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Weak mixing for locally compact quantum groups

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Abstract. We generalize the notion of weakly mixing unitary representations to locally compact quantum groups, introducing suitable extensions of all standard characterizations of weak mixing to this setting. These results are used to complement the non-commutative Jacobs–de Leeuw–Glicksberg splitting theorem of Runde and the author [Ergodic theory for quantum semigroups. *J. Lond. Math. Soc.* (2) **89**(3) (2014), 941–959]. Furthermore, a relation between mixing and weak mixing of state-preserving actions of discrete quantum groups and the properties of certain inclusions of von Neumann algebras, which is known for discrete groups, is demonstrated.

1. *Introduction*

Weak mixing is an intrinsic part of ergodic theory. It was introduced by Koopman and von Neumann in a specific setting [**33**], and then extended to amenable topological semigroups by Dye [**23**] and to locally compact groups by Bergelson and Rosenblatt [**7**]. Lying between ergodicity and (strong) mixing, this notion comes in various shapes and has an abundance of applications, an acclaimed one being Furstenberg's proof of Szemerédi's theorem on arithmetic progressions [**29**]. In operator algebras it has noticeably played a central role in Popa's deformation/rigidity theory, and in the study of singular masas in II_1-factors using the weak asymptotic homomorphism property and similar conditions; see [**48**, **47**, **49**–**52**, **56**, **53**, **57**, **32**, **9**] and their numerous sequels.

A related result is the famous Jacobs–de Leeuw–Glicksberg splitting theorem [**31**, **17**]. Considering a weakly almost periodic semigroup $\mathcal{S}$ of operators on a Banach space, the notions of *almost periodicity* and of *weak mixing* of vectors with respect to $\mathcal{S}$ are defined. The theorem says, roughly, that under a suitable amenability condition, $\mathcal{S}$ induces a splitting of the Banach space that is acted on as the direct sum of the subspace of almost periodic vectors and the subspace of weakly mixing vectors. An important example arises from a dynamical system consisting of a topological semigroup acting on a probability

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space by measure-preserving transformations. The splitting theorem then applies to the Koopman representation of the dynamical system.

This leads naturally to the question of finding non-commutative generalizations of these results. Niculescu *et al* [**43**] considered dynamical systems in which the object that was *acted* on was non-commutative, consisting of a von Neumann algebra N and an endomorphism of N preserving a faithful normal state θ; this generalizes the classical setting of a measure-preserving map on a probability space. A notion of almost periodic *operators* in N with respect to the action was introduced, and it was proved that the set of these operators is a von Neumann subalgebra N^{AP} of N. The Jacobs–de Leeuw–Glicksberg splitting has the form of a θ-preserving conditional expectation E^{AP} from N onto N^{AP} [**43**, Theorem 4.2]. The second direct summand, namely $\ker E^{\mathrm{AP}}$, has a weak mixing property described in [**43**, Proposition 5.5]. A generalization of these results to actions of more general (amenable) topological semigroups G was announced in [**67**] (the above case is that of $G = \mathbb{Z}_+$). Subsequent works that address weak mixing of actions of more general groups on von Neumann algebras include Beyers *et al* [**8**] and Duvenhage [**19**–**21**]; see also Duvenhage and Mukhamedov [**22**].

Runde and the author undertook in [**55**] to generalize the results of Niculescu *et al* to dynamical systems that were 'fully non-commutative' in the sense that the *acting* object and the object that was *acted* on were both non-commutative. This means that a quantum semigroup (a Hopf–von Neumann algebra) $\mathbb{G}$ acts on a von Neumann algebra N via an action α preserving a faithful normal state θ. We introduced the notions of completely almost periodic vectors and operators with respect to α. When $\mathbb{G}$ is amenable and co-amenable, mild assumptions yield a Jacobs–de Leeuw–Glicksberg splitting theorem as above: there exists a θ-preserving conditional expectation E^{CAP} from N onto the set (indeed, von Neumann algebra) N^{CAP} of completely almost periodic operators [**55**, Corollary 4.10]. These mild assumptions are readily fulfilled when $\mathbb{G}$ is a locally compact quantum group (LCQG) in the sense of Kustermans and Vaes [**36**, **37**]. Nevertheless, while the completely almost periodic part was fully taken care of, it was not clear at the time in what sense $\ker E^{\mathrm{CAP}}$ is 'weakly mixing'.

The purpose of the present paper is to develop a non-commutative theory of weak mixing for unitary co-representations that, in particular, will provide an answer to the question of the weak mixing nature of $\ker E^{\mathrm{CAP}}$ from the previous paragraph. The proof for the case $\mathbb{G} = \mathbb{Z}_+$ in [**43**, Proposition 5.5] relies chiefly on the classical Koopman–von Neumann spectral mixing theorem, characterizing the weakly mixing vectors with respect to a contraction in a Hilbert space [**34**, Theorem 2.3.4]. It is tailored for a single contraction, i.e. an action of $\mathbb{Z}_+$, and does not extend to more general semigroups, let alone quantum semigroups. Indeed, the general framework for weak mixing in ergodic theory deals with groups. The setting of our main results will therefore be that of LCQGs $\mathbb{G}$. It is neither more nor less restrictive than that of [**55**] for we do not require $\mathbb{G}$ to be co-amenable, and also amenability is required only for some implications. *Mixing* of unitary co-representations of LCQGs was recently introduced by Daws *et al* [**15**] and was successfully used to define the Haagerup property in this setting.

Let us recall the main characterizations of weakly mixing group representations and actions. We recommend the survey of Bergelson and Gorodnik [**6**] and the lecture notes of

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Austin [**1**] and Peterson [**45**]. Let π be a unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}$. The following conditions are equivalent:

(WM1) for every $\zeta_1, \ldots, \zeta_n \in \mathcal{H}$ and $\varepsilon > 0$, there is some $\gamma \in G$ such that $|\langle \pi(\gamma)\zeta_i, \zeta_j \rangle| < \varepsilon$ for every $1 \leq i, j \leq n$;

(WM2) the tensor product $\pi \otimes \overline{\pi}$, of π and its conjugate representation $\overline{\pi}$, is ergodic;

(WM3) 'continuity of the spectrum'/triviality of the Kronecker factor: π has no non-zero finite-dimensional sub-representations.

In fact, if π is weakly mixing, then its tensor product by any other unitary representation of G is also ergodic. Under additional hypotheses, e.g. amenability of G, formally stronger conditions are equivalent to weak mixing.

An important, intuitive special case comes from (classical) dynamical systems $(X, \mathbb{A}, \mu, G, T)$ consisting of a probability space $(X, \mathbb{A}, \mu)$, a locally compact group G and a measure-preserving action $T = (T_\gamma)_{\gamma \in G}$ of G on X. The *Koopman representation* of the system is the unitary representation π of G on $L^2(X, \mathbb{A}, \mu)$ given by $\pi(\gamma) f := f \circ T_{\gamma^{-1}}$ $(\gamma \in G,\ f \in L^2(X, \mathbb{A}, \mu))$. Let $L^2_0(X, \mathbb{A}, \mu) := \{f \in L^2(X, \mathbb{A}, \mu) : \int_X f \, d\mu = 0\}$. We say that the system is weakly mixing if the restriction of the Koopman representation to $L^2_0(X, \mathbb{A}, \mu)$ satisfies one, hence all, of conditions (WM1)–(WM3) above. The first two take the following simpler forms:

(a) for every $A_1, \ldots, A_n \in \mathbb{A}$ and $\varepsilon > 0$, there is some $\gamma \in G$ such that $|\mu(A_i \cap T_\gamma A_j) - \mu(A_i)\mu(A_j)| < \varepsilon$ for every $1 \leq i, j \leq n$;

(b) the product system $(X \times X, \mathbb{A} \times \mathbb{A}, \mu \times \mu, G, T \times T)$, where $T \times T := (T_\gamma \times T_\gamma)_{\gamma \in G}$, is ergodic.

The paper is structured as follows. In §3 we introduce the notion of weakly mixing unitary co-representations of LCQGs. All classical characterizations of this property (see above) are generalized in the paper's main result, Theorem 3.11. It is subsequently used to complete the von Neumann algebraic Jacobs–de Leeuw–Glicksberg splitting theorem of [**55**] in §4.1. In §4.2 we study the mixing and weak mixing properties of inclusions of von Neumann algebras arising from a discrete quantum group acting on a von Neumann algebra, generalizing part of results of Jolissaint and Stalder [**32**] and Cameron *et al* [**9**]. Section 5 ends the paper with several open questions.

2. *Preliminaries*

Let us begin with a few conventions. All Hilbert spaces in this paper are complex. For ζ, η in a Hilbert space $\mathcal{H}$, we let $\omega_{\zeta,\eta}$ stand for the functional in $B(\mathcal{H})_*$ given by $x \mapsto \langle x\zeta, \eta \rangle$, and let $\omega_\zeta := \omega_{\zeta,\zeta}$. By ζ^*, we mean the functional $\langle \cdot\, , \zeta \rangle \in \mathcal{H}^*$. We write $\mathbb{1}$ for the unit of a C^*-algebra (if it exists), and id for the identity map over C^*-algebras. The left and the right absolute values of an element x of a C^*-algebra are $|x| := (x^*x)^{1/2}$ and $|x|_{\mathrm{r}} := (xx^*)^{1/2}$, respectively. Representations of C^*-algebras are assumed to be non-degenerate. The flip map $a \otimes b \mapsto b \otimes a$ at the algebra level is denoted by σ. The symbols $\otimes$, $\otimes_{\min}$ and $\overline{\otimes}$ are used for the Hilbert space, minimal (spatial) C^*-algebraic and normal spatial tensor products, respectively.

Let N be a von Neumann algebra. For a weight θ on N, we denote by $(L^2(N, \theta), \Lambda_\theta)$ the associated GNS construction. When θ is normal, semi-finite and faithful (n.s.f.), we

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denote by ∇_θ and J_θ the modular operator and modular conjugation of θ, respectively, both acting on $L^2(N, \theta)$, and by σ^θ the modular automorphism group of θ [**59**, **60**].

Locally compact quantum groups (LCQGs) are a far-reaching generalization of locally compact groups. Their axiomatization, which is the product of a long list of works that go back to the 1970s, was introduced by Kustermans and Vaes [**36**, **37**], and an equivalent one by Masuda *et al* [**41**]. A LCQG is a pair $\mathbb{G} = (L^\infty(\mathbb{G}), \Delta)$ satisfying the following conditions:

(a) $L^\infty(\mathbb{G})$ is a von Neumann algebra;

(b) Δ is a co-multiplication, namely a unital normal $*$-homomorphism $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} L^\infty(\mathbb{G})$ that is co-associative in the sense that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$;

(c) there exist n.s.f. weights φ, ψ on $L^\infty(\mathbb{G})$, called the left and right Haar weights, respectively, that satisfy

$$\varphi((\omega \otimes \mathrm{id})\Delta(x)) = \varphi(x)\omega(\mathbb{1}) \quad \text{for all } \omega \in L^\infty(\mathbb{G})_*^+,\ x \in L^\infty(\mathbb{G})^+ \text{ with } \varphi(x) < \infty,$$
$$\psi((\mathrm{id} \otimes \omega)\Delta(x)) = \psi(x)\omega(\mathbb{1}) \quad \text{for all } \omega \in L^\infty(\mathbb{G})_*^+,\ x \in L^\infty(\mathbb{G})^+ \text{ with } \psi(x) < \infty.$$

We set $\nabla := \nabla_\varphi$ and $J := J_\varphi$. The predual $L^\infty(\mathbb{G})_*$ of $L^\infty(\mathbb{G})$ is denoted by $L^1(\mathbb{G})$. It is a Banach algebra when equipped with the convolution product $\omega_1 * \omega_2 := (\omega_1 \otimes \omega_2)\Delta$, $\omega_1, \omega_2 \in L^1(\mathbb{G})$.

The *dual* of $\mathbb{G}$ is a LCQG denoted by $\hat{\mathbb{G}} = (L^\infty(\hat{\mathbb{G}}), \hat{\Delta})$. We will not elaborate on the precise construction of $\hat{\mathbb{G}}$, but give only a few details we shall need. The objects associated with the dual will be denoted by adding a hat to the relevant notation, e.g. $\hat{\varphi}$, $\hat{\nabla}$, $\hat{J}$. The Hilbert spaces $L^2(L^\infty(\mathbb{G}), \varphi)$ and $L^2(L^\infty(\hat{\mathbb{G}}), \hat{\varphi})$ are canonically isometrically isomorphic, allowing us to view both $L^\infty(\mathbb{G})$ and $L^\infty(\hat{\mathbb{G}})$ as acting standardly on the same Hilbert space $L^2(\mathbb{G})$ [**30**].

The *left regular co-representation* of $\mathbb{G}$ is a unitary $W \in L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} L^\infty(\hat{\mathbb{G}})$ satisfying $\Delta(x) = W^*(\mathbb{1} \otimes x)W$ for every $x \in L^\infty(\mathbb{G})$ and $(\Delta \otimes \mathrm{id})(W) = W_{13}W_{23}$, where the subscript numbers are the customary leg numbering. Its dual object is just $\hat{W} = \sigma(W^*)$. The space $C_0(\mathbb{G}) := \overline{\mathrm{span}}^{\|\cdot\|}\{(\mathrm{id} \otimes \omega)(W) : \omega \in L^1(\hat{\mathbb{G}})\}$ is a weakly dense C^*-subalgebra of $L^\infty(\mathbb{G})$. It satisfies $\Delta(C_0(\mathbb{G})) \subseteq M(C_0(\mathbb{G}) \otimes_{\min} C_0(\mathbb{G}))$, where M stands for the multiplier algebra. We have $W \in M(C_0(\mathbb{G}) \otimes_{\min} C_0(\hat{\mathbb{G}}))$.

The *antipode* of $\mathbb{G}$ is a $*$-ultra-strongly closed, densely defined operator S over $L^\infty(\mathbb{G})$ with the property (2.2) below. It admits a polar decomposition $S = R \circ \tau_{-i/2}$, where the *unitary antipode* R is an anti-automorphism of $L^\infty(\mathbb{G})$ and the *scaling group* $\tau = (\tau_t)_{t \in \mathbb{R}}$ is a group of automorphisms of $L^\infty(\mathbb{G})$, and by $\tau_{-i/2}$ we mean the analytic generator of τ at the point $-i/2$ [**11**, **66**]. We have the useful formulas

$$\sigma(R \otimes R)\Delta = \Delta R \tag{2.1}$$

and $R(x) = \hat{J}x^*\hat{J}$, $\tau_t(x) = \hat{\nabla}^{it}x\hat{\nabla}^{-it}$ for every $x \in L^\infty(\mathbb{G})$, $t \in \mathbb{R}$.

We mention several types of LCQGs. First, *compact* quantum groups were introduced by Woronowicz [**65**] (see also Maes and Van Daele [**40**]). We say that $\mathbb{G}$ is compact if $C_0(\mathbb{G})$ is unital. This is equivalent to either of the Haar weights being finite. In this case, the Haar weights are, in fact, equal after being normalized to states, and the common value is called the *Haar state* of $\mathbb{G}$. We write $C(\mathbb{G})$ for $C_0(\mathbb{G})$ and $\mathrm{Irred}(\mathbb{G})$ for the set of equivalence classes of irreducible unitary co-representations of $\mathbb{G}$ (see below). Second,

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discrete quantum groups were introduced by Effros and Ruan [**24**] and by Van Daele [**63**]. We say that $\mathbb{G}$ is discrete if $\hat{\mathbb{G}}$ is compact. In this case, we write $c_0(\mathbb{G})$, $\ell^\infty(\mathbb{G})$, $\ell^1(\mathbb{G})$ for $C_0(\mathbb{G})$, $L^\infty(\mathbb{G})$, $L^1(\mathbb{G})$, respectively. We have $\ell^\infty(\mathbb{G}) = \ell^\infty - \bigoplus_{\gamma \in \mathrm{Irred}(\hat{\mathbb{G}})} M_{N(\gamma)}$ and $c_0(\mathbb{G}) = c_0 - \bigoplus_{\gamma \in \mathrm{Irred}(\hat{\mathbb{G}})} M_{N(\gamma)}$, where $N(\gamma)$ is the dimension of γ. There is a distinguished central minimal projection $p \in c_0(\mathbb{G})$ such that $\Delta(a)(\mathbb{1} \otimes p) = a \otimes p$ for all $a \in \ell^\infty(\mathbb{G})$. See Runde [**54**] for more information and proofs.

Kac algebras [**26**] were introduced by Enock and Schwartz and by Kac and Vainerman. These are precisely the LCQGs with a trivial scaling group ($\tau = \mathrm{id}$) that satisfy $\sigma^\varphi = \sigma^\psi$.

Example 2.1. Every locally compact group G induces two LCQGs. The first, which is identified with G, has $C_0(G)$, $L^\infty(G)$ as its underlying C^*- and von Neumann algebras, and $(\Delta(f))(t, s) := f(ts)$ for $f \in L^\infty(G)$ and $t, s \in G$. The Haar weights are given by integration against the Haar measures. Its dual LCQG $\hat{G}$ has $L^\infty(\hat{G}) = \mathrm{VN}(G)$, the left von Neumann algebra of G generated by the left translation operators $\{\lambda(g) : g \in G\}$ over $L^2(G)$, $C_0(\hat{G}) = C^*_{\mathrm{r}}(G)$, and $\hat{\Delta}$ is determined by satisfying $\hat{\Delta}(\lambda(g)) = \lambda(g) \otimes \lambda(g)$ for each $g \in G$. The left and right Haar weights of $\hat{G}$ are both equal to the Plancherel weight on $\mathrm{VN}(G)$ [**60**, §VII.3]. If G is abelian, the quantum duality between G and $\hat{G}$ reduces, up to unitary equivalence, to the Pontryagin duality.

A *co-representation* of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$ is an operator $U \in B(\mathcal{H}) \mathbin{\overline{\otimes}} L^\infty(\mathbb{G})$ such that $(\mathrm{id} \otimes \Delta)(U) = U_{12}U_{13}$. A closed subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ is called *invariant* under U if the projection $p_{\mathcal{H}_0}$ of $\mathcal{H}$ onto $\mathcal{H}_0$ satisfies $U(p_{\mathcal{H}_0} \otimes \mathbb{1}) = (p_{\mathcal{H}_0} \otimes \mathbb{1})U(p_{\mathcal{H}_0} \otimes \mathbb{1})$. The operator $U_0 := U(p_{\mathcal{H}_0} \otimes \mathbb{1}) \in B(\mathcal{H}_0) \mathbin{\overline{\otimes}} L^\infty(\mathbb{G})$ is then a co-representation of $\mathbb{G}$ on $\mathcal{H}_0$. We say that U_0 is a *sub-representation* of U, or that U *contains* U_0, and write $U_0 \leq U$. A vector $\zeta \in \mathcal{H}$ is said to be *invariant* under U if the projection p_ζ of $\mathcal{H}$ onto $\mathbb{C}\zeta$ satisfies $U(p_\zeta \otimes \mathbb{1}) = p_\zeta \otimes \mathbb{1}$. We say that U is *ergodic* if it has no non-zero invariant vector. If U, V are co-representations of $\mathbb{G}$ on Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, respectively, their *tensor product* is the co-representation $U \mathbin{⦶} V := U_{13}V_{23}$ of $\mathbb{G}$ on $\mathcal{H} \otimes \mathcal{K}$.

Unitary co-representations U of $\mathbb{G}$ on $\mathcal{H}$ have additional useful features. They satisfy $U \in M(\mathbb{K}(\mathcal{H}) \otimes_{\min} C_0(\mathbb{G}))$, where $\mathbb{K}(\mathcal{H})$ is the C^*-algebra of all compact operators over $\mathcal{H}$. Moreover,

$$(\mathrm{id} \otimes S)(U) = U^*, \tag{2.2}$$

that is, for every $\omega \in B(\mathcal{H})_*$, we have $(\omega \otimes \mathrm{id})(U) \in D(S)$ and $S((\omega \otimes \mathrm{id})(U)) = (\omega \otimes \mathrm{id})(U^*)$. As $S = R \circ \tau_{-i/2}$, this means that formally $(\mathrm{id} \otimes \tau_{-i/2})(U) = (\mathrm{id} \otimes R)(U^*)$, which should be understood similarly. If a closed subspace $\mathcal{H}_0$ is invariant under U, then, in the above notation, U commutes with $p_{\mathcal{H}_0} \otimes \mathbb{1}$, for instance by the next paragraph. The associated sub-representation of $\mathbb{G}$ on $\mathcal{H}_0$ is therefore unitary.

There is also the *universal* face of $\mathbb{G}$ [**35**]. It consists of a C^*-algebra $C_0^{\mathrm{u}}(\mathbb{G})$, a canonical surjective $*$-homomorphism $\pi : C_0^{\mathrm{u}}(\mathbb{G}) \to C_0(\mathbb{G})$ and a co-multiplication $\Delta^{\mathrm{u}} : C_0^{\mathrm{u}}(\mathbb{G}) \to M(C_0^{\mathrm{u}}(\mathbb{G}) \otimes_{\min} C_0^{\mathrm{u}}(\mathbb{G}))$ such that $(\pi \otimes \pi)\Delta^{\mathrm{u}} = \Delta\pi$. Doing this construction also for $\hat{\mathbb{G}}$, the left regular co-representation W lifts to a universal left regular co-representation, which is a unitary $\mathbb{W} \in M(C_0(\mathbb{G}) \otimes_{\min} C_0^{\mathrm{u}}(\hat{\mathbb{G}}))$ with $(\mathrm{id} \otimes \hat{\pi})(\mathbb{W}) = W$. The universality is expressed by the following fact: for every unitary co-representation U of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$, there exists a representation ρ of $C_0^{\mathrm{u}}(\hat{\mathbb{G}})$ on $\mathcal{H}$ such

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that $U = \sigma(\mathrm{id} \otimes \rho)(\mathbb{W})$. (We require the burdensome flip since we are working with the left regular co-representation, but with right unitary co-representations.) In particular, the character ϵ of $C_0^{\mathrm{u}}(\mathbb{G})$ corresponding to the trivial representation $\mathbb{1}$ of $\hat{\mathbb{G}}$ is called the *co-unit* of $\mathbb{G}$ and satisfies $(\epsilon \otimes \mathrm{id})\Delta^{\mathrm{u}} = \mathrm{id}$. There are also universal versions R^{u}, τ^{u} of R, τ.

A *left-invariant mean* for $\mathbb{G}$ is a state $m \in L^\infty(\mathbb{G})^*$ satisfying $m(\omega \otimes \mathrm{id})\Delta = \omega(\mathbb{1})m$ for every $\omega \in L^1(\mathbb{G})$. Right-invariant and two-sided invariant means are defined similarly. We say that $\mathbb{G}$ is *amenable* if it possesses a left-invariant mean. This is equivalent to $\mathbb{G}$ admitting a right-invariant, or a two-sided invariant, mean. We say that $\mathbb{G}$ is *co-amenable* if the canonical surjective $*$-homomorphism $\pi : C_0^{\mathrm{u}}(\mathbb{G}) \to C_0(\mathbb{G})$ is injective. In this case, we identify $C_0^{\mathrm{u}}(\mathbb{G})$ with $C_0(\mathbb{G})$. This is equivalent to $L^1(\mathbb{G})$ having a bounded left, right or two-sided approximate identity, so discrete quantum groups are trivially co-amenable. For other equivalent conditions for amenability and co-amenability, see Bédos and Tuset [**3**].

A unitary co-representation U of $\mathbb{G}$ on $\mathcal{H}$ is called *mixing* [**15**, Definition 4.1] if for every $\omega \in B(\mathcal{H})_*$, $(\omega \otimes \mathrm{id})(U) \in C_0(\mathbb{G})$. The left regular co-representation of $\mathbb{G}$ is mixing by definition. Additionally, all unitary co-representations of a compact quantum group are mixing.

A (right) *action* of $\mathbb{G}$ on a von Neumann algebra N is a unital normal $*$-homomorphism $\alpha : N \to N \mathbin{\overline{\otimes}} L^\infty(\mathbb{G})$ such that

$$(\alpha \otimes \mathrm{id})\alpha = (\mathrm{id} \otimes \Delta)\alpha.$$

For $\theta \in L^1(\mathbb{G})$, we say that α *preserves* θ if $(\theta \otimes \mathrm{id})\alpha = \theta(\cdot)\mathbb{1}$. If $\mathbb{G}$ is co-amenable, we adopt the convention of requiring that some bounded left approximate identity (ϵ_λ) of the Banach algebra $L^1(\mathbb{G})$ would satisfy

$$(\mathrm{id} \otimes \epsilon_\lambda)\alpha(a) \to a \text{ weakly} \quad (\text{for all } a \in N). \tag{2.3}$$

If $\mathbb{G}$ is discrete, this is the same as asking that, with $p \in c_0(\mathbb{G})$ being the distinguished central minimal projection,

$$\alpha(a)(\mathbb{1} \otimes p) = a \otimes p \quad (\text{for all } a \in N). \tag{2.4}$$

3. *Conditions for weak mixing*

In this section we introduce weak mixing of unitary co-representations of LCQGs, generalizing conditions (WM1)–(WM3) from the Introduction. There are two versions: weak mixing (Definition 3.9) and strict weak mixing (Definition 3.10), which coincide, e.g., for Kac algebras and for discrete quantum groups. The first is the natural one for LCQGs. We establish the equivalence of all conditions except for one implication, for which amenability is assumed. The reason for that is the possible lack of certain invariant means, as explored in depth by Das and Daws [**13**]. The second version is not as natural and, indeed, less is known about it without assuming amenability. The 'price' that one has to pay when working with the former version is that most of its characterizations have two parts, and some involve unbounded co-representations.

3.1. *Preliminaries on complete almost periodicity.*

We need several facts about notions discussed in [**55**]. See also Sołtan [**58**] and Daws [**14**].

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LEMMA 3.1. (Cf. [**58**, Proposition 4.11]) *Let $\mathbb{G}$ be a LCQG. Suppose that $C(\mathbb{H})$ is a unital C^*-subalgebra of $L^\infty(\mathbb{G})$ such that $\Delta(C(\mathbb{H})) \subseteq C(\mathbb{H}) \otimes_{\min} C(\mathbb{H})$ and $\mathbb{H} := (C(\mathbb{H}), \Delta|_{C(\mathbb{H})})$ is a C^*-algebraic compact quantum group in the sense of* [**65**, **40**]. *Then the antipode, unitary antipode and scaling group $S^{\mathbb{H}}$, $R^{\mathbb{H}}$, $\tau^{\mathbb{H}}$ of $\mathbb{H}$ are the restrictions of $S^{\mathbb{G}}$, $R^{\mathbb{G}}$, $\tau^{\mathbb{G}}$ to the canonical dense Hopf $*$-algebra $\mathcal{A}$ of $\mathbb{H}$. In particular, $S^{\mathbb{H}}$ is (norm) closable, $\tau^{\mathbb{H}}$ extends to an automorphism group of $C(\mathbb{H})$ and $R^{\mathbb{H}}$ extends to an anti-automorphism of $C(\mathbb{H})$.*

Such $\mathbb{H}$ is called a *compactification* of $\mathbb{G}$. The point is that as $\mathbb{H}$ is not necessarily reduced in the sense of [**36**] (cf. [**14**, §6.2]), the above properties of the antipode are not automatic; cf. [**65**, §8] and [**36**, Proposition 5.45].

Proof of Lemma 3.1. Every co-representation of $\mathbb{H}$ is a co-representation of $\mathbb{G}$. By (2.2), the antipode $S^{\mathbb{H}}$ of $\mathbb{H}$ agrees with $S^{\mathbb{G}}$ on the components of all irreducible unitary co-representations of $\mathbb{H}$, which linearly span $\mathcal{A}$. Thus, $S^{\mathbb{H}} \subseteq S^{\mathbb{G}}$, and in particular $S^{\mathbb{H}}$ is closable. Let $u \in M_n \otimes C(\mathbb{H})$ be an irreducible unitary co-representation of $\mathbb{H}$. By the classical theory [**65**], $u \in D(\mathrm{id} \otimes (S^{\mathbb{H}})^2)$, and there is a strictly positive matrix $F \in M_n$ such that $(\mathrm{id} \otimes \tau_t^{\mathbb{H}})(u) = (F^{it} \otimes \mathbb{1})u(F^{-it} \otimes \mathbb{1})$ for every $t \in \mathbb{R}$ and $(\mathrm{id} \otimes (S^{\mathbb{H}})^2)(u) = (F \otimes \mathbb{1})u(F^{-1} \otimes \mathbb{1})$. By the foregoing, $u \in D(\mathrm{id} \otimes (S^{\mathbb{G}})^2)$ and

$$(\mathrm{id} \otimes \tau_{-i}^{\mathbb{G}})(u) = (\mathrm{id} \otimes (S^{\mathbb{G}})^2)(u) = (F \otimes \mathbb{1})u(F^{-1} \otimes \mathbb{1}).$$

A classical complex analysis argument implies that $(\mathrm{id} \otimes \tau_t^{\mathbb{G}})(u) = (F^{it} \otimes \mathbb{1})u(F^{-it} \otimes \mathbb{1})$ for every t. Thus, $\mathrm{id} \otimes \tau^{\mathbb{G}}$ and $\mathrm{id} \otimes \tau^{\mathbb{H}}$ agree on u. Consequently, $\tau^{\mathbb{H}}$ is the restriction of $\tau^{\mathbb{G}}$ to $\mathcal{A}$, and so $\tau^{\mathbb{H}}$ extends to an automorphism group of $C(\mathbb{H})$. The conclusion about R follows. □

For the following definition, see [**55**, Definition 3.7], in which the setting is a little different.

Definition 3.2. Let U be a unitary co-representation of a LCQG $\mathbb{G}$ on a Hilbert space $\mathcal{H}$. A vector $\zeta \in \mathcal{H}$ is *completely periodic* with respect to U if it belongs to a finite-dimensional subspace $\mathcal{H}_0$ of $\mathcal{H}$, such that U restricts to a sub-representation u on $\mathcal{H}_0$ whose transpose u^{t} is invertible. The closed linear span $\mathcal{H}^{\mathrm{CAP}}$ of all completely periodic vectors is called the subspace of *completely almost periodic* vectors with respect to U.

Finite-dimensional unitary co-representations admitting an invertible transpose are *admissible* in the terminology of [**58**], in which this notion is used in a broader sense. Proving the invertibility of the transpose can be done as follows (cf. [**14**, Proposition 3.11]). Let u be a finite-dimensional unitary co-representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}_0$. Let $n := \dim \mathcal{H}_0$, view $B(\mathcal{H}_0)$ as M_n and denote by A the natural anti-linear isomorphism of M_n given by $(a_{ij}) \mapsto (\overline{a_{ij}})$. Since the unitary antipode satisfies $R(x) = \hat{J}x^*\hat{J}$ for every $x \in L^\infty(\mathbb{G})$, from (2.2) we get $(\mathrm{id} \otimes \tau_{-i/2})(u) = (\mathrm{id} \otimes R)(u^*) = (A \otimes \mathrm{Ad}(\hat{J}))(u^{\mathrm{t}})$. Thus, u^{t} is invertible if and only if $(\mathrm{id} \otimes \tau_{-i/2})(u)$ is. As $\tau_{-i/2}$ is multiplicative on its domain, this

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happens when $u^* \in D(\mathrm{id} \otimes \tau_{-i/2})$, that is, $u \in D(\mathrm{id} \otimes \tau_{i/2}) = D(\mathrm{id} \otimes S^{-1})$. Conversely, if u^{t} is invertible, then the unital C^*-algebra $C(\mathbb{H})$ generated by the components of u, together with the restriction of Δ to $C(\mathbb{H})$, is a (not necessarily reduced) C^*-algebraic compact quantum group $\mathbb{H}$ by [**64**], [**40**, Proposition 3.8]. From Lemma 3.1, the scaling group $\tau^{\mathbb{H}}$ of $\mathbb{H}$ is a restriction of $\tau = \tau^{\mathbb{G}}$. From the general theory [**65**], the components of u are analytic for $\tau^{\mathbb{H}}$ and thus for $\tau^{\mathbb{G}}$.

Using the idea of the canonical Kac quotient of a compact quantum group, it is proved in [**58**, §4.3] that, for a *discrete* quantum group $\mathbb{G}$, the components of all finite-dimensional unitary co-representations of $\mathbb{G}$ linearly span the same subspace as those of the ones having an invertible transpose. The previous paragraph thus implies the following; cf. [**14**, Corollary 6.6].

PROPOSITION 3.3. *Every finite-dimensional unitary co-representation of a discrete quantum group has an invertible transpose.*

It is unknown whether this holds true for all LCQGs; see [**14**, Conjectures 7.1 and 7.2].

3.2. $\overline{U}$ *and* U'.

Definition 3.4. Let $\mathbb{G}$ be a LCQG and let $U \in B(\mathcal{H}) \mathbin{\overline{\otimes}} L^\infty(\mathbb{G})$ be a unitary co-representation of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$.

(a) (cf. [**3**]) Fixing some anti-unitary $\mathcal{J}$ from $\mathcal{H}$ onto another Hilbert space $\mathcal{J}\mathcal{H}$, consider the $*$-anti-isomorphism $j : B(\mathcal{H}) \to B(\mathcal{J}\mathcal{H})$ given by $j(x) := \mathcal{J}x^*\mathcal{J}^*$ for all $x \in B(\mathcal{H})$, and define the unitary co-representation of $\mathbb{G}$ *conjugate* to U to be $\overline{U} := (j \otimes R)(U) \in B(\mathcal{J}\mathcal{H}) \mathbin{\overline{\otimes}} L^\infty(\mathbb{G})$.

(b) Let U' be the formal object $(\mathrm{id} \otimes \tau_{-i/2})(U) = (\mathrm{id} \otimes R)(U^*)$ (see (2.2)). Rigorously, recalling that $B(L^1(\mathbb{G}), B(\mathcal{H})) \cong B(B(\mathcal{H})_*, L^\infty(\mathbb{G}))$ canonically, we view U' either as the element $L^1(\mathbb{G}) \ni \omega \mapsto (\mathrm{id} \otimes (\omega \circ R))(U^*)$ of $B(L^1(\mathbb{G}), B(\mathcal{H}))$ or, equivalently, as the matching element $B(\mathcal{H})_* \ni \rho \mapsto \tau_{-i/2}((\rho \otimes \mathrm{id})(U)) = R((\rho \otimes \mathrm{id})(U^*))$ of $B(B(\mathcal{H})_*, L^\infty(\mathbb{G}))$, thus defining the expressions $(\mathrm{id} \otimes \omega)(U')$ and $(\rho \otimes \mathrm{id})(U')$.

When the maps defining U' happen to be completely bounded, we get an element of $B(\mathcal{H}) \mathbin{\overline{\otimes}} L^\infty(\mathbb{G})$, as this operator space is naturally identified with $\mathcal{CB}(L^1(\mathbb{G}), B(\mathcal{H})) \cong \mathcal{CB}(B(\mathcal{H})_*, L^\infty(\mathbb{G}))$ [**25**, Ch. 7]. For instance, when the scaling group of $\mathbb{G}$ is trivial (e.g. if $\mathbb{G}$ is a Kac algebra), we have $U' = U$.

Plainly, $\overline{U}$ depends on $\mathcal{J}$ only up to unitary equivalence. Using (2.1), one verifies that $\overline{U}$ is indeed a unitary co-representation of $\mathbb{G}$ on $\mathcal{J}\mathcal{H}$, and that U' is formally an unbounded co-representation of $\mathbb{G}$ on $\mathcal{H}$ in the sense that $[\mathrm{id} \otimes ((\omega_1 * \omega_2) \circ R)](U^*) = (\mathrm{id} \otimes (\omega_1 \circ R))(U^*)(\mathrm{id} \otimes (\omega_2 \circ R))(U^*)$ for all $\omega_1, \omega_2 \in L^1(\mathbb{G})$.

LEMMA 3.5. *Let U be a unitary co-representation of a LCQG $\mathbb{G}$ on a Hilbert space $\mathcal{H}$. For every $\zeta, \eta \in \mathcal{H}$, we have*

$$(\omega_{\mathcal{J}\zeta, \mathcal{J}\eta} \otimes \mathrm{id})(\overline{U}) = R[(\omega_{\eta,\zeta} \otimes \mathrm{id})(U)] = (\omega_{\zeta,\eta} \otimes \mathrm{id})(U')^*. \tag{3.1}$$

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In addition, for every Hilbert space $\mathcal{K}$ *and* $\Xi \in \mathcal{J}\mathcal{H} \otimes \mathcal{K}$, *the operator* $T_\Xi \in B(\mathcal{H}, \mathcal{K})$ *given by* $T_\Xi \zeta := ((\mathcal{J}\zeta)^* \otimes \mathbb{1})(\Xi)$, $\zeta \in \mathcal{H}$, *satisfies*

$$(\omega_{(\mathbb{1}\otimes\xi^*)(\Xi),\mathcal{J}\zeta} \otimes \mathrm{id})(\overline{U}^*) = (\omega_{\zeta,T_\Xi^*\xi} \otimes \mathrm{id})(U') \tag{3.2}$$

for every $\zeta \in \mathcal{H}$ *and* $\xi \in \mathcal{K}$.

Proof. Equation (3.1) is obtained from the equality $\omega_{\mathcal{J}\zeta,\mathcal{J}\eta} \circ j = \omega_{\eta,\zeta}$. Notice that $T_\Xi^*\xi = \mathcal{J}(\mathbb{1} \otimes \xi^*)(\Xi)$ for each $\xi \in \mathcal{K}$. Hence, taking adjoints in (3.1) with $\eta := T_\Xi^*\xi$, we establish (3.2). □

We now explain the nomenclature used in Definitions 3.9 and 3.10.

Definition 3.6. Consider unitary co-representations U, V of a LCQG $\mathbb{G}$ on Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, respectively. Denote by $\mathcal{BR}(V \textcircled{T} U')$ the set of all closed subspaces $\mathcal{F}$ of $\mathcal{K} \otimes \mathcal{H}$ such that, with $p_\mathcal{F}$ being the projection of $\mathcal{K} \otimes \mathcal{H}$ onto $\mathcal{F}$, we have $U_{23}(p_\mathcal{F} \otimes \mathbb{1}) \in D(\mathrm{id} \otimes \mathrm{id} \otimes \tau_{-i/2})$, and $Z := V_{13}(\mathrm{id} \otimes \mathrm{id} \otimes \tau_{-i/2})(U_{23}(p_\mathcal{F} \otimes \mathbb{1}))$ is a co-representation of $\mathbb{G}$ commuting with $p_\mathcal{F} \otimes \mathbb{1}$. In particular, if $\mathcal{F} = \mathbb{C}\Xi$ for a vector $\Xi \in \mathcal{K} \otimes \mathcal{H}$ and Z is equal to $p_{\mathbb{C}\Xi} \otimes \mathbb{1}$, we say that Ξ *is invariant under* $V \textcircled{T} U'$.

Evidently, if the scaling group of $\mathbb{G}$ is trivial (so that $U' = U$), then $\mathcal{K} \otimes \mathcal{H}$ itself belongs to $\mathcal{BR}(V \textcircled{T} U')$, and invariance of a vector under $V \textcircled{T} U'$ has the usual meaning.

For general LCQGs, invariance of vectors under $V \textcircled{T} U'$ has a more concrete interpretation.

LEMMA 3.7. *Let* U, V *be unitary co-representations of a LCQG* $\mathbb{G}$ *on Hilbert spaces* $\mathcal{H}$, $\mathcal{K}$, *respectively, and* $\Xi \in \mathcal{K} \otimes \mathcal{H}$. *Then* Ξ *is invariant under* $V \textcircled{T} U'$ *if and only if for all* $\zeta \in \mathcal{K}$, $\xi \in \mathcal{H}$,

$$(\omega_{(\zeta^*\otimes\mathbb{1})(\Xi),\xi} \otimes \mathrm{id})(U') = (\omega_{(\mathbb{1}\otimes\xi^*)(\Xi),\zeta} \otimes \mathrm{id})(V^*). \tag{3.3}$$

Proof. Write $p_{\mathbb{C}\Xi}$ for the projection of $\mathcal{K} \otimes \mathcal{H}$ onto $\mathbb{C}\Xi$. The implication ($\Longrightarrow$) is obtained by applying $\omega_{\Xi,\zeta\otimes\xi} \otimes \mathrm{id}$ to both sides of the equality $(\mathrm{id} \otimes \mathrm{id} \otimes \tau_{-i/2})(U_{23}(p_{\mathbb{C}\Xi} \otimes \mathbb{1})) = V_{13}^*(p_{\mathbb{C}\Xi} \otimes \mathbb{1})$. For ($\Longleftarrow$), approximate $U_{23}(p_{\mathbb{C}\Xi} \otimes \mathbb{1})$ by operators of the form

$$\sum_{\alpha,\beta}((f_\beta \otimes e_\alpha) \otimes \Xi^*) \otimes (\omega_{\Xi,f_\beta\otimes e_\alpha} \otimes \mathrm{id})(U_{23})$$
$$= \sum_{\alpha,\beta}((f_\beta \otimes e_\alpha) \otimes \Xi^*) \otimes (\omega_{(f_\beta^*\otimes\mathbb{1})(\Xi),e_\alpha} \otimes \mathrm{id})(U),$$

where (e_α), (f_β) are orthonormal bases of $\mathcal{H}$, $\mathcal{K}$, respectively, and the sums are finite. By assumption, applying $\mathrm{id} \otimes \mathrm{id} \otimes \tau_{-i/2}$ to this operator gives $\sum_{\alpha,\beta}((f_\beta \otimes e_\alpha) \otimes \Xi^*) \otimes (\omega_{(\mathbb{1}\otimes e_\alpha^*)(\Xi),f_\beta} \otimes \mathrm{id})(V^*)$. The closedness of $\mathrm{id} \otimes \mathrm{id} \otimes \tau_{-i/2}$ thus yields the desired conclusion. □

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3.3. *Weak mixing.*

Definition 3.8. Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. Vectors $\Xi, \Upsilon \in \mathcal{H} \otimes \mathcal{K}$ are said to be *matched* if for every $\zeta \in \mathcal{H}$ and $\xi \in \mathcal{K}$:

- $\langle (\zeta^* \otimes \mathbb{1})\Xi, (\zeta^* \otimes \mathbb{1})\Upsilon \rangle, \langle (\mathbb{1} \otimes \xi^*)\Xi, (\mathbb{1} \otimes \xi^*)\Upsilon \rangle \in \mathbb{R}$; and
- $(\zeta^* \otimes \mathbb{1})\Xi = 0 \iff (\zeta^* \otimes \mathbb{1})\Upsilon = 0, (\mathbb{1} \otimes \xi^*)\Xi = 0 \iff (\mathbb{1} \otimes \xi^*)\Upsilon = 0.$

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, $\mathcal{J} : \mathcal{H} \to \mathcal{J}\mathcal{H}$ an anti-unitary and $\Xi, \Upsilon \in \mathcal{J}\mathcal{H} \otimes \mathcal{K}$. Define $T_\Xi, T_\Upsilon \in B(\mathcal{H}, \mathcal{K})$ by $T_\Xi \zeta := ((\mathcal{J}\zeta)^* \otimes \mathbb{1})(\Xi)$, $\zeta \in \mathcal{H}$, and similarly for T_Υ. Notice that $T_\Xi^* T_\Upsilon \in B(\mathcal{H})$ and $T_\Xi T_\Upsilon^* \in B(\mathcal{K})$ are selfadjoint if and only if the first condition of Definition 3.8 holds, while the second is equivalent to $\ker T_\Xi = \ker T_\Upsilon$ and $\ker T_\Xi^* = \ker T_\Upsilon^*$. Hence, by Lemma A.2, Ξ, Υ are matched if and only if there exists a (generally unbounded) injective selfadjoint operator B over $\mathcal{K}$ commuting with $T_\Xi T_\Xi^*$ such that $T_\Upsilon = B T_\Xi$. The latter just means that $\Upsilon = (\mathbb{1} \otimes B)\Xi$.

We use the convention that for $y \in L^\infty(\mathbb{G})$ and $c \geq 0$, we write $\|\tau_{i/2}(y)\| > c$ to mean either that y does not belong to $D(\tau_{i/2})$ or that it does and genuinely $\|\tau_{i/2}(y)\| > c$.

Definition 3.9. Let $\mathbb{G}$ be a LCQG and let U, V be unitary co-representations of $\mathbb{G}$ on Hilbert spaces $\mathcal{H}, \mathcal{K}$, respectively. We introduce the following *weak mixing* conditions.

(WM) For every $\zeta_1, \ldots, \zeta_n \in \mathcal{H}$ such that some element of $\mathcal{BR}(\overline{U} \mathbin{⦶} U')$ contains all vectors $\mathcal{J}\zeta_i \otimes \zeta_i$, $1 \leq i \leq n$, and every $\varepsilon > 0$, *either* there is a state $\omega \in L^1(\mathbb{G})$ such that $\omega(|(\omega_{\zeta_i, \zeta_j} \otimes \mathrm{id})(U)|) \leq \varepsilon$ for each $1 \leq i, j \leq n$ *or* there exist $1 \leq i, j \leq n$ and $\rho_1, \rho_2 \in L^1(\mathbb{G})$ of norm 1 such that $\|\tau_{i/2}[(\omega_{(\mathrm{id} \otimes \rho_1)(U^*)\zeta_i, (\mathrm{id} \otimes \rho_2)(U^*)\zeta_j} \otimes \mathrm{id})(U)]\| > \varepsilon^{-1}$.

(PSE) The tensor product co-representations $V \mathbin{⦶} U$ and $V \mathbin{⦶} U'$ are *simultaneously ergodic*: they do not admit non-zero *matched* invariant vectors in $\mathcal{K} \otimes \mathcal{H}$.

(PEB) The tensor product co-representation $V \mathbin{⦶} U'$ does not admit a non-zero invariant vector $\Xi \in \mathcal{K} \otimes \mathcal{H}$ such that $(\omega_{(\zeta^* \otimes \mathbb{1})\Xi, (\xi^* \otimes \mathbb{1})\Xi} \otimes \mathrm{id})(U) \in D(\tau_{i/2})$ for all $\zeta, \xi \in \mathcal{K}$.

(NCAP) There are no non-zero completely (almost) periodic vectors with respect to U, that is, U does not admit a non-zero finite-dimensional sub-representation with an invertible transpose.

If $\mathbb{G}$ is amenable and $m \in L^\infty(\mathbb{G})^*$ is a right-invariant mean for $\mathbb{G}$, we introduce the following strengthening of (WM).

(WMa) For every $\zeta_1, \ldots, \zeta_n \in \mathcal{H}$ such that some element of $\mathcal{BR}(\overline{U} \mathbin{⦶} U')$ contains all vectors $\mathcal{J}\zeta_i \otimes \zeta_i$, $1 \leq i \leq n$, *either* for each $1 \leq i, j \leq n$ we have $m(|(\omega_{\zeta_i, \zeta_j} \otimes \mathrm{id})(U)|) = 0$ *or* for every $\varepsilon > 0$ there exist $1 \leq i, j \leq n$ and $\rho_1, \rho_2 \in L^1(\mathbb{G})$ of norm 1 such that $\|\tau_{i/2}[(\omega_{(\mathrm{id} \otimes \rho_1)(U^*)\zeta_i, (\mathrm{id} \otimes \rho_2)(U^*)\zeta_j} \otimes \mathrm{id})(U)]\| > \varepsilon^{-1}$.

Definition 3.10. Let $\mathbb{G}$ be a LCQG and let U, V be unitary co-representations of $\mathbb{G}$ on Hilbert spaces $\mathcal{H}, \mathcal{K}$, respectively. We introduce the following *strict weak mixing* conditions.

(sWM) For every $\zeta_1, \ldots, \zeta_n \in \mathcal{H}$ such that some element of $\mathcal{BR}(\overline{U} \mathbin{⦶} U')$ contains all vectors $\mathcal{J}\zeta_i \otimes \zeta_i$, $1 \leq i \leq n$, and every $\varepsilon > 0$, there is a state $\omega \in L^1(\mathbb{G})$ such that $\omega(|(\omega_{\zeta_i, \zeta_j} \otimes \mathrm{id})(U)|) \leq \varepsilon$ for each $1 \leq i, j \leq n$.

(PE) The tensor product co-representation $V \mathbin{⦶} U'$ is *ergodic*.

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(NFDS) There is no non-zero finite-dimensional sub-representation of U.

If $\mathbb{G}$ is amenable and $m \in L^\infty(\mathbb{G})^*$ is a right-invariant mean for $\mathbb{G}$, define:

(sWMa) for every $\zeta_1, \ldots, \zeta_n \in \mathcal{H}$ such that some element of $\mathcal{BR}(\overline{U} \textcircled{T} U')$ contains all vectors $\mathcal{J}\zeta_i \otimes \zeta_i$, $1 \le i \le n$, we have $m(|(\omega_{\zeta_i,\zeta_j} \otimes \mathrm{id})(U)|) = 0$ for each $1 \le i, j \le n$.

The following is the main result of this section.

THEOREM 3.11. *Let $\mathbb{G}$ be a LCQG and let U be a unitary co-representation of $\mathbb{G}$.*

(a) *The conditions ((PSE) for $V := \overline{U}$), ((PSE) holding for every V), ((PEB) for $V := \overline{U}$), ((PEB) holding for every V) and (NCAP) are equivalent, and are implied by (WM).*

(b) *We have (sWM) $\Longrightarrow$ (NFDS) $\Longrightarrow$ ((PE) holding for every V) $\Longrightarrow$ ((PE) for $V := \overline{U}$).*

(c) *If $\mathbb{G}$ is amenable, then all conditions in (a) are equivalent to one another and to (WMa), and all conditions in (b) are equivalent to one another and to (sWMa).*

(d) *If the scaling group of $\mathbb{G}$ is trivial, then all conditions in (a) and in (b) are equivalent to one another.*

Remark 3.12.

(a) It is clear that (NFDS) $\Longrightarrow$ (NCAP) $\Longrightarrow$ ergodicity. When $\mathbb{G}$ is non-compact, mixing implies (NFDS): every non-trivial finite-dimensional sub-representation has entries in $C_0(\mathbb{G})$ by mixing, so its unitarity forces $C_0(\mathbb{G})$ to be unital, which is a contradiction. When $\mathbb{G}$ is compact, every unitary co-representation is trivially mixing, but never satisfies (NCAP) by the general theory [**65**].

(b) The advantage of (PSE) over (PEB) is that the former can be written without any reference to the scaling group.

(c) When $\mathbb{G}$ is a locally compact group, our definition of weak mixing reduces to the classical one.

(d) When $\mathbb{G}$ is discrete, (NCAP) is equivalent to (NFDS) by Proposition 3.3.

(e) Let $\mathbb{H}$ be a closed quantum subgroup of $\mathbb{G}$ in the sense of Woronowicz [**16**, Definition 3.2 and Theorem 3.6] and let $\pi : C_0^{\mathrm{u}}(\mathbb{G}) \to C_0^{\mathrm{u}}(\mathbb{H})$ be the associated surjective $*$-homomorphism. It follows from [**42**, Proposition 3.10 and Theorem 4.8] that $R^{\mathrm{u},\mathbb{H}} \circ \pi = \pi \circ R^{\mathrm{u},\mathbb{G}}$ and $\tau_t^{\mathrm{u},\mathbb{H}} \circ \pi = \pi \circ \tau_t^{\mathrm{u},\mathbb{G}}$ for every $t \in \mathbb{R}$. Assume that $\mathbb{G}$, $\mathbb{H}$ are co-amenable. If U is a unitary co-representation of $\mathbb{G}$ on $\mathcal{H}$, then its 'restriction to $\mathbb{H}$' $U_{\mathbb{H}} := (\mathrm{id} \otimes \pi)(U)$ (view π as a $C_0(\mathbb{G}) \to C_0(\mathbb{H})$ map) is a unitary co-representation of $\mathbb{H}$ on $\mathcal{H}$. If either of (NFDS) or (NCAP) holds for $U_{\mathbb{H}}$, then it holds for U.

In the course of the proof of Theorem 3.11, which is divided into several steps, we assume that U is a unitary co-representation on a Hilbert space $\mathcal{H}$ and fix an anti-unitary $\mathcal{J} : \mathcal{H} \to \mathcal{J}\mathcal{H}$. Notice first that when $\mathbb{G}$ is amenable, (WMa) $\Longrightarrow$ (WM) and (sWMa) $\Longrightarrow$ (sWM). Also, (NFDS) and (NCAP) are equivalent if the scaling group of $\mathbb{G}$ is trivial by §3.1.

The following 'mean ergodic theorem' is elementary.

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LEMMA 3.13. *Let* $V \in B(\mathcal{H}) \overline{\otimes} L^\infty(\mathbb{H})$ *(respectively,* $V \in M(\mathbb{K}(\mathcal{H}) \otimes_{\min} C(\mathbb{H}))$*) be a co-representation of an amenable LCQG (respectively, a not necessarily reduced* C^**-algebraic compact quantum group* [**65**, **40**]*)* $\mathbb{H}$ *on a Hilbert space* $\mathcal{H}$*. Then* $(\mathrm{id} \otimes m)(V)$ *is an idempotent whose image consists of all vectors invariant under* V*, where* m *is a left-invariant mean for (respectively, the Haar state of)* $\mathbb{H}$*.*

PROPOSITION 3.14. *(WM)* $\vee$ *((PSE) with* $V := \overline{U}$*)* $\vee$ *((PEB) with* $V := \overline{U}$*)* $\Longrightarrow$ *(NCAP), and (sWM)* $\Longrightarrow$ *(NFDS).*

Proof. Assume first that (NFDS) does not hold, and let $\mathcal{H}_0$ be a non-zero finite-dimensional subspace of $\mathcal{H}$ that is invariant under U. Denote by $u \in B(\mathcal{H}_0) \otimes L^\infty(\mathbb{G})$ the ambient sub-representation of U. One checks that $\mathcal{H}_0$, $\mathcal{J}\mathcal{H}_0$ are invariant under U', $\overline{U}$, respectively, where the invariance under U' means that $(\mathrm{id} \otimes \tau_{-i/2})(U(p_{\mathcal{H}_0} \otimes \mathbb{1}))$ commutes with $p_{\mathcal{H}_0} \otimes \mathbb{1}$. Let u', $\overline{u}$ be the corresponding (bounded, the former generally not unitary) sub-representations. We have $\mathcal{J}\mathcal{H}_0 \otimes \mathcal{H}_0 \in \mathcal{BR}(\overline{U} \mathbin{⦶} U')$. Let $\zeta_1, \ldots, \zeta_n$ be an orthonormal basis of $\mathcal{H}_0$. Write $u = (u_{ij})_{i,j=1}^n$ with respect to this basis, that is, $u_{ji} = (\omega_{\zeta_i, \zeta_j} \otimes \mathrm{id})(U)$ for every $1 \le i, j \le n$. Since u is unitary, for every state $\omega \in L^1(\mathbb{G})$ there exist $1 \le i, j \le n$ such that $\omega(|u_{ij}|) \ge \omega(|u_{ij}|^2) \ge 1/n$. Hence, (sWM) fails.

Suppose henceforth that (NCAP) does not hold. So, we can assume that u has an invertible transpose or, equivalently, that $u \in D(\mathrm{id} \otimes \tau_{i/2})$ (§3.1). As u is finite dimensional, there is $M < \infty$ such that for all $1 \le i, j \le n$ and $\rho_1, \rho_2 \in L^1(\mathbb{G})$ of norm 1, $y := (\omega_{(\mathrm{id} \otimes \rho_1)(U^*)\zeta_i, (\mathrm{id} \otimes \rho_2)(U^*)\zeta_j} \otimes \mathrm{id})(U) = (\omega_{(\mathrm{id} \otimes \rho_1)(u^*)\zeta_i, (\mathrm{id} \otimes \rho_2)(u^*)\zeta_j} \otimes \mathrm{id})(u)$ satisfies $y \in D(\tau_{i/2})$ and $\|\tau_{i/2}(y)\| \le M$. In conclusion, (WM) fails.

Denote by $\mathbb{H}$ the C^*-algebraic compact quantum group induced by u as in §3.1 and by h the Haar state on $\mathbb{H}$. Notice that h is not necessarily faithful because $\mathbb{H}$ is not necessarily reduced. Nonetheless, there is no ambiguity about the scaling group and unitary antipode by Lemma 3.1. The components of u belong to $C(\mathbb{H})$ by definition, so u, u', $\overline{u}$ are co-representations of $\mathbb{H}$.

We may and do assume that u is irreducible. Hence, there exists a strictly positive matrix $F \in M_{\dim \mathcal{H}_0}$, which we view as an operator on $\mathcal{H}_0$, such that $(\mathrm{id} \otimes \tau_z)(u) = (F^{-iz} \otimes \mathbb{1})\, u\, (F^{iz} \otimes \mathbb{1})$ for every $z \in \mathbb{C}$ [**65**, §7].

Since $\overline{u}_{13} u'_{23}$ is a co-representation of $\mathbb{H}$ on $\mathcal{J}\mathcal{H}_0 \otimes \mathcal{H}_0$, every vector in the image of $(\mathrm{id} \otimes \mathrm{id} \otimes h)(\overline{u}_{13} u'_{23})$ is invariant under $\overline{u}_{13} u'_{23}$ by Lemma 3.13 and thus under $\overline{U} \mathbin{⦶} U'$ (see Lemma 3.7). Fix $0 \ne \zeta \in \mathcal{H}_0$ and consider the invariant vector $\Xi := ((\mathrm{id} \otimes \mathrm{id} \otimes h)(\overline{u}_{13} u'_{23}))(\mathcal{J}\zeta \otimes \zeta)$. To show that it is non-zero, let $\eta \in \mathcal{H}_0$ be such that $x := (\omega_{\eta, \zeta} \otimes \mathrm{id})(u) \ne 0$. From Lemma 3.5, (3.1), we obtain

$$\begin{aligned} \langle \Xi, \mathcal{J}\eta \otimes \eta \rangle &= \langle ((\mathrm{id} \otimes \mathrm{id} \otimes h)(\overline{u}_{13} u'_{23}))(\mathcal{J}\zeta \otimes \zeta), \mathcal{J}\eta \otimes \eta \rangle \\ &= h[(\omega_{\mathcal{J}\zeta, \mathcal{J}\eta} \otimes \mathrm{id})(\overline{u})(\omega_{\zeta, \eta} \otimes \mathrm{id})(u')] = h(R(x^*x)) = h(x^*x). \end{aligned}$$

By [**65**, Proposition 4.2], this number is strictly positive. Therefore, $\Xi \ne 0$, and ((PEB) with $V := \overline{U}$) fails.

Similarly, $\overline{u}_{13} u_{23}$ is also a co-representation of $\mathbb{H}$ on $\mathcal{J}\mathcal{H}_0 \otimes \mathcal{H}_0$. The vector $\Upsilon := ((\mathrm{id} \otimes \mathrm{id} \otimes h)(\overline{u}_{13} u_{23}))(\mathcal{J}\zeta \otimes F^{1/2}\zeta)$ is invariant under $\overline{u}_{13} u_{23}$ and thus under $\overline{U} \mathbin{⦶} U$,

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by Lemma 3.13, and

$$(\mathrm{id}\otimes F^{-1/2})\Upsilon = ((\mathrm{id}\otimes\mathrm{id}\otimes h)(\overline{u}_{13}(\mathbb{1}\otimes F^{-1/2}\otimes\mathbb{1})u_{23}(\mathbb{1}\otimes F^{1/2}\otimes\mathbb{1})))(\mathcal{J}\zeta\otimes\zeta)$$
$$= ((\mathrm{id}\otimes\mathrm{id}\otimes h)(\overline{u}_{13}u'_{23}))(\mathcal{J}\zeta\otimes\zeta) = \Xi.$$

Defining $T_\Xi\in B(\mathcal{H}_0)$ by $T_\Xi\xi := ((\mathcal{J}\xi)^*\otimes\mathrm{id})(\Xi)$, $\xi\in\mathcal{H}_0$, we get as above

$$\langle T_\Xi\xi,\,\xi'\rangle = h((\omega_{\xi',\zeta}\otimes\mathrm{id})(u)^*(\omega_{\xi,\zeta}\otimes\mathrm{id})(u))$$

for all $\xi,\,\xi'\in\mathcal{H}_0$. Thus,

$$\langle T_\Xi F^{-1/2}\xi,\,F^{1/2}\xi'\rangle$$
$$= h((\omega_{\xi',\zeta}\otimes\mathrm{id})(u(F^{1/2}\otimes\mathbb{1}))^*(\omega_{\xi,\zeta}\otimes\mathrm{id})(u(F^{-1/2}\otimes\mathbb{1})))$$
$$= h((\omega_{\xi',F^{1/2}\zeta}\otimes\mathrm{id})((\mathrm{id}\otimes\tau_{-i/2})(u))^*(\omega_{\xi,F^{-1/2}\zeta}\otimes\mathrm{id})((\mathrm{id}\otimes\tau_{i/2})(u))).$$

Taking the original ζ to be an eigenvector of F and using that h is invariant under τ, we conclude that $\langle T_\Xi F^{-1/2}\xi,\,F^{1/2}\xi'\rangle = \langle T_\Xi\xi,\,\xi'\rangle$, namely that T_Ξ and $F^{1/2}$ commute. Thus, defining $B\in B(\mathcal{H})$ to be $F^{1/2}$ on $\mathcal{H}_0$ and the identity on $\mathcal{H}_0^\perp$, we infer that Ξ, Υ are matched, and ((PSE) with $V := \overline{U}$) fails. □

In the particular case of $\mathbb{G}$ with a trivial scaling group (where $U' = U$), the next result was proved by Kyed and Sołtan [**39**, Theorem 2.6], Das and Daws [**13**, Proposition 7.2] and Chen and Ng [**10**, Proposition 3.5].

PROPOSITION 3.15. *For every unitary co-representation V of $\mathbb{G}$, (NCAP) $\Longrightarrow$ (PSE), (PEB) and (NFDS) $\Longrightarrow$ (PE).*

Proof. Let us replace V by $\overline{V}$ for convenience. Fix an anti-unitary $\mathcal{J}_V:\mathcal{K}\to\mathcal{J}_V\mathcal{K}$. Assume that (PE) does not hold, and let $\Xi\in\mathcal{J}_V\mathcal{K}\otimes\mathcal{H}$ be a non-zero invariant vector for $\overline{V}\mathbin{⦶}U'$ in the sense of Definition 3.6. It induces a Hilbert–Schmidt operator $T_\Xi:\mathcal{K}\to\mathcal{H}$ given by $T_\Xi\zeta := ((\mathcal{J}_V\zeta)^*\otimes\mathbb{1})(\Xi)$, $\zeta\in\mathcal{K}$. For all $\zeta\in\mathcal{K}$ and $\xi\in\mathcal{H}$, by invariance of Ξ under $\overline{V}\mathbin{⦶}U'$ (replacing V by $\overline{V}$ in Lemma 3.7) and Lemma 3.5, (3.2) with V instead of U,

$$(\omega_{T_\Xi\zeta,\xi}\otimes\mathrm{id})(U') = (\omega_{(\mathbb{1}\otimes\xi^*)(\Xi),\mathcal{J}_V\zeta}\otimes\mathrm{id})(\overline{V}^*) = (\omega_{\zeta,T_\Xi^*\xi}\otimes\mathrm{id})(V'). \tag{3.4}$$

Formally, this means that $U'(T_\Xi\otimes\mathbb{1}) = (T_\Xi\otimes\mathbb{1})V'$. Since $U' = (\mathrm{id}\otimes\tau_{-i/2})(U)$ and the same for V, and since $\tau_{-i/2}$ is injective, (3.4) holds with U, V in place of U', V', respectively; thus, $U(T_\Xi\otimes\mathbb{1}) = (T_\Xi\otimes\mathbb{1})V$. Hence, the compact, non-zero, positive operator $T_\Xi T_\Xi^*$ over $\mathcal{H}$ intertwines U with itself. Fixing a strictly positive eigenvalue, the associated finite-dimensional spectral subspace $\mathcal{H}_0$ of $T_\Xi T_\Xi^*$ is invariant under U, so that (NFDS) does not hold.

We should now show that under additional assumptions, the restriction u of U to $\mathcal{H}_0$ has an invertible transpose. If (PSE) fails, then there is $\Upsilon\in\mathcal{J}_V\mathcal{K}\otimes\mathcal{H}$ invariant under $\overline{V}\mathbin{⦶}U$ such that Ξ, Υ are matched. As above, the induced Hilbert–Schmidt operator $T_\Upsilon:\mathcal{K}\to\mathcal{H}$ given by $T_\Upsilon\zeta := ((\mathcal{J}_V\zeta)^*\otimes\mathbb{1})(\Upsilon)$, $\zeta\in\mathcal{K}$, satisfies, for all $\zeta\in\mathcal{K}$ and $\xi\in\mathcal{H}$,

$$(\omega_{T_\Upsilon\zeta,\xi}\otimes\mathrm{id})(U) = (\omega_{(\mathbb{1}\otimes\xi^*)(\Upsilon),\mathcal{J}_V\zeta}\otimes\mathrm{id})(\overline{V}^*) = (\omega_{\zeta,T_\Upsilon^*\xi}\otimes\mathrm{id})(V') \tag{3.5}$$

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using Lemma 3.5, (3.2) with V instead of U. Thus, formally, $U(T_\Upsilon \otimes \mathbb{1}) = (T_\Upsilon \otimes \mathbb{1})V'$. As Ξ, Υ are matched, there is, by Lemma A.2, a generally unbounded, injective, selfadjoint operator B over $\mathcal{H}$ commuting with $T_\Xi T_\Xi^*$ such that $T_\Upsilon = BT_\Xi$ (see the paragraph succeeding Definition 3.8). Consequently, B maps $\mathcal{H}_0$ onto itself. If now $\xi_1, \xi_2 \in \mathcal{H}_0$ and $\zeta \in \mathcal{K}$ is such that $T_\Xi \zeta = B^{-1}\xi_1$, then from (3.5) we deduce that

$$(\omega_{\xi_1,\xi_2} \otimes \mathrm{id})(U) = (\omega_{\zeta, T_\Upsilon^* \xi_2} \otimes \mathrm{id})(V') = \tau_{-i/2}[(\omega_{\zeta, T_\Upsilon^* \xi_2} \otimes \mathrm{id})(V)] \in D(\tau_{i/2}).$$

Consequently, $u \in D(\mathrm{id} \otimes \tau_{i/2})$, that is, u^{t} is invertible (see §3.1).

If (PEB) fails, then we may assume that $(\omega_{T_\Xi \zeta, T_\Xi \xi} \otimes \mathrm{id})(U) \in D(\tau_{i/2})$ for all $\zeta, \xi \in \mathcal{K}$. In particular, $u \in D(\mathrm{id} \otimes \tau_{i/2})$ again. This completes the proof. □

Remark 3.16. In the next proof we use the following simple observation. Suppose that $\mathcal{F} \in \mathcal{BR}(\overline{U} \mathbin{⦶} U')$ and $\zeta \in \mathcal{H}$ is such that $\mathcal{J}\zeta \otimes \zeta \in \mathcal{F}$. Write $p_\mathcal{F}$ for the projection of $\mathcal{J}\mathcal{H} \otimes \mathcal{H}$ onto $\mathcal{F}$, $p_{\mathbb{C}\zeta}$ for the projection of $\mathcal{H}$ onto $\mathbb{C}\zeta$ and $Z := \overline{U}_{13}(\mathrm{id} \otimes \mathrm{id} \otimes \tau_{-i/2})(U_{23}(p_\mathcal{F} \otimes \mathbb{1}))$. Then $(\mathrm{id} \otimes \mathrm{id} \otimes \tau_{-i/2})(U_{23}(p_\mathcal{F} \otimes \mathbb{1}))(\mathcal{J}\zeta \otimes \zeta \otimes \eta) = \mathcal{J}\zeta \otimes (\mathrm{id} \otimes \tau_{-i/2})(U(p_{\mathbb{C}\zeta} \otimes \mathbb{1}))(\zeta \otimes \eta)$ for all $\eta \in L^2(\mathbb{G})$ and, therefore, for every $\xi, \xi' \in \mathcal{H}$ and $\omega \in L^1(\mathbb{G})$, we have

$$\begin{aligned}(\omega_{\mathcal{J}\zeta, \mathcal{J}\xi} \otimes \omega_{\zeta,\xi'} \otimes \omega)(Z) &= \omega[(\omega_{\mathcal{J}\zeta,\mathcal{J}\xi} \otimes \mathrm{id})(\overline{U})\tau_{-i/2}((\omega_{\zeta,\xi'} \otimes \mathrm{id})(U))] \\ &= \omega[(\omega_{\mathcal{J}\zeta,\mathcal{J}\xi} \otimes \mathrm{id})(\overline{U})(\omega_{\zeta,\xi'} \otimes \mathrm{id})(U')]\end{aligned}$$

by the definition of U'.

PROPOSITION 3.17.

(a) *Assume that $\mathbb{G}$ is amenable. Then the condition ((PEB) with $V := \overline{U}$), respectively ((PE) with $V := \overline{U}$), implies (WMa), respectively (sWMa).*

(b) *Assume that the scaling group of $\mathbb{G}$ is trivial. Then the condition ((PE) with $V := \overline{U}$) implies (sWM).*

Proof. All implications are proved by contraposition. Hence, we are given $\zeta_1, \ldots, \zeta_n \in \mathcal{H}$ without the respective property. Out of them we obtain a vector Ξ invariant under $\overline{U} \mathbin{⦶} U'$ by averaging, as follows. Let $\Theta := \sum_{i=1}^n \mathcal{J}\zeta_i \otimes \zeta_i$.

Case I. $\mathbb{G}$ is amenable and (sWMa) does not hold. Letting $\mathcal{F}$ be the given element of $\mathcal{BR}(\overline{U} \mathbin{⦶} U')$, we have $\mathcal{J}\zeta_i \otimes \zeta_i \in \mathcal{F}$ for every $1 \le i \le n$, so $\Theta \in \mathcal{F}$. Let $p_\mathcal{F}$ be the projection of $\mathcal{J}\mathcal{H} \otimes \mathcal{H}$ onto $\mathcal{F}$ and $Z := \overline{U}_{13}(\mathrm{id} \otimes \mathrm{id} \otimes \tau_{-i/2})(U_{23}(p_\mathcal{F} \otimes \mathbb{1}))$. Then Z is a (bounded, generally not unitary) co-representation of $\mathbb{G}$ on $\mathcal{J}\mathcal{H} \otimes \mathcal{H}$ commuting with $p_\mathcal{F} \otimes \mathbb{1}$. If $m \in L^\infty(\mathbb{G})^*$ is the given right-invariant mean for $\mathbb{G}$, then $m' := m \circ R$ is a left-invariant mean for $\mathbb{G}$ by (2.1). Let $\Xi := ((\mathrm{id} \otimes \mathrm{id} \otimes m')(Z))\Theta \in \mathcal{F}$. By Lemma 3.13, Ξ is invariant under Z and thus under $\overline{U} \mathbin{⦶} U'$ in the sense of Definition 3.6.

Case II. The scaling group is trivial (so $U' = U$) and (sWM) does not hold. Consider the closed convex set

$$K := \overline{\{((\mathrm{id} \otimes \mathrm{id} \otimes \omega)(\overline{U} \mathbin{⦶} U))\Theta : \omega \text{ is a state in } L^1(\mathbb{G})\}}.$$

Since $\overline{U} \mathbin{⦶} U$ is a unitary co-representation of $\mathbb{G}$, K is invariant under $\overline{U} \mathbin{⦶} U$, because if $\omega_1, \omega_2 \in L^1(\mathbb{G})$ are states, then $(\mathrm{id} \otimes \mathrm{id} \otimes \omega_1)(\overline{U} \mathbin{⦶} U)(\mathrm{id} \otimes \mathrm{id} \otimes \omega_2)(\overline{U} \mathbin{⦶} U) = (\mathrm{id} \otimes \mathrm{id} \otimes (\omega_1 * \omega_2))(\overline{U} \mathbin{⦶} U)$. Let Ξ be the unique element of minimal norm in

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K. Then Ξ is invariant under $\overline{U}\mathbin{⑦}U$ because for every state ω in $L^1(\mathbb{G})$, $((\mathrm{id}\otimes\mathrm{id}\otimes\omega)(\overline{U}\mathbin{⑦}U))\Xi\in K$ has norm at most $\|\Xi\|$. Let $Z:=\overline{U}\mathbin{⑦}U$.

Treating both cases together, we show that $\Xi\neq 0$. Let ω be a state of $L^\infty(\mathbb{G})$. By Remark 3.16 and Lemma 3.5, (3.1), for every $\zeta\in\{\zeta_1,\ldots,\zeta_n\}$ and $\xi\in\mathcal{H}$,

$$\begin{aligned}\langle((\mathrm{id}\otimes\mathrm{id}\otimes\omega)(Z))(\mathcal{J}\zeta\otimes\zeta),\mathcal{J}\xi\otimes\xi\rangle &= \omega[(\omega_{\mathcal{J}\zeta,\mathcal{J}\xi}\otimes\mathrm{id})(\overline{U})(\omega_{\zeta,\xi}\otimes\mathrm{id})(U')]\\ &= (\omega\circ R)[(\omega_{\xi,\zeta}\otimes\mathrm{id})(U)^*(\omega_{\xi,\zeta}\otimes\mathrm{id})(U)].\end{aligned}$$

Writing $\Xi_\omega:=((\mathrm{id}\otimes\mathrm{id}\otimes\omega)(Z))\Theta$, we infer that

$$\langle\Xi_\omega,\Theta\rangle=\sum_{i,j=1}^{n}(\omega\circ R)(|(\omega_{\zeta_j,\zeta_i}\otimes\mathrm{id})(U)|^2). \tag{3.6}$$

By assumption, in Case I, there exist i, j such that $m(|(\omega_{\zeta_j,\zeta_i}\otimes\mathrm{id})(U)|^2)>0$ and hence $\langle\Xi,\Theta\rangle>0$ by setting $\omega:=m'$ in (3.6). In Case II, there is $\varepsilon_0>0$ such that for every state ω in $L^1(\mathbb{G})$ there exist i, j such that $(\omega\circ R)(|(\omega_{\zeta_j,\zeta_i}\otimes\mathrm{id})(U)|^2)\geq\varepsilon_0^2$ and hence $\langle\Xi_\omega,\Theta\rangle\geq\varepsilon_0^2$ by (3.6). Therefore, $\Xi\neq 0$ either way, failing (PE) with $\overline{U}$ for V.

Now suppose that $\mathbb{G}$ is amenable and (WMa) does not hold (with respect to the same $\zeta_1,\ldots,\zeta_n\in\mathcal{H}$). Fix $\zeta,\zeta'\in\mathcal{H}$. To fail (PEB) with $\overline{U}$ for V, we need to establish that $(\omega_{T_\Xi\zeta,T_\Xi\zeta'}\otimes\mathrm{id})(U)\in D(\tau_{i/2})$. Let ω,ω' be states in $L^1(\mathbb{G})$. Write $\rho_i:=\omega\cdot(\omega_{\mathcal{J}\zeta_i,\mathcal{J}\zeta}\otimes\mathrm{id})(\overline{U})\in L^1(\mathbb{G})$, $1\leq i\leq n$. From Remark 3.16, we get

$$\begin{aligned}T_{\Xi_\omega}\zeta &= ((\mathcal{J}\zeta)^*\otimes\mathrm{id})(\Xi_\omega)=\sum_{i=1}^{n}((\mathcal{J}\zeta)^*\otimes\mathbb{1})[((\mathrm{id}\otimes\mathrm{id}\otimes\omega)(Z))(\mathcal{J}\zeta_i\otimes\zeta_i)]\\ &=\sum_{i=1}^{n}((\mathrm{id}\otimes\rho_i)(U'))\zeta_i=\sum_{i=1}^{n}(\mathrm{id}\otimes(\rho_i\circ R))(U^*)\zeta_i.\end{aligned}$$

Similarly, $T_{\Xi_{\omega'}}\zeta'=\sum_{j=1}^{n}(\mathrm{id}\otimes(\rho'_j\circ R))(U^*)\zeta_j$ with suitable $\rho'_j\in L^1(\mathbb{G})$, $1\leq j\leq n$. Let

$$a_{\omega,\omega'}:=(\omega_{T_{\Xi_\omega}\zeta,T_{\Xi_{\omega'}}\zeta'}\otimes\mathrm{id})(U)=\sum_{i,j=1}^{n}(\omega_{(\mathrm{id}\otimes(\rho_i\circ R))(U^*)\zeta_i,(\mathrm{id}\otimes(\rho'_j\circ R))(U^*)\zeta_j}\otimes\mathrm{id})(U).$$

By assumption, $a_{\omega,\omega'}$ belongs to $D(\tau_{i/2})$ and satisfies

$$\|\tau_{i/2}(a_{\omega,\omega'})\|\leq\varepsilon_0^{-1}\sum_{i,j=1}^{n}\|\rho_i\circ R\|\|\rho'_j\circ R\|\leq\varepsilon_0^{-1}\sum_{i,j=1}^{n}\|\zeta_i\|\|\zeta_j\|\|\zeta\|\|\zeta'\|. \tag{3.7}$$

Choose a net (ω_ι) of states in $L^1(\mathbb{G})$ that is w^*-convergent to m'. Hence, $\Xi_{\omega_\iota}\to\Xi$ weakly. By (3.7), $a_{\omega_\iota,\omega_k}$ belongs to $D(\tau_{i/2})$ for every ι,κ and $(\|\tau_{i/2}(a_{\omega_\iota,\omega_k})\|)_{\iota,\kappa}$ is bounded. When κ is fixed, $a_{\omega_\iota,\omega_k}\to a_{\omega_\kappa}:=(\omega_{T_\Xi\zeta,T_{\Xi_{\omega_\kappa}}\zeta'}\otimes\mathrm{id})(U)$ weakly. From Lemma A.4, $a_{\omega_\kappa}\in D(\tau_{i/2})$ and $(\tau_{i/2}(a_{\omega_\kappa}))_\kappa$ is bounded. Now $a_{\omega_\kappa}\to(\omega_{T_\Xi\zeta,T_\Xi\zeta'}\otimes\mathrm{id})(U)$ weakly, and Lemma A.4 is used again to conclude that $(\omega_{T_\Xi\zeta,T_\Xi\zeta'}\otimes\mathrm{id})(U)\in D(\tau_{i/2})$, as desired. □

This completes the proof of Theorem 3.11.

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COROLLARY 3.18. *Let $\mathbb{G}$ be a LCQG with trivial scaling group and let U be a unitary co-representation of $\mathbb{G}$. If U is weakly mixing, then so is $U^{\textcircled{T}n}$ for every $n > 1$.*

Proof. This follows easily from (PE) with $V := \overline{U}$ and with arbitrary V being equivalent. □

4. *Applications*

4.1. *The non-commutative Jacobs–de Leeuw–Glicksberg splitting theorem.* Suppose that a LCQG $\mathbb{G}$ acts on a von Neumann algebra N via an action $\alpha : N \to N \overline{\otimes} L^\infty(\mathbb{G})$ that preserves a faithful normal state θ of N. The *Koopman co-representation* of the dynamical system $(N, \theta, \mathbb{G}, \alpha)$ is the unitary implementation $U \in B(L^2(N, \theta)) \overline{\otimes} L^\infty(\mathbb{G})$ of α, given by

$$(\omega_{\Lambda_\theta(a),\Lambda_\theta(b)} \otimes \mathrm{id})(U) = (\theta \otimes \mathrm{id})((b^* \otimes \mathbb{1})\alpha(a)) \quad \text{or equivalently} \tag{4.1}$$

$$(\omega_{\Lambda_\theta(a),\Lambda_\theta(b)} \otimes \mathrm{id})(U^*) = (\theta \otimes \mathrm{id})(\alpha(b^*)(a \otimes \mathbb{1})) \tag{4.2}$$

for all $a, b \in N$. Since $\Lambda_\theta(\mathbb{1})$ is invariant under U, so is $L_0^2(N, \theta) := L^2(N, \theta) \ominus \mathbb{C}\Lambda_\theta(\mathbb{1})$, and we can apply the above results to the restriction of U to $L_0^2(N, \theta)$. When this restriction is (weakly) mixing, the action α is said to be (weakly) mixing. From Lemma A.1, we obtain $\overline{U} = U$ when using J_θ as $\mathcal{J}$ (meaning that as in the classical setting, 'U is induced by the orthogonal co-representation of the dynamical system'). This simplifies a little most of the conditions for weak mixing.

The next theorem is a fundamental consequence of Theorem 3.11. Runde and the author generalized in [**55**] the Jacobs–de Leeuw–Glicksberg splitting theorem of [**43**, **67**] to state-preserving actions of quantum semigroups on von Neumann algebras. We introduced the notion of *completely almost periodic operators* [**55**, Definition 3.7 and Theorem 4.5], and established, under suitable assumptions, the existence of a conditional expectation E^{CAP} from the von Neumann algebra that was acted on onto its subalgebra consisting of these operators. The kernel of E^{CAP} was conjectured to have a weakly mixing nature, but how to put this in exact terms was left open [**55**, Corollary 4.10 and the preceding paragraph]. This issue is settled in the following result, which is also the complement of [**55**, Corollary 3.14] relating complete almost periodicity to recurrence.

Recall that actions of co-amenable LCQGs are assumed to satisfy (2.3).

THEOREM 4.1. *Let $\mathbb{G}$ be a co-amenable, amenable LCQG that acts on a von Neumann algebra N via an action $\alpha : N \to N \overline{\otimes} L^\infty(\mathbb{G})$ that preserves a faithful normal state θ of N. Denote by U the unitary implementation of α and by N^{CAP} the von Neumann subalgebra of N consisting of all completely almost periodic operators with respect to α, and let $m \in L^\infty(\mathbb{G})^*$ be a left-invariant mean for $\mathbb{G}$. Then the unique θ-preserving conditional expectation E^{CAP} from N onto N^{CAP} has the following property.*

If $a_1, \ldots, a_n \in \ker E^{\mathrm{CAP}}$ are such that some element of $\mathcal{BR}(U \textcircled{T} U')$ contains all vectors $\Lambda_\theta(a_i) \otimes J_\theta \Lambda_\theta(a_i)$, $1 \le i \le n$, then either *for each $1 \le i, j \le n$ we have $m(|(\theta \otimes \mathrm{id})((a_j^* \otimes \mathbb{1})\alpha(a_i))|_{\mathrm{r}}) = 0$* or *for every $\varepsilon > 0$ there exist $1 \le i, j \le n$ and $\rho_1, \rho_2 \in L^1(\mathbb{G})$ of norm 1 such that $y := (\theta \otimes \mathrm{id})[((\mathrm{id} \otimes \rho_2)\alpha(a_j)^* \otimes \mathbb{1})\alpha((\mathrm{id} \otimes \rho_1)\alpha(a_i))]$ satisfies $\|\tau_{i/2}(y)\| > \varepsilon^{-1}$.*

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Proof. The novelty is the weak mixing property of $\ker E^{\mathrm{CAP}}$, the rest being a special case of [**55**, Corollary 4.10]. The set $L^2(N,\theta)^{\mathrm{CAP}}$ of all completely almost periodic vectors with respect to α is a closed subspace of $L^2(N,\theta)$ invariant under the unitary implementation U of α [**55**, Definition 3.7]. The restriction of U to $L^2(N,\theta)\ominus L^2(N,\theta)^{\mathrm{CAP}}$ satisfies (NCAP) by definition, and so Theorem 3.11 applies, and we infer that (WMa) holds. Observe that each $a\in N$ belongs to $\ker E^{\mathrm{CAP}}$ if and only if $\Lambda_\theta(a)\in L^2(N,\theta)\ominus L^2(N,\theta)^{\mathrm{CAP}}$ because Λ_θ intertwines E^{CAP} and the projection of $L^2(N,\theta)$ onto $L^2(N,\theta)^{\mathrm{CAP}}$. Moreover, $L^2(N,\theta)\ominus L^2(N,\theta)^{\mathrm{CAP}}$ is invariant under J_θ. To get the desired result, take $(J_\theta\Lambda_\theta(a_i))_{i=1}^n$ for $(\zeta_i)_{i=1}^n$ in (WMa), use (4.1) and the formulas

$$\begin{aligned}(\omega_{J_\theta\zeta,J_\theta\eta}\otimes\mathrm{id})(U)&=R((\omega_{\eta,\zeta}\otimes\mathrm{id})(U)),\\ J_\theta(\mathrm{id}\otimes\overline{\rho\circ R})(U^*)J_\theta&=(\mathrm{id}\otimes\rho)(U)\end{aligned}\qquad(\text{for all }\zeta,\eta\in L^2(N,\theta),\ \rho\in L^1(\mathbb{G}))$$

derived from Lemma A.1 to get $|(\omega_{J_\theta\Lambda_\theta(a_j),J_\theta\Lambda_\theta(a_i)}\otimes\mathrm{id})(U)|=R(|(\theta\otimes\mathrm{id})((a_j^*\otimes\mathbb{1})\alpha(a_i))|_{\mathrm{r}})$ and

$$\begin{aligned}&(\omega_{(\mathrm{id}\otimes\overline{\rho_2\circ R})(U^*)J_\theta\Lambda_\theta(a_j),(\mathrm{id}\otimes\overline{\rho_1\circ R})(U^*)J_\theta\Lambda_\theta(a_i)}\otimes\mathrm{id})(U)\\ &\quad=R((\omega_{(\mathrm{id}\otimes\rho_1)(U)\Lambda_\theta(a_i),(\mathrm{id}\otimes\rho_2)(U)\Lambda_\theta(a_j)}\otimes\mathrm{id})(U))\\ &\quad=R\{(\theta\otimes\mathrm{id})[((\mathrm{id}\otimes\rho_2)\alpha(a_j)^*\otimes\mathbb{1})\alpha((\mathrm{id}\otimes\rho_1)\alpha(a_i))]\}.\end{aligned}$$

Then notice that $m\circ R$ is right invariant, and use the commutativity of τ and R. □

4.2. *Relation to (weak) mixing of inclusions in crossed products by discrete quantum groups.* A notion that is tightly related to weak mixing is the *weak asymptotic homomorphism property* of a masa in a II_1-factor, which turned out to be equivalent to (strong) singularity of the masa, and provided many examples of such masas (see Sinclair and Smith [**56**], Robertson *et al* [**53**] and Sinclair *et al* [**57**]). This was generalized and studied further by Jolissaint and Stalder [**32**] and by Cameron *et al* [**9**]. In particular, they proved that in certain cases, when a discrete group G acts on a finite von Neumann algebra N and preserves a trace, the inclusion of $\mathrm{VN}(G)$ in the crossed product $N\rtimes G$ is (weakly) mixing in appropriate senses if and only if the action is (weakly) mixing ([**32**, Propositions 2.2 and 3.6] and [**9**, Proposition 1.1]). In this subsection we show that, in the very general setting of discrete quantum group actions, one direction of each of these implications holds with respect to our definition of weakly mixing actions and the definition of mixing in [**15**]. First, we extend the notion of (weakly) mixing inclusions of von Neumann algebras [**9**, p. 344] beyond the finite case.

Definition 4.2. Let $A\subseteq B$ be an inclusion of von Neumann algebras with a faithful normal conditional expectation E from B onto A.

- The inclusion $A\subseteq B$ is called *E-weakly mixing* if for every finite subset $F\subseteq B$ there exists a sequence $(v_n)_{n=1}^\infty$ of unitaries in A such that
$$(\text{for all } x,\ y\in F)\quad E(xv_ny)-E(x)v_nE(y)\xrightarrow[n\to\infty]{}0\quad\text{strongly.}$$
- The inclusion $A\subseteq B$ is called *E-mixing* if there exists a subspace C of B, dense in the bounded $*$-strong topology, such that for every sequence $(v_n)_{n=1}^\infty$ of unitaries in A converging weakly to zero, we have
$$(\text{for all } x\in C,\ y\in B)\quad E(xv_ny)-E(x)v_nE(y)\xrightarrow[n\to\infty]{}0\quad\text{strongly.}\qquad(4.3)$$

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Remark 4.3. If B is a finite von Neumann algebra, so that the adjoint map is strongly continuous on bounded sets, (4.3) holds as it is if and only if it holds for all $x, y \in B$.

Let $\mathbb{G}$ be a discrete quantum group acting on a von Neumann algebra N via an action α that preserves a faithful normal state θ of N. Recall that the *crossed product* $N \rtimes_\alpha \mathbb{G}$ [**61**, pp. 434–435] is the von Neumann subalgebra of $N \overline{\otimes} B(\ell^2(\mathbb{G}))$ generated by $\alpha(N)$ and $\mathbb{C}\mathbb{1} \otimes L^\infty(\hat{\mathbb{G}})'$. We view $L^\infty(\hat{\mathbb{G}})'$ as embedded in $N \rtimes_\alpha \mathbb{G}$. Since $C := \operatorname{span}\{(\mathbb{1} \otimes \hat{x}')\alpha(a) : a \in N, \hat{x}' \in L^\infty(\hat{\mathbb{G}})'\}$ is dense in $N \rtimes_\alpha \mathbb{G}$ in the bounded $*$-strong topology [**61**, Lemma 3.3 and its proof] and $(\theta \otimes \mathrm{id}_{B(\ell^2(\mathbb{G}))})[(\mathbb{1} \otimes \hat{x}')\alpha(a)] = \theta(a)\hat{x}'$ for every $a \in N$, $\hat{x}' \in L^\infty(\hat{\mathbb{G}})'$, the map $E_{L^\infty(\hat{\mathbb{G}})'} := \theta \otimes \mathrm{id}_{B(\ell^2(\mathbb{G}))}$ is a faithful normal conditional expectation from $N \rtimes_\alpha \mathbb{G}$ onto $L^\infty(\hat{\mathbb{G}})'$. Denoting by $\tilde{\theta}$ the dual weight of θ [**61**, Definition 3.1], this n.s.f. weight on $N \rtimes_\alpha \mathbb{G}$ is actually a state as θ is a state and $\mathbb{G}$ is discrete, and $E_{L^\infty(\hat{\mathbb{G}})'}$ is just the unique $\tilde{\theta}$-preserving conditional expectation from $N \rtimes_\alpha \mathbb{G}$ onto $L^\infty(\hat{\mathbb{G}})'$.

Remark 4.4. We deal with $L^\infty(\hat{\mathbb{G}})'$ rather than $L^\infty(\hat{\mathbb{G}})$ because α is a *right* action. This is a mere technical matter.

PROPOSITION 4.5. *In the above setting, we have the following implications:*

(a) *if the inclusion $L^\infty(\hat{\mathbb{G}})' \subseteq N \rtimes_\alpha \mathbb{G}$ is $E_{L^\infty(\hat{\mathbb{G}})'}$-weakly mixing, then α satisfies (NFDS);*

(b) *if α is mixing, then the inclusion $L^\infty(\hat{\mathbb{G}})' \subseteq N \rtimes_\alpha \mathbb{G}$ is $E_{L^\infty(\hat{\mathbb{G}})'}$-mixing.*

Proof. Recall that $\Lambda_{\hat{\varphi}}(\mathbb{1}) = \Lambda_\varphi(p)$ and thus, for every $b \in N$, (2.4) implies that $\alpha(b)(\Lambda_\theta(\mathbb{1}) \otimes \Lambda_{\hat{\varphi}}(\mathbb{1})) = (\Lambda_\theta \otimes \Lambda_\varphi)(\alpha(b)(\mathbb{1} \otimes p)) = (\Lambda_\theta \otimes \Lambda_\varphi)(b \otimes p) = (b \otimes \mathbb{1})(\Lambda_\theta(\mathbb{1}) \otimes \Lambda_{\hat{\varphi}}(\mathbb{1}))$. Hence, for $a, b \in N$ and $\hat{x} \in L^\infty(\hat{\mathbb{G}})$,

$$\begin{aligned} E_{L^\infty(\hat{\mathbb{G}})'}[\alpha(a)(\mathbb{1} \otimes \hat{J}\hat{x}\hat{J})\alpha(b)]\Lambda_{\hat{\varphi}}(\mathbb{1}) &= (\theta \otimes \mathrm{id}_{B(\ell^2(\mathbb{G}))})[\alpha(a)(\mathbb{1} \otimes \hat{J}\hat{x}\hat{J})\alpha(b)]\Lambda_{\hat{\varphi}}(\mathbb{1}) \\ &= (\theta \otimes \mathrm{id}_{B(\ell^2(\mathbb{G}))})[\alpha(a)(\mathbb{1} \otimes \hat{J}\hat{x}\hat{J})(b \otimes \mathbb{1})]\Lambda_{\hat{\varphi}}(\mathbb{1}) \\ &= (\theta \otimes \mathrm{id}_{B(\ell^2(\mathbb{G}))})[\alpha(a)(b \otimes \mathbb{1})]\hat{J}\Lambda_{\hat{\varphi}}(\hat{x}). \qquad (4.4) \end{aligned}$$

(a) Suppose that α fails (NFDS). Denote by U the unitary implementation of α, and let u be a finite-dimensional sub-representation of U on a subspace of $L^2(N, \theta) \ominus \mathbb{C}\Lambda_\theta(\mathbb{1})$ spanned by the orthonormal set $\{\zeta_1, \ldots, \zeta_n\}$. Write $u_{ij} := (\omega_{\zeta_j, \zeta_i} \otimes \mathrm{id})(U)$ $(1 \le i, j \le n)$. Let $\varepsilon > 0$. Pick $a_1, \ldots, a_n \in N$ such that $\|\zeta_i - \Lambda_\theta(a_i)\| < \varepsilon$ and $\theta(a_i) = 0$ for $1 \le i \le n$. Then for every $\hat{v} \in L^\infty(\hat{\mathbb{G}})$ of norm at most 1 and $1 \le i, j \le n$, we have from (4.4) and (4.2) that

$$\begin{aligned} E_{L^\infty(\hat{\mathbb{G}})'}[\alpha(a_i^*)(\mathbb{1} \otimes \hat{J}\hat{v}\hat{J})\alpha(a_j)]\Lambda_{\hat{\varphi}}(\mathbb{1}) &= (\omega_{\Lambda_\theta(a_j), \Lambda_\theta(a_i)} \otimes \mathrm{id})(U^*)\hat{J}\Lambda_{\hat{\varphi}}(\hat{v}) \\ &\approx_{(2+\varepsilon)\varepsilon} (\omega_{\zeta_j, \zeta_i} \otimes \mathrm{id})(U^*)\hat{J}\Lambda_{\hat{\varphi}}(\hat{v}) = u_{ji}^*\hat{J}\Lambda_{\hat{\varphi}}(\hat{v}). \end{aligned}$$

Fix $1 \le j \le n$. Applying u_{ji} to the right-hand side and summing for $i = 1, \ldots, n$, we get $\hat{J}\Lambda_{\hat{\varphi}}(\hat{v})$ by the unitarity of u. Thus, if $(\hat{v}_m)$ are unitaries in $L^\infty(\hat{\mathbb{G}})$ with $E_{L^\infty(\hat{\mathbb{G}})'}[\alpha(a_i^*)(\mathbb{1} \otimes \hat{J}\hat{v}_m\hat{J})\alpha(a_j)] \longrightarrow_{m\to\infty} 0$ strongly for every $1 \le i, j \le n$, then, as $\hat{J}\Lambda_{\hat{\varphi}}(\hat{v}_m)$ is a unit vector for every m, we obtain $1 \le n(2 + \varepsilon)\varepsilon$. Taking ε small enough yields a contradiction. Therefore, $L^\infty(\hat{\mathbb{G}})' \subseteq N \rtimes_\alpha \mathbb{G}$ cannot be $E_{L^\infty(\hat{\mathbb{G}})'}$-weakly mixing.

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(b) Assume that α is mixing, that is, $(\theta \otimes \mathrm{id})[\alpha(a)(b \otimes \mathbb{1})] \in c_0(\mathbb{G})$ for every $a, b \in \ker\theta$ (see (4.1)). To prove that $L^\infty(\hat{\mathbb{G}})' \subseteq N \rtimes_\alpha \mathbb{G}$ is $E_{L^\infty(\hat{\mathbb{G}})'}$-mixing, fix a bounded sequence $(\hat{v}_n)$ in $L^\infty(\hat{\mathbb{G}})$ converging weakly to zero, not necessarily of unitaries. Since $\ell^\infty(\mathbb{G})$ is an ℓ^∞-direct sum of (finite-dimensional) matrix algebras, $c_0(\mathbb{G})$ is the corresponding c_0-direct sum and $\ell^\infty(\mathbb{G})$ is in standard form on $\ell^2(\mathbb{G})$, every $z \in c_0(\mathbb{G})$ is compact, so that $z\hat{J}\Lambda_{\hat{\varphi}}(\hat{v}_n) \longrightarrow_{n\to\infty} 0$. In particular, if $a, b \in \ker\theta$, then from (4.4) we get

$$E_{L^\infty(\hat{\mathbb{G}})'}[\alpha(a)(\mathbb{1} \otimes \hat{J}\hat{v}_n\hat{J})\alpha(b)]\Lambda_{\hat{\varphi}}(\mathbb{1}) = (\theta \otimes \mathrm{id})[\alpha(a)(b \otimes \mathbb{1})]\hat{J}\Lambda_{\hat{\varphi}}(\hat{v}_n) \xrightarrow[n\to\infty]{} 0.$$

Hence, $E_{L^\infty(\hat{\mathbb{G}})'}[\alpha(a)(\mathbb{1} \otimes \hat{J}\hat{v}_n\hat{J})\alpha(b)] \to 0$ strongly. Thus, (4.3) holds for every $x \in C$ and $y \in C^*$. As C is dense in $N \rtimes_\alpha \mathbb{G}$ in the $*$-strong topology, (4.3) holds for every $x \in C$ and $y \in N \rtimes_\alpha \mathbb{G}$. □

Remark 4.6. What we established in the proof of (b) is formally stronger than mixing since the operators $(\hat{v}_n)$ are not assumed to be unitary. However, this is not surprising in light of [**9**, Theorem 3.3]. It is interesting to check this result for general von Neumann algebras.

5. *Open questions*

The results of §3 may open the door to solving several questions.

In his celebrated proof of Szemerédi's theorem, Furstenberg [**29**] established a multiple recurrence result that significantly extended the Poincaré recurrence theorem, and led to a breadth of works on related convergence questions. Weak mixing was a key idea in Furstenberg's paper. Beyers *et al* [**8**] and Austin *et al* [**2**] considered the non-commutative case, namely in which the object that is acted on is a finite von Neumann algebra, obtaining very interesting partial results.

Question 5.1. Is it possible to generalize results of [**8**, **2**], and in particular [**2**, Theorem 1.17], to actions of LCQGs?

The main ingredients in the proof of [**2**, Theorem 1.17] include the von Neumann algebraic Jacobs–de Leeuw–Glicksberg theorem of [**43**] and the classical van der Corput estimate. The starting point for answering Question 5.1 can be Theorem 4.1 and a possible generalization of the van der Corput estimate (for locally compact groups, this was done in [**8**]). Note that even the case of groups other than $\mathbb{Z}$, or actions on infinite von Neumann algebras, is still unknown.

Two close questions concern formal strengthenings of the definition of property (T) for LCQGs, whose statements or proofs are related to weak mixing. In both, some separability assumption needs to be made. Property (T) for discrete quantum groups was introduced by Fima [**27**] and studied further by Kyed [**38**]. For general LCQGs, see [**15**, §6] and [**10**].

Question 5.2. Does the Connes–Weiss theorem [**12**], [**4**, Theorem 6.3.4], characterizing property (T) in terms of strong ergodicity of measure-preserving ergodic actions, generalize to LCQGs?

Generalizing the original proof, for example, would require a modification of [**10**, Proposition 3.2, (T2)] combined with the construction of Vaes [**62**, Proposition 3.1] that

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produces, from a unitary co-representation 'arising from an orthogonal co-representation', an action on a free Araki–Woods factor preserving the free quasi-free state.

Question 5.3. Does the Bekka–Valette characterization [**5**, Theorem 1], [**4**, Theorem 2.12.9] generalize to LCQGs? That is, are the following conditions equivalent for a LCQG $\mathbb{G}$?

(a) $\mathbb{G}$ has property (T).

(b) Every unitary co-representation of $\mathbb{G}$ that has almost-invariant vectors is not (strictly) weakly mixing.

In [**44**], Okayasu *et al* introduced strict mixing of bimodules (correspondences) over von Neumann algebras. To every unitary co-representation of a LCQG $\mathbb{G}$ they associated an $L^\infty(\hat{\mathbb{G}})$-$L^\infty(\hat{\mathbb{G}})$ bimodule, and vice versa, and established that mixing of an element of either of these classes implies mixing of the associated element of the other [**44**, Proposition 14].

Question 5.4. Can one introduce a notion of weak mixing of bimodules over von Neumann algebras (see e.g. Peterson and Sinclair [**46**, Definition 2.3]) that would be consistent as above with that of weak mixing for unitary co-representations of LCQGs?

A. *Appendix. Auxiliary results*

The following simple results, which are probably known, are used in the paper. For a lucid account of unbounded operators on Hilbert spaces, see the classic of Dunford and Schwartz [**18**, Ch. XII]. Recall that over a given Hilbert space, a bounded operator b is said to *commute* with an unbounded normal operator n when $bn \subseteq nb$; equivalently, when b commutes with all spectral projections of n (see Fuglede [**28**]).

LEMMA A.1. *Let α be an action of a LCQG $\mathbb{G}$ on a von Neumann algebra N that preserves a faithful normal state θ of N. Then the unitary implementation $U \in B(L^2(N,\theta)) \mathbin{\overline{\otimes}} L^\infty(\mathbb{G})$ of α satisfies*

$$(J_\theta \otimes \hat{J})U(J_\theta \otimes \hat{J}) = U^*.$$

Proof. This is another by-product of the proof of [**55**, Theorem A.1] (see also the lines that precede it). Indeed, for $a \in D(\sigma^\theta_{-i})$ and $b \in D(\sigma^\theta_i)$, we have

$$(\omega_{\Lambda_\theta(a),\Lambda_\theta(b)} \otimes \mathrm{id})(U)\hat{\nabla}^{1/2}\hat{J} \subseteq \hat{\nabla}^{1/2}\hat{J}(\omega_{\Lambda_\theta(a^*),\Lambda_\theta(\sigma^\theta_i(b)^*)} \otimes \mathrm{id})(U^*)$$

(see [**55**, equation succeeding (A.4) and (A.1), (A.2)]; we use $\hat{J}$, $\hat{\nabla}$ in place of I, L). This was used to prove that U commutes with $\nabla_\theta \otimes \hat{\nabla}^{-1}$, which implies that

$$\hat{\nabla}^{1/2}(\omega_{\Lambda_\theta(\sigma^\theta_{-i/2}(a)),\Lambda_\theta(\sigma^\theta_{i/2}(b))} \otimes \mathrm{id})(U)\hat{J} = \hat{\nabla}^{1/2}\hat{J}(\omega_{\Lambda_\theta(a^*),\Lambda_\theta(\sigma^\theta_i(b)^*)} \otimes \mathrm{id})(U^*)$$

on $D(\hat{\nabla}^{1/2}\hat{J}) = D(\hat{\nabla}^{-1/2})$. Since $\hat{\nabla}^{1/2}$ is strictly positive, we get

$$(\omega_{\Lambda_\theta(\sigma^\theta_{-i/2}(a)),\Lambda_\theta(\sigma^\theta_{i/2}(b))} \otimes \mathrm{id})(U)\hat{J} = \hat{J}(\omega_{\Lambda_\theta(a^*),\Lambda_\theta(\sigma^\theta_i(b)^*)} \otimes \mathrm{id})(U^*).$$

Recall that for every $c \in D(\sigma^\theta_{i/2})$, $J_\theta\Lambda_\theta(c) = J_\theta\nabla_\theta^{1/2}\nabla_\theta^{-1/2}\Lambda_\theta(c) = J_\theta\nabla_\theta^{1/2}\Lambda_\theta(\sigma^\theta_{i/2}(c)) = \Lambda_\theta(\sigma^\theta_{-i/2}(c^*))$. As a result,

$$\hat{J}(\omega_{J_\theta\Lambda_\theta(a^*),J_\theta\Lambda_\theta(\sigma^\theta_i(b)^*)} \otimes \mathrm{id})(U)\hat{J} = (\omega_{\Lambda_\theta(a^*),\Lambda_\theta(\sigma^\theta_i(b)^*)} \otimes \mathrm{id})(U^*).$$

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The density of $\{\Lambda_\theta(a^*) : a \in D(\sigma^\theta_{-i})\}$ and $\{\Lambda_\theta(\sigma^\theta_i(b)^*) : b \in D(\sigma^\theta_i)\}$ in $L^2(N, \theta)$ gives the result. □

LEMMA A.2. *Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $a, c \in B(\mathcal{H}_1, \mathcal{H}_2)$. Then a^*c, ac^* are selfadjoint,* $\ker a \subseteq \ker c$ *and* $\ker a^* = \ker c^*$ *if and only if there exists a (generally unbounded) injective selfadjoint operator b over $\mathcal{H}_2$ commuting with aa^* such that $c = ba$.*

Proof. Define $b(a\zeta) := c\zeta$, $\zeta \in \mathcal{H}_1$. By the selfadjointness of a^*c, the (generally unbounded) operator b is symmetric, and thus closable, over the Hilbert space $\mathcal{K} := \overline{\operatorname{Im} a} = \overline{\operatorname{Im} c} \subseteq \mathcal{H}_2$. As a^*c, ac^* are selfadjoint, one calculates that $aa^*b \subseteq baa^*$ and so $aa^*\bar{b} \subseteq \bar{b}aa^*$. If $\eta \in D(b^*)$, then for all $\zeta \in \mathcal{H}_1$, $\langle c\zeta, \eta\rangle = \langle ba\zeta, \eta\rangle = \langle \zeta, a^*b^*\eta\rangle$, so that $c^*\eta = a^*b^*\eta$. If now $b^*\eta = \pm i\eta$, then $\|c^*\eta\|^2 = \mp i\langle c^*\eta, a^*\eta\rangle = \mp i\langle ac^*\eta, \eta\rangle \in i\mathbb{R}$ as ac^* is selfadjoint. Hence, $\eta \in \ker c^*$, so that $\eta = 0$ as $\eta \in \mathcal{K}$. Therefore, b is essentially selfadjoint by von Neumann's decomposition of $D(b^*)$ [**18**, Lemma XII.4.10]. Since the range of b is dense in $\mathcal{K}$, its closure $\overline{b}$ is injective. The desired b can now be defined as $\overline{b}$ on $\mathcal{K}$ and $\mathbb{1}$ on $\mathcal{K}^\perp$.

Conversely, if such b exists, then clearly $\ker a = \ker c$ and a^*c is selfadjoint. We have $\overline{\operatorname{Im} c} = \overline{\operatorname{Im} ba} \supseteq \overline{\operatorname{Im} baa^*} = \overline{\operatorname{Im} aa^*b} = \overline{\operatorname{Im} aa^*} = \overline{\operatorname{Im} a}$ since b commutes with aa^* and is injective. Also, for the same reasons, for each $\zeta \in \mathcal{H}_1$ there exists a sequence (η_n) in $D(b)$ such that $ca^*\eta_n = baa^*\eta_n = aa^*b\eta_n \to a\zeta$, implying that $\overline{\operatorname{Im} a} \subseteq \overline{\operatorname{Im} c}$. Thus, $\ker c^* = \ker a^*$. Furthermore, $ca^* = baa^*$ is selfadjoint as aa^*, b are commuting selfadjoint operators. □

LEMMA A.3. *Let T be a closed, densely defined operator on a Hilbert space $\mathcal{H}$. If (η_ι), a net in $D(T)$, and $\eta \in \mathcal{H}$ are such that $\eta_\iota \to \eta$ weakly in $\mathcal{H}$ and $(T\eta_\iota)$ is bounded by C, then $\eta \in D(T)$ and $\|T\eta\| \le C$.*

Proof. For every $\zeta \in D(T^*)$, $\langle T^*\zeta, \eta\rangle = \lim_\iota \langle \zeta, T\eta_\iota\rangle$ has absolute value at most $C\|\zeta\|$. Since $T = T^{**}$, we infer that $\eta \in D(T)$ and $\|T\eta\| \le C$. □

LEMMA A.4. *Let τ be an action of $\mathbb{R}$ on a von Neumann algebra M and let $z \in \mathbb{C}$. If a net (a_ι) in $D(\tau_z)$ converges weakly to $a \in M$ and $(\tau_z(a_\iota))$ is bounded by C, then $a \in D(\tau_z)$ and $\|\tau_z(a)\| \le C$.*

Proof. Representing M standardly on a Hilbert space $\mathcal{H}$, there exists a (generally unbounded) strictly positive operator T on $\mathcal{H}$ such that $\tau_t = \mathrm{Ad}(T^{it})$ for all $t \in \mathbb{R}$ (by [**30**, Corollary 3.6] and Stone's theorem). Let $\zeta \in D(T^{-iz})$. For every ι, let $\eta_\iota := a_\iota T^{-iz}\zeta = T^{-iz}\tau_z(a_\iota)\zeta$. Then $\eta_\iota \to aT^{-iz}\zeta$ weakly and the net $(T^{iz}\eta_\iota) = (\tau_z(a_\iota)\zeta)$ is bounded by $C\|\zeta\|$. By the previous lemma, $aT^{-iz}\zeta \in D(T^{iz})$ and $\|T^{iz}aT^{-iz}\zeta\| \le C\|\zeta\|$. Consequently, $b := \overline{T^{iz}aT^{-iz}}$ satisfies $\|b\| \le C$. It is standard that $b \in M$. For every $\zeta \in D(T^{-iz})$, we have $aT^{-iz}\zeta = T^{-iz}T^{iz}aT^{-iz}\zeta = T^{-iz}b\zeta$, whence $a \in D(\tau_z)$ and $\tau_z(a) = b$. □

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REFERENCES

[**1**] T. Austin. *Lecture Notes on Ergodic Theory*. http://cims.nyu.edu/∼tim.
[**2**] T. Austin, T. Eisner and T. Tao. Nonconventional ergodic averages and multiple recurrence for von Neumann dynamical systems. *Pacific J. Math.* **250**(1) (2011), 1–60.
[**3**] E. Bédos and L. Tuset. Amenability and co-amenability for locally compact quantum groups. *Internat. J. Math.* **14**(8) (2003), 865–884.
[**4**] B. Bekka, P. de la Harpe and A. Valette. *Kazhdan's Property (T) (New Mathematical Monographs, 11)*. Cambridge University Press, Cambridge, 2008.
[**5**] M. E. B. Bekka and A. Valette. Kazhdan's property (T) and amenable representations. *Math. Z.* **212**(2) (1993), 293–299.
[**6**] V. Bergelson and A. Gorodnik. Weakly mixing group actions: a brief survey and an example. *Modern Dynamical Systems and Applications*. Cambridge University Press, Cambridge, 2004, pp. 3–25.
[**7**] V. Bergelson and J. Rosenblatt. Mixing actions of groups. *Illinois J. Math.* **32**(1) (1988), 65–80.
[**8**] C. Beyers, R. Duvenhage and A. Ströh. The Szemerédi property in ergodic W^*-dynamical systems. *J. Operator Theory* **64**(1) (2010), 35–67.
[**9**] J. Cameron, J. Fang and K. Mukherjee. Mixing subalgebras of finite von Neumann algebras. *New York J. Math.* **19** (2013), 343–366.
[**10**] X. Chen and C.-K. Ng. Property T for locally compact quantum groups. *Internat. J. Math.* **26**(3) (2015), 1550024, 13 pp.
[**11**] I. Ciorănescu and L. Zsidó. Analytic generators for one-parameter groups. *Tôhoku Math. J.* (2) **28**(3) (1976), 327–362.
[**12**] A. Connes and B. Weiss. Property T and asymptotically invariant sequences. *Israel J. Math.* **37**(3) (1980), 209–210.
[**13**] B. Das and M. Daws. Quantum Eberlein compactifications and invariant means. *Indiana Univ. Math. J.*, to appear, *Preprint*, 2014, arXiv:1406.1109v2.
[**14**] M. Daws. Remarks on the quantum Bohr compactification. *Illinois J. Math.* **57**(4) (2013), 1131–1171.
[**15**] M. Daws, P. Fima, A. Skalski and S. White. The Haagerup property for locally compact quantum groups. *J. Reine Angew. Math.*, to appear, *Preprint*, doi:10.1515/crelle-2013-0113.
[**16**] M. Daws, P. Kasprzak, A. Skalski and P. M. Sołtan. Closed quantum subgroups of locally compact quantum groups. *Adv. Math.* **231**(6) (2012), 3473–3501.
[**17**] K. de Leeuw and I. Glicksberg. Applications of almost periodic compactifications. *Acta Math.* **105** (1961), 63–97.
[**18**] N. Dunford and J. T. Schwartz. *Linear Operators. Part II. Spectral Theory (Wiley Classics Library)*. John Wiley, New York, 1988.
[**19**] R. Duvenhage. Bergelson's theorem for weakly mixing C^*-dynamical systems. *Studia Math.* **192**(3) (2009), 235–257.
[**20**] R. Duvenhage. Ergodicity and mixing of W*-dynamical systems in terms of joinings. *Illinois J. Math.* **54**(2) (2010), 543–566.
[**21**] R. Duvenhage. Relatively independent joinings and subsystems of W*-dynamical systems. *Studia Math.* **209**(1) (2012), 21–41.
[**22**] R. Duvenhage and F. Mukhamedov. Relative ergodic properties of C*-dynamical systems. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **17**(1) (2014), 1450005, 26 pp.
[**23**] H. A. Dye. On the ergodic mixing theorem. *Trans. Amer. Math. Soc.* **118** (1965), 123–130.
[**24**] E. G. Effros and Z.-J. Ruan. Discrete quantum groups. I. The Haar measure. *Internat. J. Math.* **5**(5) (1994), 681–723.
[**25**] E. G. Effros and Z.-J. Ruan. *Operator Spaces (London Mathematical Society Monographs, New Series, 23)*. Clarendon Press, Oxford University Press, New York, 2000.
[**26**] M. Enock and J.-M. Schwartz. *Kac Algebras and Duality of Locally Compact Groups*. Springer, Berlin, 1992.
[**27**] P. Fima. Kazhdan's property T for discrete quantum groups. *Internat. J. Math.* **21**(1) (2010), 47–65.

https://doi.org/10.1017/etds.2015.115 Published online by Cambridge University Press

[**28**] B. Fuglede. A commutativity theorem for normal operators. *Proc. Natl. Acad. Sci. USA* **36**(1) (1950), 35–40.
[**29**] H. Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. Anal. Math.* **31** (1977), 204–256.
[**30**] U. Haagerup. The standard form of von Neumann algebras. *Math. Scand.* **37**(2) (1975), 271–283.
[**31**] K. Jacobs. Ergodentheorie und fastperiodische Funktionen auf Halbgruppen. *Math. Z.* **64** (1956), 298–338.
[**32**] P. Jolissaint and Y. Stalder. Strongly singular MASAs and mixing actions in finite von Neumann algebras. *Ergod. Th. & Dynam. Sys.* **28**(6) (2008), 1861–1878.
[**33**] B. O. Koopman and J. von Neumann. Dynamical systems of continuous spectra. *Proc. Natl. Acad. Sci. USA* **18**(3) (1932), 255–263.
[**34**] U. Krengel. *Ergodic Theorems (de Gruyter Studies in Mathematics, 6)*. Walter de Gruyter, Berlin, 1985.
[**35**] J. Kustermans. Locally compact quantum groups in the universal setting. *Internat. J. Math.* **12**(3) (2001), 289–338.
[**36**] J. Kustermans and S. Vaes. Locally compact quantum groups. *Ann. Sci. Éc. Norm. Supér.* (4) **33**(6) (2000), 837–934.
[**37**] J. Kustermans and S. Vaes. Locally compact quantum groups in the von Neumann algebraic setting. *Math. Scand.* **92**(1) (2003), 68–92.
[**38**] D. Kyed. A cohomological description of property (T) for quantum groups. *J. Funct. Anal.* **261**(6) (2011), 1469–1493.
[**39**] D. Kyed and P. M. Sołtan. Property (T) and exotic quantum group norms. *J. Noncommut. Geom.* **6**(4) (2012), 773–800.
[**40**] A. Maes and A. Van Daele. Notes on compact quantum groups. *Nieuw Arch. Wiskd.* (4) **16**(1–2) (1998), 73–112.
[**41**] T. Masuda, Y. Nakagami and S. L. Woronowicz. A C^*-algebraic framework for quantum groups. *Internat. J. Math.* **14**(9) (2003), 903–1001.
[**42**] R. Meyer, S. Roy and S. L. Woronowicz. Homomorphisms of quantum groups. *Münster J. Math.* **5** (2012), 1–24.
[**43**] C. P. Niculescu, A. Ströh and L. Zsidó. Noncommutative extensions of classical and multiple recurrence theorems. *J. Operator Theory* **50**(1) (2003), 3–52.
[**44**] R. Okayasu, N. Ozawa and R. Tomatsu. Haagerup approximation property via bimodules, *Preprint*, 2015, arXiv:1501.06293v2.
[**45**] J. Peterson. *Lecture Notes on Ergodic Theory*. http://www.math.vanderbilt.edu/∼peters10/teaching/Spring 2011/math390Blecturenotes.html.
[**46**] J. Peterson and T. Sinclair. On cocycle superrigidity for Gaussian actions. *Ergod. Th. & Dynam. Sys.* **32**(1) (2012), 249–272.
[**47**] S. Popa. On a class of type II_1 factors with Betti numbers invariants. *Ann. of Math. (2)* **163**(3) (2006), 809–899.
[**48**] S. Popa. Some rigidity results for non-commutative Bernoulli shifts. *J. Funct. Anal.* **230**(2) (2006), 273–328.
[**49**] S. Popa. Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups. I. *Invent. Math.* **165**(2) (2006), 369–408.
[**50**] S. Popa. Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups. II. *Invent. Math.* **165**(2) (2006), 409–451.
[**51**] S. Popa. Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups. *Invent. Math.* **170**(2) (2007), 243–295.
[**52**] S. Popa and S. Vaes. Strong rigidity of generalized Bernoulli actions and computations of their symmetry groups. *Adv. Math.* **217**(2) (2008), 833–872.
[**53**] G. Robertson, A. M. Sinclair and R. R. Smith. Strong singularity for subalgebras of finite factors. *Internat. J. Math.* **14**(3) (2003), 235–258.
[**54**] V. Runde. Characterizations of compact and discrete quantum groups through second duals. *J. Operator Theory* **60**(2) (2008), 415–428.
[**55**] V. Runde and A. Viselter. Ergodic theory for quantum semigroups. *J. Lond. Math. Soc.* (2) **89**(3) (2014), 941–959.
[**56**] A. M. Sinclair and R. R. Smith. Strongly singular masas in type II_1 factors. *Geom. Funct. Anal.* **12**(1) (2002), 199–216.
[**57**] A. M. Sinclair, R. R. Smith, S. A. White and A. Wiggins. Strong singularity of singular masas in II_1 factors. *Illinois J. Math.* **51**(4) (2007), 1077–1084.
[**58**] P. M. Sołtan. Quantum Bohr compactification. *Illinois J. Math.* **49**(4) (2005), 1245–1270.
[**59**] Ş. Strătilă. *Modular Theory in Operator Algebras*. Abacus Press, Tunbridge Wells, UK, 1981.

https://doi.org/10.1017/etds.2015.115 Published online by Cambridge University Press

[**60**] M. Takesaki. *Theory of Operator Algebras. II (Encyclopaedia of Mathematical Sciences, 125)*. Springer, Berlin, 2003.
[**61**] S. Vaes. The unitary implementation of a locally compact quantum group action. *J. Funct. Anal.* **180**(2) (2001), 426–480.
[**62**] S. Vaes. Strictly outer actions of groups and quantum groups. *J. Reine Angew. Math.* **578** (2005), 147–184.
[**63**] A. Van Daele. Discrete quantum groups. *J. Algebra* **180**(2) (1996), 431–444.
[**64**] S. L. Woronowicz. A remark on compact matrix quantum groups. *Lett. Math. Phys.* **21**(1) (1991), 35–39.
[**65**] S. L. Woronowicz. Compact quantum groups. *Symétries Quantiques (Les Houches, 1995)*. North-Holland, Amsterdam, 1998, pp. 845–884.
[**66**] L. Zsidó. Spectral and ergodic properties of the analytic generators. *J. Approx. Theory* **20**(1) (1977), 77–138.
[**67**] L. Zsidó. Splitting noncommutative dynamical systems in almost periodic and weakly mixing parts. Presented at *Int. Conf. on Operator Theory (Timişoara, Romania, 2012)*.

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