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Realizability in ordered combinatory algebras with adjunction

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In this work, we continue our consideration of the constructions presented in the paper *Krivine's Classical Realizability from a Categorical Perspective* by Thomas Streicher. Therein, the author points towards the interpretation of the classical realizability of Krivine as an instance of the categorical approach started by Hyland. The present paper continues with the study of the basic algebraic set-up underlying the categorical aspects of the theory. Motivated by the search of a full adjunction, we introduce a new closure operator on the subsets of the stacks of an abstract Krivine structure that yields an adjunction between the corresponding application and implication operations. We show that all the constructions from ordered combinatory algebras to triposes presented in our previous work can be implemented, *mutatis mutandis*, in the new situation and that all the associated triposes are equivalent. We finish by proving that the whole theory can be developed using the ordered combinatory algebras with full adjunction or strong abstract Krivine structures as the basic set-up.

1. Introduction

1. Krivine's classical realizability was introduced in the mid 90's as a total reformulation of Kleene's intuitionistic realizability. Instead of translating classical logic into intuitionistic logic, Krivine uses control instructions as an interpretation for classical reasoning (see Krivine 2001, 2003, 2008, 2009). The idea of relating classical reasoning to control instructions can be traced back to the seminal paper of Griffin (1990): *A Formulæ-as-Types Notion of Control.*

   In a series of papers from 2011 to 2015 (see Krivine 2011, 2012, 2016), Krivine generalizes his own notion of classical realizability models to a variant of the well-known notion of combinatory logic (see Curry and Feys 1958), defining the so-called

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'Realizability Algebras'. For instance, in these papers Krivine shows how to transform into programs the proofs of the existence of a non-trivial ultrafilter on $\mathbb{N}$, the existence of a well order on $\mathbb{R}$ (Krivine 2011) and the consistency relatively to Zermelo Frankel Set Theory (ZF) of the existence of an infinite sequence of subsets of $\mathbb{R}$ whose cardinals are strictly decreasing (Krivine 2012). As far as we know, a proof of the last result has not been produced yet by means of forcing. Krivine's classical realizability was initially developed independently of the well-established tradition of categorical models, as evolved from the works of Hyland (1982), Hyland *et al.* (1980), Hofstra and van Oosten (2004), and van Oosten (2008).

In the paper by Streicher (2013): 'Krivine's Classical Realizability from a Categorical Perspective', the author shows that it is possible to reconcile both perspectives. With that purpose in mind, Streicher axiomatizes the reduction relation and simplifies the combinators of the realizability algebras, defining the concept of 'Abstract Krivine Structure' (AKS). Furthermore, the author builds for each AKS an ordered combinatory algebra (OCA, see, Hofstra and van Oosten 2004), whose elements belong to $\mathcal{P}(\Pi)$, the powerset of the set of the stacks of the AKS. Those OCAs have application and implication operations and their elements are used directly as truth values. One of the technical difficulties Streicher has to deal with, is that the OCA he obtains from an AKS does not have the adjunction property. To deal with that peculiarity, he finds an η-expansor in the OCA whose definition depends upon the original AKS.

Even if the work of Streicher mentions the OCAs as a tool to build a Heyting preorder (HPO) and from it a tripose, the proof he presents, uses the fact that the OCA is constructed from an AKS.

In the paper, 'Ordered Combinatory Algebras and Realizability' the authors together with J. Frey and A. Miquel, prove that Streicher's construction can be put in an axiomatic platform as it can be performed directly from IOCAs that are a special kind of OCAs.

Therein, the KOCAs (that are IOCAs with a combinator that realizes Peirce's law) appear as the common generalization of classical realizability and Cohen's forcing. Moreover, since the elements of the KOCAs are also combinatory objects, they can be used as realizers, thus defining the internal classical realizability associated to a KOCA. Therefore, we guarantee that KOCAs can be used as the basic building bricks to understand classical realizability from a categorical perspective.

In the current paper, we attempt to improve and simplify some aspects of the constructions introduced in Ferrer Santos *et al.* (2015). In order to find a mathematical structure more suitable for modelling implication and substitution, we introduce the notion of FOCA, i.e., an OCA with a full adjunction. We also show that FOCAs are closely related to A. Miquel's implicative algebras as in Miquel (2014) and Miquel (2016).

At the end of this work, we revisit in this new context, the constructions between AKSs, OCAs and indexed Heyting preorders (IHPO) performed in Ferrer Santos *et al.*

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(2015). In the situation of the current manuscript, we replace the closure operator given double orthogonality, by its maximal Alexandroff approximation. This operator is compatible with infinite unions, and hence we can use it to produce FOCAs based now in the set of its closed subsets.

All our observations show that the family of triposes obtained from the stronger structure of an FOCA is the same as the one obtained from IOCAs. Thus, the choice of FOCAs as a set-up of the theory, can be made without any loss of generality.

We also observe that the construction of an OCA from an AKS can be generalized to the construction of a family of OCAs parametarized by a family of closure operators on $\mathcal{P}(\Pi)$. We prove that the closure operators of this family that satisfy the Alexandroff condition leads to FOCAs. In particular, being the identity an Alexandroff closure operator, it leads to an FOCA which is defined over the whole set $\mathcal{P}(\Pi)$. The double orthogonal operator is not in general an Alexandroff closure operator and hence it defines an IOCA, which is not an FOCA, though its underlying set is comparable to the phase space defined as a semantic for linear logic.

Being our new operator, the maximal Alexandroff closure operator which is (pointwise) below to the double orthogonal closure operator, our construction gives the minimal FOCA that contains the truth values of the base space of the Streicher model.

II. In what follows, we describe briefly the content of each of the sections of this paper. For the basic nomenclature, see Ferrer Santos *et al.* (2015) and Streicher (2013).

In Section 2 in order to establish notations, we recall general concepts of the theory of ordered structures and indexed ordered structures. In particular, we write the definitions of preorder, meet-semilattice and HPO and its indexed versions. As a particular case of indexed HPOs, we recall the crucial definition of tripos (see Hyland *et al.* 1980) and the equivalence of tripos.

We also recall some topological tools on preorders, in particular the definitions of closure and interior operators and their induced topologies. Special attention is given to the case in which the induced topologies are Alexandroff.

In Section 3, we present the classes of ordered combinatory algebras that we need in the paper, quasi-implicative ordered combinatory algebras (QOCAs); implicative ordered combinatory algebras (IOCAs) and full adjunction implicative ordered combinatory algebras (FOCAs).

We show that our FOCAs are equivalent to the implicative algebras with separator defined by Miquel (2014).

Finally, we also show, almost in the same manner that in Ferrer Santos *et al.* (2015), Section 6 for IOCAs, how to define the internal realizability of an FOCA for higher order arithmetic.

In Section 4, in preparation for the definition of AKS and using standard constructions in lattice theory (see Birkhoff 1995), we define realizability lattices and operations on

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the power sets of the set of terms and of stacks of the lattice. We present the definition by steps.

First, we consider a pair of sets equipped with a subset of its cartesian product – the so called 'pole' – and the induced orthogonal map on the family of subsets of the original sets – the so called 'polarity'. We consider two closure operators associated to the polarity and the corresponding sets of closed subsets which are: the double perpendicular introduced by Streicher in Streicher (2013) and the one introduced in this paper denoted by $(-)^{\frown}$, that is closely related to the first. We also consider another closure operator, which is not associated to the polarity: the identity map. Those three closures define their own families of closed sets, which are denoted, respectively, $\mathcal{P}_{\perp}(\Pi)$, $\mathcal{P}_{\bullet}(\Pi)$ and $\mathcal{P}(\Pi)$. For this sets, we have $\mathcal{P}_{\perp}(\Pi) \subseteq \mathcal{P}_{\bullet}(\Pi) \subseteq \mathcal{P}(\Pi)$.

We observe that the identity and the $(-)^{\frown}$ are Alexandroff closures, which means that their associated closed sets constitutes a topology. On the other hand, we observe that the Streicher's closure is not an Alexandroff closure.

Second, in the case that the above structure is equipped with an additional *push* map, we are in the presence of what we call a *realizability lattice* – similarly to the situation treated in Ferrer Santos *et al.* (2015, Section 2, Streicher's Abstract Krivine Structures). In this context, we define the basic operations of *conductor* and *application*, which are defined between subsets of the set of stacks. Using the three closure operators that we consider, we obtain three pairs of conductor and application operations – including the one used by Streicher. We observe that the applications and implications induced by an Alexandroff closure have the full adjunction property in contraposition with what happens with the Streicher's operations.

Finally, we observe that the choice of the new closure $(-)^{\frown}$ leads to a family of closed sets $\mathcal{P}_{\bullet}(\Pi)$, that is minimal with respect to the property of being Alexandroff and containing $\mathcal{P}_{\perp}(\Pi)$.

In Section 5, we incorporate the remaining pieces of the structure: the additional application operation in the set of terms, the set of quasi-proofs with its distinguished elements, and the axioms relating the application with the push through perpendicularity.

Along the way and as a technical device, we define another operation, called the *diamond* operator, denoted as $\diamond$, and a distinguished term called the 'adjunctor'.

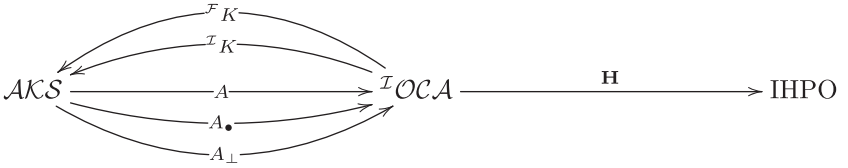
Then, we use the three pairs of application/implication–like operations to construct from the given AKS called K, different OCAs denoted as $A_{\perp}(K) \subseteq A_{\bullet}(K) \subseteq A(K)$[†]. The new OCA, named as $A_{\bullet}(K)$, has the full adjunction property, and we propose it as a candidate to construct the tripos we are searching for, in a simpler manner than

[†] The reader should be aware of the abuse of notation we commit here and later, as the above inequalities mean that the underlying basic set that supports the OCAs are the ones that satisfy the inclusion relation.

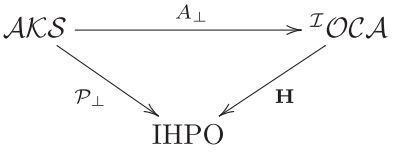
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the original construction via the double perpendicular closure appearing in Streicher (2013).

In Section 6, we present the main results of the paper that concern the constructions from OCAs to AKSes and to triposes. We depict below two diagrams: the first does not involve any categorical considerations, but only a map of the constructions is introduced here as well as the ones presented in Ferrer Santos *et al.* (2015) – and that are recalled for the sake of completeness and reference.



The second is a commutative triangle (as studied in Ferrer Santos *et al.* 2015) and links our constructions with Streicher's by identifying some elements of the above diagram. It involves the map considered by Streicher (named in Ferrer Santos *et al.* (2015) as $\mathcal{P}_\perp$) that produces an IHPO from an AKS, and the factorization of $\mathcal{P}_\perp$ introduced in Ferrer Santos *et al.* (2015, Theorem 5.11).



After defining $A_\bullet(K)$ for a general AKS called K (Definition 6.1) we show in Theorem 6.4 that is indeed an FOCA.

The three constructions $A_\perp(K) \subseteq A_\bullet(K) \subseteq A(K)$ when composed with $\mathbf{H}$ produce the indexed HPOs $\mathbf{H}_{A_\perp(K)}, \mathbf{H}_{A_\bullet(K)}, \mathbf{H}_{A(K)}$ that are equivalent (see Theorem 6.6).

The reverse construction $^{\mathcal{F}}K$ that was introduced in Definition 6.7 is similar to the analogous map $^{\mathcal{I}}K$ that was defined in Ferrer Santos *et al.* (2015). It is important to notice that the AKSes of the form $^{\mathcal{F}}K(A)$ are always strong (Theorem 6.8).

Concerning this construction, we know (as proved in Ferrer Santos *et al.* 2015) that the process of starting from an FOCA called B and applying first $^{\mathcal{I}}K$ and then $A_\perp$, produces an OCA isomorphic to B. Doing the same process with $^{\mathcal{F}}K$ and $A_\bullet$, we obtain two FOCAs – B and $A_\bullet(^{\mathcal{F}}K(B))$ – that are related by a Galois injection (Theorem 6.11).

We end our considerations by proving the expected result that guarantees that for an FOCA called A, then $\mathbf{H}_A$ and $\mathbf{H}_{A_\bullet(^{\mathcal{F}}K(A))}$ are equivalent tripos.

We finish this introduction by thanking the referees. Their exhaustive comments have been instrumental in improving the original manuscript in its mathematical and grammatical rigor.

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2. Ordered structures, indexed ordered structures, triposes

In this section – in order to set notations and vocabulary – we briefly recall some general definitions related to indexed ordered structures. For more details, we refer the reader to the corresponding section in Ferrer Santos *et al.* (2015).

III. We list some definitions and basic results.

Definition 2.1.

1. A preorder $(D, \leqslant)$ is a pair given by a set D together with a reflexive and transitive relation $\leqslant$. A morphism of preorders is a monotonic map between the sets.
2. A monotonic map $f : (C, \leqslant) \to (D, \leqslant)$ is an *equivalence*, if there exists $g : (D, \leqslant) \to (C, \leqslant)$ monotonic such that $g \circ f \cong \mathrm{id}_C$, and $f \circ g \cong \mathrm{id}_D$.
3. Given monotonic maps $f : (C, \leqslant) \to (D, \leqslant)$, $g : (D, \leqslant) \to (C, \leqslant)$, we say that '$f$ is left adjoint to g', or 'g is right adjoint to f', and write $f \dashv g$, if $\mathrm{id}_C \leqslant g \circ f$ and $f \circ g \leqslant \mathrm{id}_D$. In this context, the terminology of Galois connection is used for the pair (f, g) and if the additional condition $gf \cong \mathrm{id}_C$ holds, we say that the pair (f, g) is a 'Galois injection'.
4. We call **Ord** the category of preorders and *order preserving maps*, i.e., monotonic maps.

Observation 2.2. In what follows, we will use a standard result in category theory adapted to the context of posets that reads:
A monotonic map $f : (C, \leqslant) \to (D, \leqslant)$ is an equivalence if and only if:

— f is *order reflecting*: $\forall c, c' \in C, f(c) \leqslant f(c') \Rightarrow c \leqslant c'$

— f is *essentially surjective*: $\forall d \in D\ \exists c \in C, f(c) \cong d$.

Definition 2.3.

— We call **Ord** the category of preorders and monotonic maps.

— We call **SLat** the category of meet-semilattices, and *meet preserving monotonic maps*, i.e., monotonic maps $f : (C, \leqslant) \to (D, \leqslant)$ such that $f(c) \wedge f(c') \cong f(c \wedge c')$ and $f(\top) \cong \top$.

— We call **HPO**, the category of HPOs a.k.a. HPO and its morphisms, i.e., monotonic maps $f : (A, \leqslant) \to (B, \leqslant)$ such that $f(\top) \cong \top$, $f(a \wedge b) \cong f(a) \wedge f(b)$, $f(a \to b) \cong f(a) \to f(b)$ for all $a, b \in A$.

IV. **Definition 2.4.** An indexed category is a pseudo-functor $\mathbf{F} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$, where **Cat** is the category of all small categories.

Example 2.5. Replacing **Cat** by **Ord**, **Slat** and **HPO**, we obtain the classes of indexed categories, IOrd, ISlat and IHPO, respectively.

Definition 2.6. Given indexed preorders $\mathbf{C}, \mathbf{D} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ an *indexed monotonic map* $\sigma : \mathbf{C} \to \mathbf{D}$ is a family $\sigma_I : \mathbf{C}(I) \to \mathbf{D}(I)$ $(I \in \mathbf{Set})$ of monotonic functions, such that for all functions $f : J \to I$ and predicates $\varphi \in \mathbf{C}(I)$:

$$\sigma_J(f^*(\varphi)) \cong f^*(\sigma_I(\varphi)), \tag{2.6.1}$$

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where the symbol f^* denotes $\mathbf{C}(f)$ when on the left side of the equation and $\mathbf{D}(f)$ when on the right side.

Definition 2.7. Let $\mathbf{C}, \mathbf{D} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Ord}$ be indexed preorders.

1. For indexed monotonic maps $\sigma, \tau : \mathbf{C} \to \mathbf{D}$, we define

$$\sigma \leqslant \tau \quad \Leftrightarrow \quad \forall I \in \mathbf{Set}\ \sigma_I \leqslant \tau_I \quad \Leftrightarrow \quad \forall I \in \mathbf{Set} \text{ and } \forall d \in \mathbf{C}(I)\,,\ \sigma_I(d) \leqslant \tau_I(d).$$

   We say that σ and τ are *isomorphic*, and write $\sigma \cong \tau$, if $\sigma \leqslant \tau$ and $\tau \leqslant \sigma$.
2. An indexed monotonic map $\sigma : \mathbf{C} \to \mathbf{D}$ is called an *equivalence*, if there exists a indexed monotonic map $\tau : \mathbf{D} \to \mathbf{C}$ such that $\tau \circ \sigma \cong \mathrm{id}_{\mathbf{C}}$, and $\sigma \circ \tau \cong \mathrm{id}_{\mathbf{D}}$. In this case, τ is called an *(indexed) weak inverse* of σ.
3. We say that $\mathbf{C}$ and $\mathbf{D}$ are *equivalent*, and write $\mathbf{C} \simeq \mathbf{D}$, if there exists an equivalence $\sigma : \mathbf{C} \to \mathbf{D}$.

The next Lemma follows from the considerations above, see III.

Lemma 2.8. An indexed monotonic map $\sigma : \mathbf{C} \to \mathbf{D}$ is an equivalence, if and only if for every set I, the monotonic map $\sigma_I : \mathbf{C}(I) \to \mathbf{D}(I)$ is order reflecting and essentially surjective.

V. Next, we consider a special kind of indexed HPOs, called *triposes*, see Hyland *et al.* (1980).

Definition 2.9. A *tripos* is a pseudo-functor $\mathbf{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HPO}$ such that:

1. For every function $f : J \to I$, the reindexing map $f^* = \mathbf{P}(f) : \mathbf{P}(I) \to \mathbf{P}(J)$ has a right adjoint ‡ $\forall_f : \mathbf{P}(J) \to \mathbf{P}(I)$, which means that $f^*(\varphi) \leqslant \psi \Leftrightarrow \varphi \leqslant \forall_f(\psi)$ for all $\varphi \in \mathbf{P}(I)$ and $\psi \in \mathbf{P}(J)$.
2. If

$$\begin{array}{ccc} L & \xrightarrow{h} & K \\ \downarrow k & & \downarrow g \\ J & \xrightarrow{f} & I \end{array} \tag{2.9.2}$$

   is a pullback square of sets and functions, then $\forall_h(k^*(\varphi)) \cong g^*(\forall_f(\varphi))$ for all $\varphi \in \mathbf{P}(J)$ (this is the *Beck–Chevalley condition*).
3. $\mathbf{P}$ has a *generic predicate*, i.e., there exists a set Prop, and a $\mathsf{tr} \in \mathbf{P}(\mathsf{Prop})$ such that for every set I and $\varphi \in \mathbf{P}(I)$ there exists a (not necessarily unique) function $\chi_\varphi : I \to \mathsf{Prop}$ with $\varphi \cong \chi_\varphi^*(\mathsf{tr})$.

Remark 2.10. Assume that $\mathbf{C}$ and $\mathbf{P}$ are equivalent indexed preorders, and that $\mathbf{P}$ is a tripos, then so is $\mathbf{C}$.

VI. We recall a few definitions concerning partially ordered sets (posets).

Definition 2.11. Let $(A, \leqslant)$ be a partially ordered set.

‡ Notice that the adjoint does not need to preserve the whole HPO structure.

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1. A *closure operator* in $(A, \leqslant)$ is a map $\mathrm{c} : A \to A$ such that:
   — For all $a \in A$, $a \leqslant \mathrm{c}(a)$.
   — For all $a \in A$, $\mathrm{c}(\mathrm{c}(a)) = \mathrm{c}(a)$.
   — For all $a, b \in A$, $a \leqslant b$ implies that $\mathrm{c}(a) \leqslant \mathrm{c}(b)$.
   In the situation that A is also a distributive lattice with top and bottom, the operator is said to be a *topological closure operator* if:
   — For all $a, b \subseteq A$, then $\mathrm{c}(a \vee b) = \mathrm{c}(a) \vee \mathrm{c}(b)$.
   — $\mathrm{c}(\bot) = \bot$
   In the above situation, it is customary to define the set of c–closed elements of A as $A_{\mathrm{c}} = \{a \in A : \mathrm{c}(a) = a\}$
2. An *interior operator* $\mathrm{i} : A \to A$ is a closure operator for the opposite order (meaning that all the inequalities are reversed and $\vee$ and $\wedge$ are interchanged). The concept of *topological interior* operator is defined dually. The set of open elements defined by an interior operator i is: $A_{\mathrm{i}} = \{a \in A : \mathrm{i}(a) = a\}$.
3. In the case that the lattice is complete the closure operator is said to satisfy the *Alexandroff condition* if for every $\{a_i : i \in I\} \in A$ we have $\mathrm{c}(\bigvee\{a_i : i \in I\}) = \bigvee\{\mathrm{c}(a_i) : i \in I\}$. The Alexandroff condition for an interior operator is defined by duality.

Observation 2.12. The Alexandroff property of an operator guarantees that the family of closed elements is closed under infinite joins. Observe also that if $\{a_i : i \in I\} \subseteq A$ is a family of c-closed elements, then $\bigwedge_i a_i$ is also closed. Hence, in this situation A_{c} is also the family of open subsets of a topology that has an interior operator associated that we call i_{c}.
In this situation, given $B = A_{\mathrm{c}} = A_{\mathrm{i}_{\mathrm{c}}}$ we have that: $\mathrm{c}(a) = \bigwedge\{b : b \in B, a \leqslant b\}$ and $\mathrm{i}(a) = \bigvee\{b : b \in B, b \leqslant a\}$.

Definition 2.13. Assume that A is as poset and that c is a closure operator. The *Alexandroff approximation* of c is – in case it exists – the maximal element of the set of Alexandroff closure operators $\mathrm{d} : A \to A$ such that $\mathrm{id} \leqslant \mathrm{d} \leqslant \mathrm{c}$. In case, it exists it will be denoted as c_A.

Observation 2.14. It is easy to prove that to give a Galois injection is equivalent to give an interior operator $\imath$ on B with family of open sets $A = B_\imath$.

3. Implicative and full adjunction ordered combinatory algebras

VII. We start by recalling the following basic structure defined in Ferrer Santos *et al.* (2015, Section 3.2). We refer the reader to Hofstra and van Oosten (2004) for the definition of the similar concept of Ordered partial combinatory algebras.

Definition 3.1.
Let $(A, \leqslant)$ be an inf-complete partially ordered set equipped with a top element denoted as $\top$– i.e., the inf of the empty set – and define the following conditions:

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(OP) binary operations

$$\text{app} : A \times A \to A, \qquad (a, b) \mapsto ab,$$

called *application*, monotonic in both arguments, and

$$\text{imp} : A \times A \to A, \qquad (a, b) \mapsto a \to b,$$

called *implication*, anti-monotonic in the first argument and monotonic in the second;

(CO) distinguished elements $\mathtt{s}, \mathtt{k} \in A$ such that for all $a, b, c \in A$ the following holds:

(PK) $\mathtt{k}ab \leqslant a$

(PS) $\mathtt{s}abc \leqslant ac(bc)$

(PA) If $a \leqslant b \to c$, then $ab \leqslant c$.

(FI) a subset $\Phi \subseteq A$ (called *filter*), which is closed under application and such that $\mathtt{s}, \mathtt{k} \in \Phi$. Moreover, Φ is upper closed: if $f \leqslant g, f \in \Phi$ and $g \in A$, then $g \in \Phi$.

In the above context, we establish the following definitions:

1. The structure $\mathcal{A} = (A, \leqslant, \text{app}, \text{imp}, \mathtt{s}, \mathtt{k}, \Phi)$ that satisfies all the above conditions is called a QOCA. We denote the class of all QOCAs with the symbol ${}^{\mathcal{Q}}\mathcal{OCA}$.
2. If there exists a distinguished element $\mathtt{e} \in \Phi$ such that:

$$\text{(PE)} \qquad \text{If} \quad ab \leqslant c, \quad \text{then} \quad \mathtt{e}a \leqslant b \to c,$$

we say that $\mathcal{A} = (A, \text{app}, \text{imp}, \mathtt{s}, \mathtt{k}, \mathtt{e}, \Phi)$ is an IOCA and the class denoted as ${}^{\mathcal{I}}\mathcal{OCA}$.

3. If for all $a, b, c \in A$

$$\text{(PE)}' \qquad \text{If} \quad ab \leqslant c, \quad \text{then} \quad a \leqslant b \to c,$$

we say that $\mathcal{A} = (A, \leqslant, \text{app}, \text{imp}, \mathtt{s}, \mathtt{k}, \Phi)$ is a FOCA and the class denoted ${}^{\mathcal{F}}\mathcal{OCA}$.

Notation 3.2. Notice that the application $\text{app}(a, b)$ is denoted as ab (cf. 3.1(OP)) and occasionally (to avoid a cumbersome text as in the comment that follows) ab will be denoted $a \circ b$.
When operating with the function app, we associate to the left and when operating with the function imp we associate to the right, so that abc means $(ab)c$ and $a \to b \to c$ means $a \to (b \to c)$.

The name FOCA is due to the fact that conditions (CO)(PA) and (PE)' are equivalent to the following assertion: for every $b \in A$ the map $- \circ b : A \to A$ is left adjoint of the map $b \to - : A \to A$.
We refer the reader to Ferrer Santos *et al.* (2015, Section 3) for motivation and more details concerning the definition of IOCA, in particular, we mention that the element $\mathtt{e}$ is called an *adjunctor*. Notice that in the present work, we assume that the filter is upward closed in contrast with the notion of filter in Ferrer Santos *et al.* (2015).

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The QOCAs obtained from AKSes in this paper, will have filters that will satisfy this property.
When there is no danger of confusion we denote $\mathcal{A} = (A, \mathrm{app}, \mathrm{imp}, \mathsf{s}, \mathsf{k}, \mathsf{e}, \Phi)$ as A.

Lemma 3.3. Assume that A is a poset with two operations $\rightarrow, \circ : A \times A \rightarrow A$ with the same variance properties as in Definition 3.1.

1. $\forall a, b \in A$, $(a \rightarrow b) \circ a \leqslant b$ if and only if $\forall a, b, c \in A$, $a \leqslant b \rightarrow c \Rightarrow a \circ b \leqslant c$.
2. For all $e \in A$ the following are equivalent:
   - $\forall a, b \in A \quad e \circ a \leqslant b \rightarrow (a \circ b)$.
   - $\forall a, b, c \in A \quad a \circ b \leqslant c \Rightarrow e \circ a \leqslant b \rightarrow c$.
3. The following are equivalent:
   - for all $a, b, c \in A$ $a \circ b \leqslant c$ if and only if $a \leqslant b \rightarrow c$.
   - for all $a, b, c \in A$ $a \leqslant b \rightarrow (a \circ b)$ and $(b \rightarrow c) \circ b \leqslant c$.

Proof.

1. In the right -hand side implication, if we take $a = b \rightarrow c$ and substitute in the rightmost inequality, we obtain that $(b \rightarrow c) \circ b \leqslant c$.
   The proof of (2) and (3) follows the same pattern. □

Observation 3.4. Notice that, given an FOCA called A, the element $\mathsf{i} = \mathsf{skk} \in \Phi$ satisfies for all $a, b, c \in A$ if $ab \leqslant c$ then $\mathsf{i}a \leqslant a \leqslant b \rightarrow c$ and thus A is also an IOCA with adjunctor i. Another special element of $\mathcal{A}$ is the 'bottom': $\bot = \inf(A)$.

VIII. We use the inf-completeness property to establish the following definition – that yields an operation that is adjoint to the original implication map.

Definition 3.5. Let A be a QOCA, for $a, b \in A$ define

$$a \sharp b = \inf\{c : a \leqslant (b \rightarrow c)\}.$$

Theorem 3.6.

1. If A is a QOCA, then for all $a, b \in A$, $ab \leqslant a \sharp b$.
2. If A is an IOCA with adjunctor e, we have that for all $a, b \in A$: $(\mathsf{e}\, a) \sharp b \leqslant ab$.
3. If A is an FOCA we have that for all $a, b \in A$: $a \sharp b = ab$.

Proof.

1. Follows from condition (PA): if $a \leqslant (b \rightarrow c)$ we have that $ab \leqslant c$, and then $ab \leqslant a \sharp b$.
2. From $ab \leqslant ab$, we obtain that $\mathsf{e}\, a \leqslant b \rightarrow (ab)$, and this implies that $(\mathsf{e}\, a) \sharp b \leqslant ab$.
3. Is proved as (2), but without e. □

Example 3.7. Consider the following FOCA that we denote as $\mathcal{P}$ and is defined in the following manner:

1. The basic set is $[0, \infty] = \mathbb{R}^{\geqslant 0} \cup \{\infty\}$ equipped with the usual order,

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2. The operation app is commutative and $\mathrm{app}(a,\infty)=\infty$ if $a\neq 0$; $\mathrm{app}(\infty,\infty)=\infty$; $\mathrm{app}(0,\infty)=0$ and otherwise $\mathrm{app}(a,b)=ab$–the usual product of $\mathbb{R}$.
3. Moreover, the imp map is defined as: for $b\neq 0,\infty$, $\mathrm{imp}(b,\infty)=\infty$; $\mathrm{imp}(\infty,b)=0$; $\mathrm{imp}(\infty,\infty)=\infty$; $\mathrm{imp}(0,b)=\infty$; $\mathrm{imp}(b,0)=0$; $\mathrm{imp}(0,0)=\infty$; $\mathrm{imp}(0,\infty)=\infty$; $\mathrm{imp}(\infty,0)=0$; otherwise $\mathrm{imp}(b,c)=c/b$.
4. $\mathsf{k}=\mathsf{s}=0$.
5. Take as filter $\Phi=[0,\infty]$.

It is easy to check that $\mathcal{P}$ is an FOCA, and thus $a\sharp b=\mathrm{app}(a,b)$.
We can perturb the application map as follows: $\overline{\mathrm{app}}(a,b)=\mathrm{app}(a,b)/2$ and if we maintain the rest of the original structure we obtain a new QOCA, where $\overline{\mathrm{app}}$ and $\sharp$ are different maps.

IX. An *implicative algebra* is a complete meet-semilattice $(A,\leqslant)$ equipped with a binary operation $\to\colon A\times A\to A$ that is contravariant in the first variable and covariant in the second, with the additional property that it commutes with arbitrary meets in the second variable, i.e., for all $a\in A$ and $C\subseteq A$, $a\to\inf\{c : c\in C\}=\inf\{a\to c : c\in C\}$. A subset $S\subseteq A$ is called a separator if it is upward closed, $\top\in S$ the elements $\mathsf{k}:=\inf\{a\to b\to a : a,b\in A\}$, $\mathsf{s}:=\inf\{(a\to b\to c)\to(a\to b)\to a\to c : a,b,c\in A\}$ belong to S, and if $a\in S$ and $a\to b\in S$ then $b\in S$. These concepts were defined by A. Miquel in 2014 or earlier (see for example: Miquel (2014)).
As a direct consequence of Freyd's theorem concerning existence of an adjoint functor, one can show that the concept of FOCA is equivalent with the concept of *implicative algebra* with separator. As in the case of ordered structures, the proofs are very easy, we sketch them below.
Using the completeness of A, we can define an operation $a\circ b=\inf\{c : a\leqslant(b\to c)\}$. It is easy to prove that for an implicative algebra the associated structure of $(A,\leqslant,\circ,\to,\mathsf{k},\mathsf{s},S)$ is an FOCA with filter S. For example, the crucial property of the full adjunction can be proved as follows: (a) by definition, it is clear that if $a\leqslant b\to c$ then as $a\circ b$ is the infimum of all such c's, then $a\circ b\leqslant c$; (b) $b\to(a\circ b)=b\to\inf\{c : a\leqslant(b\to c)\}=\inf\{b\to c : a\leqslant(b\to c)\}\geqslant a$, and if $a\circ b\leqslant c$, then $a\leqslant(b\to c)$. Conversely, in any FOCA it is clear that if we fix the first variable as $b\in A$, the map $c\mapsto(b\to c)$ given by the implication, commutes with infinite meets – being a right adjoint –. In more explicit terms, we have that from the adjunction condition we deduce that for all $a,c\in A$, $(a\to c)a\leqslant c$ an then $\inf\{a\to c : c\in C\}a\leqslant\inf\{c : c\in C\}$. Then, $\inf\{a\to c : c\in C\}\leqslant a\to\inf\{c : c\in C\}$. The other inequality is obvious. All the other conditions for an implicative algebra are satisfied, and we have shown the equivalence of both concepts.

X. A very important property of IOCAs is that they are combinatory complete. More precisely, we have a translation $()^*$ from λ-calculus into IOCAs in such a way that $t\to_\beta u$ by a weak-head reduction implies that $t^*\leqslant u^*$. We briefly recall this translation – see for example Ferrer Santos *et al.* (2013) or (Ferrer Santos *et al.* 2015, Theorem 3.4) for details.

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For any finite set of variables $\{x_1,\dots,x_k,y\}$, we denote by $A[x_1,\dots,x_k,y]$ the set of all polynomials in $\{x_1,\dots,x_k,y\}$. The order relation $\leqslant$ of A can be extended to $A[x_1,\dots,x_k]$ as the least partial order $\leqslant'$ satisfying:

1. if $a \leqslant b$ then $a \leqslant' b$.
2. $\forall p,q,r,s \in A[x_1,\dots,x_k]$ if $p \leqslant' q$ and $r \leqslant' s$ then $pr \leqslant' qs$.

Since $\leqslant'$ is an extension of $\leqslant$, there is no danger of confusion in denoting $\leqslant'$ as $\leqslant$.

Theorem 3.8. There exists a function $\lambda^* y : A[x_1,\dots,x_k,y] \to A[x_1,\dots,x_k]$ satisfying the following property: If $t \in A[x_1,\dots,x_k,y]$ and $u \in A[x_1,\dots,x_k]$, then $(\lambda^* y(t))u \leqslant t\{y := u\}$.

Proof. The function $\lambda^* y$ is defined recursively on the length of their arguments: (a) If $y \neq p$ where $p \in A[x_1,\dots,x_k,y]$, then $\lambda^* y(p) := \mathsf{k}\,p$; (b) $\lambda^* y(y) := \mathsf{s}\,\mathsf{k}\,\mathsf{k}$; (c) if p,q are polynomials, then: $\lambda^* y(pq) := \mathsf{s}\,(\lambda^* y(p))\lambda^* y(q)$. □

We will write $\lambda^* y.t$ instead of $\lambda^* y(t)$. The significance of the theorem is that it allows one to translate (by means of $\lambda^* y$) every λ-term depending on the variables $\{x_1,\dots,x_k\}$ into a polynomial of $A[x_1,\dots,x_k]$. In particular, every closed λ-term is translated into an element of A.

Observation 3.9. Programming in λ-calculus and translating the λ-terms we obtain combinators i, b, p, p_0 and p_1 satisfying the following properties (cf. (Ferrer Santos *et al.* 2015, Definition 3.5)): for every a,b and $c \in A$

— $\mathsf{i}\,a \leqslant a$.

— $\mathsf{b}\,abc \leqslant a(bc)$.

— $\mathsf{p}_0\,(\mathsf{p}\,ab) \leqslant a$.

— $\mathsf{p}_1\,(\mathsf{p}\,ab) \leqslant b$.

Finally, we define functions $\mathsf{a} : \Phi \times \Phi \to \Phi$ and $\mathsf{d} : \Phi \to \Phi$ satisfying:

— For all $r,s \in \Phi$ and for all $c,a,b \in A$, if $rc \leqslant a$ and $sc \leqslant b$, then $\mathsf{a}\,(r,s)c \leqslant \mathsf{p}\,ab$.

— For all $r \in \Phi$ and for all $a \in A$, $\mathsf{d}\,(r)a \leqslant r(\mathsf{p}_0\,a)(\mathsf{p}_1\,a)$.

XI. Assuming that A is an IOCA we want to consider two IOCAs – structures in the sets of the form A^I for I an arbitrary set.[§] This will be used later when we present the construction of triposes associated to OCAs.

Definition 3.10. Let I be an arbitrary set and $A = (A, \leqslant, \mathrm{app}, \mathrm{imp}, \mathsf{s}, \mathsf{k}, \Phi)$ be a ${}^{\mathbb{Q}}\mathcal{OCA}$. Define in the set of set theoretical functions $A^I = \{\varphi : I \to A\}$ the following items –for $a \in A$ we denote as $\kappa_a \in A^I$ the constant function $\kappa_a(i) = a$, $\forall i \in I$.

1. $\varphi \leqslant_I \psi$ if and only if $\varphi(i) \leqslant \psi(i), \forall i \in I$;
2. $\mathrm{app}_I = \circ_I : A^I \times A^I \to A^I$, $(\varphi \circ_I \psi)(i) := \varphi(i) \circ \psi(i), \forall i \in I$;

[§] We would like to thank Alexandre Miquel for making us aware of the crucial difference for the theory of realizability, between both structures on A^I.

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3. $\mathrm{imp}_I = \to_I : A^I \times A^I \to A^I$, $(\varphi \to_I \psi)(i) := \varphi(i) \to \psi(i), \forall i \in I$;
4. $\mathsf{k}_I := \kappa_{\mathsf{k}} \in A^I$, $\mathsf{k}_I(i) = \mathsf{k}\ \ \forall i \in I$;
5. $\mathsf{s}_I := \kappa_{\mathsf{s}} \in A^I$, $\mathsf{s}_I(i) = \mathsf{s}\ \ \forall i \in I$.

If A has an adjunctor e, we define:

6. $\mathsf{e}_I := \kappa_{\mathsf{e}} \in A^I$, $\mathsf{e}_I(i) = \mathsf{e}\ \ \forall i \in I$.

Define the following two subsets Φ^I, $\Delta(\Phi) \subseteq A^I$ as follows:

1 $\Phi^I := \{\varphi \in A^I : \forall i \in I,\ \varphi(i) \in \Phi\}$;

2 $\Delta(\Phi) := \{\varphi \in A^I : \exists f \in \Phi : \kappa_f \leqslant_I \varphi\}$.

Also consider the following structures:

1 $P_I(A) := (A^I, \leqslant_I, \mathrm{app}_I, \mathrm{imp}_I, \mathsf{s}_I, \mathsf{k}_I, \Phi^I)$.

2 $\Delta_I(A) := (A^I, \leqslant_I, \mathrm{app}_I, \mathrm{imp}_I, \mathsf{s}_I, \mathsf{k}_I, \Delta(\Phi))$.

It is clear that $\Delta(\Phi) \subseteq \Phi^I \subseteq A^I$, as we are assuming that Φ is an upper closed subset of A. Obviously in the case that I is a singleton, there is no difference between both filters in $A^I = A$.

The proof of the lemma that follows is direct and will be omitted.

Lemma 3.11. Let A be a ${}^{\mathcal{Q}}\mathcal{OCA}$ and I an arbitrary set, then:

1. $P_I(A)$ and $\Delta_I(A)$ are also ${}^{\mathcal{Q}}\mathcal{OCA}$s.
2. If A is an ${}^{\mathcal{I}}\mathcal{OCA}$, so are $P_I(A)$ and $\Delta_I(A)$, and an adjunctor for both is e_I.
3. If A is an ${}^{\mathcal{F}}\mathcal{OCA}$ so are $P_I(A)$ and $\Delta_I(A)$.

Observation 3.12. It is clear, for an arbitrary $\mathsf{c} \in \Phi \subseteq A$, that $\kappa_{\mathsf{c}} \in \Delta(\Phi) \subseteq \Phi^I \subseteq A^I$ has the same reduction rules in both $P_I(A)$ and $\Delta_I(A)$.

XII. *From* ${}^{\mathcal{I}}\mathcal{OCA}$ *to* IHPO

Our aim is to recall a general construction from ${}^{\mathcal{I}}\mathcal{OCA}$ to IHPO that appears (except for some minor modifications for the sake of clarifying some concepts) in Ferrer Santos *et al.* (2015).

We start with the case the indexing set is a singleton and this will be the basis of the general construction.

If A is an IOCA based on a set A with filter Φ and maximum element $\top \in \Phi$, define:

1. The relation $\sqsubseteq$ in A: $a \sqsubseteq b$, if and only if $\exists f \in \Phi$ such that $fa \leqslant b$,
2. A map $\wedge : A \times A \to A$ as $a \wedge b := \mathsf{p}\,ab$.

If A is a IOCA then $\mathcal{H}(A) := (A, \sqsubseteq, \wedge, \to)$, is a HPO, where $\sqsubseteq, \wedge$ are as above and $\to$ is the original implication of A. For the basic properties of p, see Observation 3.9. See Ferrer Santos *et al.* (2015, Section 4, Lemma 4.14) for the proof that the above construction produces a meet-semilattice and (Ferrer Santos *et al.* 2015, Section 4, Theorem 4.15) for the rest of the proof.

An element f as above is said to be 'a realizer of the relation $a \sqsubseteq b$' and we write this assertion as $f \Vdash a \sqsubseteq b$.

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Lemma 3.13. If $\mathcal{H}(A)$ is the HPO induced by an IOCA named as A, the following holds:

1. If $a, b \in A$, then $a \sqsubseteq b$ if and only if $a \to b \in \Phi$ and in that case: $(a \to b) \Vdash a \sqsubseteq b$. Moreover, if $f \in \Phi$ realizes $a \sqsubseteq b$, then $\mathsf{e}f \leqslant a \to b$ with $a \to b \in \Phi$. In the case that A is an FOCA, we have that the set of elements that realize $a \sqsubseteq b$ is the set $\downarrow (a \to b) \cap \Phi$.
2. In $\mathcal{H}(A)$ all the elements of Φ are isomorphic to $\top$.

Proof.

1. From the relation $fa \leqslant b$ we deduce that $\mathsf{e}f \leqslant a \to b$. Being both elements $\mathsf{e}, f \in \Phi$ it follows that $\mathsf{e}f \in \Phi$. The rest of the proof is clear.
2. From Definition 3.1, we see that for all $a \in A$ and $f \in \Phi$, $\mathsf{k}fa \leqslant f$ and that means that $a \sqsubseteq f$. Hence, all the elements of the filter are isomorphic with the top element. □

In the lemma that follows, we compare the $\sqsubseteq$ preorders in $P_I(A)$ and in $\Delta_I(A)$.

Lemma 3.14. Let A be an IOCA and consider the associated IOCAs called $P_I(A)$ and $\Delta_I(A)$. Then:

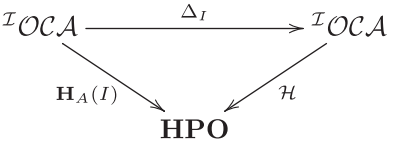
1. $\varphi \sqsubseteq_{A^I} \psi$ if and only if $\forall i \in I, \varphi(i) \to \psi(i) \in \Phi$;
2. $\varphi \sqsubseteq_{\Delta_I(A)} \psi$ if and only if $\inf\{\varphi(i) \to \psi(i) : i \in I\} \in \Phi$.

Proof.

1. By definition $\varphi \sqsubseteq_{A^I} \psi$ if and only if $\forall i \in I\,, \varphi(i) \sqsubseteq_A \psi(i)$ and by Lemma 3.13 it is equivalent to $\forall i \in I\ \varphi(i) \to \psi(i) \in \Phi$.
2. By definition $\varphi \sqsubseteq_{\Delta_I(A)} \psi$ if and only if there is a $f \in \Phi$ such that $\forall i \in I\,, f \leqslant \varphi(i) \to \psi(i)$. As we are supposing that A is inf complete, we deduce that $f \leqslant \inf\{\varphi(i) \to \psi(i) : i \in I\}$ and then – as we take the filter to be upper closed – that $\inf\{\varphi(i) \to \psi(i) : i \in I\} \in \Phi$. □

Definition 3.15. In A^I the order $\sqsubseteq_{\Delta_I(A)} \subseteq A^I \times A^I$ will be denoted as $\vdash_I := \sqsubseteq_{\Delta_I(A)}$ and called the entailment relation, or the entailment preorder. Frequently, we simply write $\vdash$ instead of $\vdash_I$, omitting the subscript.

XIII. Consider the triangle of constructions, depicted in the family of diagrams that follows.



Definition 3.16. Let A be an $^{\mathcal{I}}\mathcal{OCA}$ and consider the indexed HPO, $\mathbf{H}_A : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HPO}$ defined as

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1. $\mathbf{H}_A(I) := \mathcal{H}(\Delta_I(A)) = (A^I, \vdash, \wedge, \rightarrow)$;
2. $f : J \rightarrow I$, $\mathbf{H}_A(f) = f^* : \mathbf{H}_A(I) \rightarrow \mathbf{H}_A(J)$: $f^* : (A^I, \vdash) \rightarrow (A^J, \vdash)$, $\varphi \overset{f^*}{\mapsto} \varphi \circ f$.

For future reference, we recall the theorem that follows that appears in (Ferrer Santos *et al.* 2015, Section 5, Theorem 5.8).

Theorem 3.17. Let $\mathcal{A} = (A, \leqslant, \mathrm{app}, \mathrm{imp}, \Phi, \mathsf{k}, \mathsf{s})$ be an IOCA, then $\mathbf{H}_A$ is a tripos.

XIV. In Ferrer Santos *et al.* (2015, Section 6), we illustrated that we can define realizability for IOCAs and thus, to define realizability in higher order arithmetics. In what follows, we mention briefly how to adapt the constructions therein, to the context of an FOCA, and hence we show that the ordered combinatory algebras considered above, can be taken as an adequate platform to do realizability.

The adaptations are minor: the kinds, the language and the type system are the same. We have to slightly modify the typing rule of the implication that becomes:

$$\frac{\Gamma, x : A^o \vdash p : B^o}{\Gamma \vdash \lambda^* x.p : (A^o \Rightarrow B^o)^o}(\rightarrow_i)$$

The interpretation of $\mathcal{L}^\omega$ is defined in the same manner as in Ferrer Santos *et al.* (2015), and the adequacy follows also in the same manner, the only new problem to deal with is the adequacy for the implication rules. This follows directly using that the FOCA satisfies the full adjunction property. Indeed, for the implication rule: $(\rightarrow)_i$ we assume $\mathcal{A} \models \Gamma, x : A^o \vdash p : B^o$ where $\Gamma = x_1 : A_1^o, \ldots, x_k : A_k^o$. Consider an assignment $\mathfrak{a}$ and $b_1, \ldots, b_k \in A$ such that $b_i \leqslant [[A_i^o]]$. We get:

$$\begin{aligned}(\lambda^* x.p)\{x_1 := b_1, \ldots, x_k := b_k\} \circ [[A^o]] \;=\; & \lambda^* x.(p\{x_1 := b_1, \ldots, x_k := b_k\}) \circ [[A^o]] \leqslant \\ & p\{x_1 := b_1, \ldots, x_k := b_k, x := [[A^o]]\} \leqslant \\ & [[B^o]]\end{aligned}$$

for the first inequality, see Paragraph X, the last inequality follows directly from the assumption that $\mathcal{A} \models \Gamma, x : A^o \vdash p : B^o$. Applying the full adjunction relation between $\circ$ and $\rightarrow$ we deduce that $\lambda^* x.p\{x_1 := b_1, \ldots, x_k := b_k\} \leqslant [[(A^o \Rightarrow B^o)^o]]$. As this is proved for all assignments $\mathfrak{a}$, we conclude that $\mathcal{A} \models \Gamma \vdash \lambda^* x.p : (A^o \Rightarrow B^o)^o$.

4. Operations in sets of terms and stacks

In this Section, we first recapitulate briefly the definitions and basic operations in a Realizability lattice a.k.a. (RL) and then introduce new operations. Compare with Ferrer Santos *et al.* (2015) where some aspects of the work of J.L. Krivine and T. Streicher are reformulated (see Krivine 2008 and Streicher (2013), respectively).

XV. *Polarities and closure operations.* A triple of sets $(\Lambda, \Pi, \perp\!\!\!\perp)$ with $\perp\!\!\!\perp \subseteq \Lambda \times \Pi$ being a subset or *a relation* –called a 'pole'–(see (Birkhoff 1995, Chapter V, Section 7)), induces a *polarity* as illustrated below. A generic triple as above will be denoted as $\mathcal{R} = (\Lambda, \Pi, \perp\!\!\!\perp)$.

The elements of Λ and of Π are called, respectively, *terms* and *stacks* and the elements of $\Lambda \times \Pi$ are called *processes*. The processes are pairs (t, π), usually denoted by $t \star \pi$

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(c.f.: Krivine (2003)). If $t \star \pi \in \perp\!\!\!\perp$, we write $t \perp \pi$, and say that 't is perpendicular to π' or that 't realizes π'. If $P \subseteq \Pi$ and $t \perp \pi$ for all $\pi \in P$ we say that 't realizes P' and write $t \perp P$. Symmetrically if $t \star \pi \in \perp\!\!\!\perp$, we say that '$\pi$ is perpendicular to t'. If $L \subseteq \Lambda$ and $t \perp \pi$ for all $t \in L$ we say that π is perpendicular to L.

Definition 4.1. Given a triple $\mathcal{R}$ as above, we define the following order reversing maps and families of sets:

$$\begin{aligned}(\quad)^{\perp} &: \mathcal{P}(\Lambda) \longrightarrow \mathcal{P}(\Pi) \\ &L \longmapsto L^{\perp} = \{\pi \in \Pi |\ \forall t \in L, t \perp \pi\};\end{aligned}$$

$$\begin{aligned}{}^{\perp}(\quad) &: \mathcal{P}(\Pi) \longrightarrow \mathcal{P}(\Lambda) \\ &P \longmapsto {}^{\perp}P = \{t \in \Lambda |\ \forall \pi \in P, t \perp \pi\}.\end{aligned}$$

The maps are called the polar maps and the subsets $L^{\perp}$ and ${}^{\perp}P$ are called the polars –or perpendiculars– of L and P, respectively.

Observation 4.2. The maps $(\quad)^{\perp} : \mathcal{P}(\Lambda) \to \mathcal{P}(\Pi)$ and ${}^{\perp}(\quad) : \mathcal{P}(\Pi) \to \mathcal{P}(\Lambda)$ are order reversing.
For an arbitrary $L \in \mathcal{P}(\Lambda)$ and $P \in \mathcal{P}(\Pi)$, one has that ${}^{\perp}(L^{\perp}) \supseteq L$ and $({}^{\perp}P)^{\perp} \supseteq P$.
For an arbitrary $L \in \mathcal{P}(\Lambda)$ and $P \in \mathcal{P}(\Pi)$, one has that $({}^{\perp}(L^{\perp}))^{\perp} = L^{\perp}$ and ${}^{\perp}(({}^{\perp}P)^{\perp}) = {}^{\perp}P$.

The roles of Λ and Π up to now, are completely symmetric. For reasons that will be clear later, for the next considerations we work with Π, i.e., with subsets of stacks, noting that the corresponding definitions for Λ are identical.

Definition 4.3. The set $(({}^{\perp}P)^{\perp})$ is abbreviated as $\overline{P}$ (when P is a wide expression, we write $(P)^{-}$ instead of $\overline{P}$).
The map $P \mapsto \mathrm{c}(P) := (({}^{\perp}P)^{\perp}) = \overline{P}$ is called the *closure operation* associated to the polarity, see Definition 2.11. The family of closed subsets associated to the closure operation (for the general definition of closure operator, see Birkhoff (1995, Chapter V, Section 7, Theorem 19) and also Definition 2.11) is denoted as follows:

$$\mathcal{P}_{\perp}(\Pi) = \{P \subseteq \Pi : \overline{P} = P\}.$$

In general, a closure (or an interior operator with the opposite order) gives rise a reflective subcategory (see Observation 4.4 and 4.10).

Observation 4.4. The basic properties of the inclusion $\mathcal{P}_{\perp}(\Pi) \subseteq \mathcal{P}(\Pi)$ can be expressed by saying that the first is a reflective subcategory of the second. The map $(P \mapsto \overline{P}) : \mathcal{P}(\Pi) \to \mathcal{P}_{\perp}(\Pi)$ is the reflector: i.e.,

$$P \subseteq Q \Leftrightarrow \overline{P} \subseteq Q$$

for $P \in \mathcal{P}(\Pi)$ and $Q \in \mathcal{P}_{\perp}(\Pi)$(see Borceaux 2008a). Notice that the reflective property can also be expressed using the $\perp$ operator as follows:

$$P \subseteq Q \Leftrightarrow \overline{P} \subseteq Q \Leftrightarrow {}^{\perp}Q \subseteq {}^{\perp}P.$$

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Observe that the polar maps are anti-monotonic bijections between $\mathcal{P}_{\perp}(\Lambda)$ and $\mathcal{P}_{\perp}(\Pi)$, each inverse of the other.

Notation 4.5.
We notice that as we will need to use the order reverse to the inclusion in $\mathcal{P}(\Pi)$, we introduce the notation: for $P, Q \in \mathcal{P}(\Pi)$, $P \leqslant Q$ if and only if $P \supseteq Q$. If $f \perp P, f \in \Lambda$ then $F := \{f\}^{\perp} \supseteq \overline{P} \supseteq P$. Hence, if $f \in \Lambda$ and $P \subseteq \Pi$, then the inequality $F \leqslant P$ in $\mathcal{P}_{\perp}(\Pi)$ is equivalent to the assertion $f \perp P$.

Below, we define another closure operator named as c_A. See Definition 2.11 for the basic concepts related to closure operators. The notation c_A is due to the fact that it is an Alexandroff operator canonically associated to c – see Definition 2.13 and Theorem 4.12.

Definition 4.6.

$$P \mapsto \mathrm{c}_A(P) := \widehat{P} := \bigcup \{\overline{\pi} : \pi \in P\} : \mathcal{P}(\Pi) \to \mathcal{P}(\Pi).$$

Where we use $\overline{\pi}$ as a shorthand for $\overline{\{\pi\}}$. Also, when it is needed, we write $(P)^{\wedge}$ instead of $\widehat{P}$.
The family of closed subsets associated to this closure operation is denoted as follows:

$$\mathcal{P}_{\bullet}(\Pi) = \{P \subseteq \Pi : \widehat{P} = P\}.$$

We define now an interior operator as follows:

Definition 4.7.

$$P \mapsto \mathrm{i}_{\mathrm{c}_A}(P) := \widetilde{P} := \{\pi \in P : \overline{\pi} \subseteq P\} : \mathcal{P}(\Pi) \to \mathcal{P}(\Pi).$$

When P is a too wide expression, we write $(P)\widetilde{\ }$ instead of $\widetilde{P}$.
Notice –see the Theorem 4.12– that we also have

$$\mathcal{P}_{\bullet}(\Pi) = \{P \subseteq \Pi : \widetilde{P} = P\}.$$

Observation 4.8. The closure operator $\mathrm{c} = \overline{(-)}$ defines what is called the *specialization order* in Π: $\pi \leqslant \rho :\Leftrightarrow \pi \in \overline{\rho}$. A subset P of Π is called a lower set with respect to the specialization order if $\pi \in P$ whenever $\pi \leqslant \rho$ and $\rho \in P$. It is clear that $\mathrm{c}_A(P)$ is the smallest lower set containing P and $\mathrm{i}_{\mathrm{c}_A}$ is the largest lower set contained in P.

Observation 4.9. In explicit terms one can write

$$\widehat{P} = \bigcap \{Q \in \mathcal{P}_{\bullet}(\Pi), P \subseteq Q\} \quad ; \quad \widetilde{P} = \bigcup \{Q \in \mathcal{P}_{\bullet}(\Pi), Q \subseteq P\}.$$

Observation 4.10.
We have the following reflective properties:
if $P \in \mathcal{P}(\Pi)$ and $Q \in \mathcal{P}_{\bullet}(\Pi)$, then

$$P \subseteq Q \Leftrightarrow \widehat{P} \subseteq Q.$$

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$P \in \mathcal{P}_\bullet(\Pi)$ and $Q \subseteq \mathcal{P}(\Pi)$, then

$$P \subseteq Q \Leftrightarrow P \subseteq \widetilde{Q}.$$

Observation 4.11. The operators that appear in Definitions 4.3, 4.6, 4.7, satisfy the following:

1. $\mathrm{c}, \mathrm{c}_A : \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ are closure operators; c_A satisfies Alexandroff condition and hence it is a topological closure operator;
2. $\mathrm{i}_{\mathrm{c}_A} : \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ is the interior operator whose open sets are the same than the closed sets for c_A. Moreover, the topology in Π associated to the closure operator c_A is an *Alexandroff topology* (cf. Definition 2.11).

The proof of these assertions follows directly from the adjunction properties mentioned above, of the operators: $P \mapsto \overline{P}, P \mapsto \widehat{P}, P \mapsto \widetilde{P}$.

Theorem 4.12. Consider the operators $\mathrm{c}, \mathrm{c}_A, \mathrm{i}_{\mathrm{c}_A}$ on $\mathcal{P}(\Pi)$ an let $P \subseteq \Pi$ be an arbitrary subset:

1. $\mathrm{i}_{\mathrm{c}_A} \leqslant \mathrm{id} \leqslant \mathrm{c}_A \leqslant \mathrm{c}$ or equivalently $\widetilde{P} \subseteq P \subseteq \widehat{P} \subseteq \overline{P}$;
2. The operator c_A is the largest Alexandroff closure operator d such that $\mathrm{id} \leqslant \mathrm{d} \leqslant \mathrm{c}$.
3. $\mathcal{P}_\perp(\Pi) \subseteq \mathcal{P}_\bullet(\Pi) \subseteq \mathcal{P}(\Pi)$;
4. $\mathrm{c} = \mathrm{c}\,\mathrm{c}_A = \mathrm{c}_A\,\mathrm{c}$ or equivalently ${}^\perp P = {}^\perp(\widehat{P})$ and $\overline{P} = \overline{\widehat{P}}$,
5. The following are equivalent:
   (a) $P \in \mathcal{P}_\bullet(\Pi)$
   (b) $\pi \in P \Leftrightarrow \overline{\pi} \subseteq P$.
   (c) $\widetilde{P} = P$
6. If $P \subseteq \Pi$, then $P \in \mathcal{P}_\perp(\Pi) \Leftrightarrow \widetilde{P} = P = \widehat{P} = \overline{P}$.

Proof.

1. We only need to prove that $\mathrm{c}_A \leqslant \mathrm{c}$. Observe that ${}^\perp P = \bigcap\{{}^\perp\{\pi\} : \pi \in P\}$. Thus $\overline{P} = ({}^\perp P)^\perp = (\bigcap\{{}^\perp\{\pi\} : \pi \in P\})^\perp \supseteq \bigcup\{\overline{\pi} : \pi \in P\} = \widehat{P}$.
2. Assume that d is an Alexandroff operator as in the hypothesis. Hence, $\mathrm{d}(P) = \mathrm{d}(\bigcup\{\{x\} : x \in P\} = \bigcup\{\mathrm{d}(\{x\}) : x \in P\} \subseteq \bigcup\{\mathrm{c}(\{x\}) : x \in P\} = \mathrm{c}_A(P)$.
3. It follows directly from (4.12) that if $\mathrm{c}(P) = P$, then $\mathrm{c}_A(P) = P$.
4. It follows directly from (4.12) and the fact that $\mathrm{c}^2 = \mathrm{c}$.

The proof of the remaining items is direct. □

XVI. *Order and Completeness.* If $\mathcal{R} = (\Lambda, \Pi, \perp\!\!\!\perp)$ is as above, all the three sets: $\mathcal{P}_\perp(\Pi) \subseteq \mathcal{P}_\bullet(\Pi) \subseteq \mathcal{P}(\Pi)$ are complete with respect to the order $\subseteq$. Indeed, as we mentioned before, the functors $(-)^\wedge : \mathcal{P}(\Pi) \to \mathcal{P}_\bullet(\Pi)$ and $(-)^- : \mathcal{P}(\Pi) \to \mathcal{P}_\perp(\Pi)$ are reflections of $\mathcal{P}(\Pi)$ into $\mathcal{P}_\bullet(\Pi)$ and $\mathcal{P}_\perp(\Pi)$, respectively, and the considerations that follow about the form of the product and coproduct (or inf and sup) in $\mathcal{P}_\bullet(\Pi)$ and $\mathcal{P}_\perp(\Pi)$ are particular cases of general results about the construction of limits and colimits in reflective subcategories – see Borceaux (2008a) and Borceaux (2008b) pg. 118 and pg. 196, respectively–.

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1. It is clear that for $\mathcal{X} \subseteq \mathcal{P}(\Pi)$, $\sup(\mathcal{X}) = \bigcup \mathcal{X}$, $\inf(\mathcal{X}) = \bigcap \mathcal{X}$.
2. For $\mathcal{X} \subseteq \mathcal{P}_\bullet(\Pi)$, $\sup_\bullet(\mathcal{X}) = \bigcup \mathcal{X}$, $\inf_\bullet(\mathcal{X}) = \bigcap \mathcal{X}$. The property for the $\inf_\bullet$ follows by general considerations, the one concerning $\sup_\bullet$ follows from Alexandroff condition.
3. For $\mathcal{X} \subseteq \mathcal{P}_\perp(\Pi)$, $\sup_\perp(\mathcal{X}) = \overline{\bigcup \mathcal{X}}$, $\inf_\perp(\mathcal{X}) = \bigcap \mathcal{X}$. This again follows from the general facts mentioned before.

XVII. We present the following example:

Example 4.13. Consider the triple $(\Lambda, \Pi, \perp\!\!\!\perp)$ with $\Lambda = \Pi = \mathbb{R}^3$, and $\perp\!\!\!\perp$ being the usual perpendicularity relation.
In this context for $P \subseteq \mathbb{R}^3$, $\overline{P}$ is the linear subspace generated by P and $\widehat{P}$ is the cone with vertex in the origin generated by P. Then, $\mathcal{P}_\perp(\mathbb{R}^3)$ is the set of linear subspaces and $\mathcal{P}_\bullet(\mathbb{R}^3)$ is the set of cones.

XVIII. We recall the following definition:

Definition 4.14. *Realizability lattice.* A quadruple $\left(\Lambda, \Pi, \perp\!\!\!\perp, \text{push}\right)$ consisting of a pair of sets Λ – set of terms – Π – set of stacks –, a subset $\perp\!\!\!\perp \subseteq \Lambda \times \Pi$ and a function $\text{push} : \Lambda \times \Pi \to \Pi$ is called a *realizability lattice* a.k.a. RL. The family of the realizability lattices is denoted as $\mathcal{RL}$ and one of its generic elements will be called $\mathcal{R}$. We also use the notation $\text{push}(t, \pi) = t \cdot \pi$ for $t \in \Lambda$ and $\pi \in \Pi$.

XIX. *The operations of application and implication.*
Below we look at three pairs of operations. As the role of the first and third operations in the formalization by Streicher of realizability theory (see Streicher (2013)) have been considered in (Ferrer Santos *et al.* 2015, Section 2), we concentrate our attention upon the new set $\mathcal{P}_\bullet(\Pi)$ and the new operations therein that we call application ($\circ_\bullet$) and implication ($\to_\bullet$). Both operations are defined in terms of the original push map in the RL, the implication is just the push composed with the perpendicularity, and the application is a *conductor* associated to the push.

Definition 4.15. For $P, Q \in \mathcal{P}(\Pi)$ we define the following subsets and maps, given in terms of the *push* operation and called the implication maps:

$$\begin{aligned}
\to &: \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi) \ , \ P \to Q &:= \{t \cdot \pi : t \perp P, \pi \in Q\} = {}^\perp P \cdot Q \\
\to_\bullet &: \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}_\bullet(\Pi) \ , \ P \to_\bullet Q &:= (P \to Q)^\wedge = \bigcup \{\overline{t \cdot \pi} : t \perp P, \pi \in Q\} \\
\to_\perp &: \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}_\perp(\Pi) \ , \ P \to_\perp Q &:= (P \to Q)^-.
\end{aligned}$$

For $P, Q \in \mathcal{P}(\Pi)$ we define the following *conductor–type* operations of associated to ${}^\perp Q$ and P and called the application maps:

$$\begin{aligned}
\circ &: \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi), \ P \circ Q &:= \{\pi \in \Pi : Q \to \pi \subseteq P\} = \{\pi \in \Pi : {}^\perp Q \cdot \pi \subseteq P\} \\
\circ_\bullet &: \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}_\bullet(\Pi), \ P \circ_\bullet Q &:= \{\pi \in \Pi : Q \to \overline{\pi} \subseteq P\} = \{\pi \in \Pi : {}^\perp Q \cdot \overline{\pi} \subseteq P\} \\
\circ_\perp &: \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}_\perp(\Pi), \ P \circ_\perp Q &:= \{\pi \in \Pi : Q \to \pi \subseteq P\}^- = (P \circ Q)^-
\end{aligned}$$

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Observation 4.16.

1. By definition the operation $\to$ is monotonic in the right argument and antimonotonic in the left argument and hence $\to_\bullet$ as well as $\to_\perp$ have the same properties.
2. By definition the operation $\circ$ is monotonic in both variables and hence $\circ_\bullet$ as well as $\circ_\perp$ have the same properties.

The following 'stability lemma' is important for future properties of the above operations.

Lemma 4.17. For $P, Q \in \mathcal{P}(\Pi)$ we have that

1. $P \to Q = \widehat{P} \to Q = \overline{P} \to Q$, and similarly for the operations $\to_\bullet$ and $\to_\perp$.
2. $P \circ Q = P \circ \widehat{Q} = P \circ \overline{Q}$, and similarly for the operations $\circ_\bullet$ and $\circ_\perp$.

Proof. Both parts follow directly from the fact that for $R \subseteq \Pi$, $R = {}^\perp\widehat{R} = {}^\perp\overline{R}$ (see Theorem 4.12, (4.12)) and the fact that the definitions of the operations in the corresponding variables involve the perpendicular of the argument. □

Next theorem addresses the question of the compatibility of the implications with respect to arbitrary unions, in fact the operations $\to$ and $\to_\bullet$ are compatible with arbitrary sup – i.e., infinite unions – and the operation $\to_\perp$ is not. In this last case, the arbitrary sup is the closure by double perpendicularity of the union of the sets.

Theorem 4.18. Let $Q \in \mathcal{P}(\Lambda)$ and $P_i \in \mathcal{P}(\Pi)$, $i \in I$, then $Q \to \bigcup_i P_i = \bigcup_i (Q \to P_i)$, $Q \to_\bullet \bigcup_i P_i = \bigcup_i (Q \to_\bullet P_i)$ and $Q \to_\perp (\bigcup_i P_i)^- \supseteq \big(\bigcup_i (Q \to_\perp P_i)\big)^-$.

Proof. For Q and P_i as above: $Q \to \bigcup_i P_i = \{t \cdot \pi : t \in Q, \pi \in \bigcup_i P_i\} = \bigcup_i \{t \cdot \pi : t \in Q, \pi \in P_i\} = \bigcup_i (Q \to P_i)$. Also, $Q \to_\bullet \bigcup_i P_i = \{t \cdot \pi : t \in Q, \pi \in \bigcup_i P_i\}^\wedge = (\bigcup_i \{t \cdot \pi : t \in Q, \pi \in P_i\})^\wedge = \bigcup_i (Q \to P_i)^\wedge = \bigcup_i (Q \to_\bullet P_i)$. The inclusion $Q \to_\perp (\bigcup_i P_i)^- \supseteq \big(\bigcup_i (Q \to_\perp P_i)\big)^-$ follows from the monotonicity of $\to$ in the second variable. □

Notice that the preservation of limits shown above, guarantees that the implication maps have the applications as left adjoints.

Observation 4.19. In general $Q \to_\perp (\bigcup_i P_i)^- \neq \big(\bigcup_i (Q \to_\perp P_i)\big)^-$. This is illustrated by the example: $\Lambda = \Pi = \mathbb{R}^3$, $Q = {}^\perp(\mathbb{R}e_3)$, $P_1 = \mathbb{R}e_1$, $P_2 = \mathbb{R}e_2$, $P_3 = \mathbb{R}(e_1 + e_2 + e_3)$, push $: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is $v \cdot w := \langle v, w \rangle v \wedge w$ the first is the usual scalar product and the second the vector product. $Q \to_\perp P_1 = Q \to_\perp P_2 = \{0\}$, $Q \to_\perp P_3 = \mathbb{R}(e_2 - e_1)$. Hence: $(\bigcup_i (Q \to_\perp P_i))^- = \mathbb{R}(e_2 - e_1)$. Also, $(P_1 \cup P_2 \cup P_3)^- = \mathbb{R}^3$. Then, $Q \to_\perp (\bigcup_i P_i)^- = \mathbb{R}e_1 + \mathbb{R}e_2$.

Observation 4.20. It is important to remark that there is a crucial difference in the definition of the pair of operations $(\circ_\perp, \to_\perp)$ and the pair $(\circ_\bullet, \to_\bullet)$ (see the table below). In the first row of the table, the operations are given in terms of the closure c while in the second row we use both the *interior* operator $\mathrm{i}_{\mathrm{c}_A}$ and the associated closure c_A.

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Conductor/Application	*Push/Implication*
$P \circ_{\perp} Q = \mathrm{c}(P \circ Q)$	$P \to_{\perp} Q = \mathrm{c}(P \to Q)$
$P \circ_{\bullet} Q = \mathrm{i}_{\mathrm{c}_A}(P \circ Q)$	$P \to_{\bullet} Q = \mathrm{c}_A(P \to Q)$

Observation 4.21.

1. Applying (4.20) and (4.12)(4.12), we get the following table:

Conductor/Application	*Push/Implication*
$P \circ_{\perp} Q \supseteq P \circ Q \supseteq P \circ_{\bullet} Q$	$P \to_{\perp} Q \supseteq P \to_{\bullet} Q \supseteq P \to Q$

2. The reflection properties of the closures and of the interior operator – see Paragraph XV, observations (4.4), (4.10) and (4.10) – leads to the following tables:

   (a) If $R \in \mathcal{P}_{\perp}(\Pi)$ and $P, Q \in \mathcal{P}(\Pi)$:

Conductor/Application	*Push/Implication*
$R \supseteq P \circ Q \Leftrightarrow R \supseteq P \circ_{\perp} Q$	$R \supseteq P \to Q \Leftrightarrow R \supseteq P \to_{\perp} Q$

   (b) If $R \in \mathcal{P}_{\bullet}(\Pi)$ and $P, Q \in \mathcal{P}(\Pi)$:

Conductor/Application	*Push/Implication*
$P \circ Q \supseteq R \Leftrightarrow P \circ_{\bullet} Q \supseteq R$	$R \supseteq P \to Q \Leftrightarrow R \supseteq P \to_{\bullet} Q$

XX. *The adjunction of the new operators: First approximation.* We have the following adjunction result for $\mathcal{R}$, a realizability lattice.

1. The maps $\circ : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ and $\to : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$, satisfy the *full adjunction property*: $P \supseteq Q \to R$ if and only if $P \circ Q \supseteq R$.
2. (a) The maps $\circ_{\perp} : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}_{\perp}(\Pi)$ and $\to_{\perp} : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}_{\perp}(\Pi)$ satisfy the *half adjunction property*: $P, R \in \mathcal{P}_{\perp}(\Pi)$ and $Q \in \mathcal{P}(\Pi)$, and $P \supseteq Q \to_{\perp} R$, then $P \circ_{\perp} Q \supseteq R$.

   (b) The other *half* of the adjunction property can be partially retrieved using the combinator E as follows – see Ferrer Santos *et al.* (2015, Section 2,§2.3, Theorem 2.13) (and for another proof with a different adjunctor, see Theorem 5.12 below):

$$\text{If} \quad P \circ_{\perp} Q \supseteq R \quad \text{then} \quad \{\mathsf{E}\}^{\perp} \circ_{\perp} P \supseteq Q \to_{\perp} R.$$

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Theorem 4.22. The maps $\circ_\bullet : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}_\bullet(\Pi)$, $\to_\bullet : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}_\bullet(\Pi)$ satisfy the *adjunction property*: If $P, R \in \mathcal{P}_\bullet(\Pi)$ and $Q \in \mathcal{P}(\Pi)$, then $P \supseteq Q \to_\bullet R$ if and only if $P \circ_\bullet Q \supseteq R$.

Proof. In the above situation $P \supseteq Q \to_\bullet R$, if and only if $P \supseteq Q \to R$ by Observation 4.21, (4.21); which is equivalent to $P \circ Q \supseteq R$ by item (1) above; which is equivalent $P \circ_\bullet Q \supseteq R$ again by Observation 4.21, (4.21). □

Observation 4.23. Consider four closure operators as below:

$$\mathrm{id} \leqslant \mathrm{d} \leqslant \mathrm{c}_A \leqslant \mathrm{c},$$

where c is the operator given by double perpendicularity, and the other three are respectively, the trivial closure operator id, an arbitrary Alexandroff closure operator d and the maximal Alexandroff closure operators presented in Definition 4.6 (see also Theorem 4.12 for some of its basic properties). One has – for some of them – the corresponding interior operators:

$$\mathrm{i}_{\mathrm{c}_A} \leqslant \mathrm{i}_{\mathrm{d}} \leqslant \mathrm{id}\,.$$

Moreover, if we name as before the corresponding closed sets for the different closure operators we have that:

$$\mathcal{P}_\perp(\Pi) \subseteq \mathcal{P}_\bullet(\Pi) \subseteq \mathcal{P}_{\mathrm{d}}(\Pi) \subseteq \mathcal{P}(\Pi).$$

1. In the same manner than for c_A, one can define the operations $- \circ_{\mathrm{d}} - = \mathrm{i}_{\mathrm{d}}(- \circ -)$ and $- \to_{\mathrm{d}} - = \mathrm{d}(- \to -)$.
   See the table below

Conductor/Application	*Push/Implication*
$P \circ Q$	$P \to Q$
$P \circ_\bullet Q = \mathrm{i}_{\mathrm{c}_A}(P \circ Q)$	$P \to_\bullet Q = \mathrm{c}_A(P \to Q)$
$P \circ_{\mathrm{d}} Q = \mathrm{i}_{\mathrm{d}}(P \circ Q)$	$P \to_{\mathrm{d}} Q = \mathrm{d}(P \to Q)$

$$P \circ_\bullet Q \subseteq P \circ_{\mathrm{d}} Q \subseteq P \circ Q \subseteq P \circ_\perp Q \qquad P \to Q \subseteq P \to_{\mathrm{d}} Q \subseteq P \to_\bullet Q \subseteq P \to_\perp Q$$

2. The proof of Theorem 4.22 can be mimicked in this situation and we obtain the full adjunction for the operation $\to_{\mathrm{d}}$ and $\circ_{\mathrm{d}}$: if $P, R \in \mathcal{P}_{\mathrm{d}}(\Pi)$ and $Q \in \mathcal{P}(\Pi)$, then

   $$P \supseteq Q \to_{\mathrm{d}} R \text{ if and only if } P \circ_{\mathrm{d}} Q \supseteq R.$$

3. The family of inclusions

   $$\mathcal{P}_\perp(\Pi) \subseteq \mathcal{P}_\bullet(\Pi) \subseteq \mathcal{P}_{\mathrm{d}}(\Pi) \subseteq \mathcal{P}(\Pi)$$

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mean that if we restrict the application and implication to the different sets of subsets of Π (in the same manner as above), the minimal framework where we can obtain full adjunction and include the sets closed by double perpendicularity is $\mathcal{P}_\bullet(\Pi)$. Thus, the consideration of the closure given by the bullet construction appears as a natural continuation of Streicher's and Krivine's work.

XXI. Recall Example 4.13 and Theorem 4.18, where $\Lambda = \Pi = \mathbb{R}^3$, the push is $v \cdot w = \langle v, w\rangle v \wedge w$, $v, w \in \mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ is a positive orthonormal basis.

1. We construct an example where $P, Q \in \mathcal{P}_\bullet(\Pi)$ and $P \circ Q = P \circ_\bullet Q \neq P \circ_\perp Q$. Take $Q = {}^\perp(\mathbb{R}e_2)$ and $P = \mathbb{R}e_1$. Then: $P \circ Q = \{w \in \mathbb{R}^3 : \langle e_2, w\rangle e_2 \wedge w \in \mathbb{R}e_1\}$. If $\langle e_2, w\rangle = 0$, then $w \in \mathbb{R}e_1 + \mathbb{R}e_3$, otherwise $e_2 \wedge w \in \mathbb{R}e_1$ if and only if $w \in \mathbb{R}e_2 + \mathbb{R}e_3$. Then, $P \circ Q = P \circ_\bullet Q = (\mathbb{R}e_1 + \mathbb{R}e_3) \cup (\mathbb{R}e_2 + \mathbb{R}e_3) \neq P \circ_\perp Q = \mathbb{R}^3$.
2. Moreover, the full adjunction property does not hold in this situation for the operators $\to_\perp$ and $\circ_\perp$. Indeed, we have that if we take P, Q as above and $R = \mathbb{R}(e_1 + e_2)$ a direct computation shows that: $R = \mathbb{R}(e_1 + e_2) \subseteq P \circ_\perp Q = \mathbb{R}^3$, but $Q \to_\perp \mathbb{R}(e_1 + e_2) = \mathbb{R}e_3$ while $P = \mathbb{R}e_1$.
3. Next, we give an example where $P, Q \in \mathcal{P}_\perp(\Pi)$ but $P \circ_\bullet Q$ and $P \circ Q$ are different.
   In the same context than above, we take $\mathrm{push}(v, w) = \langle v, w - w_0\rangle v \wedge w$ where $w_0 \notin \mathbb{R}e_1 + \mathbb{R}e_3$, also consider $Q = {}^\perp(\mathbb{R}e_2)$ and $P = \mathbb{R}e_1$. Then, $P \circ_\bullet Q = \mathbb{R}e_2 + \mathbb{R}e_3$, $P \circ Q = (\mathbb{R}e_1 + \mathbb{R}e_3 + w_0) \cup (\mathbb{R}e_2 + \mathbb{R}e_3)$.

5. Abstract Krivine structures: Operations and combinators

XXII. Recall the definition of an AKS a.k.a. AKS (see J.L. Krivine Krivine (2008) and T. Streicher Streicher (2013)). We follow the notations and definitions therein with slight modifications as presented in Ferrer Santos *et al.* (2015).

Definition 5.1. Let $\mathcal{R} = (\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{push})$ be an RL, and assume that we have the following additional elements:

1. A function $\mathrm{app} : \Lambda \times \Lambda \to \Lambda$ (denoted also as app when is viewed as a map of type $\mathrm{app} : \mathcal{P}(\Lambda) \times \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda)$) such that for all $t, s \in \Lambda, \pi \in \Pi$ if $t \perp s \cdot \pi$, then $ts \perp \pi$.
2. A set $\mathrm{QP} \subseteq \Lambda$ of 'quasi–proofs', which is closed under application and contains two elements $\mathsf{k}, \mathsf{s} \in \mathrm{QP} \subseteq \Lambda$, such that:
   (a) For all $t, s \in \Lambda, \pi \in \Pi$, if $t \perp \pi$ then $\mathsf{k} \perp t \cdot s \cdot \pi$;
   (b) For all $t, s, u \in \Lambda, \pi \in \Pi$, if $tu(su) \perp \pi$ then $\mathsf{s} \perp t \cdot s \cdot u \cdot \pi$.

Frequently we abbreviate $\mathrm{app}(t, s) = ts$ for $t, s \in \Lambda$ and concerning this operation we use the convention that associates to the left when operating with more than three elements. The structure thus obtained is called an AKS –abbreviated as AKS – and

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usually $\mathcal{K}$ will denote a generic AKS.[¶] We also denote the class of all AKS with the symbol $\mathcal{AKS}$.

Definition 5.2. A *strong AKS* is an AKS satisfying the following extra condition: for all $t, s \in \Lambda, \pi \in \Pi$ if $ts \perp \pi$ then $t \perp s \cdot \pi$.

XXIII. We introduce a new operation $\diamond : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)$ that will be handy in order to deal with the basic reduction rule of an AKS: '*If* $t \perp s \cdot \pi$, *then* $ts \perp \pi$' as well as with the axioms for κ and s.

Definition 5.3. We define the operation $\diamond : \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) \to \mathcal{P}_\perp(\Pi)$ as: $P \diamond Q := ({}^\perp P {}^\perp Q)^\perp$.

Observe that $P \diamond Q$ is in $\mathcal{P}_\perp(\Pi)$ because it is the perpendicular of a subset of Λ. This operation is really the 'transfer' of the operation app from $\mathcal{P}(\Lambda)$ to $\mathcal{P}(\Pi)$ through the perpendicularity map, as the diagram below illustrates:

$$\begin{array}{ccc} \mathcal{P}(\Pi) \times \mathcal{P}(\Pi) & \xrightarrow{\diamond} & \mathcal{P}_\perp(\Pi) \\ {\scriptstyle {}^\perp(-)\times{}^\perp(-)}\downarrow & & \uparrow{\scriptstyle (-)^\perp} \\ \mathcal{P}(\Lambda) \times \mathcal{P}(\Lambda) & \xrightarrow[\text{app}]{} & \mathcal{P}(\Lambda) \end{array}$$

Lemma 5.4. The following statements hold in an AKS

1. $P \circ_\bullet Q \subseteq P \circ Q \subseteq P \circ_\perp Q \subseteq P \diamond Q$ for all $P, Q \in \mathcal{P}(\Pi)$.
2. For $P, Q \subseteq \Pi$ if $t \perp P$ and $\ell \perp Q$ then $t\ell \perp P \circ_\perp Q$.

Moreover, condition (1) implies that if $t \perp P$ and $\ell \perp Q$ then $t\ell \perp P \circ_\bullet Q$.

Proof.

1. To prove condition (1) we take $\pi \in P \circ Q$, $t \perp P$ and $\ell \perp Q$. Since $\pi \in P \circ Q$ then $\ell \cdot \pi \in P$, then $t \perp \ell \cdot \pi$. Applying the basic reduction rule of an AKS, we get $t\ell \perp \pi$ and hence $\pi \in P \diamond Q$. As $P \diamond Q$ is closed with respect to c the inclusion $P \circ_\perp Q \subseteq P \diamond Q$ follows. The other inclusion is always true.
2. It is clear that once we know that equation (1) holds, it follows that ${}^\perp P {}^\perp Q \subseteq {}^\perp(P \circ_\perp Q)$, and vice versa. □

Observation 5.5. In the situation that the AKS is strong, Lemma 5.4 (5.4), can be reinforced and we have for all $P \in \mathcal{P}_\perp(\Pi)$ and $Q \subseteq P$ the equality $P \circ_\bullet Q = P \diamond Q$. In accordance with Lemma 5.4 all we have to prove is that in the hypothesis that the AKS is strong, then $P \diamond Q \subseteq P \circ_\bullet Q$ where $P \circ_\bullet Q = \{\pi \in \Pi : {}^\perp Q \cdot \overline{\pi} \subseteq P\}$. Assume that we take $\pi \in \Pi$ such that $ts \perp \pi$ for all $t \perp P$, $s \perp Q$ and then, we need to prove that for all $\sigma \in \overline{\pi}$, ${}^\perp Q.\sigma \subseteq P$. But if $ts \perp \pi$ as $\{\sigma\}^\perp \supseteq \{\pi\}^\perp$ we have that $ts \perp \sigma$ and

[¶] *Strictu sensu*, the standard definition of *AKS* includes additional structure – see Ferrer Santos *et al.* (2013) or Streicher (2013). We only take the basic elements we need for our present considerations.

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then $t \perp s \cdot \sigma$. This last relation means – as s and t are generic in their respective sets – that ${}^{\perp}P \subseteq {}^{\perp}({}^{\perp}Q \cdot \sigma)$. Taking perpendiculars, we obtain that ${}^{\perp}Q \cdot \sigma \subseteq ({}^{\perp}P)^{\perp} = P$.[|]

Observation 5.6. Concerning the chain of inclusions $P \circ_{\bullet} Q \subseteq P \circ Q \subseteq P \circ_{\perp} Q \subseteq P \diamond Q$, we observe that the first two inclusions are related only to the original RL structure. When we furnish the original RL with the additional structure that defines an AKS and that allows the insertion of a calculus with terms, one can define the $\diamond$ operation that is larger than the original ones. The last inclusion is closely related to the basic 'reduction rule' in an AKS (see Definition 5.1).

Next we recall for future usage, the definitions and properties of the basic as well as some additional combinators that are also elements of $\mathrm{QP} \subseteq \Lambda$ (see Ferrer Santos *et al.* (2015, Section 2, 2.3, Lemma 2.12)).
For all $t, u, v \in \Lambda$, $\pi \in \Pi$ and $P, Q, R \in \mathcal{P}(\Pi)$:

K	$t \perp \pi \Rightarrow \mathsf{K} \perp t \cdot u \cdot \pi$	$\mathsf{K} \perp P \to (Q \to P)$
S	$(tu)(vu) \perp \pi \Rightarrow \mathsf{S} \perp t \cdot v \cdot u \cdot \pi$	$\mathsf{S} \perp (P \to Q \to R) \to (P \to Q) \to (P \to R)$
$\mathsf{I} = \mathsf{SKK}$	$t \perp \pi \Rightarrow \mathsf{I} \perp t \cdot \pi$	$\mathsf{I} \perp P \to P$
$\mathsf{E} = \mathsf{S}(\mathsf{KI})$	$tu \perp \pi \Rightarrow \mathsf{E} \perp t \cdot u \cdot \pi$	$\mathsf{E} \perp P \to (Q \to (P \diamond Q))$
$\mathsf{B} = \mathsf{S}(\mathsf{KS})\mathsf{K}$	$t \perp (uv) \cdot \pi \Rightarrow \mathsf{B} \perp t \cdot u \cdot v \cdot \pi$	$\mathsf{B} \perp (Q \to R) \to (P \to Q) \to P \to R$

Lemma 5.7. For all $P, Q \in \mathcal{P}(\Pi)$:

1. $\mathsf{EE} \perp P \to_{\bullet} (Q \to_{\bullet} (P \diamond Q))$.
2. $P \diamond Q \quad \subseteq \quad (\{\mathsf{EE}\}^{\perp} \circ_{\bullet} P) \circ_{\bullet} Q \quad \subseteq \quad \left(\mathsf{E}\mathsf{E}({}^{\perp}P)\right)^{\perp} \circ_{\bullet} Q$.

Proof.

1. Take $t \perp P$, $\ell \perp Q$ and $\pi \in P \diamond Q = ({}^{\perp}P {}^{\perp}Q)^{\perp}$, hence $t\ell \perp \pi$ and it follows from the above table that $\mathsf{E} \perp t \cdot \ell \cdot \pi$ that implies that $\mathsf{E}t \perp \ell \cdot \pi$ or in other words that $\mathsf{E}t \perp {}^{\perp}Q \cdot (P \diamond Q) = Q \to (P \diamond Q)$. It follows from Theorem 4.12 that for a general $R \subseteq \Pi$, ${}^{\perp}R = {}^{\perp}R^{\frown}$ and in our situation this implies that $\mathsf{E}t \perp Q \to_{\bullet} (P \diamond Q)$, or equivalently that for $t \perp P$ and $\rho \in Q \to_{\bullet} (P \diamond Q)$ $\mathsf{E}t \perp \rho$. Repeating the procedure – in this case for E and t as before was for t and ℓ – we deduce that $\mathsf{EE} \perp P \to_{\bullet} (Q \to_{\bullet} (P \diamond Q))$.
2. If we write the perpendicularity relation just proved as: $\{\mathsf{EE}\}^{\perp} \supseteq P \to_{\bullet} (Q \to_{\bullet} (P \diamond Q))$, and then apply twice the full adjunction between $\circ_{\bullet}$ and $\to_{\bullet}$ we obtain $P \diamond Q \quad \subseteq \quad (\{\mathsf{EE}\}^{\perp} \circ_{\bullet} P) \circ_{\bullet} Q$. The second inclusion of (2) is a consequence of Lemma 5.4 applied to $\{\mathsf{EE}\}^{\perp} \circ_{\perp} P$. □

[|] The authors thank one of the referees for suggesting this result, that is stronger than the one proposed in the original version of the manuscript.

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XXIV. *The adjunctor*. For the IOCA defined over $\mathcal{P}_\perp(\Pi)$, $\{\mathsf{E}\}^\perp$ is an adjunctor (see (Streicher 2013, Lemma 3.2) and (Ferrer Santos *et al.* 2015, Definition 2.14)). In this paragraph, we sketch a different proof of the existence of an adjunctor, using the new operations $\circ_\bullet$, $\to_\bullet$.

In this manner, we obtain, similarly than in Ferrer Santos *et al.* (2015), control over the 'other half' of the adjunction between the functors $\circ_\perp$ and $\to_\perp$ – compare with (Ferrer Santos *et al.* 2015, Section 2,§ 2.3, Theorem 2.13). More precisely, if we consider the category given by the set $\mathcal{P}_\perp(\Pi)$ ordered by the reverse inclusion, we aspire to have for every Q an adjunction $- \circ_\perp Q \dashv Q \to_\perp -$.

However, we can only obtain *one half of the adjunction*

$$P \leqslant Q \to_\perp R \quad \Longrightarrow \quad P \circ_\perp Q \leqslant R :$$

and for the other half we need the adjunctor

$$P \circ_\perp Q \leqslant R \quad \Longrightarrow \quad \{\mathsf{E}\}^\perp \circ_\perp P \leqslant Q \to_\perp R.$$

Definition 5.8. For an AKS, we say that a term Ex is an *η-expansor* if it satisfies $\forall t, u \in \Lambda$ if $tu \perp \pi$ then $\mathsf{Ex} \perp t \cdot u \cdot \pi$

Observation 5.9. For a general AKS, we have that $\forall t, u \in \Lambda$, $\forall \pi \in \Pi$ if $t \perp u \cdot \pi$ then $tu \perp \pi$, but the converse proposition is not true in general and in fact its validity characterizes the class of strong AKSes. A modification of the converse proposition is obtained by introducing an η-expansor that guarantees that: if $tu \perp \pi$ then $\mathsf{Ex}\, t \perp u \cdot \pi$. The usual η-expansor of λ-calculus ($\lambda x \lambda y.xy$) when translated into a combinatory term is an η-expansor in the sense of definition 5.8 (Paragraph X, Ferrer Santos *et al.* (2015) and Streicher (2013)). Observe that if Ex is an η-expansor so is $\mathsf{Ex}\mathsf{Ex}$.

Definition 5.10. For an AKS, define a map $P \mapsto \eta P : \mathcal{P}(\Pi) \to \mathcal{P}_\bullet(\Pi)$ by the formula: $\eta P = \{\mathsf{E}\,\mathsf{E}\}^\perp \circ_\bullet P$, where the element E is the η expansor defined as $\mathsf{S}(\mathsf{K}\,\mathsf{I})$ (c.f.: item X).

Observation 5.11. Assume that $P, Q \in \mathcal{P}(\Pi)$.

1. Using the definition above the result of Lemma 5.7 can be written as follows:

$$P \diamond Q \subseteq \eta P \circ_\bullet Q.$$

2. Putting together all the inequalities already obtained, we have that

$$\begin{array}{ccccccccc} P \circ_\bullet Q & \subseteq & P \circ Q & \subseteq & P \circ_\perp Q & \subseteq & P \diamond Q & \subseteq \\ \eta P \circ_\bullet Q & \subseteq & \eta P \circ Q & \subseteq & \eta P \circ_\perp Q & \subseteq & \eta P \diamond Q \end{array}$$

XXV. Next, we show that in the case that we work on $\mathcal{P}_\perp$, we retrieve partially the adjunction property – compare with Paragraph XX. The name *adjunctor* for the combinator E is justified by its role in the inclusions below. See (Ferrer Santos *et al.* 2015, Section 2, §2.3, Theorem 2.13) for a proof that does not use the bullet product and that is more precise in the sense that uses the simpler combinator E.

Theorem 5.12. Assume that $P, Q, R \in \mathcal{P}_\perp(\Pi)$. Then

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1. If $Q \to_\perp R \subseteq P$ then $R \subseteq P \circ_\perp Q$.
2. If $R \subseteq P \circ_\perp Q$ then $Q \to_\perp R \subseteq \overline{\eta P} = \overline{\{\mathsf{EE}\}^\perp \circ_\bullet P} \subseteq \{\mathsf{EE}\}^\perp \circ_\perp P$.

Proof. We concentrate in the second assertion: it follows from Observation 5.11, that $R \subseteq P \circ_\perp Q \subseteq P \diamond Q \subseteq \eta P \circ_\bullet Q$. From the adjunction property for $\circ_\bullet$ and $\to_\bullet$ –see Paragraph XX – we deduce that $Q \to_\bullet R \subseteq \eta P$. Taking double perpendicularity, we deduce that $Q \to_\perp R \subseteq \overline{\eta P}$ and the other inclusion follows from Observation 5.11. □

Observation 5.13. It is also true that if the AKS is strong, then the full adjunction holds for $\mathcal{P}_\perp(\Pi)$, i.e., $P \circ_\perp Q \supseteq R$ if and only if $P \supseteq Q \to_\perp R$. Indeed all we need to prove (see Observation 5.5) is that $R \subseteq P \diamond Q$ implies that $Q \to_\perp R \subseteq P$. This will follow if we prove: $Q \to_\perp (P \diamond Q) \subseteq P$ which is equivalent to $Q \to (P \diamond Q) \subseteq P$ or ${}^\perp Q.({}^\perp P {}^\perp Q)^\perp \subseteq P$. It is now easy to show that this inclusion can be deduced from the condition that for all t, s, π, $ts \perp \pi$ implies that $t \perp s.\pi$.

XXVI. In accordance with Definition 3.5, the implication of a complete QOCA induces an application like operation that we called $\sharp$. The theorem that follows shows that if we perform the same construction for the operation $\to_\perp$ in $\mathcal{P}_\perp(\Pi)$ with the order given by reverse inclusion, the resulting *application like* operator $\sharp$ is the closure by double perpendicular of the operator $\circ_\bullet$.

Theorem 5.14. Given an AKS consider $\mathcal{P}_\perp(\Pi)$ with the operations $\circ_\perp$ and $\to_\perp$. For $P \in \mathcal{P}_\perp(\Pi)$ and $Q \in \mathcal{P}(\Pi)$, define $P \sharp Q = \sup_\perp\{R \in \mathcal{P}_\perp(\Pi) : Q \to_\perp R \subseteq P\}$ – we are considering the $\sup_\perp$ as in Paragraph XVI–. Then $P \sharp Q = \overline{P \circ_\bullet Q}$.

Proof. We have that $P \sharp Q = \sup_\perp\{R \in \mathcal{P}_\perp(\Pi) : Q \to_\perp R \subseteq P\} = \overline{\bigcup\{R \in \mathcal{P}_\perp(\Pi) : Q \to_\perp R \subseteq P\}}$. In our situation, as $P = \overline{P}$ we have that $Q \to_\perp R \subseteq P$ if and only if $Q \to R \subseteq P$. Hence

$$\begin{aligned} P \sharp Q &= \overline{\bigcup\{R \in \mathcal{P}_\perp(\Pi) : Q \to R \subseteq P\}} = \overline{\bigcup\{R \in \mathcal{P}_\perp(\Pi) : R \subseteq P \circ Q\}} \\ &= \overline{\bigcup\{R \in \mathcal{P}_\perp(\Pi) : R \subseteq P \circ_\bullet Q\}} \end{aligned}$$

The last equality is a consequence of two facts: (a) The elements of $\mathcal{P}_\perp(\Pi)$ are also elements of $\mathcal{P}_\bullet(\Pi)$; (b) the result of 4.21, 4.21 item (b).
Now, if $\pi \in P \circ_\bullet Q$ then $\overline{\pi} \subseteq P \circ_\bullet Q$ and it follows by the above equality that $\overline{\pi} \subseteq P \sharp Q$, and hence $P \circ_\bullet Q \subseteq P \sharp Q$ and also $\overline{P \circ_\bullet Q} \subseteq P \sharp Q$.
Conversely, it is clear that $\bigcup\{R \in \mathcal{P}_\perp(\Pi) : R \subseteq P \circ_\bullet Q\} \subseteq P \circ_\bullet Q$, and taking closures we obtain that $P \sharp Q \subseteq \overline{P \circ_\bullet Q}$ □

Observation 5.15.

1. Notice that in accordance with Example 4.13, the product $P \circ_\bullet Q$ need not be closed with respect to the bar operator. Hence in general the operations $\circ_\bullet$ and $\sharp$ are different. The above proof guarantees that for all $P, Q \in \mathcal{P}_\perp(\Pi)$, we have that $P \circ_\bullet Q \subseteq P \sharp Q$.
2. In general for all $P, Q \in \mathcal{P}_\perp(\Pi)$, we have that:

$$P \circ_\bullet Q \subseteq P \sharp Q = \overline{P \circ_\bullet Q} \subseteq P \circ_\perp Q.$$

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6. From $\mathcal{AKS}$ to ${}^{\mathcal{I}}\mathcal{OCA}$ and to indexed Heyting preorders

In this section after recalling briefly, constructions appearing in Ferrer Santos *et al.* (2015) (that in fact were partial reformulations and reorganizations of constructions and results in Krivine (2001, 2003, 2008, 2009) and specially Krivine (2004) and also Streicher 2013), we present the main constructions of this paper that are based on the Alexandroff closure operator c_A, see Definition 4.6.

XXVII. From $\mathcal{AKS}$ to ${}^{\mathcal{I}}\mathcal{OCA}$

Definition 6.1. If $K = (\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{push}, \mathsf{k}, \mathsf{s}, \mathrm{Q}P)$ is an AKS call

$A(K) := (\mathcal{P}(\Pi),\ \circ,\ \to,\ \mathsf{k},\ \mathsf{s},\ \supseteq, \Phi\ )$;

$A_\perp(K) := (\mathcal{P}_\perp(\Pi), \circ_\perp, \to_\perp, \mathsf{k}_\perp, \mathsf{s}_\perp, \supseteq, \Phi_\perp)$;

$A_\bullet(K) := (\mathcal{P}_\bullet(\Pi), \circ_\bullet, \to_\bullet, \mathsf{k}_\bullet, \mathsf{s}_\bullet, \supseteq, \Phi_\bullet)$;

where

$\Phi := \{P \in \mathcal{P}(\Pi) \mid {}^\perp P \cap \mathrm{Q}P \neq \emptyset\}$ $\mathsf{k} := \{\mathsf{k}\}^\perp$ $\mathsf{s} := \{\mathsf{s}\}^\perp$;

$\Phi_\perp := \Phi \cap \mathcal{P}_\perp(\Pi)$ $\mathsf{k}_\perp := \{\mathsf{k}\}^\perp$ $\mathsf{s}_\perp := \{\mathsf{s}\}^\perp$;

$\Phi_\bullet := \Phi \cap \mathcal{P}_\bullet(\Pi)$ $\mathsf{k}_\bullet := \{\mathsf{E}\mathsf{K}\}^\perp \mathsf{s}_\bullet := \{\mathsf{E}((\mathsf{B}\,\mathsf{E})\mathsf{S})\}^\perp$.

For the combinators, see the table displayed before Lemma 5.7 and for the operations see Definition 4.15.

Observation 6.2. The reader should take into account that the order in $A(K)$ is the opposite to the inclusion order. Hence, when we view $\mathcal{P}(\Pi)$ as the basic set of an FOCA as above, the operators c and c_A should be viewed as *interior* operators rather than *closure* operators as before. Dually, the operator $\mathrm{i}_{\mathrm{c}_A}$, is in this context an Alexandroff closure operator.

Observation 6.3.

1. The construction $A_\perp(K)$ and some elements of the construction $A(K)$ were presented in Ferrer Santos *et al.* (2015). The elements of the construction $A_\bullet(K)$ appeared in Definition 4.15. Below we prove its main properties;
2. The constructions of $A(K)$ and $A_\bullet(K)$ produce FOCAs and $A_\perp(K)$ only IOCAs;
3. We call the construction involving the operations $(\circ_\perp, \to_\perp)$ defined in $\mathcal{P}_\perp(\Pi)$ the *Streicher's construction* and its arrow is a modification of Krivine's arrow obtained by closing it with double perpendicularity;
4. In this paper, $A(K)$ is called Krivine's construction as it is based on Krivine's arrow (also called Krivine's implication) $P \to Q := {}^\perp P.Q$[††].
5. It seems that the first appearance of what we call *Krivine's construction* under another (equivalent) guise but with that name, is in Miquel (2014). Therein, the set $\mathcal{P}(\Pi)$ with Krivine's implication $\to$ and ordered with reverse inclusion, is presented as an example of an *implicative algebra* that in view of the identification mentioned in IX can be viewed as an FOCA.

[††] The name *Krivine's implication* appears in different places – see, for example, Miquel (2016).

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Theorem 6.4. Let $K = (\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{push}, \mathsf{k}, \mathsf{s}, \mathrm{QP})$ be an AKS. Then, $A_\bullet(K)$ is an FOCA.

Proof. We will refer to the axioms of Definition 3.1 and the notations thereby.

(OP) Recall that $P \circ_\bullet Q = \mathrm{i}_{c_A}(P \circ Q)$ and $P \to_\bullet Q = \mathrm{c}_A(P \to Q)$ (see Observation 4.20). Then, the conditions concerning the monotonicity of the operations follow directly from the corresponding conditions for the operations $\circ$ and $\to$ and the properties of the closure and interior operator used above.

(PA)&(PE)' The adjunction of the operations follows directly from Theorem 4.22 and Definition 4.15.

(PA)&(PS) Clearly $\mathsf{k}_\bullet, \mathsf{s}_\bullet \in \mathcal{P}_\perp(\Pi) \subseteq \mathcal{P}_\bullet(\Pi)$ because both are perpendiculars of subsets of Λ. To prove that $\mathsf{k}_\bullet$ and $\mathsf{s}_\bullet$ satisfy the required properties we proceed as follows.

1. The inequality $\mathsf{k}_\bullet \circ_\bullet P \circ_\bullet Q \supseteq P$ holds for generic subsets $P, Q \in \mathcal{P}_\bullet(\Pi)$. Indeed, if $p \in {}^\perp P$, $\pi \in P$ then: – see Definition 5.1- $\mathsf{k} \perp p \cdot q \cdot \pi$ for all $q \in \Lambda$ in particular for all $q \perp Q$ and consequently we have that in the same sets: $\mathsf{k}\, p \perp q \cdot \pi$. In other words, we have that $\mathsf{k}\, p \in {}^\perp({}^\perp Q \cdot P) = {}^\perp(Q \to P) = {}^\perp\big((Q \to P)\widehat{\ }\big) = {}^\perp(Q \to_\bullet P)$, the equalities being guaranteed by the considerations in Paragraph XV,(4.12) and by properties presented in the table previous to Lemma 5.7.
   Hence, it follows from the table before Lemma 5.7 that for all $p \perp P$, $\rho \in Q \to_\bullet P$, $\mathsf{E} \perp \mathsf{K} \cdot p \cdot \rho$ and also that $\mathsf{EK} \perp p \cdot \rho$. In the same manner than before, this implies that $\mathsf{EK} \in {}^\perp(P \to_\bullet (Q \to_\bullet P))$. This inequality is equivalent with $\mathsf{k}_\bullet \supseteq P \to_\bullet (Q \to_\bullet P)$ and the adjunction property for $(\to_\bullet, \circ_\bullet)$ implies that $\mathsf{k}_\bullet \circ_\bullet P \circ_\bullet Q \supseteq P$.

2. Now, we have to prove that $\mathsf{s}_\bullet \circ_\bullet P \circ_\bullet Q \circ_\bullet R \supseteq (P \circ_\bullet R) \circ_\bullet (Q \circ_\bullet R)$ for generic $P, Q, R \in \mathcal{P}_\bullet(\Pi)$.
   It follows from the adjunction property that the above is equivalent with $\mathsf{s}_\bullet \supseteq P \to_\bullet (Q \to_\bullet (R \to_\bullet (P \circ_\bullet R) \circ_\bullet (Q \circ_\bullet R)))$. Take $p \in {}^\perp P, q \in {}^\perp Q, r \in {}^\perp R$. We use the inequality : $X \circ_\bullet Y \subseteq X \diamond Y$ to guarantee in successive steps that $pr \perp P \circ_\bullet R, qr \perp Q \circ_\bullet R$ and also that $(pr)(qr) \perp (P \circ_\bullet R) \circ_\bullet (Q \circ_\bullet R)$. It follows from Definition 5.1, that $\mathsf{s} \perp p \cdot q \cdot r \cdot \pi$ for all p, q, r as above and for all $\pi \in (P \circ_\bullet R) \circ_\bullet (Q \circ_\bullet R)$. It follows then that $\mathsf{s}\, pq \perp r \cdot \pi$ and then in the same manner than before that $\mathsf{s}\, pq \perp R \to_\bullet (P \circ_\bullet R) \circ_\bullet (Q \circ_\bullet R)$. Let $\rho \in R \to_\bullet (P \circ_\bullet R) \circ_\bullet (Q \circ_\bullet R)$ using the properties of the combinators E, B appearing in table mentioned above (before Lemma 5.7) we deduce first that $\mathsf{E} \perp (\mathsf{s}\, p) \cdot q \cdot \rho$ for all p, q, ρ as above, and then that $\mathsf{B} \perp \mathsf{E} \cdot \mathsf{s} \cdot p \cdot q \cdot \rho$. Then, $(\mathsf{BE})\mathsf{s}\, p \perp q \cdot \rho$ and as before write down this property as: $(\mathsf{BE})\mathsf{s}\, p \perp Q \to_\bullet (R \to_\bullet (P \circ_\bullet R) \circ_\bullet (Q \circ_\bullet R))$. If call γ a generic element in $Q \to_\bullet (R \to_\bullet (P \circ_\bullet R) \circ_\bullet (Q \circ_\bullet R))$, from $(\mathsf{BE})\mathsf{s}\, p \perp \gamma$ we deduce first that $\mathsf{E} \perp ((\mathsf{BE})\mathsf{s}) \cdot p \cdot \gamma$ and then that $\mathsf{E}((\mathsf{BE})\mathsf{s}) \perp p \cdot \gamma$. As before, a perpendicularity relation like the above means that $\mathsf{E}((\mathsf{BE})\mathsf{s}) \perp P \to_\bullet (Q \to_\bullet (R \to_\bullet (P \circ_\bullet R) \circ_\bullet (Q \circ_\bullet R)))$ and then that $\{\mathsf{E}((\mathsf{BE})\mathsf{s})\} \subseteq {}^\perp(P \to_\bullet (Q \to_\bullet (R \to_\bullet (P \circ_\bullet R) \circ_\bullet (Q \circ_\bullet R))))$. Then: $\{\mathsf{E}((\mathsf{BE})\mathsf{s})\}^\perp \supseteq ({}^\perp(P \to_\bullet (Q \to_\bullet (R \to_\bullet (P \circ_\bullet R) \circ_\bullet (Q \circ_\bullet R))))^\perp \supseteq P \to_\bullet (Q \to_\bullet (R \to_\bullet (P \circ_\bullet R) \circ_\bullet (Q \circ_\bullet R)))$.

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(FI) To prove that $\Phi_\bullet \subseteq \mathcal{P}_\bullet(\Pi)$ is a filter and contains $\mathtt{k}_\bullet, \mathtt{s}_\bullet$ we proceed as follows. First we show that the subset $\Phi_\bullet$ is closed under application. Indeed, if $F, G \in \Phi_\bullet$ we know that ${}^\perp F {}^\perp G \subseteq {}^\perp(F \circ_\bullet G)$ (see Observation 5.6) and if we take elements in Λ, $t_F \in {}^\perp F \cap \mathrm{QP}$ and $t_G \in {}^\perp G \cap \mathrm{QP}$, $t_F t_G \in {}^\perp F {}^\perp G \cap \mathrm{QP} \subseteq {}^\perp(F \circ_\bullet G) \cap \mathrm{QP}$. Moreover, $\mathtt{k}_\bullet$ is the set of elements perpendicular to a fixed element of QP, and the same for $\mathtt{s}_\bullet$. It is clear –by the definition of $\Phi_\bullet$– that sets of this type are in $\Phi_\bullet$. □

XXVIII. The three main constructions produce equivalent triposes

Now, we compare the constructions given by $A_\perp, A_\bullet, A : \mathcal{AKS} \to {}^{\mathcal{I}}\mathcal{OCA}$ followed by the map **H**. The main result, we prove is that we obtain equivalent indexed preorders (that in fact are triposes), and this will be deduced from the general result appearing below. The case of $A_\perp$ and A has already been proved in Ferrer Santos *et al.* (2015), but here we state a general theorem from which one can deduce both equivalences (the one between $\mathbf{H}_{A(K)}$ and $\mathbf{H}_{A_\bullet(K)}$ and also $\mathbf{H}_{A(K)}$ and $\mathbf{H}_{A_\perp(K)}$, that was already proved in Ferrer Santos *et al.* 2015, Lemma 5.5).

Theorem 6.5. Let A and B be IOCAs and assume moreover that the following conditions are satisfied:

1. There is an interior operator $\mathrm{i} : B \to B$ with the property that $A = \{b \in B : \mathrm{i}(b) = b\}$, $a \to_A a' = \mathrm{i}(a \to_B a')$ and $b \to_B b' = \mathrm{i}(b) \to_B b'$.
2. $\mathrm{i}(\Phi_B) = \Phi_A$.

In that situation for all sets I the injections $\mathbf{H}_A(I) \subseteq \mathbf{H}_B(I)$ reflect the order and are essentially surjective. Hence, the injection is an equivalence of preorders.

Proof. Assume that $\varphi, \psi \in A^I$ and $\inf\{\varphi(i) \to_B \psi(i) : i \in I\} \in \Phi_B$ then $\mathrm{i}(\inf\{\varphi(i) \to_B \psi(i) : i \in I\}) \in \mathrm{i}(\Phi_B) = \Phi_A$. Moreover, $\mathrm{i}(\inf\{\varphi(i) \to_B \psi(i) : i \in I\}) \leqslant \inf\{\mathrm{i}(\varphi(i) \to_B \psi(i)) : i \in I\}) = \inf\{\varphi(i) \to_A \psi(i) : i \in I\}$. Hence, as Φ_A is upper closed, we deduce that $\inf\{\varphi(i) \to_A \psi(i) : i \in I\} \in \Phi_A$. Then, the inclusion reflects the order.
Concerning the essential subjectivity we proceed as follows. Take $\gamma \in B^I$ and consider $\mathrm{i}\,\gamma \in A^I$.
To finish we prove that $\mathrm{i}\,\gamma \cong_{\Delta_I(B)} \gamma$.
For $b \in B$ we have that $\mathtt{skk}b \leqslant b$ and then if $f := \mathtt{eskk} \in \Phi_B$ then $f \leqslant b \to_B b$ and $f \leqslant \mathrm{i}(b) \to_B b$. Moreover, from $f \leqslant \mathrm{i}(b) \to_B \mathrm{i}(b)$ and applying the hypothesis $b \to_B b' = i(b) \to_B b'$, we obtain that $f \leqslant b \to_B \mathrm{i}(b)$. □

Theorem 6.6. Assume that K is an AKS, and consider the IOCAs, $A_\perp(K) \subseteq A_\bullet(K) \subseteq A(K)$. Then the associated indexed HPOs called $\mathbf{H}_{A_\perp(K)}$, $\mathbf{H}_{A_\bullet(K)}$ and $\mathbf{H}_{A(K)}$, are equivalent (notice again that the indexed HPOs are in fact triposes in accordance with Theorem 3.17).

Proof. Parts of this result were already proved in (Ferrer Santos *et al.* 2015, Lemmas 4.12, 5.5) and here we prove the equivalence of $\mathcal{H}(\Delta_I(A_\bullet(K)))$ and $\mathcal{H}(\Delta_I(A(K)))$. We already observed that it is enough to check the required equivalence for each $I \in \mathbf{Set}$ – see Lemma 2.8. In accordance with Theorem 6.5, all we have to do is to prove that the $\mathcal{P}_\bullet(\Pi) \subseteq \mathcal{P}(\Pi)$, are IOCAs that satisfy the adequate hypothesis.

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Condition (1) holds if as interior operator we take c_A – the Alexandroff operator associated to the interior operator c given by double perpendicularity.[‡‡] Condition (2) follows directly from the definition of the corresponding IOCAs taking into account that ${}^{\perp}\widehat{P} = {}^{\perp}P$, see Lemma 4.17.

□

XXIX. From ${}^{\mathcal{F}}\mathcal{OCA}$ to $\mathcal{AKS}$

In Ferrer Santos *et al.* (2015, Definition 5.12) and in order to validate the goal to take the concept of IOCA as the main basis for the building of realizability theory, we performed a construction ${}^{\mathcal{I}}K : {}^{\mathcal{I}}\mathcal{OCA} \to \mathcal{AKS}$[§§], such that the associated triposes of ${}^{\mathcal{I}}K(B)$ and B were equivalent for every $B \in {}^{\mathcal{I}}\mathcal{OCA}$ (Ferrer Santos *et al.* 2015, Theorem 5.16).

In what follows, and in order to continue with the program to set up the main results in realizability in terms of QOCAs, we adapt to the current context involving the closure operator $P \mapsto \widehat{P}$, the construction of the map going backwards, this time from FOCAs to AKS. This map is called ${}^{\mathcal{F}}K$ and coincides with ${}^{\mathcal{I}}K$ except for the basic combinators.

Now the relation between the general FOCA – named B – and the three that can be constructed

$$A({}^{\mathcal{F}}K(B)) \supseteq A_{\bullet}({}^{\mathcal{F}}K(B)) \supseteq A_{\perp}({}^{\mathcal{I}}K(B)) \cong B,$$

is that of a Galois injection (see Definition 2.1). We proved in Ferrer Santos *et al.* (2015, Lemma 5.15) that if B is an IOCA, then $B \cong A_{\perp}({}^{\mathcal{I}}K(B))$ – isomorphism of posets –, compare with the weaker result of 6.11 proved in this paper.

Definition 6.7. Let $(A, \leqslant, \mathrm{app}_A = \circ_A, \mathrm{imp}_A = \to_A, \mathsf{k}, \mathsf{s}, \Phi)$ be a FOCA, we define:

1. ${}^{\mathcal{F}}K(A) = (\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{app}, \mathrm{push}, \kappa, \mathsf{s}, \mathrm{QP})$;
2. $\Lambda = \Pi := A$;
3. $\perp\!\!\!\perp := \leqslant$, i.e. $s \perp \pi :\Leftrightarrow s \leqslant \pi$;
4. $\mathrm{app}(s,t) := \mathrm{app}_A(s,t) = s \circ_A t \quad , \quad \mathrm{push}(s,\pi) := \mathrm{imp}_A(s,\pi) = s \to_A \pi$;
5. $\kappa := \mathsf{k} \quad , \quad \mathsf{s} := \mathsf{s}$;
6. $\mathrm{QP} := \Phi$.

Next theorem is closely related to the corresponding result for $A_{\perp}$ in (Ferrer Santos *et al.* 2015, Theorem 5.12). For the notion of *strong* AKS, see the considerations by Streicher in (Streicher 2013, Definition 3.1 and Theorem 4.4) where the author connects that concept with the condition that the AKSs comes from Cohen's forcing.

Theorem 6.8. In the notations of Definition 6.7, ${}^{\mathcal{F}}K(A)$ is a *strong* AKS.

[‡‡] The operators c, c_A in this context are viewed as interior operators.

[§§] In Ferrer Santos *et al.* (2015) this map was called $K_{\mathcal{A}}$ but in the current perspective it is better to call it as above.

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Proof. It is clear that QP is closed under application and contains k, s. Next, we check the axioms concerning the perpendicularity relation (see Definition 5.1). Substituting the above definitions, these axioms become:

1. $t \leqslant u \to_A \pi \quad \Rightarrow \quad t \circ_A u \leqslant \pi$
2. $t \leqslant \pi \quad \Rightarrow \quad \mathsf{k} \leqslant t \to_A u \to_A \pi$
3. $(t \circ_A v) \circ_A (u \circ_A v) \leqslant \pi \quad \Rightarrow \quad \mathsf{s} \leqslant t \to_A u \to_A v \to_A \pi$

1. It follows from Definition 3.1,(CO),(PA).
2. It is proved by the following derivation based on the definition of k (3.1,(CO),(PK)) and the full adjunction, i.e., 3.1,(PE)':

$$t \leqslant \pi \quad \Rightarrow \quad \mathsf{k} \circ_A t \circ_A u \leqslant \pi \quad \Rightarrow \quad \mathsf{k} \circ_A t \leqslant u \to_A \pi \quad \Rightarrow \quad \mathsf{k} \leqslant t \to_A u \to_A \pi.$$

3. It can be proved using repeatedly the full adjunction property as before as well as 3.1,(CO),(PS):

$$(t \circ_A v) \circ_A (u \circ_A v) \leqslant \pi \Rightarrow \mathsf{s} \circ_A t \circ_A u \circ_A v \leqslant \pi \Rightarrow \mathsf{s} \circ_A t \circ_A u \leqslant v \to_A \pi \Rightarrow$$
$$\mathsf{s} \circ_A t \leqslant u \to_A v \to_A \pi \Rightarrow \mathsf{s} \leqslant t \to_A u \to_A v \to_A \pi.$$

To prove that the corresponding AKS is strong, we have to check that in ${}^{\mathcal{F}}K_{\bullet}(A)$, if $ts \perp \pi$ then $t \perp s.\pi$. The translation of this property to our context is: if $t \circ_A s \leqslant \pi$ then $t \leqslant s \to_A \pi$ and this is the condition of the full adjunction of the original FOCA. □

Notation 6.9. To avoid confusion, when we view A as the set Π of the corresponding AKS named ${}^{\mathcal{F}}K(A)$ we write it as A_Π. If we have $C \subseteq A$ when we want to operate on it considering that it is in A_Π we write it as C_Π. We proceed similarly for elements $a \in A$, that will be denoted as a_Π. Moreover, we use the same kind of notations writing C_Λ and a_Λ in the corresponding situations.

In what follows, we list some basic properties of the perpendicularity relation corresponding to the order, and the corresponding closure operators associated in the case that the AKS is ${}^{\mathcal{F}}K(A)$, compare with Ferrer Santos *et al.* (2015, Lemma 5.15).

Lemma 6.10. Let A be an FOCA and ${}^{\mathcal{F}}K(A)$ the AKS of Definition 6.7.

1. For $C_\Pi \subseteq A_\Pi$ we have ${}^{\perp}C_\Pi = {\downarrow}(\inf C)_\Lambda$.
2. In particular for $a \in A$:
$$ {}^{\perp}\{a_\Pi\} = {}^{\perp}({\uparrow}a_\Pi) = {\downarrow}a_\Lambda \ ; \ ({}^{\perp}a_\Pi)^{\perp} = {\uparrow}a_\Pi = \overline{{\uparrow}a_\Pi} = (\widehat{{\uparrow}a_\Pi}). \tag{6.10.3}$$
3. If $C_\Pi \subseteq A_\Pi$, then $\overline{C_\Pi} = {\uparrow}(\inf C)_\Pi$ and $\widehat{C_\Pi} = \bigcup\{{\uparrow}c_\Pi : c_\Pi \in C_\Pi\}$. In particular $\inf C_\Pi = \inf(\widehat{C_\Pi}) = \inf(\overline{C_\Pi})$.
4. For $C_\Pi, D_\Pi \in \mathcal{P}(A_\Pi)$ we have $(\inf C_\Pi \to \inf D_\Pi) \leqslant \inf(C_\Pi \to_\perp D_\Pi) = \inf(C_\Pi \to_\bullet D_\Pi) = \inf(C_\Pi \to D_\Pi)$.

Proof. The verification of the first three items is clear, the remaining can be proved as follows –we omit the subindexes as there is no risk of confusion.
$C \to D = {}^{\perp}C.D \subseteq ({}^{\perp}C.D)\widehat{\ } = C \to_\bullet D \subseteq ({}^{\perp}C.D)^{-} = C \to_\perp D$, then $\inf(C \to_\perp D) = \inf(C \to_\bullet D) = \inf(C \to D)$ –see part (6.10) of this Lemma.

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Clearly by the monotonicity properties of the implication we have that: $\inf(C) \to d \leqslant x \to d \quad \forall x \leqslant \inf(C),\ \forall d \in D$, hence $\inf\{\inf(C) \to d :\ \forall d \in D\} \leqslant \inf\{x \to d :\ \forall x \leqslant \inf(C),\ \forall d \in D\}$ and $\inf(C) \to \inf(D) \leqslant \inf\{x \to d :\ \forall x \leqslant c,\ \forall c \in C,\ \forall d \in D\}$, where the last assertion follows by the preservation and by the definition of inf.
Changing notations slightly we can write: $\inf(C) \to \inf(D) \leqslant \inf\{x \to d :\ \forall x \perp c, \forall c \in C,\ \forall d \in D\}$ or $\inf(C) \to \inf(D) \leqslant \inf\{x.d :\ \forall x \in {}^{\perp}C,\ \forall d \in D\} = \inf\{{}^{\perp}C.D\} = \inf\{C \to D\}$. □

Observe that $\mathcal{P}_{\perp}(A_{\Pi})$ consists of the set of all principal filters of A and $\mathcal{P}_{\bullet}(A_{\Pi})$ is the set of upper closed subsets of A_{Π}. For a general AKS with Π as set of stacks, $\pi \in \overline{\sigma}$ in Π if and only if $\sigma \leqslant_{\mathrm{e}} \pi$ and the lemma just proved guarantees that for the AKS of the form ${}^{\mathcal{F}}K(A)$ the specialization order (see Observation 4.8) coincides with the order of A.

Theorem 6.11. Let B be an FOCA and consider the inclusions of the underlying sets of the associated QOCAs: $A({}^{\mathcal{F}}K(B)), A_{\bullet}({}^{\mathcal{F}}K(B)), A_{\perp}({}^{\mathcal{I}}K(B)), B$,

$$B \cong \mathcal{P}_{\perp}(B_{\Pi}) \subseteq \mathcal{P}_{\bullet}(B_{\Pi}) \subseteq \mathcal{P}(B_{\Pi}).$$

1. (a) The monotonic maps: $(B, \leqslant) \underset{\rho}{\overset{\iota}{\rightleftarrows}} (\mathcal{P}(B_{\Pi}), \supseteq\ \iota : B \to \mathcal{P}(B_{\Pi}), a \mapsto \uparrow a$,

   $\rho : \mathcal{P}(B_{\Pi}) \to B,\ C \mapsto \inf C$, form a Galois connection, i.e. if $a \in B,\ C \subseteq B_{\Pi}$ then:

   $$a \leqslant \rho(C) \Leftrightarrow \iota(a) \leqslant C :\Leftrightarrow \iota(a) \supseteq C. \tag{6.11.4}$$

   (b) The monotonic maps $(B, \leqslant) \underset{\rho}{\overset{\iota}{\rightleftarrows}} (\mathcal{P}_{\bullet}(B_{\Pi}), \supseteq)$ given by corestriction of ι and restriction of ρ also form a Galois connection –notice that we use the same notation for the maps.

2. The unit of the adjunction $\iota \dashv \rho$ is an isomorphism : $\mathrm{id}_B = \rho\,\iota$, while $\iota\rho = \mathrm{c}$ is the closure operator $C \to \overline{C}$ and the co-unit is the natural map $\iota\,\rho \leqslant \mathrm{id}_{\mathcal{P}_{\bullet}(B)}$: $\overline{C} = \uparrow(\inf C) \supseteq C$. This holds in both cases (a) and (b) above.
3. $\iota(B) = \mathcal{P}_{\perp}(B)$, $\rho(P_{\perp}(B)) = B$, $\iota = (\rho|_{\mathcal{P}_{\perp}(B)})^{-1}$.

Proof. (1) and (2) The image of ι lies in $\mathcal{P}_{\perp}(B_{\Pi})$, see Lemma 6.10, and then in the larger $\mathcal{P}_{\bullet}(B)$ and it is also clear that the maps ι and ρ are monotonic.
As $\inf(\uparrow a) = a$ it follows that $\rho\iota = \mathrm{id}_B$. Also for $C \subseteq B_{\Pi}$, $\iota(\rho(C)) = \uparrow(\inf C) = \overline{C} \supseteq C$. Once we have the unit and co-unit, the proof of the assertion (6.11.4) follows directly and this finishes the proof of the first two items.
The proof of the third item is clear – see also Ferrer Santos *et al.* (2015, Lemma 5.15). □

Observation 6.12.

1. We have already observed, see Observation 2.14, that for a Galois injection as above the map $\zeta := \iota\,\rho : \mathcal{P}(B) \to \mathcal{P}(B)$ is an interior operator with family of open

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sets B. In our case, the operator $\zeta : \mathcal{P}(B_\Pi) \to \mathcal{P}(B_\Pi)$ is $\zeta(C) = \overline{C}$ and its family of open sets is $\mathcal{P}_\perp(B_\Pi)$ that has been identified with B.

2. The third assertion of Theorem 6.11 can be seen as a direct consequence of the general result on restriction of general adjunctions that goes as follows. Given an adjunction, take the family of objects in the domain/codomain such that the unit/co-unit is an isomorphism. Take the 'replete' subcategories based upon these objects in the domain and the codomain, respectively. The adjunction restricts to these subcategories and is an equivalence after restriction. In our case in the domain, we still have all of B because the unit is an isomorphism and in the codomain the subcategory is $\mathcal{P}_\perp(B)$. Hence the adjunction restricts to an isomorphism from $B \cong P_\perp(B)$ that is exactly the content of Ferrer Santos *et al.* (2015, Lemma 5.15).
3. It is easy to see that the adjunction $(B, \leqslant) \underset{\rho}{\overset{\iota}{\rightleftarrows}} (\mathcal{P}_\bullet(B_\Pi), \supseteq)$ can be lifted to an adjunction $\widehat{\iota} \dashv \widehat{\rho}$ of indexed preorders, where $\widehat{\iota}$ and $\widehat{\rho}$ are given by post-composition with ι and ρ since $\widehat{\iota}_J \alpha^* \cong \alpha^* \widehat{\iota}_I$ for every $\alpha : J \to I$ in **Set** (see Paré and Grunenfelder 1978 Proposition 2.2.5 page 71).

XXX. We finish by showing that the reverse construction presented in Definition 6.7 gives rise to a tripos equivalent to the one generated by the FOCA. A proof of the following theorem is done by piecing together what we have already obtained.

Theorem 6.13. Let A be an FOCA then $\mathbf{H}_A$ and the induced tripos $\mathbf{H}_{A_\bullet({}^{\mathcal{F}}K(A))}$ are equivalent.

Proof. Combining Ferrer Santos *et al.* (2015, Theorem 5.11) with Ferrer Santos *et al.* (2015, Theorem 5.16) yields that $\mathbf{H}_A$ and $\mathbf{H}_{A_\perp({}^{\mathcal{F}}K(A))}$ are equivalent. Also, by Theorem 6.6 we know that $\mathbf{H}_{A_\perp(K)}$ and $\mathbf{H}_{A_\bullet(K)}$ are equivalent for every $K \in \mathcal{AKS}$. In particular for $K = {}^{\mathcal{F}}K(A)$, we have that $\mathbf{H}_{A_\perp({}^{\mathcal{F}}K(A))}$ and $\mathbf{H}_{A_\bullet({}^{\mathcal{F}}K(A))}$ are equivalent. From the above it follows the result.

□

We summarize some of the conclusions we have obtained.

1. $\mathbf{H}_{A_\perp(K)}$, $\mathbf{H}_{A_\bullet(K)}$ and $\mathbf{H}_{A(K)}$ are equivalent triposes.
2. The map ${}^{\mathcal{I}}K$ is a section and $A_\bullet$ is the companion retraction (see Theorem 6.11).
3. The three FOCAs: $A({}^{\mathcal{F}}K(B)), A_\bullet({}^{\mathcal{F}}K(B)), B$ induce via $\mathbf{H}$ equivalent triposes (see Theorems 6.6 and 6.13).

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