PROBLEMS IN ZERO-SUM COMBINATORICS

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ABSTRACT

Let $t(G,q)$ denote the smallest integer t such that the vertex set V of the graph G can be partitioned into t classes $V(G)=\bigcup_{i=1}^{t}V_i$ such that the number of edges in the induced subgraph $\langle V_i\rangle$ for $1\leqslant i\leqslant t$, is divisible by q. Using an algebraic theorem due to Baker and Schmidt we prove that if q is a prime power then $t(G,q)$ can be computed and a corresponding partition can be presented in a polynomial time.

1. *Introduction*

Let G be a graph having $e(G)=m$ edges and $v(G)=n$ vertices. Let f be a $\mathbb{Z}_k$ colouring of G, namely $f\colon E(G)\to\mathbb{Z}_k$, where $\mathbb{Z}_k$ is the additive group modulo k. Let $A\subset V(G)$ be a subset of vertices of G. The induced subgraph on $\langle A\rangle$ is called zero-sum (mod k) (with respect to the colouring $f\colon E(G)\to\mathbb{Z}_k$) if $\sum_{e\in E(\langle A\rangle)}f(e)=0$ in $\mathbb{Z}_k$. In the last three years many zero-sum problems have been posed and solved, most of them concerning the so-called zero-sum Ramsey numbers and the zero-sum Turan numbers. The references indicate an almost complete list of the papers written on the subject since 1989. Here we shall consider two zero-sum problems, which do not belong to either Ramsey's or to Turan's zero-sum frame. The first problem is motivated by the concept of independent sets and colourings in graphs.

PROBLEM A. Let $q\geqslant 2$ be a fixed integer, and let G be a graph. Suppose that $f\colon E(G)\to\mathbb{Z}_q$. Then Problem A is in two parts.

(1) For given $0\leqslant k<q$ find the maximum integer $\alpha=\alpha_{k,q}(G,f)$ such that there exists $A\subset V(G)$ with $|A|=\alpha$ and $\sum_{e\in\langle A\rangle}f(e)\equiv k\,(\mathrm{mod}\,q)$.

(2) Find the smallest integer $t=t(G,q,f)$ such that

$$V=\bigcup_{i=1}^{t}V_i \quad\text{and}\quad \sum_{e\in\langle V_i\rangle}f(e)\equiv 0\,(\mathrm{mod}\,q) \quad\text{for } 1\leqslant i\leqslant t.$$

Problem A (1) has been carefully considered in the case $f\equiv 1$ and $k=0$ for all q. In fact in [**1**] it is proved that for q a prime power and $f\colon E(G)\to\mathbb{Z}_q$, we always have $\alpha_{0,q}(G,f)\geqslant n-2q+2$. Our main contribution here is to extend this result to any choice of k satisfying $0\leqslant k<q$. We shall prove the following theorem.

THEOREM 1. *Let q be a prime power, G a graph on n vertices, $f\colon E(G)\to\mathbb{Z}_q$ and $0\leqslant k<q$. Then*

(1) $\alpha_{0,q}(G,f)\geqslant n-2q+2$;

(2) *for $1\leqslant k<q$ either $\alpha_{k,q}(G,f)=0$ or $\alpha_{k,q}(G,f)\geqslant n-2q+2$;*

(3) *for $0\leqslant k<q$ the computation of the exact value of $\alpha_{k,q}(G,f)$, and a presentation of $A\subset V$ that realizes $\alpha_{k,q}(G,f)$, can be implemented in polynomial time.*

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Problem A(2) is treated in [**18**]. There the case $f \equiv 1$ and q a prime power is considered, and the main results are that:

(1) $t(G,q,f \equiv 1) \leqslant \frac{3}{2}(q-1) - O(\sqrt{q})$ for q an odd prime power;
(2) $t(G,q,f \equiv 1) \leqslant 2q-1$ for $q = 2^t$ and this is sharp;
(3) $t(G,3,f \equiv 1) \leqslant 3$ and $t(G,5,f \equiv 1) \leqslant 5$;
(4) $t(G,2,f \equiv 1)$ is exactly determined for every graph.

We shall prove the following theorem.

THEOREM 2. *Let q be a prime power, G a graph and $f: E(G) \to \mathbb{Z}_k$. Then,*
(1) $t(G,q,f) \leqslant 2q-1$;
(2) *the exact value of $t(G,q,f)$ and a presentation of a partition that realizes $t(G,q,f)$ can be implemented in polynomial time.*

Thus the main conclusion from Theorems 1 and 2 is that, unlike the NP-completeness of 'real colourings' and 'real independent sets', the zero-sum related concepts are polynomial time solvable.

The second problem is motivated by the following well-known theorem of Erdös [**20**] and its many derivatives (see for example [**22**]).

THEOREM A [**20**]. *Let G be a graph. Then G contains a spanning bipartite subgraph H such that $e(H) \geqslant \frac{1}{2}e(G)$.*

We shall consider cuts in graphs. A cut (A,B) in a graph G is a partition $A \cup B = V$ (where $A = \varnothing$ is allowed), and the edges of the cut (A,B) are precisely $\{e = (u,v): u \in A, v \in B\}$. The number of edges of a cut (A,B) is denoted by $e(A,B)$. If $f: E(G) \to \mathbb{Z}_q$ and (A,B) is a cut then the sum of the edges in (A,B) is

$$f(A,B) = \sum_{e \in (A,B)} f(e),$$

and if $f(A,B) = 0$ in $\mathbb{Z}_q$ then the cut is called a *zero-sum cut* (ZS-cut, for short). Thus the theorem of Erdös mentioned above states that there is always a cut containing at least half of the edges of G. So our second problem can be phrased as follows.

PROBLEM B. How large, that is, how many edges can such a cut have, if we require that $e(A,B) \equiv 0 \pmod q$?

In the second part of this paper we shall present results concerning this problem, showing that the Erdös' bound $\frac{1}{2}e(G)$ can almost be saved.

2. *The Baker–Schmidt Theorem and proofs for Theorems 1 and 2*

The basic tool in the proofs given below is the following theorem of Baker and Schmidt.

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THEOREM B [3]. *Let q be a prime power. Let*

$$h_1(x_1, \dots, x_n), h_2(x_1, \dots, x_n), \dots, h_t(x_1, \dots, x_n)$$

be polynomials in $\mathbb{Z}[x_1, \dots, x_n]$. Suppose that $h_1(0) \equiv h_2(0) = \cdots \equiv h_t(0) \equiv 0 \pmod q$ and that $n \geqslant (\sum_{i=1}^t \deg h_i)(q-1)+1$, then there exists an $\alpha \in \{0,1\}^n$ with $\alpha \neq 0$, such that $h_1(\alpha) \equiv \cdots \equiv h_t(\alpha) \equiv 0 \pmod q$.

We shall use the following quantitative form of the Baker–Schmidt Theorem.

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0,1\}^n$ define $\|\alpha\| = \sum_{i=1}^n \alpha_i$. Thus $\|\alpha\|$ counts the numbers of ones in α. A collection of polynomials $h_i(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$ for $1 \leqslant i \leqslant t$, satisfying

(1) $h_i(0) \equiv 0 \pmod q$ for $1 \leqslant i \leqslant t$ and q a prime power,

(2) $\sum_{i=1}^t \deg h_i = d$,

will be called a *form* $\mathscr{F} = \mathscr{F}(n, d, q)$.

THEOREM 3. *Let $\mathscr{F} = \mathscr{F}(n, d, q)$ be a form. Then there exists a solution $\alpha \in \{0,1\}^n$ to the form $\mathscr{F}$ such that $\|\alpha\| \geqslant n - d(q-1)$. Moreover, for fixed d and q, a solution α which maximizes the norm $\| \ \|$ can be found in polynomial time.*

Proof. If $n - d(q-1) \leqslant 0$ we have nothing to prove, by definition. So suppose that $n - d(q-1) > 0$. By the Baker–Schmidt Theorem there exists a solution α with $\|\alpha\| \geqslant 1$. Let α be a solution to $\mathscr{F}$ that maximizes $\|\alpha\|$. If $\|\alpha\| \geqslant n - d(q-1)$ there is nothing to prove, hence we may assume that $1 \leqslant \|\alpha\| < n - d(q-1)$. Let

$$V(\alpha) = \{i\colon \alpha_i = 1, \text{ where } \alpha = (\alpha_1, \dots, \alpha_n)\}.$$

For every polynomial $h_j \in \mathscr{F}$ define $h_j^* = h_j(x_1, \dots, x_n)|_{x_i = 1,\, i \in V(\alpha)}$. Now the $\{h_j^*\}$ for $1 \leqslant j \leqslant t$ constitute a new form $\mathscr{F}^* = \mathscr{F}^*(n^*, d^*, q)$ with $n^* = n - \|\alpha\| > d(q-1)$, $d^* = \sum_{j=1}^t \deg h_j^* \leqslant d$ and clearly $h_j^*(0) \equiv 0 \pmod q$ for $1 \leqslant j \leqslant t$, because h_j^* is induced by the solution α. Apply the Baker–Schmidt Theorem to

$$\mathscr{F}^* = \mathscr{F}^*(n^*, d^*, q).$$

Since $n^* > d(q-1) \geqslant d^*(q-1)$ there exists α^* with $0 \neq \alpha^* \in \{0,1\}^{n^*}$, which is a solution to $\mathscr{F}^*$, but now, the vector $b \in \{0,1\}^n$ defined by $b = \alpha + \alpha^*$ is a solution to $\mathscr{F}$ and $\|b\| > \|\alpha\|$, a contradiction.

Observe now that, instead of checking all the 2^n vectors in $\{0,1\}^n$, we may check only those vectors for which $\|\alpha\| \geqslant n - d(q-1)$, and there are at most

$$\sum_{j=n-d(q-1)}^{n} \binom{n}{j} = \sum_{j=0}^{d(q-1)} \binom{n}{j} \leqslant \sum_{j=0}^{d(q-1)} n^j = O(n^{d(q-1)})$$

such vectors. Now assuming that the polynomials are given by standard representation using monomials, the length of $\mathscr{F}(n, d, q)$ in the input is bounded above by $O(n^{dq})$. Checking that $\mathscr{F} = \mathscr{F}(n, d, q)$ is indeed a form and checking that all the above vectors are solutions can be implemented in time $O(n^{dq})$.

Proof of Theorem 1. We may assume that $n \geqslant 2q-1$; otherwise there is nothing to prove.

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(1) Associate each vertex $v \in V$ with a variable x_v and consider the following polynomial equation:

$$h = \sum_{e \in E(G)} f(e)\, x_u\, x_v \equiv 0 \,(\mathrm{mod}\, q), \quad \text{where } e = (u, v).$$

Clearly h is a form $\mathscr{F} = \mathscr{F}(n, 2, q)$ and by Theorem 3 there exists a solution $\alpha \in \{0,1\}^n$ such that $\|\alpha\| \geqslant n - 2q + 2$. Let $V(\alpha) \subseteq V$ be defined by $V(\alpha) = \{u : x_u = 1\}$, then clearly in the subgraph H induced by $\langle V(\alpha) \rangle$, we have $\sum_{e \in E(H)} f(e) \equiv 0 \,(\mathrm{mod}\, q)$ and $|H| \geqslant n - 2q + 2$.

(2) Let m be the maximum integer such that there exists a set M of m vertices of G such that $\sum_{e \in M} f(e) \equiv k \,(\mathrm{mod}\, q)$. If $m = 0$ we are done. If $m \geqslant n - 2q + 2$ then again there is nothing to prove. So we may suppose that $0 < m < n - 2q + 2$. Hence $|V \setminus M| \geqslant 2q - 1$. For each vertex $u \in V \setminus M$ define $f(u, M) = \sum f(e)$, where the sum ranges over all edges $e = (u, v)$ with $v \in M$. Associate each vertex $u \in V \setminus M$ with a variable x_u and consider the polynomial equation:

$$h = \sum_{u \in V \setminus M} f(u, M)\, x_u + \sum_{\substack{e = (u, v) \\ u, v \in V \setminus M}} f(e)\, x_u\, x_v \equiv 0 \,(\mathrm{mod}\, q).$$

Again h is a form $\mathscr{F} = \mathscr{F}(n - m, 2, q)$, where $n - m \geqslant 2q - 1$. Hence, by Theorem 3, there exists a solution α with $0 \neq \alpha \in \{0,1\}^{n-m}$. Let $V(\alpha) = \{u \in V \setminus M : x_u = 1\}$, then clearly $\sum_{e \in M \cup V(\alpha)} f(e) \equiv k \,(\mathrm{mod}\, q)$ which contradicts the maximality of M.

(3) By the first two parts it is sufficient to check all induced subgraphs $H < G$ such that $|H| \geqslant n - 2q + 2$ and to calculate the sum of their edges, which can be done in time $O(n^{2q})$.

REMARKS. (1) The proof of Theorem 1(1) has already appeared in [**1**] in the general context of r-uniform hypergraphs, with a slightly different proof.

(2) The content and the proof of Theorem 1 can be modified to show that, given a prime power q and a graph G, there exists a nested sequence of induced subgraphs $G = H_0 > H_1 > H_2 \cdots > H_t$ such that the following hold:

(i) $e(G) = e(H_0) \equiv e(H_1) \equiv e(H_2) \cdots \equiv e(H_t) \,(\mathrm{mod}\, q)$;

(ii) $0 < |V(H_i) \setminus V(H_{i+1})| \leqslant 2q - 2$ and $|V(H_t)| \leqslant 2q - 2$.

Proof of Theorem 2. (1) This is trivial following Theorem 1(1), because $\alpha_{0,q}(G, f) \geqslant n - 2q + 2$. So take as V_1 a subset that realizes $\alpha_{0,q}(G, f)$. Then $|V \setminus V_1| \leqslant 2q - 2$ and we may take all the other vertices as singles to obtain a partition with at most $2q - 1$ classes.

(2) Our first goal is to show that $t = t(G, q, f)$ can be realized by a partition in which at most one class has at least $4q - 3$ vertices. Indeed, suppose without loss of generality that $\bigcup_{i=1}^{t} V_i = V$ is a partition that realizes $t(G, q, f)$, and that $|V_1| \geqslant |V_2| \geqslant 4q - 3$. Associate with each vertex $v \in V_2$ a variable x_v. Consider the following polynomial equations

$$h_1 = \sum_{v \in V_2} f(v, V_1)\, x_v + \sum_{\substack{e = (u, v) \\ u, v \in V_2}} f(e)\, x_u\, x_v \equiv 0 \,(\mathrm{mod}\, q)$$

with the convention that $f(v, V_1) = \sum_{e = (u, v),\, u \in V_1} f(e)$. The polynomial h_1 counts the sum of edges $(\mathrm{mod}\, q)$ being added to V_1 when transferring vertices from V_2 to V_1. Then

$$h_2 = \sum_{v \in V_2} f(v, V_2)\, x_v - \sum_{\substack{e = (u, v) \\ u, v \in V_2}} f(e)\, x_u\, x_v \equiv 0 \,(\mathrm{mod}\, q),$$

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where again $f(v, V_2) = \sum_{e=(v,u),\, u\in V_2} f(e)$. The polynomial h_2 counts the sum of edges (mod q) being deleted from V_2 when transferring vertices from V_2 to V_1.

Now h_1 and h_2 constitute a form $\mathscr{F} = \mathscr{F}(|V_2|, 4, q)$, as $\deg h_1 = \deg h_2 = 2$. So by the Baker–Schmidt Theorem there is a non-trivial solution $\alpha \in \{0,1\}^{|V_2|}$. Set $V(\alpha) = \{v \in V_2 : x_v = 1\}$, then both $\langle V_1 \cup V(\alpha)\rangle$ and $\langle V_2 \backslash V(\alpha)\rangle$ are induced zero-sum subgraphs with $|V_2 \backslash V(\alpha)| < |V_2|$. So we may continue with this transfer of vertices until all classes of the partition except of V_1 have cardinality at most $4q-4$. We pause here to mention that indeed when looking back at the polynomials h_1 and h_2, substituting only the vertices of $V(\alpha) = \{v \in V_2 : x_v = 1\}$, they ensure that both $\langle V_1 \cup V(\alpha)\rangle$ and $\langle V_2 \backslash V(\alpha)\rangle$ are zero-sum induced subgraphs.

Now by the first part of the theorem $\sum_{i=2}^{t} |V_i| \leqslant (2q-2)(4q-4) = 8(q-1)^2$ as $t \leqslant 2q-1$. Hence $|V_1| \geqslant n - 8(q-1)^2$.

But now we are done just by 'brute force'. For every $A \subseteq V$ with $|A| \geqslant |V| - 8(q-1)^2$ we have to check, in $O(n)$ time, by counting edges not in $\langle A\rangle$, that $\sum_{e\in\langle A\rangle} f(e) \equiv 0 \pmod q$ and then to find $t(\langle G\backslash A\rangle, q, f)$ to obtain both $t(G, q, f)$ and a partition that realizes it. But as $|G\backslash A| \leqslant 8(q-1)^2$, finding $t(\langle G\backslash A\rangle, q, f)$ is independent of n. So we can solve the problem in at most

$$O\left(n\left(\sum_{j=8(q-1)^2}^{n}\binom{n}{j}\right)\right) = O\left(n\sum_{j=0}^{8(q-1)^2} n^j\right) = O(n^{8(q-1)^2+1})$$

time.

REMARKS. (1) Noga Alon pointed out to me that for a fixed graph H almost every graph G contains an induced copy of H. This fact has the immediate consequence that if q is a prime power and $f: E(G) \to \mathbb{Z}_q$ is the constant function $f \equiv 1$, then for almost every graph G we have $\alpha_{k,q}(G, f) \geqslant n - 2q + 2$ where $|G| = n$. This is because we can choose $H = K_{1,k}$ for each $0 \leqslant k \leqslant q-1$. Alon also proved that in the probabilistic model $G(n, \frac{1}{2})$, the probability that $e(G) \equiv 0 \pmod q$ tends to $1/q$, and the probability that $e(G) \not\equiv 0 \pmod q$ but G has a vertex v such that

$$\deg v \equiv e(G) \pmod q$$

tends to $(q-1)/q$. It follows that $t(G, q, f \equiv 1) \leqslant 2$ for almost all graphs and that the average time complexity of computing $t(G, q, f \equiv 1)$ is $O(n^2)$.

(2) if q is an odd prime power then the bound $O(n^{8(q-1)^2+1})$ can be improved to $O(n^{6(q-1)^2+1})$ using the results of [**18**]. It would be nice to present an algorithm with running time $O(n^{cq})$ for some positive constant c.

(3) It is easy to generalize the content of this section to the case of r-uniform hypergraphs, along the direction of [**1**] and the ideas presented above.

(4) The polynomial results of Theorems 1 and 2 can be extended to composite moduli using the Baker–Schmidt Theorem for composite moduli. However the bounds involved are rather weak and are not worth mentioning.

3. *Large zero-sum cuts*

Our main results concerning Problem B on zero-sum cuts are summarized in the following theorem.

THEOREM 4. *Let G be a graph on n vertices, q a prime power and $f: E(G) \to \mathbb{Z}_q$.*

(1) *For every $0 \leqslant k \leqslant n$ there is a zero-sum cut (A, B) such that $||A| - k| \leqslant q-1$, and such a cut can be constructed in a polynomial time.*

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(2) *If* $q=2$ *and* $f\equiv 1$ *then for every* k *with* $0\leqslant k\leqslant n$ *there is a zero-sum* (mod 2) *cut* (A,B) *such that* $|A|=k$ *if and only if* G *contains a vertex of even degree.*

(3) *If* $f\equiv 1$ *then there is a zero-sum* (mod q) *cut* (A,B) *such that*

$$e(A,B)>\frac{e(G)}{2}-n\left(\frac{3q}{2}-\frac{5}{4}\right).$$

Proof. (1) Observe first that it is sufficient to prove that there exists no interval of $2q-1$ consecutive integers, say $t,\ldots,t+2q-2$, such that G contains no zero-sum cut (A,B) with $|A|\in\{t,\ldots,t+2q-2\}$. Suppose on the contrary that such intervals exist and choose the interval with the least value of t. Clearly $t\neq 0, t\neq n$ since in both cases the cuts are trivial and $e(A,B)=f(A,B)=0\equiv 0\,(\mathrm{mod}\,q)$.

Since t is minimal, there exists a zero sum cut (A,B) with $|A|=t-1$ (with the possibility that $t-1=0$ and $A=\varnothing$). Clearly $|B|=n-|A|\geqslant 2q-1$ (otherwise take $A=V, B=\varnothing$). Choose $D\subset B$ such that $|D|=2q-1$. To each vertex $v\in D$ associate a variable x_v and define $f(v,A)=\sum_{e=(v,u),\,u\in A}f(e)$ and $f(v,B)=\sum_{e=(v,u),\,u\in B}f(e)$. For $u,v\in D$ define further

$$f(u,v)=\begin{cases}0 & \text{for } (u,v)\notin E(G),\\ f(e) & \text{for } e=(u,v)\in E(G).\end{cases}$$

Consider the following polynomial equation:

$$h=\sum_{v\in D}\left(f(v,B)\,x_v-f(v,A)\,x_v\right)-\sum_{u,\,v\in D}f(u,v)\,x_u\,x_v\equiv 0\,(\mathrm{mod}\,q).$$

The polynomial h constitutes a form $\mathscr{F}=\mathscr{F}(2q-1,2,q)$, and by Theorem B there exists a non-trivial solution $\alpha\in\{0,1\}^{2q-1}$. Let $V(\alpha)=\{v\in D\colon x_v=1 \text{ in } \alpha\}$. Clearly $1\leqslant|V(\alpha)|\leqslant 2q-1$ and we claim that (A',B'), where $A'=A\cup V(\alpha)$, $B'=B\backslash V(\alpha)$, is a zero-sum cut and obviously $|A'|\in\{t,\ldots,t+2q-2\}$, a contradiction. Indeed the polynomial h counts (mod q) the change in the number of edges when we switch from the cut (A,B) to the new cut $(A'B')$, where $f(v,B)-f(v,A)$ $(x_v=1)$, is just the change when we move v from B to A, and $-\sum_{u,\,v\in D}f(u,v)\,x_u\,x_v$ is just the subtraction of the double count, of the edge $e=(u,v)$, in both $f(u,B)$ and $f(v,B)$. Hence we have proved part (1) of Theorem 4.

(2) The case $k=1$ shows that G must contain a vertex v of even degree. Observe also that the cases $k=0$, respectively $k=n$, are trivial as we can take $A=\varnothing$ respectively $B=\varnothing$. For sufficiency observe first that if $k=1$ or $k=n-1$ then we may take $A=\{v\}$, respectively $B=\{v\}$, and we are done. So from now on we may assume that $2\leqslant k\leqslant n-2$. If all the degrees in G are even then every component is Eulerian, G is the edge disjoint union of cycles and every cut is even, so we may take $A\subset V$ with $|A|=k$ without any restriction. Hence assume that $\deg v\equiv 0\,(\mathrm{mod}\,2)$ and $\deg u\equiv 1\,(\mathrm{mod}\,2)$ for two vertices $u,v\in V(G)$. Choose arbitrarily $k-1$ vertices, say $u_1,\ldots,u_{k-1}$ such that u and v are not among them and let $A=\{u_1,\ldots,u_{k-1},u\}$, $B=V\backslash A$. If $e(A,B)\equiv 0\,(\mathrm{mod}\,2)$ we are done, otherwise let $A'=(A\backslash\{u\})\cup\{v\}$, $B'=(B\backslash\{v\})\cup\{u\}$ and it is easily shown that $e(A',B')\equiv 0\,(\mathrm{mod}\,2)$ as we moved u from one side of the cut to the other.

(3) Let (A,B) be a zero-sum (mod q) cut with the maximum number of edges among all the zero-sum (mod q) cuts in G. Observe that if in A (respectively B) there are at least $2q-1$ vertices, say $u_1,\ldots,u_{2q-1}$ such that $\deg_{(A,B)}u_i\leqslant\frac{1}{2}\deg_G u_i-q$, then we may apply the 'transfer argument' of part (1) to move t of them from A to B (or vice

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versa) to obtain a new zero-sum cut (A', B'). Observe that moving u_i from A to B increases the number of edges in the cut by at least $\frac{1}{2}\deg_G u_i + q - \frac{1}{2}\deg_G u_i + q = 2q$. Hence moving t of these vertices from A to B implies that

$$e(A', B') \geqslant e(A, B) + 2tq - 2\binom{t}{2},$$

since each such vertex contributes at least q edges and transferring t of them can cause a double count of at most $\binom{t}{2}$ edges. But $2tq - 2\binom{t}{2} > 0$ because $1 \leqslant t \leqslant 2q-1$ and hence $e(A', B') > e(A, B)$, a contradiction to the maximality of (A, B).

Hence in both A and B there are altogether at most $4q-4$ vertices such that $\deg_{(A,B)} u_i \leqslant \frac{1}{2}\deg_G u_i - q$. Let D denote the collection of these vertices. Let us estimate very crudely the size of $e(A, B)$. Counting degrees on both sides of the cut (A, B) and dividing by two we infer that

$$\begin{aligned}
e(A, B) &\geqslant \frac{1}{2}\sum_{v \in V} \deg_{(A,B)} v = \frac{1}{2}\left\{ \sum_{v \in V \setminus D} \deg_{(A,B)} v + \sum_{v \in D} \deg_{(A,B)} v \right\} \\
&\geqslant \frac{1}{2}\left\{ \sum_{v \in V \setminus D} \frac{\deg v + 1 - 2q}{2} + \sum_{v \in D} \frac{\deg v + 1 - 2q}{2} - \sum_{v \in D} \frac{\deg v + 1 - 2q}{2} \right\} \\
&= \frac{1}{2}\left\{ \sum_{v \in V} \frac{\deg v + 1 - 2q}{2} - \sum_{v \in D} \frac{\deg v + 1 - 2q}{2} \right\} \\
&\geqslant \frac{1}{4}\sum_{v \in V} (\deg v + 1 - 2q) - \frac{1}{4}\sum_{v \in D} (\Delta(G) + 1) \\
&> \frac{e(G)}{2} + \frac{n}{4} - \frac{nq}{2} - qn + n = \frac{e(G)}{2} - n\left(\frac{3q}{2} - \frac{5}{4}\right).
\end{aligned}$$

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