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# Natural boundary for the susceptibility function of generic piecewise expanding unimodal maps

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*Abstract*. For a piecewise expanding unimodal interval map $f$ with unique absolutely continuous invariant probability measure $\mu$, a perturbation $X$, and an observable $\varphi$, the susceptibility function is $\Psi_\varphi(z) = \sum_{k=0}^{\infty} z^k \int X(x)\varphi'(f^k)(x)(f^k)'(x)\,d\mu$. Combining previous results [V. Baladi, On the susceptibility function of piecewise expanding interval maps. *Comm. Math. Phys.* **275** (2007), 839–859; V. Baladi and D. Smania, Linear response for piecewise expanding unimodal maps. *Nonlinearity* **21** (2008), 677–711] (deduced from spectral properties of Ruelle transfer operators) with recent work of Breuer–Simon [Natural boundaries and spectral theory. *Adv. Math.* **226** (2011), 4902–4920] (based on techniques from the spectral theory of Jacobi matrices and a classical paper of Agmon [Sur les séries de Dirichlet. *Ann. Sci. Éc. Norm. Supér.* (3) **66** (1949), 263–310]), we show that density of the postcritical orbit (a generic condition) implies that $\Psi_\varphi(z)$ has a strong natural boundary on the unit circle. The Breuer–Simon method provides uncountably many candidates for the outer functions of $\Psi_\varphi(z)$, associated with precritical orbits. If the perturbation $X$ is horizontal, a generic condition (Birkhoff typicality of the postcritical orbit) implies that the non-tangential limit of $\Psi_\varphi(z)$ as $z \to 1$ exists and coincides with the derivative of the absolutely continuous invariant probability measure with respect to the map ('linear response formula'). Applying the Wiener–Wintner theorem, we study the singularity type of non-tangential limits of $\Psi_\varphi(z)$ as $z \to e^{i\omega}$ for real $\omega$. An additional 'law of the iterated logarithm' typicality assumption on the postcritical orbit gives stronger results.

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1. *Statement of results*

1.1. *Standing assumptions and notation.* Our standing assumptions on the dynamics $f$ are as follows: $I = [a, b]$ is a compact interval and $f : I \to I$ is a piecewise expanding $C^3$ unimodal map. That is, $f$ is continuous on $I$ with $f(a) = f(b) = a$, and there exists $c \in (a, b)$ (the *critical point*) so that $f$ is $C^3$ with $f' > 1$ on $[a, c]$, and $f$ is $C^3$ with $f' < -1$ on $[c, b]$. We put $c_k = f^k(c)$ for $k \geq 0$. The sequence $\{c_k\}_{k\geq 1}$ is called the *postcritical orbit.* (The interval $[c_2, c_1]$ is forward-invariant.) We set

$$\lambda = \inf_{x\neq c} |f'(x)| > 1.$$

We shall assume that the critical point is not periodic (this ensures that $f$ is 'good' in the sense of [**6**]). In fact, except in Remark 1.2, we shall always assume that the postcritical orbit $\{c_k\}_{k\geq 1}$ is infinite. We let $\mu = \rho\, dx$ be the unique absolutely continuous invariant probability measure of $f$. Recall that $(f, \mu)$ is always ergodic, and [**4**, Proposition 3.6] that $(f, \mu)$ is mixing if $f$ is topologically mixing on $[c_2, c_1]$. Except in Remark 1.3, we assume throughout that $f$ is topologically mixing on $[c_2, c_1]$.

Let $X \in C^2(f(I))$ satisfy $X(a) = 0$. We say that $X$ is *horizontal* for $f$ if

$$\sum_{n=0}^{\infty} \frac{X(c_{n+1})}{(f^n)'(c_1)} = 0. \tag{1.1}$$

Consider a $C^2$ family $t \mapsto f_t$ of $C^3$ piecewise expanding interval maps (defined for $t \in (-1, 1)$, say), with $f_0 = f$, so that all $f_t$ have the same critical point $c$ and satisfy $f_t(a) = f_t(b) = a$, and so that $\partial_t f_t|_{t=0} = X \circ f$. (In other words, $X$ represents a perturbation of the dynamics.) We denote by $\mu_t$ the unique absolutely continuous invariant probability measure of $f_t$.

The main motivation for the present work is provided by the following question: What is the smoothness of the map $t \mapsto \int \varphi\, d\mu_t$ (for suitable functions $\varphi : I \to \mathbb{C}$)? In connection with this question, Ruelle [**33, 34**] proposed to study the *susceptibility function*, i.e. the formal power series associated with $f$, $X$ and $\varphi \in C^1(I)$ by

$$\Psi_\varphi(z) = \sum_{k=0}^{\infty} z^k \int X(x)\, (\varphi' \circ f^k)(x)(f^k)'(x)\, \rho(x)\, dx. \tag{1.2}$$

(See the survey [**8**].) We can assume without loss of generality that $\int \varphi\, d\mu = 0$. Integrating each term by parts and using that $\rho$ is of bounded variation, the definition of $\Psi_\varphi(z)$ is extended to $\varphi \in C^0(I)$ (see (B.1) for an explicit formula).

For generic $X$, the following function encodes the singular behaviour of $\Psi_\varphi$:

$$\sigma_\varphi(z) = \sum_{k=0}^{\infty} \varphi(c_{k+1}) z^k. \tag{1.3}$$

Indeed, the following theorem is essentially proved in [**5, 6**] (see Appendix B for details, including explicit formulas for $\mathcal{U}$, $\mathcal{V}_\varphi$, and $\Psi_\varphi^{\mathrm{hol}}$).

THEOREM 1.1. (Susceptibility function and linear response) *If $\varphi \in C^0(I)$ and $\int \varphi\, d\mu = 0$, then the susceptibility function $\Psi_\varphi$ is holomorphic in the open unit disc, and*

$$\Psi_\varphi(z) = \Psi_\varphi^{\mathrm{sing}}(z) + \Psi_\varphi^{\mathrm{hol}}(z) = \mathcal{U}(z)\sigma_\varphi(z) + \mathcal{V}_\varphi(z) + \Psi_\varphi^{\mathrm{hol}}(z), \tag{1.4}$$

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*where the functions $\mathcal{U}$ and $\mathcal{V}_\varphi$ are holomorphic in $|z| > \lambda^{-1}$, there exists $\kappa < 1$ so that $\Psi_\varphi^{\mathrm{hol}}$ is holomorphic in $|z| < \kappa^{-1}$, and the function $\mathcal{U}$ depends only on $f$ and $X$. In addition, $X \equiv 0$ if and only if $\mathcal{U} \equiv 0$, and $X$ is horizontal if and only if $\mathcal{U}(1) = 0$.*

*If $X$ is horizontal, then for any $C^2$ family $t \mapsto f_t$ of $C^3$ piecewise expanding interval maps so that $f_0 = f$ and $\partial_t f_t|_{t=0} = X \circ f$, we have*

$$\partial_t \left( \int \varphi \, d\mu_t \right) \Big|_{t=0} = \mathcal{V}_\varphi(1) + \Psi_\varphi^{\mathrm{hol}}(1). \tag{1.5}$$

Our aim in this article is to investigate the behaviour of the functions $\sigma_\varphi(z)$ and $\Psi_\varphi(z)$ near the unit circle and particularly near the point $z = 1$.

In view of (1.3), the distribution of the points $c_1, c_2, \ldots$ in $I$ will obviously play a role. The sequence $\{c_k\}_{k \geq 1}$ is generically dense in the interval $[c_2, c_1]$. In fact, Schnellmann [**36**, Theorem 6.1, comments after Corollary 2] showed that for generic† piecewise expanding $C^2$ maps $f$, the critical point $c$ is *Birkhoff typical*, that is, for all continuous functions $\varphi : I \to \mathbb{C}$ we have

$$\lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} \varphi(c_{k+1}) = \int \varphi \, d\mu.$$

(If $c$ is Birkhoff typical, its orbit is dense in $[c_2, c_1]$, because our mixing assumption implies that $\rho$ is bounded from below on its support $[c_2, c_1]$.)

Our main results are the following.

Theorem 1 in §1.2 gives a strong natural boundary for $\Psi_\varphi$, and Theorem 2 in §1.3 guarantees 'uncountably many renascent right limits', both are proved in §2 under the assumption that the critical orbit is dense.

Under the assumption that the critical orbit is Birkhoff typical, Theorem 3 and Corollary 1 in §1.4 give non-polarity of non-tangential limits of $\Psi_\varphi(z)$ at $z = e^{i\omega}$, and, assuming horizontality, the non-tangential limit at $z = 1$ (proofs in §3), while Theorem 4 in §1.5 gives non-tangential limits of derivatives of $\Psi_\varphi(z)$ at $z = 1$ (proof in §4).

Then, Theorem 5 in §1.6 refines Theorem 3, under a (possibly generic) 'iterated logarithm law' condition on the postcritical orbit (proof in §3). Finally, §5 is devoted to observables $\varphi$ which are coboundaries, the main result there is Proposition 5.1 (assuming again only Birkhoff typicality of the periodic orbit).

*Remark 1.2.* (Finite postcritical orbits) The (highly non-generic) case when the postcritical orbit is finite is much simpler [**5**, §5]. Then the susceptibility function is holomorphic in the open unit disc and meromorphic in a disc of radius $\kappa^{-1} > 1$. Its possible poles in that disc are roots of unity, and $z = 1$ is not a pole if $X$ is horizontal. The function $\sigma_\varphi$ itself is rational: $\sigma_\varphi(z) = P(z) + (Q(z)/(1 - z^p))$, where $p \geq 1$ is the primitive period of the postcritical periodic point $f^p(c_m) = c_m$, and

$$P(z) = \sum_{k=1}^{m} \varphi(c_k) z^{k-1}, \quad Q(z) = \sum_{k=m}^{m+p-1} \varphi(c_k) z^{k-1}.$$

The residue of $\sigma_\varphi$ at $e^{i\omega}$ with $e^{ip\omega} = 1$ is equal to $(1/p) \sum_{k=m}^{m+p-1} e^{i\omega(k-1)} \varphi(c_k)$. (Note that if there is a continuous $\psi$ so that $\varphi = e^{i\omega}\psi - \psi \circ f$, then this residue vanishes.)

† That is, almost all maps in a transversal family, we refer to [**36**] for a precise statement.

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*Remark 1.3.* (Non-mixing maps) If the postcritical orbit is dense in $[c_2, c_1]$ and $f$ is not topologically mixing on $[c_2, c_1]$, then there exist two intervals $J_1$ and $J_2$ with $J_1 \cap J_2 = \{y_0\}$, $f(y_0) = y_0$, and $J_1 \cup J_2 = [c_2, c_1]$, with $f(J_1) = J_2$, $f(J_2) = J_1$, and $f^2|_{J_i}$ topologically mixing for $i = 1, 2$ (see, e.g., [**11**, §VI.5, Theorem 46 and remark thereafter], noting that the dense postcritical orbit assumption ensures that $f$ is transitive on $[c_2, c_1]$). The interested reader can exploit the present remark to study non-mixing maps $f$ for which the postcritical orbit is dense (angles $\omega$ with $e^{i\omega} = -1$ must be treated separately).

1.2. *Strong natural boundary.* Following [**13**], a function $g(z)$ holomorphic in the disc $|z| < 1$ is said to have a *strong natural boundary* on the unit circle if, for every non-empty interval $(\omega_1, \omega_2)$,

$$\sup_{0<r<1} \int_{\omega_1}^{\omega_2} |g(re^{i\omega})|\, d\omega = \infty. \tag{1.6}$$

If $g$ has a strong natural boundary, then $g$ has an $L^\infty$ natural boundary, that is, $|g|$ is unbounded in every sector $\{re^{i\omega} \mid r \in (0, 1), \omega \in (\omega_1, \omega_2)\}$.

Using recent deep results of Breuer and Simon [**13**, Theorem 3.1] based on previous work of Agmon [**1**], we prove in §2 the following theorem.

THEOREM 1. (Strong natural boundary) *Assume that the postcritical orbit is dense in* $[c_2, c_1]$. *Then, for any continuous* $\varphi$ *which is not constant on* $[c_2, c_1]$ *and so that* $\int \varphi\, d\mu = 0$, *the unit circle is a strong natural boundary for* $\sigma_\varphi$, *and also for the susceptibility function* $\Psi_\varphi$ *unless* $X \equiv 0$.

For a sequence $\{a_k\}_{k=0}^\infty$ in a topological space $E$, a *right limit* [**13**] is any two-sided sequence $\{b_n\}_{n=-\infty}^\infty$ of $E$ for which there exists an increasing sequence of positive integers $\{k_j\}_{j=0}^\infty$ such that $\lim_{j\to\infty} a_{n+k_j} = b_n$ for every $n \in \mathbb{Z}$. Theorem 3.1 in [**13**] reads as follows.

> Suppose that a bounded sequence of complex numbers $\{a_k\}_{k=0}^\infty$ has two distinct right limits $\{b_n\}_{n=-\infty}^\infty$ and $\{\tilde{b}_n\}_{n=-\infty}^\infty$, and that there exists $N \in \mathbb{Z}$ such that either $b_n = \tilde{b}_n$ for all $n \geq N$, or $b_n = \tilde{b}_n$ for all $n \leq N$, then the unit circle is a strong natural boundary for the power series $\sum_{k\geq 0} a_k z^k$.

The proof of Theorem 1 will consist of exhibiting, in the case of the sequence $\varphi(c_{k+1})$, distinct right limits which coincide for $n \geq 0$, entailing a strong natural boundary for $\sigma_\varphi$ and thus for $\Psi_\varphi$.

1.3. *Renascent right limits.* In fact, we shall see in Theorem 2 that the set of right limits of the sequence $\{\varphi(c_{k+1})\}_{k\geq 0}$ is multifarious well beyond the requirement of [**13**, Theorem 3.1]. To state this result, we introduce some terminology.

A right limit $\{b_n\}_{n=-\infty}^\infty$ of a bounded sequence of complex numbers $\{a_k\}_{k=0}^\infty$ is called *renascent* if $b_n = a_n$ for all $n \geq 0$, and *weakly renascent* if there is a finite set of integers $F$ so that $b_n = a_n$ for all $n \in \mathbb{N}\backslash F$. If $\{b_n\}_{n=-\infty}^\infty$ is renascent, then the function $g_{b_-}(z) = -\sum_{n=-1}^{-\infty} b_n z^n$, which is holomorphic in $|z| > 1$ and vanishes at $\infty$, is called an *rrl-continuation* of the function $g(z) = \sum_{k=0}^\infty a_k z^k$

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Following [**13**], we say that a bounded sequence $\{b_n\}_{n\in\mathbb{Z}}$ is *reflectionless* on an arc $J = \{e^{i\omega} \mid \omega \in (\omega_1, \omega_2)\}$ of the unit circle if the function $g_+(z) = \sum_{n=0}^{\infty} b_n z^n$ admits an analytic continuation in a neighbourhood of $J$ and the value of this continuation at any $z$ with $|z| > 1$ is $g_-(z) = -\sum_{n=-1}^{-\infty} b_n z^n$.

Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ with $\sup |a_k| < \infty$ and $J = \{e^{i\omega} \mid \omega \in (\omega_1, \omega_2)\}$. According to [**13**, Theorem 1.4], if (1.6) is violated (e.g., if $g$ admits an analytic extension through $J$), then all right limits $\{b_n\}_{n=-\infty}^{\infty}$ of $\{a_k\}_{k=0}^{\infty}$ are reflectionless on $J$. In particular, if $\{a_k\}_{k=0}^{\infty}$ admits a renascent right limit $\{b_n\}_{n=-\infty}^{\infty}$, then $g_+ = g$ by definition. Thus, if there exists an arc $J$ in the neighbourhood of which $g$ admits an analytic continuation, this continuation must coincide with each rrl-continuation $g_- = g_{b_-}$ outside the closed unit disc. In such a case, all renascent right limits coincide and all the analytic continuations of $g$ through arcs of the unit circle match. (*Mutatis mutandis*, weakly renascent right limits enjoy similar properties.) If on the contrary there is no analytic continuation for $g$ across any arc of the unit circle, then the unit circle is a strong natural boundary (because the renascent right limit is not reflectionless on any arc). However, we still may think of the rrl-continuations $g_{b_-}$ as being somewhat 'connected' to $g$, as in the theories of generalized analytic continuation [**31**] or of monogenic continuation [**12, 25**]†. We refer to Appendix A for examples.

We require more terminology: a power series $\sum_{k=0}^{\infty} a_k z^k$ with $\sup |a_k| < \infty$ is called *polygenous* if the set of renascent right limits of the sequence $\{a_k\}_{k=0}^{\infty}$ contains at least two elements.

We do not know any polygenous power series admitting a Borel monogenic extension beyond the unit circle, and we are tempted to guess that such examples do not exist. An even stronger conjecture would be that, whenever a power series has a monogenic extension beyond the unit circle and an rrl-continuation, they necessarily coincide.

By [**13**, Theorem 3.1], the unit circle is a strong natural boundary for any polygenous series. The following result, which is proved in §2, is thus a reinforcement of Theorem 1: we say that a function $\varphi$ is *$f$-symmetric on* $[c_2, c_1]$ if $\varphi(x) = \varphi(y)$ for any $x$, $y$ in $[c_2, c_1]$ so that $f(x) = f(y) \in [c_2, c_1]$.

THEOREM 2. (Uncountably many renascent right limits for the postcritical orbit) *Assume that the postcritical orbit is dense in* $[c_2, c_1]$.

(i) *The right limits of the sequence* $\{c_{k+1}\}_{k\geq 0}$ *are exactly the complete orbits of* $f$ *contained in* $[c_2, c_1]$*, i.e. the two-sided sequences* $\{x_n\}_{n=-\infty}^{\infty}$ *of* $[c_2, c_1]$ *such that* $x_{n+1} = f(x_n)$ *for all* $n \in \mathbb{Z}$.

(ii) *For any continuous* $\varphi$ *which is not* $f$*-symmetric on* $[c_2, c_1]$*, the set obtained by identifying renascent right limits of* $\{\varphi(c_{k+1})\}_{k\geq 0}$ *which differ only on a finite set is uncountable.*

We say that $\{y_n\}_{n\leq -1}$ is a *precritical orbit* if $y_n \in [c_2, c_1]$ with $f(y_{n-1}) = y_n$ for all $n \leq -1$, and $y_{-1} = c$. Slightly abusing notation, we write $c_- = \{c_{n+1}\}_{n\leq -1}$ for a

† In a nutshell, Borel monogenic functions are a generalization of analytic functions of one complex variable, which allows functions to be defined on closed sets which may even have empty interior. They share most of the properties of analytic functions, especially Cauchy's integral formula, including, in some cases relevant for our scopes, being a quasianalytic space.

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precritical orbit. Theorem 2(i) implies that the renascent right limits of the sequence $\{c_{k+1}\}_{k\geq 0}$ are the complete orbits of $c_1$ contained in $[c_2, c_1]$, i.e. the two-sided sequences obtained by glueing any precritical orbit with $\{c_{k+1}\}_{k=0}^{\infty}$. The argument for Theorem 2(ii) uses the fact (proved there) that there are uncountably many precritical orbits.

*Remark 1.4.* (Renascent right limits and formal resolvents in $|z| > 1$) By Theorem 2, the rrl-continuations of $\sigma_\varphi$ are of the form

$$\sigma_{\varphi,c_-}(z) = -\sum_{n\leq -1} \varphi(c_{n+1}) z^n, \tag{1.7}$$

where $c_- = \{c_{n+1}\}_{n\leq -1}$ is any precritical orbit. Clearly, for $|z| < 1$

$$\sigma_\varphi(z) = \sum_{n=0}^{\infty} [(zU)^n \varphi](c_1) = [(1 - zU)^{-1}\varphi](c_1), \tag{1.8}$$

where $U(\varphi) = \varphi \circ f$ is the Koopman operator acting (for example) on $C^0$, which has spectral radius equal to one, but is not invertible. It is therefore hardly surprising that the rrl-continuations, which we expect to be candidates for the 'outer function' of $\sigma_\varphi$ outside of the unit disc, are nothing else than

$$\sigma_{\varphi,c_-}(z) = -\sum_{n=1}^{\infty} [(zU)^{-n}\varphi](c_1), \tag{1.9}$$

where the operator sequence $\{U^{-n}\}_{n\geq 1}$ is any sequence satisfying $U^{-n}U = U^{-n+1}$, so that (1.9) can formally be written as

$$-[(1 - (zU)^{-1})^{-1}(zU)^{-1}\varphi](c_1).$$

(This remark should be put in parallel with the manipulations in [**35**, §17].) Note that (5.2) in Proposition 5.1 below is not surprising either in view of the present discussion, since $(1 - e^{i\omega}U)\psi(c) = \varphi(c)$ for the function $\varphi$ considered there, although it is not clear how to exploit (1.8) to prove (5.2) without using Wiener–Wintner.

1.4. *Non-tangential limits: the outer functions $\sigma_{\varphi,c_-}$ and $\Psi_{\varphi,c_-}$.* We move on to results on the convergence of $\Psi_\varphi(z)$ when $z$ tends non-tangentially in the open unit disc (i.e. without leaving a fixed open sector based at $e^{i\omega}$ and contained inside the open unit disc) to some $e^{i\omega}$ with $\omega \in \mathbb{R}$, denoted $z \overset{NT}{\to} e^{i\omega}$. (Clearly, if $\lim_{z\overset{NT}{\to} e^{i\omega}} \psi(z)$ exists then the Abelian limit $\lim_{r\to 1} \psi(re^{i\omega})$ exists.) The following theorem is proved in §3 using the Wiener–Wintner results from Appendix C.

THEOREM 3. (Non-polar non-tangential limits) *Assume that the critical point is Birkhoff typical. Let $\varphi \in C^0(I)$. For any $\omega \in \mathbb{R}$ with $e^{i\omega} \neq 1$, we have*

$$\lim_{z\overset{NT}{\to} e^{i\omega}} (z - e^{i\omega})\sigma_\varphi(z) = 0 \quad \textit{and thus} \quad \lim_{z\overset{NT}{\to} e^{i\omega}} (z - e^{i\omega})\Psi_\varphi(z) = 0. \tag{1.10}$$

*If $\int \varphi \, d\mu = 0$, then the above also holds for $e^{i\omega} = 1$.*

In the horizontal case, we get a linear response interpretation of the non-tangential limit to $z = 1$ by applying the results of [**6**] recalled in Theorem 1.1.

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COROLLARY 1. (Linear response and non-tangential limit at 1) *Assume that the critical point is Birkhoff typical, $X$ is horizontal, and let $\varphi \in C^0(I)$ with $\int \varphi \, d\mu = 0$. Then*

$$\lim_{z \overset{NT}{\to} 1} \Psi_\varphi(z) = \mathcal{V}_\varphi(1) + \Psi_\varphi^{\mathrm{hol}}(1). \tag{1.11}$$

*In addition, for any $C^2$ family $t \mapsto f_t$ of $C^3$ piecewise expanding interval maps so that $f_0 = f$ and $\partial_t f_t|_{t=0} = X \circ f$, we have*

$$\lim_{z \overset{NT}{\to} 1} \Psi_\varphi(z) = \partial_t \left( \int \varphi \, d\mu_t \right) \Big|_{t=0}. \tag{1.12}$$

Recalling (1.5), if $X$ is horizontal, then $\lim_{z \overset{NT}{\to} 1} \Psi_\varphi(z) - \Psi_\varphi^{\mathrm{hol}}(1) = \mathcal{V}_\varphi(1)$ (see also the convergent resummation (B.7) for $\mathcal{V}_\varphi(1)$).

Replacing the non-tangential limit with the Abelian limit, the identity (1.12) was proved in [**6**], under a stronger assumption on $X$, see §1.5.

*Proof of Corollary 1.* By Theorem 1.1, we have $\mathcal{U}(1) = 0$. Thus, (1.10) from Theorem 3 for $e^{i\omega} = 1$ gives $\lim_{z \overset{NT}{\to} 1} \mathcal{U}(z)\sigma_\varphi(z) = 0$, which yields (1.11). Then use (1.5) from Theorem 1.1. □

If the sequence $c_- = \{c_{n+1}\}_{n \le -1}$ is a precritical orbit then $\{\varphi(c_{n+1})\}_{n \in \mathbb{Z}}$ is a renascent right limit for $\{\varphi(c_{k+1})\}_{k \ge 0}$ giving rise to the rrl-continuation $\sigma_{\varphi, c_-}$ defined in (1.7). If $\varphi$ is not $f$-symmetric on $[c_2, c_1]$, there are uncountably many such rrl-continuations by Theorem 2(ii). To each renascent right limit, we associate a candidate for the 'outer' susceptibility function by setting

$$\Psi_{\varphi, c_-}(z) = \mathcal{U}(z)\sigma_{\varphi, c_-}(z) + \mathcal{V}_\varphi(z) + \Psi_\varphi^{\mathrm{hol}}(z). \tag{1.13}$$

Note that $\sigma_{\varphi, c_-}(z)$ is holomorphic in $|z| > 1$ (it vanishes at infinity), and $\Psi_{\varphi, c_-}(z)$ is holomorphic in $1 < |z| < \kappa^{-1}$.

We say that a *precritical orbit $\{c_{n+1}\}_{n \le -1}$ is Birkhoff typical* if for all $\varphi \in C^0$ we have

$$\lim_{m \to \infty} \frac{1}{m} \sum_{n=-1}^{-m} \varphi(c_{n+1}) = \int \varphi \, d\mu.$$

Theorem 3 and Corollary 1 have an analogue for $\sigma_{\varphi, c_-}$ and $\Psi_{\varphi, c_-}$ (see §3).

THEOREM 1.5. (Non-tangential limits for outer functions) *Let $c_- = \{c_{n+1}\}_{n \le -1}$ be a Birkhoff typical precritical orbit.*

*For any $\varphi \in C^0(I)$ and any $\omega \in \mathbb{R}$ with $e^{i\omega} \ne 1$, we have*

$$\lim_{z \overset{NT}{\to} e^{i\omega}} (z^{-1} - e^{-i\omega})\sigma_{\varphi, c_-}(z^{-1}) = 0 \quad \textit{and thus} \quad \lim_{z \overset{NT}{\to} e^{i\omega}} (z^{-1} - e^{-i\omega})\Psi_{\varphi, c_-}(z^{-1}) = 0.$$

*If $\int \varphi \, d\mu = 0$, then the above also holds for $e^{i\omega} = 1$.*

*Let $\varphi \in C^0(I)$ with $\int \varphi \, d\mu = 0$, and let $X$ be horizontal. Then*

$$\lim_{z \overset{NT}{\to} 1} \Psi_{\varphi, c_-}(z^{-1}) = \lim_{z \overset{NT}{\to} 1} \Psi_\varphi(z) = \mathcal{V}_\varphi(1) + \Psi_\varphi^{\mathrm{hol}}(1). \tag{1.14}$$

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*Remark 1.6.* (Birkhoff typical precritical orbits) A precritical orbit can accumulate at a repelling periodic orbit, in which case it is not Birkhoff typical. However, we expect that generic piecewise expanding maps have infinitely many Birkhoff typical precritical orbits. Note that if there were a unique Birkhoff typical precritical orbit for $f$, then this would single out a renascent limit giving an rrl-continuation more natural than all others. See also Remark 1.8.

1.5. *Higher-order horizontality and derivatives.* In [**6**, Proposition 4.6] we gave a sufficient condition for existence of the Abelian limit at $z = 1$ of the susceptibility function in the piecewise expanding case. We asked in [**8**] whether this condition was necessary. By Corollary 1, the answer to this question is negative (horizontality with Birkhoff typicality of the postcritical orbit suffices). However, the sufficient additional condition from [**6**, Proposition 4.6],

$$\sum_{n=1}^{\infty} \frac{nX(c_{n+1})}{(f^n)'(c_1)} = 0, \tag{1.15}$$

appears in a more natural manner in Theorem 4 below. We say that $X$ is horizontal of order $H \geq 1$ if

$$\sum_{n=\ell}^{\infty} \frac{n!}{(n-\ell)!} \frac{X(c_{n+1})}{(f^n)'(c_1)} = 0 \quad \text{for all } 0 \leq \ell \leq H-1. \tag{1.16}$$

The following result is proved in §4.

THEOREM 4. (Non-tangential limits of derivatives of $\Psi$ and $\Psi_{\varphi,c_-}$ at 1) *Assume that the critical point is Birkhoff typical. Let $\varphi \in C^0(I)$ satisfy $\int \varphi \, d\mu = 0$. Assume that $X$ is horizontal of order $H$ for some $H \geq 1$.*

*Then, we have, for each $0 \leq \ell \leq H-1$,*

$$\lim_{z \overset{NT}{\to} 1} (\mathcal{U}(z)\sigma_\varphi)^{(\ell)}(z) = 0 \quad \textit{and thus} \quad \lim_{z \overset{NT}{\to} 1} \Psi_\varphi^{(\ell)}(z) = \mathcal{V}_\varphi^{(\ell)}(1) + (\Psi_\varphi^{\mathrm{hol}})^{(\ell)}(1).$$

*In addition, for any Birkhoff typical precritical orbit $c_- = \{c_{n+1}\}_{n \leq -1}$, we have*

$$\lim_{z \overset{NT}{\to} 1} (\mathcal{U}(z)\sigma_{\varphi,c_-})^{(\ell)}(z^{-1}) = 0 \quad \textit{for all } 0 \leq \ell \leq H-1,$$

*so that*

$$\lim_{z \overset{NT}{\to} 1} (\Psi_{\varphi,c_-})^{(\ell)}(z^{-1}) = \lim_{z \overset{NT}{\to} 1} \Psi_\varphi^{(\ell)}(z) = \mathcal{V}_\varphi^{(\ell)}(1) + (\Psi_\varphi^{\mathrm{hol}})^{(\ell)}(1) \quad \textit{for all } 0 \leq \ell \leq H-1.$$

*Remark 1.7.* (Elusiveness of quasi-analytic extensions) Horizontality of order $\ell$ means that $\mathcal{U}^{(k)}(1) = 0$ for $k = 0, \ldots, \ell - 1$. Since $\mathcal{U}$ is holomorphic close to $z = 1$, horizontality of all orders would mean that $\mathcal{U} \equiv 0$. In addition, since $X$ is continuous, (B.4) below implies that $\mathcal{U} \equiv 0$ is equivalent to $X \equiv 0$ if the postcritical orbit is dense. So there is no hope of proving the existence of a non-holomorphic quasi-analytic extension at $z = 1$ by using the horizontality mechanism.

1.6. *Law of the iterated logarithm.* In order to get more precise results, we shall make assumptions involving the (rotated) law of the iterated logarithm (LIL) for the postcritical orbit. Let us first recall known results. We say that $\varphi$ is a coboundary if there exists $\psi \in C^0$

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so that $\varphi = \psi - \psi \circ f$. The ordinary LIL for a piecewise expanding map $f$ says (see [**20**], where an almost sure invariance principle is proved, from which the LIL follows; see also [**14**, §6], and [**19**] and references therein) that for any $\varphi$ of bounded variation with mean zero, which is not a coboundary, there exists $C(\varphi) > 0$ so that for Lebesgue almost every $x$

$$\limsup_{m\to\infty} \left| \frac{\sum_{k=1}^{m} \varphi(f^k(x))}{\sqrt{m \log\log m}} \right| = C(\varphi). \tag{1.17}$$

It is not unreasonable to expect that the postcritical orbit is typical for the LIL, i.e. that we may take $x = c$ in (1.17). (Indeed, several experts [**10, 37**] expect that the postcritical orbit may be generically typical for the LIL in the setting of smooth unimodal maps.)

Let now $\omega \in \mathbb{R}$ be such that $e^{i\omega} \neq 1$. We say that $\varphi$ is an $\omega$-coboundary if there exists $\psi \in C^0$ so that $\varphi = \psi - e^{i\omega}\psi \circ f$. We refer to [**39**] for an $\omega$-rotated LIL in the probabilistic setting. In a deterministic dynamical setting, the rotated LIL says that for every real $\omega$ and any Hölder† $\varphi$, which is not an $\omega$-coboundary, there exists $C(\varphi, \omega) > 0$ so that for Lebesgue almost every $x$

$$\limsup_{m\to\infty} \left| \frac{\sum_{k=1}^{m} e^{ik\omega}\varphi(f^k(x))}{\sqrt{m \log\log m}} \right| = C(\varphi, \omega). \tag{1.18}$$

To obtain the above, first put together the results of [**18**] on mixing skew products $(x, \theta) \mapsto (f(x), \theta + \omega)$ and observables $\Phi(\theta, x) = e^{i\theta}\varphi(x)$ for Hölder $\varphi$ to get the almost sure invariance principle and thus the LIL for almost every $(\theta, x)$. Then, use [**28**] to remove the mixing assumption, allowing rational $\omega$. Finally, apply [**29**] to get the LIL for almost every $x$ while fixing $\theta = 0$. See also [**29**, Remark 2.5] for Young towers (which include our piecewise expanding maps) and [**30**, §5.b] for the non-degeneracy coboundary condition. (We are grateful to Melbourne for explaining his results.)

The above discussion gives hope that the following typicality condition holds for a large (at least countable dense?) set of $\omega$ if $f$ is generic and $\varphi$ is smooth.

The critical point is *typical for the upper $\omega$-rotated LIL* and $\varphi \in C^0$ if

$$\limsup_{m\to\infty} \left| \frac{\sum_{k=1}^{m} e^{ik\omega}\varphi(c_k)}{\sqrt{m \log\log m}} \right| < \infty.$$

(Observe that we may as well write 'sup' instead of 'lim sup'.)

The following theorem is proved in §3.

THEOREM 5. (Consequences of upper $\omega$-rotated LIL typicality of $c$) *Let $\varphi \in C^0(I)$ and $\omega \in \mathbb{R}$ with $e^{i\omega} \neq 1$. Assume that the critical point is typical for the upper $\omega$-rotated LIL and $\varphi$. Then, for any open sector $\mathcal{S}$ contained in $\{\lambda^{-1} < |z| < 1\}$ with vertex at $e^{i\omega}$, there exists $C(\mathcal{S}) > 0$ such that*

$$\max\{|\sigma_\varphi(z)|,\ |\Psi_\varphi(z)|\} \leq \frac{C(\mathcal{S})}{|z - e^{i\omega}|^{1/2}} \left( \log\log \frac{1}{|z - e^{i\omega}|} \right)^{1/2} \quad \text{for all } z \in \mathcal{S}. \tag{1.19}$$

*If, in addition, $\int \varphi \, d\mu = 0$, then the above also holds for $e^{i\omega} = 1$.*

† Very recent results [**17**] on the central limit theorem for rotated ergodic sums indicate that the smoothness assumption on $\varphi$ may be unnecessary if one is willing to restrict to an $f$-dependent full measure set of $\omega$.

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The above result may indicate that, generically, $\Psi_\varphi$ could have uncountably many singularities on the unit circle which are 'slightly worse' than ramifications of order two. If true, this fact could be related to the polygenous property of $\{\varphi(c_{k+1})\}_{k\geq 0}$. See also Remark 3.2 about potential lower bounds.

Theorem 5 has an obvious analogue for the outer functions $\sigma_{\varphi,c_-}$ and $\Psi_{\varphi,c_-}$ associated with precritical orbits which are typical for an upper rotated LIL. We refrain from making a formal statement.

*Remark 1.8.* (Rotated-LIL typical precritical orbits) Remark 3.6(b) in [**28**] says the following: 'passing to the natural extension [**32**], it follows from the methods in [**18**] that the law of the iterated logarithm (and much more, including the almost sure invariance principle) can be proved in backwards time'. This gives the LIL for Hölder observables and some backwards orbit of almost every $x$. It is yet unknown whether the critical point itself satisfies the LIL, but Bruin [**15**] has heuristical arguments which encourage us to expect that, generically, some precritical orbit is typical for the LIL in the piecewise expanding setting.

1.7. *Open questions.* Here are three open questions which arise from our results (see also Remark 3.2).

- If $X$ is not horizontal and $\varphi$ is not a coboundary, does the Abelian (or non-tangential) limit of $\Psi(z)$ as $z \to 1$ ever exist? (See also (5.1)†.)
- If $X$ is horizontal but not horizontal of order two, does the Abelian (or non-tangential) limit of $\Psi'(z)$ as $z \to 1$ ever exist?
- For $\omega \neq 0$, if $\varphi$ is not an $\omega$-coboundary, does the Abelian (or non-tangential) limit as $z \to e^{i\omega}$ ever converge to a finite number? (See also (5.1).) Does the answer depend on the diophantine properties of $\omega$?

We end this section by some comments on the smooth unimodal case. The piecewise expanding situation considered here can be viewed as a toy model for the more difficult smooth unimodal case [**7, 34, 35**]. In the postcritically finite (Misiurewicz–Thurston) smooth unimodal case, the susceptibility function admits a meromorphic extension in a disc of radius larger than 1. In some Misiurewicz–Thurston cases, Ruelle [**34**] was able to prove that $z = 1$ is not a pole without assuming horizontality. However, there are examples [**9**] of Misiurewicz–Thurston parameters where (Whitney) linear response is violated when horizontality does not hold. The existence of an extension which is holomorphic at $z = 1$ thus does not guarantee linear response.

When the postcritical orbit of a smooth unimodal map is infinite and slowly recurrent (Collet–Eckmann, topologically slowly recurrent, or polynomially recurrent), it is expected [**35**] that the natural boundary will be a circle of radius strictly smaller than one, while the derivative of the Sinai–Ruelle–Bowen (SRB) measure (in the horizontal case) should be related to a suitable extension‡ of $\Psi_\varphi$ evaluated at $z = 1$, at least under generic assumptions (as in Corollary 1). In the analytic Misiurewicz unimodal case an analogue $\tilde{\sigma}_\varphi$ of $\sigma_\varphi$ can

† Clearly, the series evaluated at $z = 1$ is in general divergent. In [**6**, Proposition 4.5] we proved that the resummation there diverges for some $C^\infty$ function $\varphi$.

‡ Perhaps only along the real axis? The nature of this extension remains admittedly mysterious.

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be obtained from [**35**, end of §§16(b), 17]. Horizontality would perhaps guarantee that the contribution $\widetilde{\mathcal{U}}(z)\tilde{\sigma}_{\varphi,c_-}(z)$ of any outer function corresponding to a renascent right limit continuation $\tilde{\sigma}_{\varphi,c_-}(z)$ of $\tilde{\sigma}_\varphi(z)$ would vanish at $z=1$ (in the same way as horizontality implied $\mathcal{U}(1)=0$ in the present setting). In the Misiurewicz–Thurston case, the above-mentioned result of Ruelle [**34**] implies that the singular term $\widetilde{\mathcal{U}}(z)\tilde{\sigma}_\varphi$ is meromorphic with no pole at $z=1$, but this neither implies that this singular term vanishes at $z=1$, nor that $\Psi_\varphi(z)$ is real analytic on $[0,1]$. Summarising, although we are perhaps closer to understanding the misleading behaviour [**34**] of finite postcritical orbits, the Borel monogenic extension hoped for in [**8**] remains elusive.

## 2. *Strong natural boundary and renascent right limits*

In this section, we prove Theorems 1 and 2. We begin with a simple lemma about right limits (which is implicit in [**13**]).

LEMMA 2.1. *Let $\{a_k\}_{k=0}^\infty$ be a sequence in a compact metric space $E$. Then any increasing sequence of positive integers $\{m_j\}_{j=0}^\infty$ admits a subsequence $\{k_j\}_{j=0}^\infty$ such that, for each $n\in\mathbb{Z}$, the limit $\lim_{j\to\infty} a_{n+k_j}$ exists.*

*In particular, for any accumulation point $b$ of $\{a_k\}_{k=0}^\infty$, there exists a right limit $\{b_n\}_{n=-\infty}^\infty$ with $b_0=b$.*

*Proof.* Pick an arbitrary point $e$ in $E$, and define for each $k\geq 0$ a two-sided sequence $\vec{a}^{[k]}\in E^{\mathbb{Z}}$ by $a_n^{[k]}=a_{n+k}$ if $n\geq -k$ and $e$ if $n<-k$. By Tikhonov's theorem, $E^{\mathbb{Z}}$ is compact for the product topology, hence the sequence $(\vec{a}^{[m_j]})_{j\geq 0}$ admits a subsequence $(\vec{a}^{[k_j]})_{j\geq 0}$ which converges to a limit $\vec{b}$ in $E^{\mathbb{Z}}$, which exactly means $\lim_{j\to\infty} a_{n+k_j}=b_n$ for each $n\in\mathbb{Z}$.

The last statement follows by choosing $\{m_j\}_{j=0}^\infty$ so that $\lim_{j\to\infty} a_{m_j}=b$, and applying the first part of the lemma. □

To prove Theorem 1, we shall also need a well-known consequence of our mixing assumption.

LEMMA 2.2. *For any $x_0\in(c_2,c_1)$ and any $\epsilon>0$, there exists a positive integer $\ell$ so that the set $f^{-\ell}(x_0)$ is $\epsilon$-dense in $I=[a,b]$.*

*Proof.* Since $f$ is topologically mixing on $[c_2,c_1]$, it has (see the reference to Hofbauer in [**16**, Proposition 2.6 and Appendix B], as explained in [**38**, p. 641])† the following covering property [**24**]: for every $n\geq 1$ there is $\ell(n)<\infty$ such that, for every interval of monotonicity $I_n$ of $f^n$, the set $f^{\ell(n)}(I_n)$ covers $[c_2,c_1]$ up to finitely many points (note that Hofbauer did not assume continuity of $f$). Since $f^{\ell(n)}$ is continuous, $f^{\ell(n)}(I_n)$ is an interval and, hence, contains $x_0$. Therefore, $f^{-\ell(n)}(x_0)\cap I_n\neq\emptyset$ for each interval of monotonicity of $f^n$. It suffices to take $n(\epsilon)$ so that each interval of monotonicity of $f^n$ has length $<\epsilon$ (this is possible since each such interval has length $\leq |b-a|\lambda^{-n}$). □

† See also [**21, 22**, Lemma 2] for earlier proofs of the 'weak covering' property, which by [**24**, Theorem 4.4] implies covering.

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*Proof of Theorem 1.* Let $y, \tilde{y} \in [c_2, c_1]$ be such that $\varphi(y) \neq \varphi(\tilde{y})$. Since $\varphi$ is continuous, we can take $\delta > 0$ so that

$$x \in J \text{ and } \tilde{x} \in \tilde{J} \Longrightarrow \varphi(x) \neq \varphi(\tilde{x}),$$

where $J = [y - \delta, y + \delta] \cap [c_2, c_1]$ and $\tilde{J} = [\tilde{y} - \delta, \tilde{y} + \delta] \cap [c_2, c_1]$ are non-trivial intervals. Since $f$ is mixing, by Lemma 2.2 applied to $x_0 = c$, we can find $\ell \geq 1$ such that $f^{-\ell}(c) \cap J$ and $f^{-\ell}(c) \cap \tilde{J}$ are non-empty.

Pick $x \in f^{-\ell}(c) \cap J$ and $\tilde{x} \in f^{-\ell}(c) \cap \tilde{J}$. Since $\{c_k\}_{k\geq 0}$ is dense in $[c_2, c_1]$, we can find two integer sequences $\{m_j\}$ and $\{\tilde{m}_j\}$ such that $x = \lim c_{m_j+1}$ and $\tilde{x} = \lim c_{\tilde{m}_j+1}$. By Lemma 2.1, we get two right limits $\{x_n\}_{n\in\mathbb{Z}}$ and $\{\tilde{x}_n\}_{n\in\mathbb{Z}}$ for the sequence $\{c_{k+1}\}_{k\geq 0}$ such that $x_0 = x$ and $\tilde{x}_0 = \tilde{x}$. By construction, $f^\ell(x_0) = f^\ell(\tilde{x}_0) = c$ but $\varphi(x_0) \neq \varphi(\tilde{x}_0)$.

Now the relation $c_{k+2} = f(c_{k+1})$ and the continuity of $f$ imply that $x_{n+1} = f(x_n)$ and $\tilde{x}_{n+1} = f(\tilde{x}_n)$ for every $n \in \mathbb{Z}$, hence

$$n \geq \ell \Longrightarrow x_n = \tilde{x}_n.$$

By continuity of $\varphi$, the two-sided sequences $\{b_n\}_{n\in\mathbb{Z}}$ and $\{\tilde{b}_n\}_{n\in\mathbb{Z}}$ defined by

$$b_n = \varphi(x_n), \quad \tilde{b}_n = \varphi(\tilde{x}_n), \quad n \in \mathbb{Z},$$

are right limits for the sequence $\{\varphi(c_{k+1})\}_{k\geq 0}$. They are distinct, since $b_0 \neq \tilde{b}_0$, but they coincide for $n \geq \ell$. We can thus apply [**13**, Theorem 3.1], which ensures that $\sigma_\varphi(z) = \sum_{k\geq 0} \varphi(c_{k+1}) z^k$ has a strong natural boundary on $\{|z| = 1\}$. By Theorem 1.1, this must be the case for $\Psi_\varphi(z)$ too. □

*Remark 2.3.* Our argument hinges on the non-invertibility of $f$. Breuer–Simon [**13**, Theorem 7.1] have a similar result for homeomorphisms of the circle with dense orbits, but for observables with discontinuities: the discontinuities there achieve the effect we obtain here from the non-invertibility.

Let us move on to the proof of Theorem 2. We need the following lemma.

LEMMA 2.4. *Let $(E, d)$ be a metric space and $T : E \to E$ be a continuous map. Suppose that $\gamma \in E$ has a dense orbit $\{\gamma_k = T^k(\gamma)\}_{k\geq 0}$. Then the right limits of this sequence are exactly the complete orbits of $T$, i.e. the two-sided sequences $\{x_n\}_{n=-\infty}^{\infty}$ of $E$ such that $x_{n+1} = T(x_n)$ for all $n \in \mathbb{Z}$.*

*Proof.* We first suppose that $\{x_n\}_{n\in\mathbb{Z}}$ is a right limit. Let $\{k_j\}$ be an integer sequence such that $\lim_{j\to\infty} \gamma_{k_j+n} = x_n$ for every $n \in \mathbb{Z}$. Then, for any $n \in \mathbb{Z}$, the limit $x_{n+1}$ of the sequence $\gamma_{k_j+n+1} = T(\gamma_{k_j+n})$ must coincide with $T(x_n)$ (by continuity of $T$), hence $\{x_n\}_{n\in\mathbb{Z}}$ is a complete orbit of $T$.

Suppose now that $\{x_n\}_{n\in\mathbb{Z}}$ is any complete orbit of $T$. For $m, j \geq 0$, we denote by $B_{m,j}$ the open ball of centre $x_{-m}$ and radius $1/(j + 1)$. For each $j \geq 0$, the set

$$V_j = \bigcap_{m=0}^{j} T^{-m}(B_{j-m,j}) = \bigcap_{m=0}^{j} T^{-(j-m)}(B_{m,j})$$

is open and contains $x_{-j}$ (because $T^m(x_{-j}) = x_{-(j-m)}$ for all $m$); in particular it is non-empty and must contain at least one point of our dense sequence $\{\gamma_k\}$. We can thus define

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an integer sequence $\{k_j\}$ by

$$k_j = j + \min\{k \geq 0 \mid T^k(\gamma) \in V_j\}, \quad j \geq 0.$$

This sequence tends to $\infty$ (since $k_j \geq j$) and, for any fixed $m \geq 0$,

$$\begin{aligned} j \geq m &\Longrightarrow T^{k_j - j}(\gamma) \in V_j \subset T^{-(j-m)}(B_{m,j}) \\ &\Longrightarrow T^{k_j - m}(\gamma) = T^{j-m}(T^{k_j - j}(\gamma)) \in B_{m,j}, \end{aligned}$$

hence $\lim_{j\to\infty} \gamma_{k_j - m} = x_{-m}$. In particular, $\lim \gamma_{k_j} = x_0$ and, by continuity of $T$, $\lim_{j\to\infty} \gamma_{k_j+n} = x_n$ for every $n \geq 1$. We thus have proved that $\{x_n\}_{n=-\infty}^{\infty}$ is a right limit of $\{\gamma_k\}_{k=1}^{\infty}$. □

*Proof of Theorem 2.* We can restrict $f$ to $[c_2, c_1]$, because this interval is forward-invariant and contains all the elements of any right limit of $\{c_{k+1}\}$ (indeed, all of them are accumulation points of this orbit, which is contained in $[c_2, c_1]$). Statement (i) thus follows from Lemma 2.4.

For (ii), we consider $\varphi(x) = x$ as a warmup case. Then it suffices to show that the set of complete orbits of $c_1$ contained in $[c_2, c_1]$ is uncountable. This is an immediate consequence of the fact that there are uncountably many precritical orbits. This fact can be easily proved as follows: if $y_n \in [c_2, c_1]$ is given with $n \leq -1$ and $f^{-n}(y_n) = c_1$, then $y_n < c_1$ (because we assumed that the postcritical orbit is not finite), thus it has two distinct preimages; there are one or two possibilities for $y_{n-1} \in [c_2, c_1]$ (according to whether $y_n < c_3$ or $y_n \geq c_3$), but there are always at least two possibilities for $y_{n-2} \in [c_2, c_1]$. (The reader is invited to draw a picture.)

Let us now consider $\varphi$ continuous and not $f$-symmetric on $[c_2, c_1]$. Let $x \neq y$ so that $f(x) = f(y) = v$ be a pair where symmetry is violated, i.e. $\varphi(x) \neq \varphi(y)$. We may assume that $\min(x, y) > c_3$ and $\max(x, y) < c_1$. Then, since $\varphi$ is continuous, there exist $\delta > 0$ so that, for any $\tilde{x} \neq \tilde{y}$ with $f(\tilde{x}) = f(\tilde{y})$ and $|f(\tilde{x}) - v| < \delta$, we have $\varphi(\tilde{x}) \neq \varphi(\tilde{y})$. To prove claim (ii), it suffices to show that there are uncountably many precritical orbits so that $|c_n - v| < \delta$ for infinitely many $n \leq -1$. For this, since $f$ is mixing, we may apply Lemma 2.2 for $\epsilon < \delta/2$ (say) iteratively, first for $x_0 = c$, and then, infinitely many times, to all $x_0$ in the $\epsilon$-dense set from the previous iteration which satisfy $|x_0 - v| < \delta$ (there are at least two different such $x_0$ at each step since $\epsilon < \delta/2$). □

## 3. *Non-polar non-tangential limits as $z \to e^{i\omega}$*

In this section we prove Theorems 3 and 5. (Theorem 1.5 is proved in exactly the same way as Theorem 3, replacing $z^k$ by $z^{-k}$, the iterate $f^k$ by the inverse branch corresponding to the chosen precritical orbit, and noting that the proof of Lemma C.1 shows that, if the precritical orbit is Birkhoff typical, then it is Wiener–Wintner typical.)

*Proof of Theorem 3.* For $\omega \in \mathbb{R}$, it is enough to consider $\sigma_\varphi(e^{i\omega}z)$ for $z$ in an open sector $\mathcal{S}$ contained in $\{\lambda^{-1} < |z| < 1\}$ with vertex at 1. We introduce a notation for the partial sums:

$$S_k(e^{i\omega}) = \sum_{j=0}^{k-1} e^{ij\omega}\varphi(c_{j+1}), \quad k \geq 1. \tag{3.1}$$

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Then, by Abel summation,

$$\sigma_\varphi(e^{i\omega}z) = (1-z)\sum_{k=1}^{\infty} S_k(e^{i\omega})z^{k-1} \quad \text{for } |z|<1. \tag{3.2}$$

Let $\omega \in \mathbb{R}$ (if $e^{i\omega}=1$ we assume that $\int \varphi \, d\mu = 0$). Since the critical point of $f$ is Birkhoff typical, Lemma C.1 (Wiener–Wintner) gives $S_k(e^{i\omega}) = o(k)$. On the other hand, $|S_k(e^{i\omega})| \le k \sup|\varphi|$. Therefore, for each $\epsilon > 0$, there is $K$ so that $|S_k(e^{i\omega})| \le \epsilon k$ for all $k \ge K$ and

$$\begin{aligned}\left|\sum_{k=1}^{\infty} S_k(e^{i\omega})z^{k-1}\right| &\le \sum_{k=1}^{K-1}|S_k(e^{i\omega})| + \sum_{k=K}^{\infty}|S_k(e^{i\omega})||z|^{k-1}\\ &\le K^2|\sup\varphi| + \epsilon(1-|z|)^{-2} \quad \text{for } |z|<1.\end{aligned}$$

Since there exists a constant $C(\mathcal{S}) > 0$ such that

$$z \in \mathcal{S} \implies 1-|z| \le |1-z| \le C(\mathcal{S})(1-|z|), \tag{3.3}$$

we obtain from (3.2) that

$$|(1-z)\sigma_\varphi(e^{i\omega}z)| \le |1-z|^2K^2|\sup\varphi| + C(\mathcal{S})^2\epsilon,$$

for any $z \in \mathcal{S}$, whence the conclusion follows. □

*Remark 3.1.* Our argument for proving Theorem 3 is similar to the argument in [**23**, pp. 12–13] mentioned in Remark A.3 about Hecke's example. (The difference is that, in [**23**], a non-zero Fourier coefficient caused a polar-type blowup, while here the Wiener–Wintner theorem in contrast ensures convergence to zero.)

*Proof of Theorem 5.* We now assume that the critical point of $f$ is typical for the upper $\omega$-rotated LIL and $\varphi$, i.e. $|S_k(e^{i\omega})| \le D\sqrt{k\log\log k}$ for all $k \ge 3$, with a certain $D = D(e^{i\omega}) > 0$. By (3.2), we obtain

$$|\sigma_\varphi(e^{i\omega}z)| \le |1-z|(3\sup|\varphi| + DL(|z|)) \quad \text{for } |z|<1, \tag{3.4}$$

with

$$L(r) = \sum_{k=3}^{\infty} r^{k-1}\sqrt{k\log\log k} \quad \text{for } 0 \le r < 1. \tag{3.5}$$

Assume for a moment that for any $s_0 \in (0, \log(1/(1-e^{-1})))$ there exists $M > 0$ such that

$$L(r) \le M(1-r)^{-3/2}\left(\log\log\frac{1}{1-r}\right)^{1/2} \quad \text{for } e^{-s_0} \le r < 1. \tag{3.6}$$

Then, putting together (3.4) and (3.6), we get

$$|\sigma_\varphi(e^{i\omega}z)| \le \widetilde{M}(\omega)|1-z|(1-|z|)^{-3/2}\left(\log\log\frac{1}{1-|z|}\right)^{1/2} \quad \text{for } \lambda^{-1} \le |z| < 1, \tag{3.7}$$

from which inequality (1.19) for $\sigma_\varphi$ follows by (3.3). Inequality (1.19) for $\Psi_\varphi$ then follows by Theorem 1.1.

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To complete the proof of Theorem 5, it remains to show (3.6). The condition $r \geq e^{-s_0}$ ensures that $\log\log(1/(1-r)) \geq \log\log(1/(1-e^{-s_0})) > 0$, so that the right-hand side of (3.6) is bounded from below. It is sufficient to consider arbitrarily small $s_0$. For any $s \geq 0$, we introduce the notation

$$A_s(x) = e^{-sx}\sqrt{x \log\log x} \quad \text{for } x \geq 3.$$

Let us fix $r = e^{-s}$ with $0 < s \leq s_0$, so that

$$rL(r) = \sum_{k=3}^{\infty} A_0(k) r^k = \sum_{k=3}^{\infty} A_s(k).$$

We observe that the function $A_0$ is increasing on $[3, +\infty)$, while $A_s$ is decreasing on $[s^{-1}, +\infty)$ as soon as $s_0 \leq e^{-e}$ (because the logarithmic derivative of $A_s$ is $-s + (1/2x)(1 + (1/(\log x \log\log x))) < -s + ((1+e^{-1})/2x) < 0$), hence

$$rL(r) \leq \sum_{3 \leq k < s^{-1}+1} A_0(k) + \sum_{k \geq s^{-1}+1} A_s(k) \leq \int_3^{s^{-1}+2} A_0(x)\,dx + \int_{s^{-1}}^{+\infty} A_s(x)\,dx.$$

Integrating by parts, we get

$$\int_3^{s^{-1}+2} A_0(x)\,dx = \left[\frac{2}{3}x^{3/2}(\log\log x)^{1/2}\right]_3^{s^{-1}+2} - \frac{2}{3}\int_3^{s^{-1}+2} x^{3/2}((\log\log x)^{1/2})'\,dx,$$

which is $< (2/3)(s^{-1}+2)^{3/2}(\log\log(s^{-1}+2))^{1/2} \leq \text{const}(1/s)^{3/2}(\log\log(1/s))^{1/2}$, while

$$\begin{aligned} B = \int_{s^{-1}}^{+\infty} A_s(x)\,dx &= \frac{1}{s}[e^{-sx}A_0(x)]_{+\infty}^{s^{-1}} + \int_{s^{-1}}^{+\infty} e^{-sx}\frac{A_0'(x)}{s}\,dx \\ &< \frac{e^{-1}}{s}A_0\left(\frac{1}{s}\right) + \frac{1+e^{-1}}{2}\int_{s^{-1}}^{+\infty}\frac{1}{sx}A_s(x)\,dx < \frac{e^{-1}}{s}A_0\left(\frac{1}{s}\right) + \frac{1+e^{-1}}{2}B \end{aligned}$$

(because the logarithmic derivative of $A_0$ is $< (1+e^{-1})/2x$ for $x \geq s^{-1}$). Thus, $B < (2/(e-1))(1/s)A_0(1/s) = \text{const}(1/s)^{3/2}(\log\log(1/s))^{1/2}$. Since $1 - r \sim s$, the proof is complete. □

*Remark 3.2.* (Lower bounds from the law of the iterated logarithm) Assume that the postcritical orbit is typical for the ordinary LIL (1.17). Then, in view of the lower bound analogue of (3.6), if $X$ is non-horizontal, it is not absurd to guess that $\sup_{r\in[0,1)} |\Psi_\varphi(r)| = \infty$ when $\varphi$ is not a coboundary. Since the LIL would only give a sequence of times $m_j \to \infty$ so that $\sum_{k=1}^{m_j} \varphi(c_k) > C\sqrt{m_j \log\log m_j}$, it is not clear how to transform this intuition into a rigorous proof. Presumably, some information on the sequence $m_j$ would help.

In view of studying $\omega \in \mathbb{R}$ with $e^{i\omega} \neq 1$, recall that if $\varphi$ is continuous and $c$ is typical for the Birkhoff theorem, then for Lebesgue almost every $\omega$ we have [**3**]

$$\lim_m \int \left| \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \varphi(c_k) e^{ik\omega} - \left( \int |\varphi|^2 \right)^{1/2} \right|^2 = 0,$$

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and thus for Lebesgue almost every $\omega$,

$$\limsup_m \frac{1}{\sqrt{m}} \sum_{k=1}^{m} \varphi(c_k)\Re(e^{ik\omega}) \geq \int |\varphi|^2 \, d\mu. \tag{3.8}$$

We would get (stronger) estimates for fixed $\omega$ by assuming that the critical point is typical for the $\omega$-rotated LIL (1.18) and a non-$\omega$-coboundary $\varphi$. This indicates that, for generic $f$ and $\varphi$, we may hope to get $\sup_{r\in[0,1)} |\Psi_\varphi(re^{i\omega})| = \infty$ for a large set of $\omega$ (countable dense?). Just like in the discussion for $e^{i\omega} = 1$ above, a rigorous proof would probably require information on the sequence $m_j \to \infty$ corresponding to the lim sup in (1.17). Diophantine properties of $\omega$ could play a role: possibly the non-tangential limit could exist for such angles, leading potentially to continuous extension(s) of $\sigma_\varphi$ on a subset of positive measure of the circle.

4. *Higher-order horizontality and non-tangential limits of derivatives*

*Proof of Theorem 4.* We start with the claim about the derivatives of $\mathcal{U}(z)\sigma_\varphi(z)$. It suffices to consider $H \geq 2$.

Assume first that $H = 2$. By (B.3) and (B.4) in Appendix B, if $X$ is horizontal of order two then $\mathcal{U}(z)(z-1)^{-2}$ is holomorphic in $|z| > \lambda^{-1}$. Therefore, to get $\lim_{z\overset{NT}{\to}1} (\mathcal{U}(z)\sigma_\varphi)'(z) = 0$, we only need to prove that

$$((z-1)^2\sigma_\varphi(z))' = 2(z-1)\sigma_\varphi(z) + (z-1)^2(\sigma_\varphi(z))'$$

converges to zero when $z\overset{NT}{\to}1$. The first term converges to zero as $z\overset{NT}{\to}1$ by Theorem 3. We shall next see that $(z-1)^2\sigma_\varphi'(z)$ also converges to zero.

Recalling (3.1), the proof of Theorem 3 shows that

$$\sigma_\varphi(z) = (1-z)S(z) \quad \text{with } S(z) = \sum_{k=1}^{\infty} S_k(1)z^k,$$

and that $\lim_{z\overset{NT}{\to}1}(z-1)^2S(z) = 0$. Therefore,

$$(z-1)^2\sigma_\varphi'(z) = (1-z)^3S'(z) - (z-1)^2S(z),$$

and we just need to check that $\lim_{z\overset{NT}{\to}1}(z-1)^3S'(z) = 0$. Let $\epsilon > 0$. Since $S_k(1) = o(k)$ (by Birkhoff typicality of the critical point), we can choose $K \geq 1$ such that $|S_k(1)| \leq \epsilon(k+1)$ for all $k \geq K$, hence

$$\begin{aligned} |S'(z)| = \left|\sum_{k=1}^{\infty} kS_k(1)z^{k-1}\right| &\leq \sum_{k=1}^{K} |kS_k(1)| + \epsilon \sum_{k=1}^{\infty} k(k+1)|z|^{k-1} \\ &\leq K^3 \max|\varphi| + \epsilon(1-|z|)^{-3}, \end{aligned}$$

and the conclusion follows.

Let $H \geq 3$. If $X$ is horizontal of order $H$, then (B.6) implies that $\mathcal{U}(z)(z-1)^{-H}$ is holomorphic in $|z| > \lambda^{-1}$. To get $\lim_{z\overset{NT}{\to}1}(\mathcal{U}(z)\sigma_\varphi)^{(H-1)}(z) = 0$, we only need to prove

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that $((z-1)^H\sigma_\varphi(z))^{(H-1)}$ converges to zero when $z\overset{NT}{\to}1$. The only term which needs to be considered is $(z-1)^H\sigma_\varphi^{(H-1)}(z)$. Proceeding inductively, we reduce to showing

$$\lim_{z\overset{NT}{\to}1}(z-1)^{H+1}S^{(H-1)}(z)=0.$$

Using $S^{(H-1)}(z)=\sum_{k=H-1}^{\infty}(k!/(k-(H-1))!)S_k(1)z^k$, we proceed similarly as in the case $H=2$.

The results we just proved on $(\mathcal{U}\sigma_\varphi)^{(\ell)}$ immediately give that $\Psi_\varphi^{(\ell)}(z)$ converges as $z\overset{NT}{\to}1$, for all $\ell\le H-1$.

It remains to consider the rrl-continuations associated with a Birkhoff typical precritical orbit $\{c_{n+1}\}_{n\le -1}$: replacing $z^k$ by $z^{-k}$ and $f^k$ by its appropriate inverse branch, the previous arguments give for any $\ell\le H-1$

$$\lim_{z\overset{NT}{\to}1}(z-1)^{\ell}(\sigma_{\varphi,c_-}(z))^{(\ell-1)}=0. \qquad \square$$

## 5. *Non-tangential limits as $z\to e^{i\omega}$ for $\omega$-coboundaries*

Given $\omega\in\mathbb{R}$, we say that $\varphi$ is a $C^0$ *postcritical $\omega$-coboundary* if there exists $\psi\in C^0$ so that

$$\varphi(c_k)=\psi(c_k)-e^{i\omega}(\psi\circ f)(c_k)=\psi(c_k)-e^{i\omega}\psi(c_{k+1})\quad\text{for all } k\ge 1. \tag{5.1}$$

If the postcritical orbit is dense, this is equivalent to the existence of a continuous $\psi$ so that $\varphi=\psi-e^{i\omega}\psi\circ f$. This degeneracy condition on $\varphi$ is well studied. If $e^{i\omega}=1$, it implies the vanishing of $\sum_{k=0}^{p-1}\varphi(f^k(x))$ along every periodic orbit $f^p(x)=x$, giving a closed set of infinite codimension. If $e^{i\omega}\ne 1$, see e.g. [**30**, §5.b], where the degeneracy condition is still satisfied only on a closed set of infinite codimension.

Our result below does not require horizontality, even for $e^{i\omega}=1$.

PROPOSITION 5.1. (Non-tangential limits for continuous $\omega$-coboundaries) *Assume that the critical point is Birkhoff typical. Let $\omega\in\mathbb{R}$. If $\varphi\in C^0(I)$ is a $C^0$ postcritical $\omega$-coboundary for $\psi$, then*

$$\lim_{z\overset{NT}{\to}e^{i\omega}}\sigma_\varphi(z)=\psi(c_1) \tag{5.2}$$

*and, therefore,*

$$\lim_{z\overset{NT}{\to}e^{i\omega}}\Psi_\varphi(z)=\mathcal{U}(e^{i\omega})\psi(c_1)+\mathcal{V}_\varphi(e^{i\omega})+\Psi_\varphi^{\mathrm{hol}}(e^{i\omega}).$$

For a precritical orbit $\{c_{n+1}\}_{n\le -1}$, we say that $\varphi$ is a $C^0$ *precritical $\omega$-coboundary* if there exists $\psi\in C^0$ so that

$$\varphi(c_n)=\psi(c_n)-e^{i\omega}\psi(c_{n+1})\quad\text{for all } n\le 0.$$

If the precritical orbit is dense, this is equivalent to the existence of a continuous $\psi$ so that $\varphi=\psi-e^{i\omega}\psi\circ f$. Adapting the proof of Proposition 5.1 below, one shows that if the precritical orbit $c_-=\{c_{n+1}\}_{n\le -1}$ is Birkhoff typical (and, thus, by an adaptation of the

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proof of Lemma C.1, Wiener–Wintner typical) and if $\varphi$ is a $C^0$ precritical $\omega$-coboundary, then the corresponding rrl-continuation $\sigma_{\varphi,c_-}$ of $\sigma_\varphi$ satisfies

$$\lim_{z \overset{NT}{\to} e^{i\omega}} \sigma_{\varphi,c_-}(z^{-1}) = \psi(c_1) = \lim_{z \overset{NT}{\to} e^{i\omega}} \sigma_\varphi(z).$$

For an $\omega$-coboundary $\varphi$ there is thus a (tenuous) link between $\sigma_\varphi$ and each rrl-continuation $\sigma_{\varphi,c_-}$ associated with a Birkhoff typical precritical orbit $c_-$. This can be compared with the identity $g_{b_-}(z) = g(z^{-1}) + \frac{1}{2}$ in Example A.3.

*Proof of Proposition 5.1.* Since $c$ is Birkhoff typical it is Wiener–Wintner typical by Lemma C.1. In particular, recalling (3.1), we have $S_k(e^{i\omega}) = o(k)$ for all $\omega \in \mathbb{R}$ (where we used $\int \varphi \, d\mu = 0$ if $e^{i\omega} = 1$, since $\varphi = \psi - \psi \circ f$). We first show that this Wiener–Wintner-type property, combined with the assumption that $\varphi$ is a $C^0$ postcritical $\omega$-coboundary, implies (5.2): the coboundary assumption yields $\psi \in C^0$ with $S_k(e^{i\omega}) = \psi(c_1) - e^{ik\omega}\psi(c_{k+1})$ for all $k \geq 1$. Therefore,

$$(1-z)\sum_{k=1}^{\infty} S_k(e^{i\omega}) z^{k-1} = \psi(c_1) - (1-z)\sum_{k=1}^{\infty} e^{ik\omega}\psi(c_{k+1}) z^{k-1} \tag{5.3}$$

$$= \psi(c_1) - (1-z)\sigma_\psi(e^{i\omega}z). \tag{5.4}$$

The Wiener–Wintner-type property $S_k(e^{i\omega}) = o(k)$ and the proof of Theorem 3 (replacing $\varphi$ by $\psi$) give $\lim_{z \overset{NT}{\to} 1}(1-z)\sigma_\psi(e^{i\omega}z) = 0$. Therefore, recalling (3.2), we obtain

$$\lim_{z \overset{NT}{\to} e^{i\omega}} \sigma_\varphi(z) = \psi(c_1),$$

and we have proved (5.2).

To conclude, combine Theorem 1.1 with (5.2) and the proof of Theorem 3. □

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## A. *Appendix. Examples of renascent right limits and rrl-continuations*

A.1. *Preperiodic series.* A power series with bounded coefficients, admitting an analytic continuation outside the unit circle or not, may have no renascent right limit at all. Think for instance of any non-trivial series $\sum_{k\geq 0} a_k z^k$ with $\lim_{k\to\infty} a_k = 0$. In the case of the geometric series $\sum_{k\geq 0} z^k$, there is of course only one right limit, which is

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renascent, and indeed the corresponding rrl-continuation is $-\sum_{n=-1}^{-\infty} z^n = (1-z)^{-1}$, as expected. In the case of a preperiodic sequence, $g(z) = \sum_{k=0}^{\infty} a_k z^k$ with $a_k = a_{k+p}$ for $k \geq m$ (as in Remark 1.2), it is easy to see that the existence of a renascent right limit implies that the sequence $\{a_k\}_{k=0}^{\infty}$ is in fact periodic, with $g(z) = (1/(1-z^p)) \sum_{k=0}^{p-1} a_k z^k$: a periodic sequence has a unique renascent right limit (obtained by periodic extension from $\mathbb{N} = \{0, 1, 2, \ldots\}$ to $\mathbb{Z}$), a preperiodic non-periodic sequence does not have any, but it has a single weakly renascent right limit.

A.2. *Examples with Borel monogenic extensions.* Holomorphic functions of the form

$$g(z) = \sum_{m=1}^{\infty} \frac{\rho_m}{z - \lambda_m}, \quad |z| < 1, \tag{A.1}$$

with $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ a dense subset of the unit circle and $(\rho_m)_{m\geq 1} \in \ell^1(\mathbb{N}^*, \mathbb{C})$, were considered in [**25**] for dynamical reasons, particularly in the case where $\Lambda$ consists of all roots of unity since the simplest small divisor problem gives rise to such a situation. One can show [**27**] that the function $h(z)$ defined for $|z| > 1$ by the same series of partial fractions is the unique rrl-continuation of $g$. Here the unit circle is a strong natural boundary as soon as none of the 'residues' $\rho_m$ vanishes, yet several results on the connection between $g$ and $h$ are available: under appropriate assumptions on the size of the $\rho_m$ one can cross the unit circle through Diophantine points and define a Borel monogenic function (or even a $C^\infty$-holomorphic function, with non-tangential limits for all derivatives) in a compact subset of $\mathbb{C}$ whose intersection with the unit circle has positive Haar measure [**25**, §§2.4–2.5], so that the resulting function enjoys a certain quasianalyticity property [**26**]; one can also cross the unit circle through any of the 'poles' $\lambda_m$ by means of a quasianalytic generalized Laurent series [**25**, §4.1].

A.3. *Hecke's example.* In [**13**, (1.14)] one finds Hecke's example

$$\tilde{g}(z) = \sum_{k=0}^{\infty} \{k\theta\} z^k,$$

where $\theta$ is a fixed irrational number and $\{\cdot\}$ denotes the fractional part. Let us consider

$$g(z) = \sum_{k=0}^{\infty} \left(\{k\theta\} - \frac{1}{2}\right) z^k = \tilde{g}(z) - \frac{1}{2}\frac{1}{1-z},$$

so as to deal with the zero mean-value observable $\{\cdot\} - \frac{1}{2}$ evaluated along an orbit of the $\theta$-rotation. This is an example of a function with a single renascent right limit for which the unit circle is a strong natural boundary.

Indeed, all of the right limits can be easily determined in this case: the only renascent one is $b_- = \{b_n^- = \{n\theta\} - \frac{1}{2}\}_{n\in\mathbb{Z}}$ and, due to the discontinuity of the fractional part, there is another right limit $\{b_n^+\}_{n\in\mathbb{Z}}$, with $b_0^+ = \frac{1}{2}$ and $b_n^+ = b_n^-$ for $n \in \mathbb{Z}^*$. By [**13**, Theorem 3.1] this implies strong natural boundary. (The fact that the unit circle is a natural boundary can also be deduced from Weyl's criterion, which shows that $\lim_{z \overset{NT}{\to} e^{2\pi i m\theta}} (z - e^{2\pi i m\theta}) g(z) \neq 0$

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for every non-zero integer $m$, so that there is a countable dense set of pole-like singularities on the unit circle [**23**].) Observe that the unique rrl-continuation of $g$ is

$$g_{b_-}(z) = -\sum_{n=-1}^{-\infty} \left( \{n\theta\} - \frac{1}{2} \right) z^n.$$

Since $\{-x\} - \frac{1}{2} = -(\{x\} - \frac{1}{2})$ if $x \notin \mathbb{Z}$, we find $g_{b_-}(z) = g(z^{-1}) + \frac{1}{2}$.

B. *Appendix. Deducing Theorem 1.1 from* [**5, 6**]

Observe first that the assumption in [**5, 6**] that $\varphi \in C^1$ was only used to give a meaning to (1.2) without integration by parts.

We first show how to obtain (1.4) from the arguments in [**6**, Proposition 4.6] (note that this is similar in spirit to the computation in [**35**, §17]). There exists a decomposition $\rho = \rho_{sal} + \rho_{reg}$ of the invariant density of $f$ (which is of bounded variation) into $\rho_{sal} = \sum_{n=1}^{\infty} s_n H_{c_n}$, where $H_x$ is a Heaviside function at $x$, $s_k = f'(c_k) s_{k+1}$ and $\rho'_{reg}$ is of bounded variation [**5**, Proposition 3.3]. Let $\mathcal{L}$ be the usual transfer operator of $f$ acting on $BV$ by

$$(\mathcal{L}\psi)(x) = \sum_{f(y)=x} \frac{\psi(y)}{|f'(y)|}.$$

This operator has a simple eigenvalue at 1, for $\mathcal{L}\rho = \rho$ and $\int \mathcal{L}(\psi)\, dx = \int \psi\, dx$, and the rest of the spectrum lies in a disc of radius $\kappa < 1$.

Setting $s_1 = -\lim_{x \uparrow c_1} \rho(x)$, a result of Baladi [**5**, Proposition 4.4] gives

$$\begin{aligned} \Psi_\varphi(z) = &-\sum_{j=1}^{\infty} \varphi(c_j) \sum_{k=1}^{j} z^{j-k} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)} \\ &- \int (1 - z\mathcal{L})^{-1} (X' \rho_{sal} + (X\rho_{reg})') \varphi\, dx, \end{aligned} \tag{B.1}$$

where the second term is meromorphic in the disc of radius $\kappa^{-1}$, with at most a single pole at $z = 1$, this pole being simple, but the residue of this pole vanishes here since $\int \varphi\, d\mu = 0$. Set

$$\Psi_\varphi^{\mathrm{sing}}(z) = -\sum_{j=1}^{\infty} \varphi(c_j) \sum_{k=1}^{j} z^{j-k} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)} = \sum_{n=0}^{\infty} z^n \sum_{k=1}^{\infty} \frac{s_1 X(c_k)}{(f^{k-1})'(c_1)} \varphi(c_{k+n})$$

(which is clearly holomorphic in $|z| < 1$), and

$$\Psi_\varphi^{\mathrm{hol}}(z) = -\int (1 - z\mathcal{L})^{-1} (X' \rho_{sal} + (X\rho_{reg})') \varphi\, dx. \tag{B.2}$$

To analyse $\Psi^{\mathrm{sing}}(z)$, we adapt the proof of [**6**, Proposition 4.6]. Define for $\ell \geq 1$

$$\alpha(c_\ell, z) = -\sum_{j=1}^{\infty} \frac{X(f^j(c_\ell))}{z^j (f^j)'(c_\ell)}.$$

Clearly, $z \mapsto \alpha(c_1, z)$ is analytic in $\{z \in \mathbb{C} \mid |z| \min |f'| > 1\}$ and we have

$$\partial_z \alpha(c_1, z)|_{z=1} = \sum_{j=1}^{\infty} \frac{j\, X(f^j(c_1))}{(f^j)'(c_1)}. \tag{B.3}$$

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For $(\min|f'|)^{-1} < |z| < 1$, the proof of [**6**, Proposition 4.6] gives

$$\Psi^{\text{sing}}(z) = s_1\left[(X(c_1) - \alpha(c_1, z))\sum_{j=1}^{\infty}\varphi(c_j)z^{j-1} - \sum_{j=1}^{\infty}\frac{\varphi(c_j)\,\alpha(c_j, z)}{(f^{j-1})'(c_1)}\right].$$

Finally, we set

$$\mathcal{U}(z) = s_1(X(c_1) - \alpha(c_1, z)), \tag{B.4}$$

and

$$\mathcal{V}_\varphi(z) = -s_1\sum_{j=1}^{\infty}\frac{\varphi(c_j)\,\alpha(c_j, z)}{(f^{j-1})'(c_1)}. \tag{B.5}$$

Clearly, $X$ is horizontal if and only if $\mathcal{U}(1) = 0$. Also, (B.3) implies that $X$ is horizontal of order two if and only if $\mathcal{U}(1) = \mathcal{U}'(1) = 0$. In fact, for any $1 \le \ell \le H - 1$, we have the following generalization of (B.3):

$$\partial_z^\ell\alpha(c_1, z)|_{z=1} = (-1)^{\ell-1}\sum_{j=\ell}^{\infty}\frac{j!}{(j-\ell)!}\frac{X(f^j(c_1))}{(f^j)'(c_1)}. \tag{B.6}$$

Therefore, $X$ is horizontal of order $H \ge 1$ (recall (1.16)) if and only if $\mathcal{U}(1) = \mathcal{U}'(1) = \cdots = \mathcal{U}^{(H-1)}(1) = 0$.

Note also that if $X$ is horizontal, we have (convergence of the first series implies convergence of the second)

$$\mathcal{V}_\varphi(1) = \sum_{j=1}^{\infty}\varphi(c_j)\sum_{k=j}^{\infty}\frac{s_1X(c_{k+1})}{(f^k)'(c_1)} = -\sum_{j=1}^{\infty}\varphi(c_j)\sum_{k=1}^{j}\frac{s_1X(c_k)}{(f^{k-1})'(c_1)}. \tag{B.7}$$

Finally, if $c$ is not periodic, then $f$ is 'good' in the sense of [**6**] so that [**6**, Theorem 5.1, Proposition 4.3, Lemma 4.4] give (1.5).

C. *Birkhoff and Wiener–Wintner typicality*

LEMMA C.1. (Birkhoff typical implies Wiener–Wintner typical) *Let $X$ be a compact metric space, let $T : X \to X$ be continuous, and let $\mu$ be a mixing invariant probability measure for $T$. Let $c \in X$ be so that for any continuous function $\varphi\colon X \to \mathbb{C}$ (Birkhoff typicality)*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\varphi(T^k(c)) = \int\varphi\,d\mu.$$

*Then, for any continuous function $\varphi : X \to \mathbb{C}$ (Wiener–Wintner typicality)*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}e^{i\omega k}\varphi(T^k(c)) = 0 \quad \textit{for all } \omega \in (0, 2\pi).$$

*Proof.* Our proof is inspired by [**2**, §2.4]. We shall use the van der Corput inequality (see e.g. [**2**]): for all $n \ge 1$, each sequence $(u_k, 0 \le k \le n-1)$ of complex numbers, and every integer $h$ between 1 and $n-1$, we have

$$\left|\sum_{k=0}^{n-1}u_k\right|^2 \le \frac{n+h}{h+1}\sum_{k=0}^{n-1}|u_k|^2 + \frac{2(n+h)}{(h+1)^2}\sum_{\ell=1}^{h}(h+1-\ell)\left|\sum_{k=0}^{n-\ell-1}\overline{u_{k+\ell}}u_k\right|.$$

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We first decompose for each $n$

$$\frac{1}{n}\sum_{k=0}^{n-1} e^{i\omega k}\varphi(T^k(c)) = \frac{1}{n}\sum_{k=0}^{n-1} e^{i\omega k}\left(\varphi(T^k(c)) - \int \varphi\, d\mu\right) + \frac{\int \varphi\, d\mu}{n}\sum_{k=0}^{n-1} e^{i\omega k}.$$

For any $\omega \neq 0$ we have $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1} e^{i\omega k} = \lim_{n\to\infty}(1/n)(1-e^{in\omega})/(1-e^{i\omega}) = 0$. We can thus assume $\int \varphi\, d\mu = 0$. Since $|e^{i\omega k}| = 1$, the van der Corput bound implies

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} e^{i\omega k}\varphi(T^k(c))\right|^2 \le \frac{n+h}{n^2(h+1)}\sum_{k=0}^{n-1}|\varphi(T^k(c))|^2 + \frac{2(n+h)}{n^2(h+1)^2}\sum_{\ell=1}^{h}(h+1-\ell)\left|\sum_{k=0}^{n-\ell-1}\overline{\varphi(T^{k+\ell}(c))}\varphi(T^k(c))\right|.$$

Now, on the one hand

$$\frac{n+h}{n^2(h+1)}\sum_{k=0}^{n-1}|\varphi(T^k(c))|^2 \le \frac{2}{n(h+1)}\sum_{k=0}^{n-1}|\varphi(T^k(c))|^2.$$

Since $c$ is Birkhoff typical and $|\varphi|^2$ is continuous, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|\varphi(T^k(c))|^2 = \int |\varphi|^2\, d\mu.$$

Therefore, for each $\epsilon > 0$ and all large enough $n$ (depending on $\epsilon$), we have for all $h$ between 1 and $n-1$

$$\frac{2}{n(h+1)}\sum_{k=0}^{n-1}|\varphi(T^k(c))|^2 \le \frac{2}{h+1}\left(\int |\varphi|^2\, d\mu + \epsilon\right). \tag{C.1}$$

On the other hand

$$\frac{2(n+h)}{n^2(h+1)^2}\sum_{\ell=1}^{h}(h+1-\ell)\left|\sum_{k=0}^{n-\ell-1}\overline{\varphi(T^{k+\ell}(c))}\varphi(T^k(c))\right| \le \frac{4}{n(h+1)}\sum_{\ell=1}^{h}\left|\sum_{k=0}^{n-\ell-1}\overline{\varphi(T^{k+\ell}(c))}\cdot\varphi(T^k(c))\right|.$$

Since $\varphi$ is continuous, the function $\overline{\varphi\circ T^\ell}\cdot\varphi$ is continuous for each $\ell \ge 1$. Since $c$ is Birkhoff typical, we get for all $\ell \ge 1$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-\ell-1}\overline{\varphi(T^{k+\ell}(c))}\varphi(T^k(c)) = \int \overline{\varphi\circ T^\ell}\cdot\varphi\, d\mu.$$

Thus, for each fixed $\epsilon > 0$ and $h \ge 1$, and all large enough $n > h$ (depending on $h$ and $\epsilon$)

$$\frac{4}{n(h+1)}\sum_{\ell=1}^{h}\left|\sum_{k=0}^{n-\ell-1}\overline{\varphi(T^{k+\ell}(c))}\cdot\varphi(T^k(c))\right| \le \frac{4}{h+1}\left(\sum_{\ell=1}^{h}\left|\int \overline{\varphi\circ T^\ell}\cdot\varphi\, d\mu\right| + \epsilon\right).$$

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Since $\mu$ is mixing and $\int \varphi \, d\mu = 0$, we get

$$\lim_{\ell\to\infty}\left|\int \overline{\varphi \circ T^{\ell}} \cdot \varphi \, d\mu\right| = \left|\int \varphi \, d\mu \cdot \int \bar{\varphi} \, d\mu\right| = 0,$$

and, therefore,

$$\lim_{h\to\infty} \frac{1}{h+1} \sum_{\ell=1}^{h}\left|\int \overline{\varphi \circ T^{\ell}} \cdot \varphi \, d\mu\right| = 0.$$

Thus, for each $\epsilon > 0$, all large enough $h$ (depending on $\epsilon$), and all large enough $n > h$ (depending on $h$ and $\epsilon$) we get

$$\frac{4}{n(h+1)} \sum_{\ell=1}^{h}\left|\sum_{k=0}^{n-\ell-1} \overline{\varphi(T^{k+\ell}(c))} \cdot \varphi(T^{k}(c))\right| \le 8\epsilon. \tag{C.2}$$

Take $\epsilon$ small, $h(\epsilon)$ large, $n(h, \epsilon)$ even larger, and use (C.1) and (C.2). □

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