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On a non-linear integral equation occurring in diffraction theory

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Abstract. The problem of diffraction of a plane wave by a semi-infinite grating of isotropic scatterers leads to the consideration of a non-linear integral equation. This bears a resemblance to Chandrasekhar's integral equation which arises in the study of radiative transfer through a semi-infinite atmosphere. It is shown that methods which have been used with success to solve Chandrasekhar's equation are equally useful here. The solution to the non-linear equation satisfies a more simple functional equation which may be solved by factoring (in the Wiener–Hopf sense) a given function. Subject to certain additional conditions which are dictated by physical considerations, a solution is obtained which is the unique admissible solution of the non-linear integral equation. The factors and solution are found explicitly for the case which corresponds to closely spaced scatterers.

1. *Introduction.* The non-linear integral equation

$$f(\alpha) = 1 + Qf(\alpha) \int_{S(0)} f(\psi)\, h(\psi, \alpha)\, d\psi \tag{1}$$

arises in the analysis of the diffraction of a time-harmonic plane wave by a semi-infinite grating of isotropic scatterers. It may be obtained by specializing equations (19) and (20) of (6) to the isotropic case; this derivation and the application of the following results will be given elsewhere. The unknown f determines what is called the scattering function of the grating, from which the scattered field is obtained; in fact f is proportional to the scattering function of the end element of the grating. The kernel h is defined by

$$h(\psi, \alpha) \equiv \frac{e(\psi)\, e(\alpha)}{1 - e(\psi)\, e(\alpha)}, \tag{2}$$

in which

$$e(\phi) \equiv e^{i\kappa \cos \phi}. \tag{3}$$

Here $\kappa \equiv ka$, where a denotes the constant distance between centres of successive scatterers and k is the wave-number of the incident plane wave. $S(0)$ is an infinite contour which runs from $\psi = -\frac{1}{2}\pi + i\infty$ to $\psi = \frac{1}{2}\pi - i\infty$ and passes through $\psi = 0$. It is convenient to choose $S(0)$ to be symmetrical about the origin, and to be defined by

$$\sin \tfrac{1}{2}\psi = e^{-\frac{1}{4}\pi i} \tau / \sqrt{2} \quad (-\infty < \tau < \infty). \tag{4}$$

Nevertheless, the desired solution f is to be of such a nature that the integral in (1) converges when either termination of $S(0)$ is displaced parallel to the real axis by an amount not exceeding $\frac{1}{2}\pi$.

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It is required that $f(\psi)-1$ be integrable on $S(0)$ if $\kappa > 0$, and a constant C is defined by

$$C \equiv \int_{S(0)} [f(\psi)-1]\,d\psi. \tag{5}$$

This integral, too, is to exist when the terminations of $S(0)$ are translated in the above-mentioned manner.

The parameter Q in (1) is to be related to C through the equation

$$Q = P/(1-PC), \tag{6}$$

in which P (the scattering function for an isolated, isotropic scatterer) is a given constant. It is evident that (1), (5) and (6) pose a type of eigenvalue problem.

In the present paper, it will be shown that any solution of (1), subject to (5) and (6), is a solution of the functional equation

$$g(\alpha)\,g(\alpha-\pi) = Q^{-1}p^{+}(\alpha), \tag{7}$$

where p^{+} is a given function. A unique solution of (7) will be found which satisfies the condition

$$g(\alpha) = g(-\alpha) \tag{8}$$

and for which $g(\psi)-1$ is integrable on $S(0)$ if $\kappa > 0$. (That f is an even function of its argument is a necessary consequence of (1); see section 2.) The constant Q in (7) is automatically determined. It will be verified that the function g so obtained is the unique solution of (1), (5) and (6); therefore it will be identified with the desired solution f.

Equation (1) is similar in form to Chandrasekhar's integral equation

$$H(z) = 1 + H(z)\int_0^1 \frac{z\psi(\mu)}{z+\mu}\,H(\mu)\,d\mu \tag{9}$$

which arises in the study of radiative transfer through a semi-infinite atmosphere ((1)). Both (1) and (9) are consequences of certain invariant properties of the corresponding semi-infinite configurations. Although (1) is complicated by the discrete nature of the structure, the method of solution closely parallels that employed by Crum ((2)) to solve (9); a discussion of Crum's analysis is to be found in section (40) of (1).

2. *Properties of f.* Various properties of the unknown may be deduced directly from (1). Many of these are special cases of more general results given in (6), so details will be omitted. It is assumed, and will be verified later at the conclusion of section 6, that the integral in (1) defines f as an analytic, single-valued function of α in a simply connected domain D of the α plane. The boundary of D is determined by those α for which a pole of $h(\psi,\alpha)$ traces out the curve $S(0)$ in the ψ plane. D is bounded on the left- and right-hand sides by the contours $S(-\pi)$ and $S(\pi)$, respectively; these are the translations of $S(0)$ through $\mp\pi$ parallel to the real axis.

The only singularities of f are branch points which lie on the boundary of D at $\alpha = \pm\alpha_n\,(n = \pm 1, \pm 2, \pm 3, \ldots)$; these are determined by the condition that a pole of $h(\psi,\alpha)$ occurs at $\psi = 0$. Hence

$$\kappa(\cos\alpha_n + 1) = 2n\pi \quad (n = \pm 1, \pm 2, \pm 3, \ldots). \tag{10}$$

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The branch cuts run from these points to infinity in the upper or lower half plane, and form part of the boundary of D. No branch cuts lie in the two semi-infinite strips $-\pi < \Re\alpha < 0, \Im\alpha > 0$ and $0 < \Re\alpha < \pi, \Im\alpha < 0$, which contain $S(0)$ and throughout which f is therefore analytic and single-valued.

If $\alpha \in D$, then also $-\alpha \in D$. A path may be found in D from α to $-\alpha$ and, because $h(\psi, \alpha) = h(\psi, -\alpha)$, it follows that $f(\alpha) = f(-\alpha)$. (Therefore, the analysis may be confined to those α which lie on one side or the other of $S(0)$.)

f is a periodic function of $\cos\alpha$. If $\alpha \in D$, let $\psi_m [\equiv \psi_m(\alpha)]$ denote any solution also $\in D$ of

$$\kappa(\cos\alpha - \cos\psi_m) = 2m\pi \quad (m = 0, \pm 1, \pm 2, \ldots). \tag{11}$$

Because $h(\psi, \alpha) = h(\psi, \psi_m)$, it follows that

$$f(\psi_m) = f(\psi_n) \quad (m, n = 0, \pm 1, \pm 2, \ldots). \tag{12}$$

From (1), it is apparent that $f(\alpha) \neq 0$, except possibly at $\alpha = \pi$ or at a branch point; that $f(\alpha_n) = 0$ $(n = 0, \pm 1, \pm 2, \ldots)$ is demonstrated at the end of section 6.

3. *The functional equation.* It now will be shown that the method employed by Crum ((2)) to obtain a functional equation for $H(z)$ from (9) may be used to demonstrate that f is a solution of (7). The following analysis is presented only in a formal manner; for the solution to be obtained later, the required interchanges in order of integration may be justified because of the exponential factor in the integrand when $\kappa > 0$.

Let α lie between $S(0)$ and $S(\pi)$. Then $\alpha \in D$, $\alpha - \pi \in D$, and (1) gives

$$[1 - 1/f(\alpha)][1 - 1/f(\alpha - \pi)] = Q^2 \int f(\phi)\, h(\phi, \alpha)\, d\phi \int f(\psi)\, h(\psi, \alpha - \pi)\, d\psi, \tag{13}$$

integration being along $S(0)$. Partial fraction decomposition of $h(\phi, \alpha)\, h(\psi, \alpha - \pi)$ results in

$$h(\phi, \alpha)\, h(\psi, \alpha - \pi) = \frac{e(\psi)\, e(\alpha)}{e(\psi) - 1/e(\phi)} \left[\frac{1}{e(\alpha) - 1/e(\phi)} + \frac{1}{e(\psi) - e(\alpha)} \right]. \tag{14}$$

This permits the right-hand side of (13) to be written as $I_1 + I_2$, where

$$I_1 \equiv Q^2 \int \frac{f(\phi)}{e(\alpha) - 1/e(\phi)}\, d\phi \int f(\psi)\, \frac{e(\psi)\, e(\alpha)}{e(\psi) - 1/e(\phi)}\, d\psi, \tag{15}$$

$$I_2 \equiv Q^2 \int f(\psi)\, \frac{e(\psi)\, e(\alpha)}{e(\psi) - e(\alpha)}\, d\psi \int \frac{f(\phi)}{e(\psi) - 1/e(\phi)}\, d\phi. \tag{16}$$

Then, because $\phi \in D$, (1) may be employed to yield

$$I_1 = Q \int f(\phi)\, h(\phi, \alpha)\, [1 - 1/f(\phi)]\, d\phi$$

or, again with the aid of (1),

$$I_1 = 1 - 1/f(\alpha) - Q \int h(\phi, \alpha)\, d\phi. \tag{17}$$

I_2 is reduced in a similar manner to give

$$I_2 = Q \int \frac{f(\psi) - 1}{1 - e(\psi)/e(\alpha)}\, d\psi,$$

so

$$I_2 = 1 - 1/f(\alpha - \pi) + QC - Q \int h(\psi, \alpha - \pi)\, d\psi, \tag{18}$$

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in which C is the constant defined by (5). Therefore (13), (15)–(18) imply that

$$\frac{1}{f(\alpha)f(\alpha-\pi)} = 1+QC-QI(\alpha) \quad (\alpha\in D, \alpha-\pi\in D), \tag{19}$$

where

$$I(\alpha) \equiv \int [h(\psi,\alpha)+h(\psi,\alpha-\pi)]\,d\psi \quad (\alpha\in D, \alpha-\pi\in D). \tag{20}$$

It is known [equation (22) of (5)] that $I(\alpha)$ may be represented by the following infinite series when α lies between $S(0)$ and $S(\pi)$.

$$I(\alpha) = -\pi - 2i\log\left(\frac{\gamma\kappa}{4\pi}\right) + \frac{2\pi}{\kappa}\left\{\frac{1}{\sin\alpha} + \sum_{m=-\infty}^{\infty}{}' \left[\frac{1}{\sin\psi_m} - \frac{\kappa}{2\pi i|m|}\right]\right\}. \tag{21}$$

Here ψ_m is the solution of (11) which lies between $S(0)$ and $S(\pi)$, and $\log\gamma$ ($= 0{\cdot}57721\ldots$) is Euler's constant; the prime indicates that the term corresponding to $m = 0$ is to be omitted from the sum.

All solutions of (1) for any value of Q for which the derivation may be justified will satisfy (19). In the case of the sought-after solution, Q and C must be related by (6). When C is thereby eliminated from (19), such a solution of (1) is seen to satisfy (7), in which

$$p^+(\alpha) = P/[1-PI(\alpha)]. \tag{22}$$

$p^+(\alpha)$ is the scattering function for the infinite grating of isotropic scatterers [(5), equation (19)].

Because an admissible solution of (7) must satisfy (8), and because $\pi-\alpha\in D$ if $\alpha-\pi\in D$, f will be a solution of

$$g(\alpha)\,g(\pi-\alpha) = Q^{-1}p^+(\alpha), \tag{23}$$

in which both α and $\pi-\alpha$ lie between $S(0)$ and $S(\pi)$.

Because f satisfies the functional equation (23) as well as (1), it is possible to derive a linear singular integral equation for f. This is the specialization to the case $n = 1$ of an equation satisfied by the scattering function of the nth element of the grating; it will not be considered here.

4. *Solution of the functional equation.* In the present section a solution of (23) will be derived which in section 5 will be shown to be the unique solution of (1), subject to (5) and (6). The method which is employed requires the factoring of $p^+(\alpha)$ in an appropriate manner and, ultimately, identification of the factors with $f(\alpha)$ and $f(\pi-\alpha)$. The solution of (23) is to be an even function of α. The only singularities of $g(\alpha)$ are to be branch points at $\alpha = \pm\alpha_n$ the location of which is determined by (10); $g(\alpha)$ is to be analytic and single-valued throughout the semi-infinite strips $-\pi < \Re\alpha < 0, \Im\alpha > 0$ and $0 < \Re\alpha < \pi, \Im\alpha < 0$ which contain $S(0)$. Then $g(\pi-\alpha)$ will be singular at $\alpha = \pm\alpha_n'$, where

$$\kappa(\cos\alpha_n' - 1) = 2n\pi \quad (n = \pm1, \pm2, \pm3, \ldots), \tag{24}$$

and analytic and single-valued throughout the corresponding two semi-infinite strips which contain $S(\pi)$. Zeros of $g(\alpha)$ may occur only at branch points.

It is convenient to map the domain between $S(0)$ and $S(\pi)$ within which (23) is valid into the λ plane by means of the transformation.

$$\lambda = \cos\alpha. \tag{25}$$

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Then $S(0)$ is mapped into a contour Γ which begins at $1+0+i\infty$, ends at $1-0+i\infty$, and which encloses the half line defined by $\lambda = 1+i\tau\,(0 \leqslant \tau < \infty)$. $S(\pi)$ is mapped into a loop about the half line $\lambda = -1-i\tau\,(0 \leqslant \tau < \infty)$ which starts at $-1+0-i\infty$ and terminates at $-1-0-i\infty$.

Under this transformation, let $g(\alpha) \to G(\lambda)$; then $g(\pi-\alpha) \to G(-\lambda)$. Let

$$1/p^+(\alpha) \to R(\lambda)$$

where, according to (20)–(22),

$$R(\lambda) = P^{-1} + \int_\Gamma [H(z,\lambda) + H(z,-\lambda)]\,dz$$

$$= P^{-1} + \pi + 2i\log\left(\frac{\gamma\kappa}{4\pi}\right) - \frac{2\pi}{\kappa}\left\{(1-\lambda^2)^{-\frac{1}{2}} + \sum_{m=-\infty}^{\infty}{}' \left[\{1-(\lambda-2m\pi/\kappa)^2\}^{-\frac{1}{2}} - \frac{\kappa}{2\pi i|m|}\right]\right\}. \tag{26}$$

Here
$$H(z,\lambda) \equiv \frac{e^{i\kappa(z+\lambda)}}{1-e^{i\kappa(z+\lambda)}}\frac{1}{\sqrt{(1-z^2)}}, \tag{27}$$

and in both (26) and (27) that branch of each square root is chosen which transforms into the appropriate sine function under (25). Then (23) transforms into

$$QG(\lambda)\,G(-\lambda) = 1/R(\lambda). \tag{28}$$

$R(\lambda)$ has branch points on the real axis of λ at λ_n, λ_n' $(n = 0, \pm 1, \pm 2, \ldots)$, where

$$\lambda_n = 2n\pi/\kappa - 1, \tag{29}$$

$$\lambda_n' = 2n\pi/\kappa + 1. \tag{30}$$

It is apparent that $\lambda_{-n} = -\lambda_n'$. Furthermore, if it is assumed that the two sets of branch points are distinct (so $\kappa \neq m\pi$ for any positive integer m), then it is seen that consecutive branch points in either set are separated by a member of the other set. $R(\lambda)$ is unbounded on every branch point.

If
$$K(\lambda) \equiv Q^{\frac{1}{2}}G(\lambda), \tag{31}$$

then (28) implies that
$$K(\lambda)\,K(-\lambda) = 1/R(\lambda) \tag{32}$$

in the cut λ plane. Because $S(0)$ is defined by (4), the cut in the α plane which corresponds to the branch point α_n is determined by $\cos\alpha = 2n\pi/\kappa - 1 - i\tau^2\,(-\infty < \tau < \infty)$. Therefore in the λ plane the cut which corresponds to λ_n is given by

$$\lambda = 2n\pi/\kappa - 1 - i\tau^2 \quad (-\infty < \tau < \infty), \tag{33}$$

while the cut for λ_n' is defined by

$$\lambda = 2n\pi/\kappa + 1 + i\tau^2 \quad (-\infty < \tau < \infty). \tag{34}$$

Thus all branch cuts in the λ plane are half lines parallel to the imaginary axis; that associated with λ_n lies in the lower half plane while that which corresponds to λ_n' is in the upper half plane.

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The only singularities of $K(\lambda)$ are to be the branch points $\lambda = \lambda_n (n = 0, \pm 1, \pm 2, \ldots)$; $K(-\lambda)$ will possess the branch points $\lambda = \lambda_n' (n = 0, \pm 1, \pm 2, \ldots)$. Equation (32) implies that

$$K(\lambda_n) = 0 \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{35}$$

Evidently there is ambiguity in the sign of $K(\lambda)$ as determined by (32). There can be no such ambiguity about $f(\alpha)$ and, therefore, about the desired solution $G(\lambda)$ in (28). Thus the ambiguity in K is compensated for by a corresponding indeterminacy in the sign of $Q^{\frac{1}{2}}$. In the following analysis, consideration is given only to one of the two possible values of $K(\lambda)$.

The only singularities of R in the finite λ plane are the branch points

$$\lambda_n, \lambda_n' \quad (n = 0, \pm 1, \pm 2, \ldots).$$

R is an even function of λ, with period $2\pi/\kappa$. Furthermore, because the constant P is very small (see (73)), one may argue that $p^+(\alpha)$ has no poles, so $R(\lambda) \neq 0$ in the cut λ plane; thus it is possible to define the logarithm of R in an unambiguous manner throughout the cut plane. (An exceptional case will be discussed elsewhere.)

Suppose that λ lies in the upper half plane, and define R_+, analytic everywhere in the upper half plane, by

$$R_+(\lambda) = \frac{\lambda}{2\pi i} \int_{-\infty}^{\infty} \frac{\log R(z)}{z^2 - \lambda^2} dz \quad (\Im \lambda > 0), \tag{36}$$

where the logarithm is made definite by choosing the branch for which $\log 1 = 0$. The cut in the complex R plane runs from 0 to ∞ outside the domain containing the values $R(z)$. The contour of integration is the real axis, suitably indented at the branch points. Convergence of the integral is assured because $\log R$ is a periodic function of its argument. Furthermore, because $R(z) \neq 0$ for z in the cut plane, $\log R$ is analytic in the cut z plane and R_+ can be continued analytically into the lower half λ plane by a downward deformation of the integration contour between two consecutive branch points. One then sees that

$$R_+(\lambda) + R_+(-\lambda) = \log R(\lambda), \tag{37}$$

and a function R_- which is an analytic function of λ in the lower half plane may be defined by

$$R_-(\lambda) \equiv R_+(-\lambda), \tag{38}$$

$$R_-(\lambda) = -\frac{\lambda}{2\pi i} \int_{-\infty}^{\infty} \frac{\log R(z)}{z^2 - \lambda^2} dz \quad (\Im \lambda < 0). \tag{39}$$

Equation (37) provides the necessary factorization of $R(\lambda)$. For

$$R(\lambda) = \exp R_+(\lambda) \exp R_-(\lambda). \tag{40}$$

The first factor is analytic for $\Im \lambda \geqslant 0$ except at the branch points $\lambda_n (n = 0, \pm 1, \pm 2 \ldots)$, while the second is analytic in $\Im \lambda \leqslant 0$ with the branch points $\lambda_n' (n = 0, \pm 1, \pm 2, \ldots)$ excepted.

[It may be noted here that Hills and Karp ((3)) factor essentially the same function as that considered above, in their solution to the grating problem. Under the transformation $\rho = e^{i\kappa\lambda}$, the factors $\exp R_+$ and $\exp R_-$ become functions which are analytic inside and outside the unit circle in the ρ plane, respectively; except for constant multipliers, these are identical with the factors in (3).]

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From (32) and (40), one obtains for all λ in the cut plane

$$\log K(\lambda)+\log K(-\lambda) = -R_+(\lambda)-R_-(\lambda). \tag{41}$$

Now, $\log K(\lambda)$ is to be analytic in $\Im\lambda \geqslant 0$, except at the branch points λ_n ($n = 0, \pm 1, \pm 2, \ldots$), while $\log K(-\lambda)$ must be analytic in $\Im\lambda \leqslant 0$, the points λ_n' ($n = 0, \pm 1, \pm 2, \ldots$) being excepted.

Equation (41) may be written in the form

$$\log K(\lambda)+R_+(\lambda) = -\log K(-\lambda)-R_-(\lambda), \tag{42}$$

valid for all λ in the cut plane. The left-hand side of this equation is analytic in the upper half plane and on the real axis except possibly at the points λ_n ($n = 0, \pm 1, \pm 2, \ldots$); the right-hand side is analytic in the lower half plane and on the real axis except possibly on the branch points λ_n' ($n = 0, \pm 1, \pm 2, \ldots$). Either side of (42) provides the analytic continuation of the other. The two sets of branch points are distinct. It follows that each side of (42) is equal to an integral function which will be denoted by E. Then

$$K(\lambda) = \exp[E(\lambda)-R_+(\lambda)], \tag{43}$$

while from (38) and (42) it may be seen that

$$E(\lambda) = -E(-\lambda). \tag{44}$$

Next it will be demonstrated that only if $E(\lambda) \equiv 0$ can $K(\lambda)$ lead to a valid solution of (1). For it has been shown in section 2 that $f(\alpha)$ has period $\delta \equiv 2\pi/\kappa$ in $\cos\alpha$. Therefore an admissible solution $K(\lambda)$ must have period δ in λ. It is easy to verify that $R_+(\lambda)$, defined by (36), possesses this periodicity by virtue of the periodicity of $\log R(z)$. Therefore $L(\lambda)[\equiv \exp E(\lambda)]$ must exhibit the same periodicity. If $\Im\lambda$ is finite, it follows that $L(\lambda)$ remains bounded as $|\Re\lambda| \to \infty$. Then, because $L(\lambda)$ ($\not\equiv 0$) is an integral function of λ, either $L(\lambda)$ is constant or $|L(\lambda)| \to \infty$ as $|\Im\lambda| \to \infty$ somewhere in every periodic strip of width δ. Therefore, unless $E(\lambda) \equiv 0$ (for this is the only constant permitted by (44)), $\Re E(\lambda) \to +\infty$ or $-\infty$ as $\Im\lambda \to \infty$ somewhere in every periodic strip.

Furthermore, in order that the constant C (defined by (5)) exist, it is necessary that

$$\int_1^{1+i\infty} \frac{G(\lambda)-1}{\sqrt{(1-\lambda^2)}}\,d\lambda \tag{45}$$

be bounded. Moreover, the integral is to converge if the upper limit $1+i\infty$ is replaced by $\infty e^{i\theta}$ ($0 < \theta < \frac{1}{2}\pi$); see the discussion following (4).

It will be shown later that $R_+(\lambda)$ tends to a constant as $\Im\lambda \to \infty$ while Q is to be a finite constant. Therefore, the above arguments indicate that unless $E(\lambda) \equiv 0$, $|G(\lambda)|$ must tend to 0 or ∞ as $\Im\lambda \to \infty$ somewhere in the first quadrant, and the integral (45) will not exist when its upper limit is replaced by an arbitrary point at infinity in that domain. Thus

$$E(\lambda) \equiv 0, \tag{46}$$

so

$$K(\lambda) = \exp[-R_+(\lambda)]. \tag{47}$$

The existence of (45) also suffices to determine Q. For because $R_+(\lambda)$ tends to a constant as $\Im\lambda \to \infty$ it follows that $Q^{\frac{1}{2}}$ must be given by

$$Q^{\frac{1}{2}} = \lim \exp[-R_+(\lambda)] \quad (\Im\lambda \to \infty), \tag{48}$$

and $G(\lambda) \to 1$ as $\Im\lambda \to \infty$. A more explicit formula for Q will be given in section 6.

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Although in general a discussion of K necessitates careful study of R_+, it is interesting to observe that certain values of K may be found without recourse to the factorization. Because $K(\lambda)$ has period $2\pi/\kappa$, (32) at once gives

$$[K(n\pi/\kappa)]^2 = 1/R(n\pi/\kappa) \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{49}$$

Equivalent results have been given by Hills and Karp ((3)).

5. *Verification of solution.* Under the transformation (25), $f(\alpha) \to F(\lambda)$ and (1) becomes

$$F(\lambda) = 1 - QF(\lambda) \int_\Gamma F(z)\, H(z, \lambda)\, dz, \tag{50}$$

where H is defined by (27) and Γ is a loop about $z = 1 + i\tau$ $(0 \leqslant \tau < \infty)$. In the present section it will be shown that the unique solution of (50), subject to (5) and (6) and whose only singularities are the branch points λ_n $(n = 0, \pm 1, \pm 2, \ldots)$ is

$$\begin{aligned} F(\lambda) &\equiv G(\lambda) \\ &= Q^{-\frac{1}{2}} \exp[-R_+(\lambda)], \end{aligned} \tag{51}$$

in which $Q^{\frac{1}{2}}$ is determined by (48).

A direct verification that (51) is the required solution seems to be difficult. For this reason, an indirect method, of the type employed by Crum ((2)), is adopted.

In the λ plane, cut as before in $\Im\lambda < 0$ from the points λ_n $(n = 0, \pm 1, \pm 2 \ldots)$ to infinity, define a function Φ by

$$\Phi(\lambda) \equiv 1/G(\lambda) - 1 - Q \int_\Gamma G(z)\, H(z, \lambda)\, dz, \tag{52}$$

where G is determined by (31) and (47), and Q by (48). Because G is analytic on Γ, it follows that Φ is analytic in the cut plane, and may have branch points at

$$\lambda = \lambda_n \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Furthermore, G satisfies (28), so

$$\Phi(\lambda) = QG(-\lambda)\, R(\lambda) - 1 - Q \int_\Gamma G(z)\, H(z, \lambda)\, dz \tag{53}$$

and, by employing the first of (26) to eliminate $R(\lambda)$, it is seen that

$$\Phi(\lambda) = QG(-\lambda)\left[P^{-1} + \int_\Gamma H(z, -\lambda)\, dz\right] - 1 - Q \int_\Gamma [G(z) - G(-\lambda)]\, H(z, \lambda)\, dz. \tag{54}$$

Now $G(-\lambda)$ has branch points at $\lambda = \lambda_n'$ $(n = 0, \pm 1, \pm 2, \ldots)$ and is analytic in the λ plane cut from each of these points to infinity in $\Im\lambda > 0$. Moreover, the two integrals in (54) have these same properties; the integrand of the second integral is not singular when $z = -\lambda + 2n\pi/\kappa$, so the integral does not possess branch points at $\lambda = \lambda_n$ $(n = 0, \pm 1, \pm 2, \ldots)$.

Equation (52) shows that the only possible singularities of $\Phi(\lambda)$ occur at

$$\lambda = \lambda_n \quad (n = 0, \pm 1, \pm 2, \ldots)$$

and Φ is analytic in the λ plane cut in $\Im\lambda < 0$. Equation (54) likewise demonstrates that the only possible singularities of $\Phi(\lambda)$ are at $\lambda = \lambda_n'$ $(n = 0, \pm 1, \pm 2, \ldots)$ and Φ is analytic

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in the plane, cut in $\Im\lambda > 0$. Because the two sets of branch points are distinct, it may be concluded that Φ is an analytic function of λ everywhere in the finite plane, and therefore is an integral function of λ. $\Phi(\lambda)$ has period $2\pi/\kappa$ in λ and, provided $\Im\lambda$ is finite, $\Phi(\lambda)$ remains bounded as $|\Re\lambda| \to \infty$. Furthermore, $G(\lambda) \to 1$ as $\Im\lambda \to \infty$ while the integral in (52) tends to zero as $\Im\lambda \to \infty$. Therefore

$$\Phi(\lambda) \to 0 \quad (\Im\lambda \to \infty). \tag{55}$$

Now, let $\Im\lambda \to -\infty$ in (54). Then the first integral tends to zero (see (27)) and

$$\Phi(\lambda) \to Q/P - 1 - Q \lim \int_\Gamma [G(z) - G(-\lambda)]\, H(z,\lambda)\, dz \quad (\Im\lambda \to -\infty). \tag{56}$$

But

$$[G(z) - G(-\lambda)]\, H(z,\lambda) = -\frac{[G(z)-1]}{\sqrt{(1-z^2)}} + \frac{[G(z)-1]}{1-e^{i\kappa(z+\lambda)}} \frac{1}{\sqrt{(1-z^2)}} + [1 - G(-\lambda)]\, H(z,\lambda), \tag{57}$$

so the integral in (56) may be written as the sum of three terms. The results of the following section (in particular (64) and (66)) enable one to show that $G(z) - 1 = O(e^{i\kappa z})$ as $\Im z \to \infty$. It may be assumed that Γ is deformed so that $e^{i\kappa(z+\lambda)} \neq 1$. Moreover, as z ranges along this contour, $|[1 - e^{i\kappa(z+\lambda)}]\sqrt{(1-z^2)}|$ will have an absolute minimum for z of the order of $-\lambda$. Then, on the contour, $|[1 - e^{i\kappa(z+\lambda)}]\sqrt{(1-z^2)}|^{-1} < M|1-\lambda^2|^{-\frac{1}{2}}$ for some constant $M > 0$. Therefore the integral of the second term in (57) tends to zero as $\Im\lambda \to -\infty$. In a similar manner, it is seen that the integral of the third term is of order λ^{-1} and vanishes as $\Im\lambda \to -\infty$. It then follows that

$$\Phi(\lambda) \to Q/P - 1 + Q \int_\Gamma \frac{[G(z)-1]}{\sqrt{(1-z^2)}}\, dz \quad (\Im\lambda \to -\infty). \tag{58}$$

Thus the integral function Φ is bounded everywhere and so is a constant. According to (55), this constant is zero, and (52) implies that G is a solution of (50). Furthermore, (58) now shows that Q, defined for the present arguments by (48), also satisfies the relation

$$Q/P - QC = 1,$$

or

$$Q = P/(1 - PC), \tag{59}$$

in which

$$C = -\int_\Gamma \frac{G(z)-1}{\sqrt{(1-z^2)}}\, dz. \tag{60}$$

Therefore it may be concluded that $g(\alpha) \equiv G(\cos\alpha)$ is a solution of (1) with the prescribed singularities, subject to (5) and (6) and, moreover, this solution is unique. For every solution of (1) which satisfies (5) and (6) is a solution of (23). But the solution of (23) with these properties is unique and the conclusion follows.

6. *The functions R_+ and R_-.* R_+ and R_- are related through (38), so attention will be confined to R_+ which is defined for $\Im\lambda > 0$ by (36). Because $R(z)$ is periodic, it is possible to express $R_+(\lambda)$ as an integral over a finite interval. Moreover, for the same reason, it is necessary only to consider those λ which lie in a strip of width $\delta(= 2\pi/\kappa)$ parallel to the imaginary axis. (The following operations, although formal, may be

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justified because the series involved is uniformly convergent and its terms analytic on the contour of integration when $\Im\lambda > 0$.) Then

$$\int_{-\infty}^{\infty} \frac{\log R(z)}{z^2-\lambda^2}\,dz = \int_{-\frac{1}{2}\delta}^{\frac{1}{2}\delta} \sum_{n=-\infty}^{\infty} \frac{1}{(z+n\delta)^2-\lambda^2} \log R(z)\,dz, \tag{61}$$

where the interval of integration $(-\frac{1}{2}\delta, \frac{1}{2}\delta)$ is chosen for later convenience, and is understood in the integrals below.

This series may be summed by a well-known, complex variable method (see, for example, Titchmarsh ((7), section 3·3)); its sum is given also by Jolley ((4)) as series number 827.

$$\sum_{-\infty}^{\infty} \frac{1}{(z+n\delta)^2-\lambda^2} = -\frac{\pi}{2\delta\lambda}\{\cot[\pi(\lambda-z)/\delta] + \cot[\pi(\lambda+z)/\delta]\}. \tag{62}$$

Therefore (36), (61) and (62) give

$$R_+(\lambda) = \frac{i}{4\delta}\int \{\cot[\pi(\lambda-z)/\delta] + \cot[\pi(\lambda+z)/\delta]\} \log R(z)\,dz, \tag{63}$$

when $\Im\lambda > 0$. Because $R(-z) = R(z)$, one easily sees that (63) is the sum of two equal terms, and

$$R_+(\lambda) = \frac{i}{2\delta}\int \cot[\pi(\lambda-z)/\delta] \log R(z)\,dz \quad (\Im\lambda > 0). \tag{64}$$

Moreover,

$$\cot[\pi(\lambda-z)/\delta] \to -i \quad (\Im\lambda \to \infty).$$

Thus

$$R_+(\lambda) \to \frac{1}{2\delta}\int \log R(z)\,dz \quad (\Im\lambda \to \infty), \tag{65}$$

and (48) becomes

$$Q^{\frac{1}{2}} = \exp\left[-\frac{1}{2\delta}\int \log R(z)\,dz\right]. \tag{66}$$

The periodicity of $R(z)$ makes possible the derivation from (64) of an alternative representation for $R_+(\lambda)$. For $\Im z = 0$, let

$$\log R(z) = b_0 + \frac{1}{2}\sum_{\nu=-\infty}^{\infty}{}' b_\nu e^{i2\pi\nu z/\delta}, \tag{67}$$

in which $b_{-\nu} = b_\nu$ because $\log R$ is an even function of z and the prime implies omission of the term for $\nu = 0$. The second equation of (26) shows that $R(z)$ behaves like $(z-\lambda_n)^{-\frac{1}{2}}$ or $(z-\lambda_n')^{-\frac{1}{2}}$ (plus analytic terms) as z tends to a branch point λ_n or λ_n', so $\log R$ possesses logarithmic singularities at these points; this will be reflected in the form of the coefficients b_ν as $\nu \to \infty$. On one set of branch points, the terms in (67) for $\nu \to +\infty$ will cause divergence while the terms for $\nu \to -\infty$ will bring about divergence of the series on the other set. Moreover, the series $\sum_0^{\infty} b_\nu e^{i2\pi\nu z/\delta}$ and $\sum_0^{-\infty} b_\nu e^{i2\pi\nu z/\delta}$ represent functions which are analytic in the upper and lower half planes of z, respectively. Thus it would not be surprising if the functions $R_\pm$ were related to the series $\sum_0^{\pm\infty}$, and this is indeed true.

It is not difficult to find the Fourier coefficients of $R_+(\lambda)$ by inserting the representation (67) into (64) and integrating term-by-term; the integrals may be evaluated

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readily with the aid of the transformation $e^{i2\pi z/\delta} = \rho$, followed by integration about the unit circle in the ρ plane. One finds that

$$R_+(\lambda) = \tfrac{1}{2}\sum_0^\infty b_\nu e^{i2\pi\nu\lambda/\delta} \quad (\Im\lambda > 0), \tag{68}$$

and the series will converge on the real axis except at the branch points

$$\lambda_n (n = 0, \pm 1, \pm 2, \ldots).$$

From either (66) and (67), or from (48) and (68), it follows that

$$Q^{\frac{1}{2}} = \exp(-\tfrac{1}{2}b_0), \tag{69}$$

so

$$F(\lambda) = \exp\left(-\frac{1}{2}\sum_1^\infty b_\nu e^{i2\pi\nu\lambda/\delta}\right) \tag{70}$$

and

$$f(\alpha) = \exp\left[-\frac{1}{2}\sum_1^\infty b_\nu e^{i(2\pi\nu\cos\alpha)/\delta}\right]. \tag{71}$$

The periodicity of $F(\lambda)$ as well as its exponential approach to unity as $\Im\lambda \to \infty$ are obvious in (70). The fact that $f(\alpha) = f(-\alpha)$ is an immediate consequence of (71). The series in (70) will diverge only on the branch points $\lambda_n (n = 0, \pm 1, \pm 2, \ldots)$ because the terms for $\nu < 0$ do not appear. Equation (35) implies that

$$F(\lambda_n) = f(\alpha_n) = 0 \quad (n = 0, \pm 1, \pm 2, \ldots),$$

so the real parts of the series in (70) and (71) must tend to $+\infty$ at these points.

$f(\alpha)$ is analytic in a neighbourhood of $S(0)$ and tends to unity as $|\alpha| \to \infty$ on $S(0)$. Thus, as assumed in the opening paragraph of section 2, the integral in (1) does define f as an analytic, single-valued function of α throughout D.

7. *The solution for* $\kappa \to 0$. For general values of κ, the determination of $R_+(\lambda)$ appears to be possible only by numerical methods based on (64) or (68). At least for real values of λ this should be quite feasible. Hills and Karp ((3)) have obtained an approximate analytical expression for, in essence, $R_+(\lambda)$ which is valid when $\kappa \gg 1$. Thus the case of small κ is excluded from their considerations.

However, for the interesting limiting case $\kappa \to 0$, it is possible to determine $R_+(\lambda)$ explicitly, as well as the constants Q and C. It turns out that the integral in (5) does not exist when $\kappa \to 0$, and $R_+(\lambda)$ does not tend to a constant as $\Im\lambda \to \infty$. Thus (48) does not give the limiting form of $Q^{\frac{1}{2}}$, nor does (5) give a useful representation for C. However, $Q^{\frac{1}{2}}$ may be found from (66) and C from (6).

The limiting form of $R(\lambda)$ as $\kappa \to 0$ ($\delta \to \infty$) is easily found from (26).

$$R(\lambda) \sim -\delta(1-\lambda^2)^{-\frac{1}{2}} \quad (\delta \to \infty). \tag{72}$$

Although $R(\lambda)$ contains a term in $\log\delta$, the neglected term on the right-hand side of (72) is bounded and has no effect on the following results. For the constant P is given by

$$P^{-1} = -2i\log(\tfrac{1}{2}\gamma kb) - \pi, \tag{73}$$

where b is some characteristic dimension of the small cylinder; if the cylinders are circular, then b is the radius. [For a circular cylinder, equation (73) follows readily

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from the definition of P as the scattering function of a cylinder centred at the origin of coordinates. Then the scattered field at a point distant r from the origin is $\pi P H_0^{(1)}(kr)$. The incident field at (r, θ) is $e^{ikr\cos(\theta-\alpha)}$, where α is the angle of incidence. The total field must vanish on the surface $(r = b)$ of the cylinder and, for small values of kb, this is expressed by (73).]

In any event, because each cylinder scatters isotropically, kb will be small and of the same order of magnitude as κ. Thus the logarithmic term in P^{-1} cancels with $\log\kappa$ in (26).

The limiting form of $R_+(\lambda)$ is found most easily from (36). If one keeps in mind the fact that all logarithms are principal values, then it follows that

$$R_+(\lambda) \sim \frac{\lambda}{2\pi i}\int_{-\infty}^{\infty} \frac{[\log\delta + i\pi - \log\sqrt{(1+z)} - \log\sqrt{(1-z)}]}{z^2-\lambda^2}\,dz \quad (\Im\lambda > 0). \tag{74}$$

Here $\sqrt{(1-z)}$ is analytic in the z plane cut from 1 to $1+i\infty$, $\sqrt{(1+z)}$ is analytic in the plane cut from -1 to $-1-i\infty$, while both are real and positive for $-1 < z < 1$. Thus $\log\sqrt{(1+z)}$ is analytic above the contour of integration and $\log\sqrt{(1-z)}$ is analytic below. The integrals may be evaluated by closing the contour in the upper or lower half plane as the case may be. It is found that

$$R_+(\lambda) \sim \tfrac{1}{2}\log\delta + \tfrac{1}{2}\pi i - \log\sqrt{(1+\lambda)} \quad (\Im\lambda > 0) \tag{75}$$

and, according to (47),

$$K(\lambda) \sim -i\delta^{-\frac{1}{2}}\sqrt{(1+\lambda)} \quad (\Im\lambda > 0). \tag{76}$$

Although (75) and (76) have been proved under the restriction $\Im\lambda > 0$, by employing (37)–(39) it may be verified that they are valid everywhere.

The limiting form of $Q^{\frac{1}{2}}$ is found from (66). It is convenient to replace the interval of integration by $(1-\delta, 1)$ which is justifiable because of periodicity of the integrand. Upon inserting the approximation (72), it is found that all the resulting integrals may be evaluated explicitly to give

$$\int_{1-\delta}^{1} \log R(z)\,dz \sim \delta(1+\tfrac{1}{2}\pi i) + \log\delta + 1 - \log 2 + \pi i \tag{77}$$

and

$$Q^{\frac{1}{2}} \sim e^{-\frac{1}{2}}e^{-\frac{1}{4}\pi i}. \tag{78}$$

Then from (6) and (73), one finds

$$C \sim -2i\log(\tfrac{1}{2}\gamma kb) - \pi - ei \tag{79}$$

and this will become unbounded as $\delta \to \infty$ for $kb \to 0$ in these circumstances; the unboundedness of C is consistent with the divergence of the integral in (5).

Finally, the limiting form of $F(\lambda)$ may be found from (51).

$$F(\lambda) \sim \sqrt{e}\,e^{-\frac{1}{4}\pi i}\delta^{-\frac{1}{2}}\sqrt{(1+\lambda)}, \tag{80}$$

so

$$f(\alpha) \sim \sqrt{(2e)}\,e^{-\frac{1}{4}\pi i}\delta^{-\frac{1}{2}}\cos\tfrac{1}{2}\alpha. \tag{81}$$

It is possible also to consider the limiting form of (1) as $\kappa \to 0$. The same methods that have been used in section 3 lead to the limiting form which corresponds to (23); this is identical with the limiting form of (23). The solution is obtained easily and is in agreement with the above results.

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