BIPARTITE PARTITIONS

By V. S. NANDA

Received 23 August 1954

Asymptotic theory of partitions of unipartite numbers has been extensively investigated after the classic work of Hardy and Ramanujan on the subject. The more complicated problem of bipartite partitions has been only recently broached. Auluck(1) has deduced asymptotic formulae for $p(m, n)$, the number of partitions of the bipartite number (m, n), both for m fixed and $n \to \infty$, and for both m and n tending to infinity in such a way that $m/n = O(1)$ and $n/m = O(1)$. It will be shown here that the formula obtained for the first case is actually valid for $m = o(n^{\frac{1}{2}})$. This proves the conjecture made earlier by the author(3).

The generating function for $p(m, n)$ is

$$\sum_m \sum_n p(m,n)\, x^n y^m = \frac{1}{(1-x)(1-y)(1-x^2)(1-xy)(1-y^2)\dots}. \tag{1}$$

By separation of variables this can be written in the form

$$\sum_n \sum_m p(m,n)\, x^n y^m = \frac{1}{\prod_r (1-x^r)} \{1 + A_1 y + A_2 y^2 + \dots\}, \tag{2a}$$

where

$$A_m = \Sigma a_{p_1} a_{p_2} \dots a_{p_s},$$

the sum being over all solutions of $\Sigma r p_r = m$, and

$$a_r = \frac{1}{(1-x)(1-x^2)\dots(1-x^r)}.$$

From $(2a)$ it follows that

$$\sum_n p(m,n)\, x^n = \frac{A_m}{(1-x)(1-x^2)\dots(1-x^r)\dots}. \tag{2b}$$

It can be easily seen that there is one-to-one correspondence between the terms in A_m and the partitions of m, whose number will be denoted by $p(m)$ in the sequel. We divide the partitions of m into classes such that those having the same number of parts are placed in the same class. Correspondingly, the terms of A_m having the same number of factors are collected in the same group. The $(r+1)$th group has $P_{m-r}(m)$ terms with $(m-r)$ factors each, where $P_k(m)$ stands for the number of partitions of m into exactly k parts. In the first group there is only one term

$$\frac{1}{(1-x)(1-x^2)\dots(1-x^m)},$$

and it has the highest number of factors. The successive groups contain terms in which the number of factors decreases by one as we pass to the next group. On the basis of the above grouping, it will be convenient to regard the corresponding terms in $(2b)$ as sub-generating functions (S.G.F.'s) grouped accordingly.

https://doi.org/10.1017/S0305004100032291 Published online by Cambridge University Press

To find the coefficient of x^n in (2b), we make use of the Hardy–Ramanujan formula (2)

$$p(n) = \left(\frac{1}{4\sqrt{3}N} - \frac{1}{4\pi\sqrt{2}N^{\frac{3}{2}}}\right)\exp\{\pi\sqrt{(\tfrac{2}{3}N)}\} + O(\exp(Hn^{\frac{1}{2}})), \tag{3}$$

where $H < \pi\sqrt{(\frac{2}{3})}$ and $N = n - \frac{1}{24}$. Consider first the generating function

$$\sum_n \omega^{(1)}(n)\,x^n = \frac{1}{(1-x)}\prod_{r=1}(1-x^r)^{-1}. \tag{4}$$

Since
$$\sum_n p(n)x^n = \prod_{r=1}^{\infty}(1-x^r)^{-1},$$

we at once obtain the relation

$$\omega^{(1)}(n) = \sum_{r=0}^{n} p(n-r).$$

Making use of equation (3), we have

$$\omega^{(1)}(n) = \sum_{r=0}^{n}\left[\left\{\frac{1}{4\sqrt{3}(N-r)} - \frac{1}{4\pi\sqrt{2}(N-r)^{\frac{3}{2}}}\right\}\exp\{\pi\sqrt{(\tfrac{2}{3})}(N-r)^{\frac{1}{2}}\}\right] + O(\exp Hn^{\frac{1}{2}}).$$

We now break the summation into two parts Σ_1 and Σ_2, where Σ_1 consists of the terms with $r \leqslant n^{\frac{3}{5}}$ and Σ_2 of the others. On carrying out the summation, we have

$$\Sigma_1 = \phi(n)\left[\frac{\exp\{\pi/\sqrt{(6N)}\}}{\exp\{\pi/\sqrt{(6N)}\}-1} + O(n^{-\frac{1}{2}})\right]$$

and
$$\Sigma_2 = O(\exp\{\pi\sqrt{(\tfrac{2}{3}n)} - \pi n^{\frac{1}{10}}/\sqrt{6}\}),$$

where
$$\phi(n) = \frac{1}{4\sqrt{3}N}\exp\{\pi\sqrt{(\tfrac{2}{3}N)}\}.$$

This shows that

$$\omega^{(1)}(n) = \phi(n)\frac{\sqrt{(6N)}}{\pi}\left\{1 + \left(c_1 - \frac{3}{\pi^2}\right)\frac{\pi}{\sqrt{(6N)}} + O(n^{-1})\right\}, \tag{5}$$

where $c_r = \frac{1}{2} - \frac{3}{\pi^2}\left(1 - \frac{2}{r}\right)$, r being a positive integer. Next, consider the generating function

$$\sum_n \omega^{(2)}(n)\,x^n = \frac{1}{(1-x)(1-x^2)}\prod_r (1-x^r)^{-1}.$$

Using equation (4), we have the recurrence relation

$$\omega^{(2)}(n) = \sum_r \omega^{(1)}(n-2r).$$

Substituting the expression for $\omega^{(1)}(n-2r)$ from (5) and carrying out the summation as before, we have

$$\omega^{(2)}(n) = \phi(n)\left\{\frac{\sqrt{(6N)}}{\pi}\right\}^2\frac{1}{2!}\left\{1 + \left(c_1 + 2c_2 - \frac{3}{\pi^2}\right)\frac{\pi}{\sqrt{(6N)}} + O(n^{-1})\right\}. \tag{6}$$

Consider now the S.G.F.

$$\sum_n \omega^{(m)}(n)\,x^n = \frac{\prod_{r=1}^{\infty}(1-x^r)^{-1}}{(1-x)(1-x^2)\ldots(1-x^m)},$$

https://doi.org/10.1017/S0305004100032291 Published online by Cambridge University Press

and assume for $\omega^{(m)}(n)$ the form

$$\omega^{(m)}(n) = \phi(n)\left\{\frac{\sqrt{(6N)}}{\pi}\right\}^m \frac{1}{m!}\left\{1 + \frac{\pi}{\sqrt{(6N)}}\left(\sum_{r=1}^{m} rc_r - \frac{3}{\pi^2}\right) + \frac{1}{N}\sum_{r=1}^{m}\beta_r\right\}, \tag{7}$$

where $\beta_r = O(r^2)$. Substituting the expression for $\omega^{(m)}(n)$ in the recurrence relation

$$\omega^{(m+1)}(n) = \sum_r \omega^{(m)}(n - \overline{m+1}\,.\,r),$$

an expression for $\omega^{(m+1)}(n)$ can be obtained. As before we break the summation into two parts Σ_1 and Σ_2, where Σ_1 consists of the terms with $r \leqslant n^{\frac{3}{5}}/(m+1)$ and Σ_2 of the others. To evaluate Σ_1 we have to carry out three summations. The first sum is

$$\sum_r \phi(n - \overline{m+1}\,.\,r)\left\{\frac{6(N - \overline{m+1}\,.\,r)}{\pi^2}\right\}^{\frac{1}{2}m}$$

$$= \left(\frac{6N}{\pi^2}\right)^{\frac{1}{2}m}\phi(n)\left[\left\{\sum_r e^{-(m+1)r\pi/\sqrt{(6N)}}\left(1 - \frac{(1+m)^2 - 3(m+1)}{2N}r - \frac{\pi}{\sqrt{(6N)}}\frac{r^2(m+1)^2}{4N}\right)\right\} + O(mn^{-\frac{1}{2}})\right].$$

The summation over r within the curly brackets might be extended to ∞, the error involved being $\left[O\exp\left(-\frac{\pi n^{\frac{1}{10}}}{\sqrt{6}}\right)\right]$. Noting that

$$\frac{1}{1 - \exp(-m\theta)} = \frac{1}{m\theta} + \tfrac{1}{2} + O(m\theta),$$

and carrying out the summations, we have

$$\sum_r \phi(n - \overline{m+1}\,.\,r)\left\{\frac{6(N - \overline{m+1}\,.\,r)}{\pi^2}\right\}^{\frac{1}{2}m}$$

$$= \frac{1}{m+1}\left(\frac{6N}{\pi^2}\right)^{\frac{1}{2}(m+1)}\phi(n)\left[1 + (m+1)\,c_{m+1}\left(\frac{\pi}{\sqrt{(6N)}}\right) + O(m^2n^{-1})\right].$$

Similarly, the second sum is

$$\sum_r \phi(n - \overline{m+1}\,.\,r)\left\{\frac{6(N - \overline{m+1}\,.\,r)}{\pi^2}\right\}^{\frac{1}{2}(m-1)} = \frac{1}{m+1}\left(\frac{6N}{\pi^2}\right)^{\frac{1}{2}m}\phi(n)[1 + O(mn^{-\frac{1}{2}})],$$

while the third sum is

$$\sum_r \phi(n - \overline{m+1}\,.\,r)\left\{\frac{6(N - \overline{m+1}\,.\,r)}{\pi^2}\right\}^{\frac{1}{2}(m-2)} = \frac{1}{m+1}\left(\frac{6N}{\pi^2}\right)^{\frac{1}{2}(m-1)}\phi(n)[1 + O(mn^{-\frac{1}{2}})].$$

Finally, we have therefore

$$\Sigma_1 = \phi(n)\left(\frac{6N}{\pi^2}\right)^{\frac{1}{2}(m+1)}\frac{1}{(m+1)!}\left\{1 + \left(\sum_{r=1}^{m+1} rc_r - \frac{3}{\pi^2}\right)\frac{\pi}{\sqrt{(6N)}} + \frac{1}{n}\sum_{r=1}^{m+1}\beta_r\right\}.$$

https://doi.org/10.1017/S0305004100032291 Published online by Cambridge University Press

Noting that $\left(1-\frac{\overline{m+1}.r}{N}\right)^{\frac{1}{2}} < 1-\frac{\overline{m+1}.r}{2N}$, we find

$$\Sigma_2 = \left(\frac{6N}{\pi^2}\right)^{\frac{1}{2}m}\frac{1}{m!}\sum_{r>n^{\frac{1}{2}}/(m+1)}\left\{1-(m+1)\frac{r}{N}\right\}^{\frac{1}{2}(m-2)}\frac{1}{4\sqrt{3}}\exp\left\{\pi\sqrt{(\tfrac{2}{3}N)}\left(1-\frac{\overline{m+1}.r}{N}\right)^{\frac{1}{2}}\right\}$$

$$< \left(\frac{6N}{\pi^2}\right)^{\frac{1}{2}m}\frac{1}{m!}\phi(n)\sum e^{-(\overline{m+1}.r\pi/\sqrt{(6N)})}$$

$$= \left(\frac{6N}{\pi^2}\right)^{\frac{1}{2}(m+1)}\frac{1}{(m+1)!}\phi(n)\exp\left(-\frac{\pi n^{\frac{1}{10}}}{\sqrt{6}}\right)\{1+O(mn^{-\frac{1}{2}})\}.$$

From the evaluations of Σ_1 and Σ_2, we get

$$\omega^{(m+1)}(n) = \left(\frac{6N}{\pi^2}\right)^{\frac{1}{2}(m+1)}\frac{1}{(m+1)!}\phi(n)\left\{1+\frac{\pi}{\sqrt{(6N)}}\left(\sum_{r=1}^{m+1}rc_r-\frac{3}{\pi^2}\right)+\frac{1}{n}\sum_{r=1}^{m+1}\beta_r\right\}. \quad (8a)$$

This is precisely the same result as we would get by replacing m by $m+1$ in equation (7). From equations (5) and (6) it is clear that the result is true for $m=1$ and $m=2$. Hence, by the method of induction, it holds for all m for which the terms of order (m^3n^{-1}) are negligible. In particular, for $m = o(n^{\frac{1}{4}})$ we can write

$$\omega^{(m)}(n) \sim \left(\frac{6N}{\pi^2}\right)^{\frac{1}{2}m}\frac{1}{m!}\phi(n). \quad (8b)$$

It will be noticed that $\omega^{(m)}(n)$ is the contribution to the coefficient of x^n from the only S.G.F. in the first group from among the m groups in equation $(2b)$. The contribution from the following groups will now be examined.

The $(r+1)$th group of S.G.F.'s has the form

$$\frac{\prod_{r=1}^{\infty}(1-x^r)^{-1}}{(1-x)^{\alpha_1+1}(1-x^2)^{\alpha_2+1}\dots(1-x^{m-r-t})^{\alpha_{m-r-t}+1}}, \quad (9)$$

where α's satisfy the condition $\sum_{p=1}^{m-r-t}\alpha_p = t$. Using formula (3) for evaluating the coefficient of x^n according to the procedure followed previously, we have the expression

$$R_n = \left(\frac{6N}{\pi^2}\right)^{\frac{1}{2}(m-r)}\frac{\phi(n)}{(m-r-t)!\prod_{k=1}^{m-r-t}(k)^{\alpha_k}}$$

$$\times\left\{1+\frac{\pi}{\sqrt{(6N)}}\left(\sum_{p=1}^{m-r-t}p(\alpha_p+1)c_p-\frac{3}{\pi^2}\right)+\frac{1}{n}\sum_p(\alpha_p+1)\beta_p\right\}. \quad (10)$$

It is clear that R_n is larger for higher values of t and that for given t it is a maximum for $\alpha_2=\alpha_3=\dots=0$. For each group of terms in A_m there is a maximum value of t which we shall denote by $t_{\max.}$. To determine for our purpose a suitable value of $t \geqslant t_{\max.}$, consider the partitions of m into exactly k parts in which unity does not occur more than d times. The generating function for $P_k(m,d)$, the number of such partitions, is

$$\sum_m\sum_k P_k(m,d)x^mz^k = \frac{\{1-(zx)^{d+1}\}}{(1-zx)(1-zx^2)\dots}, \quad (11)$$

https://doi.org/10.1017/S0305004100032291 Published online by Cambridge University Press

whence we get the recurrence relation

$$p_k(m,d) = P_k(m) - P_{k-d-1}(m-d-1).$$

This shows that $P_k(m,d) = 0$ for $d < (m-2k)$ and $k > \frac{1}{2}m$. Hence, unity occurs at least $(m-2r)$ times in every partition of m into exactly $(m-r)$ parts for $r < \frac{1}{2}m$. Thus for this case the factor $(1-x^{m-2r})$ occurs in each term of $(r+1)$th group in A_m. Therefore, $t_{\max.} \leqslant r$ for $r < \frac{1}{2}m$, while for $r \geqslant \frac{1}{2}m$ we have obviously $t_{\max.} \leqslant m-(r+1)$ as $(1-x)$ is a factor of every term in A_m. Since in $(2b)$ there are $P_{m-r}(m)$ S.G.F.'s in the $(r+1)$th group, it follows that their contribution to the coefficient of x^n is less than

$$S_n = (cn)^{\frac{1}{2}(m-r)} \frac{\phi(n)}{f_{m-2r}} p(r), \tag{12}$$

where $c > 6/\pi^2$ and f_{m-2r} is equal to $(m-2r)!$ for $r < \frac{1}{2}m$ and is unity for $r \geqslant \frac{1}{2}m$.

The contribution from the single S.G.F. in the first group is given by $\omega^{(m)}(n)$, which has been evaluated in (8). The condition under which the combined contribution of the remaining groups can be neglected is easily seen to be

$$\sum_{r=1}^{m-1} \frac{p(r)}{f_{m-2r}(n)^{r/2}} = o(1). \tag{13}$$

By considering the order of various terms on the left, we find that relation (13) is true for $m = o(n^{\frac{1}{4}})$. Thus we can write, for $m = o(n^{\frac{1}{4}})$,

$$p(m,n) \sim \left(\frac{6n}{\pi^2}\right)^{\frac{1}{2}m} \left(\frac{1}{m!}\right) \frac{1}{4\sqrt{3}\,n} \exp\{\pi\sqrt{(\tfrac{2}{3}n)}\}. \tag{14}$$

Some physical applications of bipartite partitions will be discussed in a subsequent paper.

My thanks are due to Prof. D. S. Kothari for his interest and to Prof. F. C. Auluck for his suggestions and criticisms. I am also grateful to Prof. H. N. V. Temperley for going through the first draft and suggesting some improvements in the presentation of the proof.

REFERENCES

(1) Auluck, F. C. *Proc. Camb. Phil. Soc.* 49 (1953), 72.
(2) Hardy, G. H. *Ramanujan* (Cambridge, 1940).
(3) Nanda, V. S. Statistical mechanics and partition theory of numbers (Dissertation, Delhi, 1951).

Delhi University
Delhi, India

https://doi.org/10.1017/S0305004100032291 Published online by Cambridge University Press