A pretty series revisited

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In the May 1954 issue of the *Gazette* Daniel F. Ferguson challenged readers to devise their own proof for what he described as a curious and somewhat pleasing sum (see [1])

$$\sum_{n=1}^{\infty} \binom{3n+1}{n} \left(\frac{2}{27}\right)^2 = \frac{1}{2}. \tag{1}$$

The sum is no doubt intriguing and, interestingly, its proposer is the same Ferguson who, a decade earlier, had found an error in the 528th decimal place of the monumental 707 decimal place calculation of π completed by William Shanks in 1873 (see [2, 3]).

We shall refer to the sum appearing in (1) as Ferguson's sum. The Maclaurin series corresponding to Ferguson's sum is

$$\sum_{n=1}^{\infty} \binom{3n+1}{n} x^n.$$

The ratio test establishes that the Maclaurin series converges for $|x| < \frac{4}{27}$. Raabe's test can be used to show the series diverges at $x = \frac{4}{27}$ while the alternating version of Raabe's test [4] can be used to show the series converges conditionally at $x = -\frac{4}{27}$. Also, note that

$$\sum_{n=0}^{\infty} \binom{3n+1}{n} x^n = 1 + \sum_{n=1}^{\infty} \binom{3n+1}{n} x^n. \tag{2}$$

The first to respond to Ferguson's challenge was George N. Watson (see [5]), the Watson of *Whittaker & Watson* fame (see [6]). Suspecting a (Gaussian) hypergeometric function to be lurking in the background, Watson expressed the series appearing in (2) in terms of this special function. To find the value of the sum when $x = \frac{2}{27}$ he then made use of Kummer's quadratic transformation before applying Gauss' identity for the hypergeometric function when its argument is equal to unity. Expressed in terms of four gamma functions this allowed Watson to arrive at the desired value of $\frac{1}{2}$ for Ferguson's sum. Fearing his proof would be open to accusations of being too 'high-brow', he encouraged readers to construct simpler proofs.

Two years later W. Hope-Jones, a former past-President of The Mathematical Association (for the year 1938), using a method of solution he attributed to a Mr G. H. Stevens, found an expression for the series appearing in (2) valid for $0 < x < \frac{4}{27}$ in terms of elementary functions; in this case the sine function and its inverse, using what he described as nothing 'beyond school algebra' [7]. In fact what Hope-Jones does is to make use of series reversion. In doing so he is led to a certain cubic equation whose roots he then finds in trigonometric form. At the conclusion of his

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paper an editorial note states that similar solutions to the one given by Hope-Jones had been received from A. G. D. Watson, L. Carlitz, and R. Coles, a pupil at St Albans School no less! Later that year G. P. M. Heselden found an expression for (2) using contour integral representations for the binomial coefficient. The result found depended on the root z to the cubic equation $z(1 - z)^2 = x$ with modulus less than $\frac{1}{3}$ having to be found [8].

In this Article we intend to revisit this intriguing series by constructing a different proof for Ferguson's sum not considered by others. Relying on methods of integration, a closed-form expression for (2) valid for $|x| < \frac{4}{27}$ in terms of elementary functions will be found. Using this expression, Ferguson's sum together with several other closely related sums will be found.

We begin by writing the binomial coefficient appearing in (2) in terms of an integral. We achieve this by expressing the binomial coefficient in terms of the beta function $\mathrm{B}(x, y)$. From the triplication formula for the gamma function [9, p. 23]

$$\Gamma(3z) = \frac{3^{3z-\frac{1}{2}}}{2\pi}\Gamma(z)\Gamma\left(z + \frac{1}{3}\right)\Gamma\left(z + \frac{2}{3}\right),$$

setting $z = n + \frac{2}{3}$ where n is a non-negative integer, we see that

$$(3n + 1)! = \Gamma(3n + 2) = \frac{3^{3n+\frac{3}{2}}}{2\pi}\Gamma(n + 1)\Gamma\left(n + \frac{2}{3}\right)\Gamma\left(n + \frac{4}{3}\right).$$

Thus

$$\binom{3n + 1}{n} = \frac{(3n + 1)!}{n!\,(2n + 1)!} = \frac{3^{3n+\frac{3}{2}}}{2\pi}\frac{\Gamma\left(n + \frac{2}{3}\right)\Gamma\left(n + \frac{4}{3}\right)}{\Gamma(2n + 2)}$$

$$= \frac{3^{3n+\frac{3}{2}}}{2\pi}\mathrm{B}\left(n + \frac{2}{3}, n + \frac{4}{3}\right),$$

where the well-known relation between the beta and gamma functions, $\mathrm{B}(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$ has been used. From the following integral representation for the beta function

$$\mathrm{B}(x, y) = \int_0^1 u^{x-1}(1 - u)^{y-1}\,du, \qquad x, y > 0, \tag{3}$$

as an integral the binomial coefficient appearing in (2) can be expressed as

$$\binom{3n + 1}{n} = \frac{3^{3n+\frac{3}{2}}}{2\pi}\int_0^1 u^{n-\frac{1}{3}}(1 - u)^{n+\frac{1}{3}}\,du.$$

Using this result we can rewrite (2) as

$$\sum_{n=0}^{\infty}\binom{3n + 1}{n}x^n = \frac{3\sqrt{3}}{2\pi}\sum_{n=0}^{\infty}\left(3^{3n}\int_0^1 u^{n-\frac{1}{3}}(1 - u)^{n+\frac{1}{3}}\,du\right)x^n,$$

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or

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}x^n = \frac{3\sqrt{3}}{2\pi}\int_0^1\left(u^{-\frac{1}{3}}(1-u)^{\frac{1}{3}}\sum_{n=0}^{\infty}(27xu(1-u))^n\right)du,$$

after the order of the integration and the summation have been interchanged, an operation that can be justified by applying the dominated convergence theorem. Recognising the series that has appeared is an infinite geometric sum, we have

$$\sum_{n=2}^{\infty}\binom{3n+1}{n}x^n = \frac{3\sqrt{3}}{2\pi}\int_0^1\frac{u^{-\frac{1}{3}}(1-u)^{\frac{1}{3}}}{1-4a^2u(1-u)}\,du.$$

Here we have introduced $a = 3\sqrt{3x}/2$ where we restrict x to $\left[0, \frac{4}{27}\right)$ such that $0 \leqslant a < 1$. Making a substitution of $u = \sin^2\theta$ gives

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}x^n = \frac{3\sqrt{3}}{\pi}\int_0^{\pi/2}\frac{\sin^{1/3}\theta\cos^{5/3}\theta}{1-4a^2\sin^2\theta\cos^2\theta}\,d\theta.$$

Then making next a substitution of $t = \tan\theta$ leads to

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}x^n = \frac{3\sqrt{3}}{\pi}\int_0^{\infty}\frac{t^{1/3}}{1+2\left(1-2a^2\right)t^2+t^4}\,dt. \tag{4}$$

We are now in a position to find the value for Ferguson's sum. When $x = \frac{2}{27}$ we have $a = \frac{1}{\sqrt{2}}$, so from (4) we immediately see that

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\left(\frac{2}{27}\right)^n = \frac{3\sqrt{3}}{\pi}\int_0^{\infty}\frac{t^{1/3}}{1+t^4}\,dt.$$

The convenient cancellation of the t^2 term appearing in the denominator of (4) allows the resultant integral to be readily found in terms of a beta function. Making a substitution of $t \to \sqrt[4]{t}$ one has

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{3n+1}{n}\left(\frac{2}{27}\right)^n &= \frac{3\sqrt{3}}{4\pi}\int_0^{\infty}\frac{t^{-2/3}}{1+t}\,dt = \frac{3\sqrt{3}}{4\pi}\,\mathrm{B}\left(\frac{1}{3},\frac{2}{3}\right)\\ &= \frac{3\sqrt{3}}{4\pi}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{3\sqrt{3}}{4\pi}\Gamma\left(1-\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)\\ &= \frac{3\sqrt{3}}{4\pi}\frac{\pi}{\sin\frac{\pi}{3}} = \frac{3}{2},\end{aligned}$$

with the value of $\frac{1}{2}$ for Ferguson's sum following on subtracting 1 from the result. Note in obtaining this result the following integral representation for the beta function of

$$\mathrm{B}(x,y) = \int_0^{\infty}\frac{t^{x-1}}{(1+t)^{x+y}}\,dt, \qquad x, y > 0$$

was used and comes from substituting $u = t/(1+t)$ into (3), together with

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Euler's well-known reflexion formula for the gamma function of

$$\Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin \pi z}.$$

Of course the real advantage of the formulation presented leading to (4) is it provides one with a means for finding a closed-form expression for the Maclaurin series given in (2), on the condition one is able to evaluate the integral appearing in (4). That this can indeed be done we now show. Making a substitution of $t \to \sqrt{t}$ in (4) one obtains

$$\sum_{n=0}^{\infty} \binom{3n + 1}{n} x^n = \frac{3\sqrt{3}}{2\pi} \int_0^{\infty} \frac{t^{-1/3}}{1 + 2bt + t^2}\, dt. \tag{5}$$

Here $b = 1 - 2a^2$ such that $|b| < 1$. To find the integral consider

$$g(q; \alpha) = \int_0^{\infty} \frac{x^q}{x^2 + 2x \cos \alpha + 1}\, dx,$$

where $|q| < 1$ and $-\pi < \alpha < \pi$. Writing

$$g(q; \alpha) = \int_0^1 \frac{x^q}{x^2 + 2x \cos \alpha + 1}\, dx + \int_1^{\infty} \frac{x^q}{x^2 + 2x \cos \alpha + 1}\, dx,$$

in the second of the integrals appearing to the right of the equality, making a substitution of $x \to \frac{1}{x}$ yields

$$g(q; \alpha) = \int_0^1 \frac{x^q + x^{-q}}{x^2 + 2x \cos \alpha + 1}\, dx. \tag{6}$$

Since

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1 - z}, \qquad |z| < 1, \tag{7}$$

setting $z = -xe^{i\alpha}$ in (7), where $|x| < 1$ and i is the imaginary unit, and equating the imaginary part on either side of the equation yields

$$\frac{x \sin \alpha}{x^2 + 2x \cos \alpha + 1} = \sum_{n=1}^{\infty} (-1)^{n+1} x^n \sin n\alpha.$$

So the function $g(q; \alpha)$ in (6) can be written as

$$\begin{aligned} g(q; \alpha) &= \frac{1}{\sin \alpha} \sum_{n=1}^{\infty} (-1)^{n+1} \sin n\alpha \int_0^1 \left(x^{n+q-1} + x^{n-q-1}\right) dx \\ &= \frac{1}{\sin \alpha} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n \sin n\alpha}{n^2 - q^2}, \end{aligned}$$

after the order of the integration and the summation has been interchanged, an operation that can be justified on applying the dominated convergence theorem. The Fourier series expansion for the function $\sin q\alpha$, $|q| < 1$, $\alpha \in (-\pi, \pi)$ can readily be found. The result is

$$\sin q\alpha = \frac{\sin q\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n \sin n\alpha}{n^2 - q^2}.$$

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Thus we immediately see the function $g(q;\,\alpha)$ reduces to

$$g(q;\,\alpha) \;=\; \frac{\pi\,\sin q\alpha}{\sin q\pi\,\sin\alpha}. \tag{8}$$

Note that as an alternative to the approach given, $g(q;\,\alpha)$ can be evaluated using contour integration.

In terms of the function g the integral appearing in (5) corresponds to $g\left(-\frac{1}{3};\,\cos^{-1}(b)\right)$. Setting $q \;=\; -\frac{1}{3}$ and $\alpha \;=\; \cos^{-1}(b)$ in (8), an expression for the Maclaurin series we seek is

$$\sum_{n=0}^{\infty}\binom{3n\,+\,1}{n}x^n \;=\; \frac{3\,\sin\left(\frac{1}{3}\,\cos^{-1}b\right)}{\sin\left(\cos^{-1}b\right)} \;=\; \frac{3\,\sin\left(\frac{1}{3}\,\cos^{-1}b\right)}{\sqrt{1\,-\,b^2}}.$$

But since $b \;=\; 1\,-\,2a^2$, this gives

$$\sum_{n=0}^{\infty}\binom{3n\,+\,1}{n}x^n \;=\; \frac{3\,\sin\left(\frac{1}{3}\,\cos^{-1}(1\,-\,2a^2)\right)}{\sqrt{4a^2\,(1\,-\,a^2)}} \;=\; \frac{3}{2}\frac{\sin\left(\frac{2}{3}\,\sin^{-1}a\right)}{a\sqrt{1\,-\,a^2}},$$

where we have made use of the following inverse trigonometric identity

$$\sin^{-1}a \;=\; \tfrac{1}{2}\,\cos^{-1}\left(1\,-\,2a^2\right), \qquad 0\;\leqslant\;a\;\leqslant\;1.$$

Finally, since $a \;=\; \frac{3}{2}\sqrt{3x}$, one can write

$$\sum_{n=0}^{\infty}\binom{3n\,+\,1}{n}x^n \;=\; \frac{2}{\sqrt{3}}\,\frac{\sin\left(\frac{2}{3}\,\sin^{-1}\left(\frac{3}{2}\sqrt{3x}\right)\right)}{\sqrt{4\,-\,27x}\,\sqrt{x}}, \qquad 0\;\leqslant\;x\;<\;\frac{4}{27}. \tag{9}$$

To find a closed-form expression for (2) valid for $-\frac{4}{27}\;<\;x\;<\;0$ we set $x \;=\; -z$ (9) where $0\;<\;z\;<\;\frac{4}{27}$. For positive z, as $\sqrt{-z} \;=\; i\sqrt{z}$, $\sin^{-1}(iz) \;=\; i\,\sinh^{-1}z$, and $\sin(iz) \;=\; i\,\sinh z$, we see that

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}(-1)^n z^n = \frac{2}{\sqrt{3}}\,\frac{\sinh\left(\frac{2}{3}\sinh^{-1}\left(\frac{3}{2}\sqrt{3z}\right)\right)}{\sqrt{4+27z}\sqrt{z}}, \qquad 0\leqslant z<\frac{4}{27}. \tag{10}$$

Combining the two results appearing in (9) and (10), a closed-form expression for our Maclaurin series may be written as

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}x^n = \begin{cases} \dfrac{2}{\sqrt{3}\sqrt{4x-27x^2}}\,\sin\left(\dfrac{2}{3}\,\sin^{-1}\left(\dfrac{3}{2}\sqrt{3x}\right)\right), & 0\leqslant x<\frac{4}{27}, \\[2ex] \dfrac{2}{\sqrt{3}\sqrt{27x^2-4x}}\,\sinh\left(\dfrac{2}{3}\,\sinh^{-1}\left(\dfrac{3}{2}\sqrt{-3x}\right)\right), & -\frac{4}{27}\leqslant x<0. \end{cases}$$

As an example, if $x \;=\; -\frac{1}{27}$ we have the rather golden sum of

$$\sum_{n=0}^{\infty}\binom{3n\,+\,1}{n}\frac{(-1)^n}{27^n} \;=\; \frac{3}{\sqrt{5}}\left(\varphi^{3/2}\,-\,\varphi^{-3/2}\right) \;=\; 3\sqrt{\frac{2\left(\sqrt{5}\,-\,1\right)}{5}},$$

where $\varphi \;=\; \frac{1}{2}\left(1\,+\,\sqrt{5}\right)$ is the golden ratio.

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One further pretty series closely related to Ferguson's sum can be found. Replacing x with t in (9) before integrating from 0 to x we have

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\frac{x^{n+1}}{n+1} = \frac{2}{\sqrt{3}}\int_0^x \frac{1}{\sqrt{4t-27t^2}}\sin\left[\frac{2}{3}\sin^{-1}\left(\frac{3}{2}\sqrt{3t}\right)\right]dt.$$

Note here that the interchange made between the integration and the summation on the left can again be justified by applying the dominated convergence theorem. Making a substitution of $y = \frac{2}{3}\sin^{-1}\left(\frac{3}{2}\sqrt{3t}\right)$, as $\frac{dy}{\sqrt{3}} = \frac{dy}{\sqrt{4t^2-27t^2}}$, we see that

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\frac{x^{n+1}}{n+1} = \frac{2}{3}\int_0^z \sin y\,dy,$$

where $z = \frac{2}{3}\sin^{-1}\left(\frac{3}{2}\sqrt{3x}\right)$. The integral is elementary. We have

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\frac{x^{n+1}}{n+1} = \frac{2}{3}(1-\cos z) = \frac{4}{3}\sin^2\frac{z}{2},$$

where we have made use of the trigonometric identity $2\sin^2\theta = 1-\cos 2\theta$, or

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\frac{x^{n+1}}{n+1} = \frac{4}{3x}\sin^2\left(\frac{1}{3}\sin^{-1}\left(\frac{3}{2}\sqrt{3x}\right)\right), \qquad 0 \leqslant x \leqslant \frac{4}{27}. \tag{11}$$

Note that Raabe's test can be used to show the series now converges at the end point $x = \frac{4}{27}$. A similar thing can be done for the negative branch. Here it can be shown that

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\frac{x^{n+1}}{n+1} = -\frac{4}{3x}\sinh^2\left(\frac{1}{3}\sinh^{-1}\left(\frac{3}{2}\sqrt{-3x}\right)\right), \qquad -\frac{4}{27} \leqslant x < 0.$$

Setting $x = \frac{4}{27}$ in (11) we find a similar attractive sum of

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\left(\frac{4}{27}\right)^n\frac{1}{n+1} = \frac{9}{4}.$$

Lastly, a sum that is almost pretty can be found. I say almost as the sum contains a logarithmic term. If we replace x with t in (11) before integrating from 0 to x we obtain

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\frac{x^{n+1}}{(n+1)^2} = \frac{4}{3}\int_0^x \frac{1}{t}\sin^2\left(\frac{1}{3}\sin^{-1}\left(\frac{3\sqrt{3t}}{2}\right)\right)dt.$$

Here again the interchange made between the integration and the summation on the left can be justified by applying the dominated convergence theorem. Making a substitution of $y = \frac{1}{3}\sin^{-1}\left(\frac{3}{2}\sqrt{3t}\right)$, as $t = \frac{4}{27}\sin^2 3y$ and $dt = \frac{8}{9}\sin 3y\cos 3y\,dy$ we see that

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\frac{x^{n+1}}{(n+1)^2} = 8\int_0^z \frac{\sin^2 y\cos 3y}{\sin 3y}\,dy,$$

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where $z = \frac{1}{3}\sin^{-1}\left(\frac{3}{2}\sqrt{3x}\right)$. The trigonometric integral that has appeared is elementary. We show how it can be found. From the angle addition formula for sine,

$$\sin 3y = \sin(y + 2y) = \sin y \cos 2y + \cos y \sin 2y = \sin y\left(2\cos^2 y + \cos 2y\right),$$

where the double-angle formula for sine has been used, and $\sin 3y = \sin y\,(2\cos 2y + 1)$ since $2\cos^2 y = 1 + \cos 2y$. Also employing one of the product-to-sum rules, we have

$$2\sin y \cos 3y = \sin 4y - \sin 2y = \sin 2y\left(2\cos 2y - 1\right),$$

where the double-angle formula for sine has again been used. The trigonometric function appearing in the integrand may therefore be written as

$$\frac{\sin^2 y \cos 3y}{\sin 3y} = \frac{\sin 2y(2\cos 2y - 1)}{2\left(2\cos 2y + 1\right)} = \frac{\sin 2y(2\cos 2y + 1) - 2\sin 2y}{2\left(2\cos 2y + 1\right)}$$

$$= \frac{1}{2}\sin 2y - \frac{\sin 2y}{2\cos 2y + 1}.$$

Thus

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\frac{x^{n+1}}{(n+1)^2} = 4\int_0^z\left(\sin 2y - \frac{\sin 2y}{2\cos 2y + 1}\right)dy$$

$$= 2\left(\log\left(2\cos 2z + 1\right) - \cos 2z + 1 - \log 3\right),$$

or

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\frac{x^n}{(n+1)^2} = \frac{2}{x}\left\{\log\left[2\cos\left(\frac{2}{3}\sin^{-1}\left(\frac{3}{2}\sqrt{3x}\right)\right) + 1\right]\right.$$

$$\left. - \cos\left[\frac{2}{3}\sin^{-1}\left(\frac{3}{2}\sqrt{3x}\right)\right] + 1 - \log 3\right\}, \qquad (12)$$

and is valid for $0 \leqslant x \leqslant \frac{4}{27}$. A similar thing can be done for the negative branch. Here it can be shown that

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\frac{x^n}{(n+1)^2} = \frac{2}{x}\left\{\log\left[2\cosh\left(\frac{2}{3}\sinh^{-1}\left(\frac{3}{2}\sqrt{-3x}\right)\right) + 1\right]\right.$$

$$\left. - \cosh\left[\frac{2}{3}\sinh^{-1}\left(\frac{3}{2}\sqrt{-3x}\right)\right] + 1 - \log 3\right\},$$

and is valid for $-\frac{4}{27} \leqslant x < 0$. Setting $x = \frac{4}{27}$ in (12) we find a sum that comes tantalisingly close to being pretty:

$$\sum_{n=0}^{\infty}\binom{3n+1}{n}\left(\frac{4}{27}\right)^n\frac{1}{(n+1)^2} = \frac{27}{4}\left(1 - \log\frac{9}{4}\right).$$

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