_t + \sigma_t\bar{Z}_t) \\ \qquad\qquad - (1-e^{-\gamma\phi_t\beta_t})(\tilde{\lambda}_t\bar{Y}_t + \tilde{\lambda}_t\bar{U}_t)\Big\}dt, \\ \bar{Y}_T = \exp(-\gamma\xi), \end{cases} \tag{5.7}$$

  then $\bar{Y}_t = \bar{J}^{\xi}(t)$ a.s. for any $t \in [0,T]$.
- *$\bar{J}^{\xi}(t) = \lim_{k\to\infty} \downarrow \bar{J}^{\xi,k}(t)$, with $\bar{J}^{\xi,k}(t) = \operatorname{ess\,inf}_{\phi\in\mathcal{A}^k_{\mathbb{G},t}} \mathbb{E}[\exp(-\gamma(X_T^{t,\phi}+\xi))|\mathcal{G}_t]$ and $\mathcal{A}^k_{\mathbb{G},t}$ is the set of strategies of $\mathcal{A}_{\mathbb{G},t}$ which are uniformly bounded by k.*
- *Let $(\bar{Y}^k,\bar{Z}^k,\bar{U}^k)$ be the unique solution in $\mathcal{S}_{\mathbb{G}}^2 \times L^2(\bar{W}) \times L^2(\bar{M})$ of the following BSDE*

$$\begin{cases} -d\bar{Y}^k_t = -\bar{Z}^k_t d\bar{W}_t - \bar{U}^k_t d\bar{M}_t + \operatorname*{ess\,inf}_{\phi\in\mathcal{A}^k_{\mathbb{G}}} \Big\{ \frac{\gamma^2}{2}|\phi_t\sigma_t|^2\bar{Y}^k_t - \gamma\phi_t(\tilde{\mu}_t\bar{Y}^k_t + \sigma_t\bar{Z}^k_t) \\ \qquad\qquad - (1-e^{-\gamma\phi_t\beta_t})(\tilde{\lambda}_t\bar{Y}^k_t + \tilde{\lambda}_t\bar{U}^k_t)\Big\}dt, \\ \bar{Y}^k_T = \exp(-\gamma\xi), \end{cases} \tag{5.8}$$

  then $\bar{Y}^k_t = \bar{J}^{\xi,k}(t)$ a.s., for any $t \in [0,T]$.

We can now define the indifference price of the contingent claim ξ. The Hodges approach to pricing of unhedgeable claims is a utility-based approach and can be summarized as follows: the issue at hand is to assess the value of some claim ξ as seen from the perspective of an investor who optimizes her behavior relative to some utility function, in our case we use the exponential utility function. The investor has two choices:

- she only invests in the risk-free asset and in the risky assets, in this case the associated optimization problem is

$$V(x,0) = \sup_{\phi\in\mathcal{A}_{\mathbb{G}}(x)} \mathbb{E}\big[-\exp\Big(-\gamma X_T^{x,\phi}\Big)\big];$$

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– she also invests in the contingent claim, whose price is $\bar{p}$ at 0, in this case the associated optimization problem is

$$V(x-\bar{p},\xi) = \sup_{\phi\in\mathcal{A}_{\mathbb{G}}(x-\bar{p})} \mathbb{E}\left[-\exp\left(-\gamma\left(X_T^{x-\bar{p},\phi}+\xi\right)\right)\right].$$

DEFINITION 5.5: *For a given initial capital x, the Hodges buying price of a defaultable claim ξ is the price $\bar{p}$ such that the investor's value functions are indifferent between holding and not holding the contingent claim, that is,*

$$V(x,0) = V(x-\bar{p},\xi).$$

The Hodges price $\bar{p}$ can be derived explicitly by applying the results of Theorem 5.3. If the agent buys the contingent claim at the price $\bar{p}$ and invests the rest of her wealth in the risk-free asset and in the risky assets, the value function is equal to

$$V(x-\bar{p},\xi) = -\exp(-\gamma(x-\bar{p}))\bar{J}^{\xi}(0).$$

If she invests all her wealth in the risk-free asset and in the risky assets, the value function is equal to

$$V(x,0) = -\exp(-\gamma x)\bar{J}^{0}(0).$$

The Hodges price for the contingent claim ξ is clearly given by the formula

$$\bar{p} = \frac{1}{\gamma}\ln\left(\frac{\bar{J}^{0}(0)}{\bar{J}^{\xi}(0)}\right).$$

Remark 5.3: If we restrict the admissible strategies to the bounded set $\mathcal{A}_{\mathbb{G}}^{k}$, the indifference price $\bar{p}^{k}$ can also be defined by the same method. More precisely,

$$\bar{p}^{k} = \frac{1}{\gamma}\ln\left(\frac{\bar{J}^{0,k}(0)}{\bar{J}^{\xi,k}(0)}\right),$$

where $\bar{J}^{\xi,k}(0)$ is defined in Theorem 5.3.

Note that

$$\bar{p} = \lim_{k\to\infty}\bar{p}^{k}.$$

This allows us to approximate the indifference price by numerical computation, since the value functions $(\bar{J}^{\xi,k}(.))_{k\in\mathbb{N}}$ are the solution of Lipschitz BSDEs and the results of Kharroubi and Lim [17] can be applied.

To conclude this section we assume that there are two kinds of agents in the market: the insider agents and the classical agents. We define the information price d for a contingent claim as the difference between the buying price for a classical agent and the buying price for an insider agent. The buying price, if the agent knows the full information, is defined by (see Lim and Quenez [22])

$$p = \frac{1}{\gamma}\ln\left(\frac{J^{0}(0)}{J^{\xi}(0)}\right),$$

where (J^{ξ},Z,U,K) is the maximal supersolution of the BSDE (5.7) with $(\bar{W},\bar{M},\tilde{\mu},\tilde{\lambda})$ replaced by (W,M,μ,λ).

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Then the benefit of an insider agent who has a full information is the information price

$$d = \bar{p} - p.$$

This price can be computed as the limit of the information prices $(d^k)_{k\in\mathbb{N}}$, where d^k is the information price if we restrict the admissible strategies to the bounded sets $\mathcal{A}^k_{\mathbb{G}}$ and $\mathcal{A}^k_{\mathbb{F}}$

$$d^k = \frac{1}{\gamma}\left(\ln\left(\frac{\bar{J}^{0,k}(0)}{J^{0,k}(0)}\right) - \ln\left(\frac{\bar{J}^{\xi,k}(0)}{J^{\xi,k}(0)}\right)\right),$$

where $J^{\xi,k}$ is the solution of the BSDE (5.8) with $(\bar{W}, \bar{M}, \tilde{\mu}, \tilde{\lambda})$ replaced by (W, M, μ, λ).

Then we have

$$d = \lim_{k\to\infty} d^k.$$

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References

1. Bellamy, N. (2001). Wealth optimization in an incomplete market driven by a jump-diffusion process. *Journal of Mathematical Economics* 35: 259–287.
2. Callegaro, G., Di Masi, G., & Runggaldier, W. (2006). Portfolio optimization in discontinuous markets under incomplete information. *Asia-Pacific Financial Markets* 13: 373–394.
3. Dellacherie, C. & Meyer, P.-A. (1980). *Probabilités et potentiel*. Théorie des martingales, Hermann.
4. Detemple, J. (1986). Asset pricing in a production economy with incomplete information. *Journal of Finance* 41: 383–391.
5. Dothan, M.-U. & Feldman, D. (1986). Equilibrium interest rates and multiperiod bonds in a partially observable economy. *Journal of Finance* 41: 369–382.
6. Frey, R. & Runggaldier, W. (1999). Risk-minimizing hedging strategies under restricted information: the case of stochastic volatility models observable only at discrete random times. *Mathematical Methods of Operations Research* 50: 339–350.
7. Gennotte, G. (1986). Optimal portfolio choice under incomplete information. *Journal of Finance* 41: 733–746.
8. Hu, Y., Imkeller, P. & Muller, M. (2005). Utility maximization in incomplete markets. *The Annals of Applied Probability* 15: 1691–1712.
9. Jeanblanc, M. & Pontier, M. (1990). Optimal portfolio for a small investor in a market with discontinuous prices. *Applied Mathematics and Optimization* 22: 287–310.
10. Jeanblanc, M. & Song, S., (2012). *Martingale representation property in progressively enlarged filtrations*. arXiv:1203.1447.
11. Jeulin, T. (1980). Semimartingales et grossissements d'une filtration. *Lecture Notes in Mathematics 833*. Berlin: Springer.
12. Jeulin, T. & Yor, M. (1985). Grossissement de filtration: exemples et applications. *Lecture Notes in Mathematics 1118*. Berlin: Springer.
13. Jiao, Y. & Pham, H. (2011). Optimal investment with counterparty risk: a default-density modeling approach. *Finance and Stochastics* 15: 725–753.
14. Karatzas, I., Lehoczky, J.-P., & Shreve, S. (1987). Optimal portfolio and consumption decisions for a small investor on a finite horizon. *SIAM Journal on Control and Optimization* 25: 1557–1586.
15. Karatzas, I., Lehoczky, J.-P., Shreve, S., & Xu, G. (1991). Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and Optimization* 29: 702–730.
16. Karatzas, I. & Shreve, S., (1991). *Brownian Motion and Stochastic Calculus*. New York: Springer-Verlag.
17. Kharroubi, I. & Lim, T. (2015). A decomposition approach for the discrete-time approximation of FBSDEs with a jump. *Random Operators and Stochastic Equations* 23: 81–109.
18. Kramkov, D. & Schachermayer, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *The Annals of Applied Probability* 9: 904–950.

https://doi.org/10.1017/S0269964815000194 Published online by Cambridge University Press

19. Lakner, P. (1995). Utility maximization with partial information. *Stochastic Processes and their Applications* 56: 247–273.
20. Lakner, P. (1998). Optimal trading strategy for an investor: the case of partial information. *Stochastic Processes and their Applications* 76: 77–97.
21. Lasry, J.-M. & Lions, P.-L. (1999). Stochastic control under partial information and applications to finance. *Stochastic Processes and their Applications* 56: 247–273.
22. Lim, T. & Quenez, M.-C. (2011). Utility maximization in incomplete market with default. *Electronic Journal of Probability* 16: 1434–1464.
23. Merton, R.-C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* 3: 373–413.
24. Neveu, J. (1975). Discrete-parameter martingales. English translation, Amsterdam: North-Holland; and New York: American Elsevier.
25. Pardoux, E. (1989). Filtrage non linéaire et équations aux dérivées partielles stochastiques associées. Lecture Notes in Mathematics, Vol. 1464. Berlin: Springer-Verlag, 67–163.
26. Pham, H. & Quenez, M.-C. (2001). Optimal portfolio in partially observed stochastic volatility models. *Annals of Applied Probability* 11: 210–238.
27. Protter, P. (1990). *Stochastic Integration and Differential Equations*. Berlin: Springer-Verlag.
28. Roland, S. (2007). *Partially observed jump diffusions*. Ph.D. Thesis, Evry University.

APPENDIX A. PROOF OF PROPOSITIONS 4.2 AND 4.3

The proofs of these two propositions are based on the following lemma:

LEMMA A.1: *The set* $\{J_t^\pi,\ \pi \in \mathcal{A}_{\mathbb{F},t}\}$ *is stable by supremum for any* $t \in [0,T]$, *that is, for any* $\pi^1,\ \pi^2 \in \mathcal{A}_{\mathbb{F},t}$, *there exists* $\pi \in \mathcal{A}_{\mathbb{F},t}$ *such that* $J_t^\pi = J_t^{\pi^1} \vee J_t^{\pi^2}$.

Furthermore, for any $t \in [0,T]$, *there exists a sequence* $(\pi^n)_{n\in\mathbb{N}}$ *of* $\mathcal{A}_{\mathbb{F},t}$ *such that*

$$J(t) = \lim_{n\to\infty} \uparrow J_t^{\pi^n} \quad a.s.$$

PROOF: Let us introduce the set $E = \{J_t^{\pi^1} \geq J_t^{\pi^2}\}$ which belongs to $\mathcal{F}_t$. Let us define the strategy π by the formula $\pi_s = \pi_s^1 \mathbb{1}_E + \pi_s^2 \mathbb{1}_{E^c}$ for any $s \in [t,T]$. It is obvious that $\pi \in \mathcal{A}_{\mathbb{F},t}$. Moreover by construction of π, it is clear that $J_t^\pi = J_t^{\pi^1} \vee J_t^{\pi^2}$.

The second part of the lemma follows by classical results on the essential supremum (see Neveu [24]). ■

We first prove the process $(X^\pi)^\gamma J(.)$ is a supermartingale for any $\pi \in \mathcal{A}_\mathbb{F}$. For that it is sufficient to show for any $0 \leq s \leq t \leq T$ that

$$\mathbb{E}\big[(X_t^{s,\pi})^\gamma J(t)\big|\mathcal{F}_s\big] \leq J(s) \quad a.s.$$

By Lemma A.1, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}$ of $\mathcal{A}_{\mathbb{F},t}$ such that $J(t) = \lim \uparrow J_t^{\pi^n}$ a.s. We define the strategy $\tilde{\pi}^n$ by $\tilde{\pi}_u^n = \pi_u \mathbb{1}_{[s,t]}(u) + \pi_u^n \mathbb{1}_{(t,T]}(u)$, which is clearly admissible. By the monotone convergence theorem and using the definition of $J(.)$, one can easily show that

$$\mathbb{E}\big[(X_t^{s,\pi})^\gamma J(t)\big|\mathcal{F}_s\big] = \lim_{n\to\infty} \uparrow \mathbb{E}\big[U(X_T^{s,\tilde{\pi}^n})\big|\mathcal{F}_s\big] \leq J(s) \quad a.s.$$

Hence, the process $(X^\pi)^\gamma J(.)$ is a supermartingale for any $\pi \in \mathcal{A}_\mathbb{F}$.

Second, we prove $J(.)$ is the smallest process satisfying $(X^\pi)^\gamma J(.)$ is a supermartingale for any $\pi \in \mathcal{A}_\mathbb{F}$ with $J(T) = 1/\gamma$. For that we suppose $\bar{J}$ is an $\mathbb{F}$-adapted process such that $(X^\pi)^\gamma \bar{J}$ is a supermartingale for any $\pi \in \mathcal{A}_\mathbb{F}$ with $\bar{J}_T = 1/\gamma$. Fix $t \in [0,T]$. For any $\pi \in \mathcal{A}_\mathbb{F}$, we have $\mathbb{E}[U(X_T^\pi)|\mathcal{F}_t] \leq (X_t^\pi)^\gamma \bar{J}_t$ a.s. This inequality is equivalent to $\mathbb{E}[U(X_T^{t,\pi})|\mathcal{F}_t] \leq \bar{J}_t$, which implies

$$\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{\mathbb{F},t}} \mathbb{E}\big[U(X_T^{t,\pi})\big|\mathcal{F}_t\big] \leq \bar{J}_t \quad a.s.,$$

which clearly gives that $J_t \leq \bar{J}_t$ a.s.

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At last, we prove the optimality criterion, that is Proposition 4.3. Suppose the strategy $\hat{\pi}$ is an optimal strategy; hence we have

$$J(0) = \sup_{\pi \in \mathcal{A}_{\mathbb{F}}} \mathbb{E}\big[U\left(X_T^{\pi}\right)\big] = \mathbb{E}\big[U\left(X_T^{\hat{\pi}}\right)\big].$$

Since the process $(X^{\hat{\pi}})^{\gamma} J(.)$ is a supermartingale by Proposition 4.2 and $J(0) = \mathbb{E}[U(X_T^{\hat{\pi}})]$, the process $(X^{\hat{\pi}})^{\gamma} J(.)$ is a martingale.
To show the converse, suppose the process $(X^{\hat{\pi}})^{\gamma} J(.)$ is a martingale, then $\mathbb{E}[U(X_T^{\hat{\pi}})] = J(0)$. Moreover $\mathbb{E}[(X_t^{\pi})^{\gamma} J(t)] \leq J(0)$ for any $\pi \in \mathcal{A}_{\mathbb{F}}$ by Proposition 4.2. Which implies that

$$J(0) = \sup_{\pi \in \mathcal{A}_{\mathbb{F}}} \mathbb{E}\big[U\big(X_T^{\pi}\big)\big] = \mathbb{E}\big[U\big(X_T^{\hat{\pi}}\big)\big].$$

APPENDIX B. PROOF OF THEOREM 4.1

The proof of this theorem is based on Propositions 4.2 and 4.3, on Doob–Meyer's decomposition and on the martingale representation property.

Since the process J is a supermartingale, it can be written under the following form by using Doob–Meyer's decomposition (see Dellacherie and Meyer [3]) and the martingale representation property

$$dJ_t = Z_t dW_t + U_t dM_t - dA_t,$$

with $Z \in L^2_{\text{loc}}(W)$, $U \in L^1_{\text{loc}}(M)$ and A is a non-decreasing $\mathbb{F}$-adapted process with $A_0 = 0$.
From product rule, the derivative of process $(X^{\pi})^{\gamma} J$ can be written under the form

$$d((X_t^{\pi})^{\gamma} J_t) = (X_{t-}^{\pi})^{\gamma} \left(dA_t^{\pi} + dM_t^{\pi}\right),$$

with $A_0^{\pi} = 0$ and

$$\begin{cases} dA_t^{\pi} = \Big[\gamma \pi_t(\mu_t J_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2} \pi_t^2 \sigma_t^2 J_t + \lambda_t((1+\pi_t \beta_t)^{\gamma} - 1)(J_t + U_t)\Big] dt - dA_t, \\ dM_t^{\pi} = (\gamma \pi_t \sigma_t J_t + Z_t) dW_t + (U_t + ((1+\pi_t \beta_t)^{\gamma} - 1)(J_t + U_t)) dM_t. \end{cases} \tag{B.1}$$

From Proposition 4.2, we have $dA_t^{\pi} \leq 0$ for any $\pi \in \mathcal{A}_{\mathbb{F}}$, which implies

$$dA_t \geq \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{\mathbb{F}}} \Big\{\gamma \pi_t(\mu_t J_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2} \pi_t^2 \sigma_t^2 J_t + \lambda_t((1+\pi_t \beta_t)^{\gamma} - 1)(J_t + U_t)\Big\} dt.$$

From Kramkov and Schachermayer [18], there exists an optimal strategy $\hat{\pi} \in \mathcal{A}_{\mathbb{F}}$ to the optimization problem and from Proposition 4.3, we get

$$dA_t = \Big[\gamma \hat{\pi}_t(\mu_t J_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2} \hat{\pi}_t^2 \sigma_t^2 J_t + \lambda_t((1+\hat{\pi}_t \beta_t)^{\gamma} - 1)(J_t + U_t)\Big] dt.$$

Which imply that

$$dA_t = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{\mathbb{F}}} \Big\{\gamma \pi_t(\mu_t J_t + \sigma_t Z_t) + \frac{\gamma(\gamma-1)}{2} \pi_t^2 \sigma_t^2 J_t + \lambda_t((1+\pi_t \beta_t)^{\gamma} - 1)(J_t + U_t)\Big\} dt. \tag{B.2}$$

Therefore, the process (J, Z, U) is a solution of the BSDE (4.5).

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We now prove it is the minimal solution. Let $(\bar{J}, \bar{Z}, \bar{U})$ be a solution of the BSDE (4.5). Let us prove $(X^\pi)^\gamma \bar{J}$ is a supermartingale for any $\pi \in \mathcal{A}_\mathbb{F}$. From the product rule, we can write the derivative of this process under the form

$$d\left((X_t^\pi)^\gamma \bar{J}_t\right) = (X_{t^-}^\pi)^\gamma \left[d\bar{M}_t^\pi + d\bar{A}_t^\pi - d\bar{A}_t\right], \tag{B.3}$$

where $\bar{A}$ and $\bar{M}^\pi$ are given by (B.1) and (B.2) with (J, Z, U) replaced by $(\bar{J}, \bar{Z}, \bar{U})$, and $\bar{A}_0^\pi = 0$ and

$$d\bar{A}_t^\pi = \left[\gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) + \frac{\gamma(\gamma-1)}{2}\pi_t^2\sigma_t^2\bar{J}_t + \lambda_t((1+\pi_t\beta_t)^\gamma - 1)(\bar{J}_t + \bar{U}_t)\right]dt.$$

By integrating (B.3), we get

$$(X_t^\pi)^\gamma \bar{J}_t - \bar{J}_0 = \int_0^t (X_{s^-}^\pi)^\gamma d\bar{M}_s^\pi - \int_0^t (X_s^\pi)^\gamma (d\bar{A}_s - d\bar{A}_s^\pi).$$

Since $d\bar{A}_s \geq d\bar{A}_s^\pi$, we have $\int_0^t (X_{s^-}^\pi)^\gamma d\bar{M}_s^\pi \geq (X_t^\pi)^\gamma \bar{J}_t - \bar{J}_0 \geq -\bar{J}_0$. It implies $\bar{M}^\pi$ is a supermartingale, since it is a lower bounded local martingale. Hence, the process $(X^\pi)^\gamma \bar{J}$ is a supermartingale for any $\pi \in \mathcal{A}_\mathbb{F}$, because it is the sum of a supermartingale and a non-increasing process. Proposition 4.2 implies $J_t \leq \bar{J}_t,\ \forall t \in [0,T]$ a.s., which ends this proof.

APPENDIX C. PROOF OF THEOREM 4.2

We first remark that J^k satisfies the following property:

LEMMA C.1: *The process J^k is the smallest $\mathbb{F}$-adapted process such that $(X^\pi)^\gamma J^k$ is a supermartingale for any $\pi \in \mathcal{A}_\mathbb{F}^k$ with $J_T^k = 1/\gamma$.*

To prove this lemma, we use exactly the same arguments as in the proof of Proposition 4.2, since Lemma A.1 is still true with $\mathcal{A}_{\mathbb{F},t}^k$ instead of $\mathcal{A}_{\mathbb{F},t}$.

Fix $t \in [0,T]$. It is obvious with the definition of sets $\mathcal{A}_{\mathbb{F},t}$ and $\mathcal{A}_{\mathbb{F},t}^k$ that $\mathcal{A}_{\mathbb{F},t}^k \subset \mathcal{A}_{\mathbb{F},t}$ for each $k \in \mathbb{N}$, which implies

$$J_t^k \leq J_t \quad a.s. \tag{C.1}$$

Moreover, since $\mathcal{A}_{\mathbb{F},t}^k \subset \mathcal{A}_{\mathbb{F},t}^{k+1}$ for each $k \in \mathbb{N}$, it follows that the positive sequence $(J^k)_{k\in\mathbb{N}}$ is non-decreasing. Let us define the random variable

$$\tilde{J}(t) = \lim_{k\to\infty} \uparrow J_t^k \quad a.s.$$

It is obvious that $\tilde{J}(t) \leq J_t$ a.s. from (C.1) and this holds for any $t \in [0,T]$. It remains to prove $J_t \leq \tilde{J}(t)$ a.s. for any $t \in [0,T]$. As in the proof of Theorem 4.2 of the companion paper [22], we first prove the process $\tilde{J}(.^+)$ is càd-làg and satisfies $\tilde{J}(.^+) \leq \tilde{J}(.)$ a.s. The process $(X^\pi)^\gamma \tilde{J}(.^+)$ is a supermartingale for any bounded strategy $\pi \in \mathcal{A}_\mathbb{F}$. In the sequel, we shall denote $\bar{J}_t$ instead of $\tilde{J}(t^+)$. We now prove $\bar{J}_t \geq J_t$ a.s. for any $t \in [0,T]$. Since $\bar{J}$ is a càd-làg supermartingale, it admits the following Doob–Meyer's decomposition

$$d\bar{J}_t = \bar{Z}_t dW_t + \bar{U}_t dM_t - d\bar{A}_t,$$

with $\bar{Z} \in L^2_{\text{loc}}(W)$, $\bar{U} \in L^1_{\text{loc}}(M)$ and $\bar{A}$ is a non-decreasing $\mathbb{F}$-adapted process with $\bar{A}_0 = 0$. As before, we use the fact that the process $(X^\pi)^\gamma \bar{J}$ is a supermartingale for any bounded strategy $\pi \in \mathcal{A}_\mathbb{F}$ to give some conditions satisfied by the process $\bar{A}$. Let $\pi \in \mathcal{A}_\mathbb{F}$ be a uniformly bounded strategy, the product rule gives

$$d((X_t^\pi)^\gamma \bar{J}_t) = (X_{t^-}^\pi)^\gamma \left(d\bar{A}_t^\pi + d\bar{M}_t^\pi\right),$$

where $\bar{A}^\pi$ and $\bar{M}^\pi$ are given by (B.1) with (J, Z, U, A) replaced by $(\bar{J}, \bar{Z}, \bar{U}, \bar{A})$.

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Let $\bar{\mathcal{A}}_{\mathbb{F},t}$ be the subset of uniformly bounded strategies of $\mathcal{A}_{\mathbb{F},t}$. Since the process $(X^\pi)^\gamma \bar{J}$ is a supermartingale for any $\pi \in \bar{\mathcal{A}}_{\mathbb{F}}$, we have

$$d\bar{A}_t \geq \operatorname*{ess\,sup}_{\pi \in \bar{\mathcal{A}}_{\mathbb{F}}} \left\{ \gamma \pi_t (\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) + \frac{\gamma(\gamma-1)}{2} \pi_t^2 \sigma_t^2 \bar{J}_t + \lambda_t ((1+\pi_t \beta_t)^\gamma - 1)(\bar{J}_t + \bar{U}_t) \right\} dt. \quad \textbf{(C.2)}$$

It is not possible to give an exact expression of $\bar{A}_t$ as in the previous proof, because we do not know if $\hat{\pi} \in \bar{\mathcal{A}}_{\mathbb{F}}$. But this inequality is sufficient for the demonstration. Now, the following equality holds $dt \otimes d\mathbb{P}$ *a.s.*

$$\begin{aligned} &\operatorname*{ess\,sup}_{\pi \in \bar{\mathcal{A}}_{\mathbb{F}}} \left\{ \gamma \pi_t (\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) + \frac{\gamma(\gamma-1)}{2} \pi_t^2 \sigma_t^2 \bar{J}_t + \lambda_t ((1+\pi_t \beta_t)^\gamma - 1)(\bar{J}_t + \bar{U}_t) \right\} \\ &\quad = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{\mathbb{F}}} \left\{ \gamma \pi_t (\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) + \frac{\gamma(\gamma-1)}{2} \pi_t^2 \sigma_t^2 \bar{J}_t + \lambda_t ((1+\pi_t \beta_t)^\gamma - 1)(\bar{J}_t + \bar{U}_t) \right\}. \quad \textbf{(C.3)} \end{aligned}$$

We now want to show $(X^\pi)^\gamma \bar{J}$ is a supermartingale for any $\pi \in \mathcal{A}_{\mathbb{F}}$. Fix $\pi \in \mathcal{A}_{\mathbb{F}}$ (not necessarily uniformly bounded), we get

$$(X_t^\pi)^\gamma \bar{J}_t - \bar{J}_0 = \int_0^t (X_{s^-}^\pi)^\gamma d\bar{M}_s^\pi + \int_0^t (X_s^\pi)^\gamma d\bar{A}_s^\pi,$$

with $\bar{A}^\pi$ and $\bar{M}^\pi$ given by (B.1) with (J, Z, U, A) replaced by $(\bar{J}, \bar{Z}, \bar{U}, \bar{A})$.

Inequality (C.2) and equality (C.3) imply that $d\bar{A}_t^\pi \leq 0$ a.s. Therefore, we have

$$\int_0^t (X_{s^-}^\pi)^\gamma d\bar{M}_s^\pi \geq (X_t^\pi)^\gamma \bar{J}_t - \bar{J}_0 \geq -\bar{J}_0.$$

Thus, $\bar{M}^\pi$ is a supermartingale, since it is a lower bounded local martingale. As $\bar{M}^\pi$ is a supermartingale and $\bar{A}^\pi$ is non-increasing, the process $(X^\pi)^\gamma \bar{J}$ is a supermartingale and this holds for any $\pi \in \mathcal{A}_{\mathbb{F}}$. Since J is the smallest process (see Proposition 4.2) satisfying these properties, we have $J_t \leq \bar{J}_t$ a.s. Which ends the proof.

APPENDIX D. PROOF OF LEMMA 5.2

First, recall Bayes formula: for any $t \in [0,T]$ and $X \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})$, one has

$$\mathbb{E}\big[X \big| \mathcal{G}_t\big] = \frac{\mathbb{E}_{\mathbb{Q}}\big[\Lambda_t X \big| \mathcal{G}_t\big]}{\tilde{\Lambda}_t}. \quad \textbf{(D.1)}$$

Let ξ be the optional projection of the $\mathbb{P}$-martingale L to $\mathbb{G}$, so

$$\xi_t = \mathbb{E}\big[L_t \big| \mathcal{G}_t\big].$$

By applying relation (D.1) to $X = L_t$, we immediately obtain $\xi_t = 1/\tilde{\Lambda}_t$ and thus

$$\xi_t = \exp\left(- \int_0^t \tilde{\theta}_s d\bar{W}_s - \frac{1}{2} \int_0^t |\tilde{\theta}_s|^2 ds \right).$$

Let m be a $(\mathbb{P}, \mathbb{G})$-local martingale. From the Bayes rule, the process $\tilde{m}$ defined by

$$\tilde{m}_t = m_t \xi_t^{-1}, \quad t \in [0,T],$$

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is a $(\mathbb{Q}, \mathbb{G})$-local martingale. From Remark 5.4 and Assumption 4.1, there exists a couple of processes $(\tilde{a}, \tilde{b})$ with $\tilde{a} \in L^2_{\text{loc}}(\tilde{W})$ and $\tilde{b} \in L^1_{\text{loc}}(\bar{M})$ such that

$$\tilde{m}_t = \int_0^t \tilde{a}'_s d\tilde{W}_s + \int_0^t \tilde{b}'_s d\bar{M}_s, \quad t \in [0, T].$$

By Itô's formula applied to $m_t = \tilde{m}_t \xi_t$, definition of $\bar{W}$ and $\bar{M}$ (see (5.3)), we obtain that

$$m_t = \int_0^t a'_s d\bar{W}_s + \int_0^t b'_s d\bar{M}_s,$$

with $a_t = \xi_t \tilde{a}_t - \tilde{m}_t \xi_t \tilde{\rho}_t$ and $b_t = \xi_{t^-} \tilde{b}_t$.

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