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Entropy dimension for deterministic walks in random sceneries

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Abstract. Entropy dimension is an entropy-type quantity which takes values in [0, 1] and classifies different levels of intermediate growth rate of complexity for dynamical systems. In this paper, we consider the complexity of skew products of irrational rotations with Bernoulli systems, which can be viewed as deterministic walks in random sceneries, and show that this class of models can have any given entropy dimension by choosing suitable rotations for the base system.

Key words: complexity, deterministic walk, entropy dimension, sub-exponential growth
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1. *Introduction*

Complexity for dynamical systems describes the growth rate of orbits under the iteration of the actions. We measure the complexity via the iterated partitions in a probability measure space and via the iterated open covers in a topological setting. If a system has positive entropy, then it has exponential growth rate. Positive entropy systems have been studied and much understood in recent decades and they are known to be chaotic and unpredictable. Hence the word complexity is used mostly for entropy zero systems. Many examples of polynomial growth rate, such as irrational rotations, adding machines, interval exchange maps and some billiards on polygons are well known. However, the systems of sub-exponential but super-polynomial growth rate (called intermediate growth rate) are not well understood yet, and hence are wide open to further development.

In the study of more general group actions, for example, $\mathbb{Z}^2$-systems, the sub-exponential growth arises naturally and some of their properties have been investigated in

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[19, 20] and also in [15]. The notions of slow entropy or entropy dimension have been introduced to distinguish different levels of intermediate growth rate of zero entropy systems [7, 12, 13, 16]. For $\mathbb{Z}$-actions, several definitions of entropy dimensions for topological and measurable dynamical systems are introduced and investigated in [3, 7, 9, 10, 11, 12].

In [6], a class of uniformly recurrent infinite words with given intermediate complexity was constructed, and it was shown in [3] that the systems generated by the words are uniquely ergodic. Moreover, it was shown in [3] that entropy dimension does not have the property of variational principle. In [9, 10], systems of uniform entropy dimension (u.d., for short) were constructed. They have the property that all finite open covers have the same topological entropy dimension which corresponds to topological property of uniformly positive entropy (u.p.e.) or all finite measurable partitions have the same metric entropy dimension which corresponds to the metric property of K-mixing. Since the study of systems with intermediate complexity is still at the beginning stage, we have many interesting and basic questions. Apart from the properties mentioned above, very little is known in the way of general results and the developments have focused on examples. Some examples related to intermediate complexity can be found in [14, 18], but all of these examples are via constructive methods to have desired properties. Hence we would like to provide examples of intermediate growth rate from more familiar models.

Aaronson showed the relative complexity of intermediate growth rate for a class of random walks in random sceneries [1]. However, the model itself has positive entropy (exponential growth rate) since the base is Bernoulli. We consider a class of models which we call deterministic walks in random sceneries (DWRS, for short). The deterministic walks are chosen to be the irrational rotations of the unit circle $\mathbf{T}$ and the random sceneries are chosen to be the orbits of Bernoulli systems (or, more generally, positive entropy systems). One can refer to [2, 4, 8] for DWRS on $\mathbb{Z}$.

In our DWRS models, the unit circle of the base is divided into two subintervals of the same length. If, after a rotation, a base point lies in the first interval, it will walk forward along its scenery; it will walk backward otherwise. Due to the construction, at time n, the point can only walk $o(n)$ distance away from the starting point along the orbit of the scenery system, and hence the entropy should be zero. The scenery supplies the 'randomness' for this skew-product system and the base controls the growth rate of the complexity. By choosing suitable irrational rotation number α of the base, we will show that our DWRS model has intermediate growth rate. More precisely, for any $\tau \in [0, 1]$, there exists α such that the skew-product system has the upper entropy dimension τ.

We should note that there are several ways to give the definitions of entropy dimensions. In this paper, we proceed with those appearing in [3, 9, 10, 11] via entropy generating sequences. Due to the choice of taking lim sup or lim inf in the definitions of entropy dimensions, we have upper entropy dimensions and lower entropy dimensions. It is shown in [9, 10] that both of them are isomorphism invariants. However, it is not hard to construct non-isomorphic systems of the same upper entropy dimension. In most of the constructive examples, we can the increase the upper dimension by adding more independencies and we can also decrease the lower dimension by adding more repetitions. In fact, making the upper and lower dimensions agree demands more careful choice of parameters at the successive steps [10, 11, 12]. We remark that this is unlike computing the entropy of a

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system, which can never increase in longer blocks (names). When we consider factor maps between systems, the so-called localization theory in dynamics or the u.d. property, the upper entropy dimension appears to carry more meaning [9, 10]. Hence, in this paper, when we talk about entropy dimension, it refers to the upper entropy dimension.

The paper is organized as follows. In §2, we give the definitions of entropy dimensions via entropy generating sequences in both topological settings and measure-theoretic settings. Together with a brief review on the best approximation and continued fraction expansion of irrationals, we will also introduce the construction and some preliminaries of the DWRS models. In §3, we will discuss some recurrence properties of the irrational rotation and, in §§4 and 5, we will show the existence of any given topological and metric entropy dimensions by choosing the rotations properly.

2. *Preliminaries*

2.1. *Entropy dimension.* Let $S = \{s_1 < s_2 < \cdots\}$ be an increasing sequence of integers. We define the *upper dimension* and the *lower dimension* of the sequence S by

$$\overline{D}(S) = \limsup_{n\to\infty} \frac{\log n}{\log s_n}$$

and

$$\underline{D}(S) = \liminf_{n\to\infty} \frac{\log n}{\log s_n},$$

respectively. Equivalently,

$$\overline{D}(S) = \inf\left\{\tau \geq 0 : \limsup_{n\to\infty} \frac{n}{(s_n)^\tau} = 0\right\} = \sup\left\{\tau \geq 0 : \limsup_{n\to\infty} \frac{n}{(s_n)^\tau} = \infty\right\}$$

and

$$\underline{D}(S) = \inf\left\{\tau \geq 0 : \liminf_{n\to\infty} \frac{n}{(s_n)^\tau} = 0\right\} = \sup\left\{\tau \geq 0 : \liminf_{n\to\infty} \frac{n}{(s_n)^\tau} = \infty\right\}.$$

Clearly, $0 \leq \underline{D}(S) \leq \overline{D}(S) \leq 1$. When $\overline{D}(S) = \underline{D}(S) = \tau$, we say that the sequence S has dimension τ. For example, the dimension of the sequence $\{1, 4, 9, \ldots, n^2, \ldots\}$ is $\frac{1}{2}$ and, more generally, the dimension of the sequence $\{n^t\}$ (t is a positive integer) is $1/t$.

By a topological dynamical system (TDS, for short) we mean a pair (X, T), where X is a compact metric space and T is a continuous onto self-map on X. Let (X, T) be a TDS and let $\mathcal{U}$ be a finite open cover of X. We say that an increasing sequence of positive integers $S = \{s_1 < s_2 < \cdots\}$ is an *entropy generating sequence* of $\mathcal{U}$ if

$$\liminf_{n\to\infty} \frac{1}{n} \log N\left(\bigvee_{i=1}^{n} T^{-s_i}\mathcal{U}\right) > 0,$$

where $N(\mathcal{V})$ denotes the minimal cardinality of subcovers of the open cover $\mathcal{V}$ of X.

Then the *(upper) entropy dimension* of $\mathcal{U}$ is defined by

$$\overline{D}(T, \mathcal{U}) = \sup\{\overline{D}(S) : S \text{ is an entropy generating sequence of } \mathcal{U}\}$$

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and the *(upper) topological entropy dimension* of the TDS (X, T) is defined by

$$\overline{D}(X, T) = \sup_{\mathcal{U}} \overline{D}(T, \mathcal{U}),$$

where the supremum is taken over all the finite open covers $\mathcal{U}$ of X.

Remark 2.1. The following results will be used for the computation of topological entropy dimension which appeared in [9].

(1) There are alternative equivalent definitions for $\overline{D}(T, \mathcal{U})$: that is,

$$\begin{aligned}\overline{D}(T, \mathcal{U}) &= \inf\left\{\tau \geq 0 : \limsup_{n\to\infty} \frac{1}{n^\tau} N\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right) = 0\right\} \\ &= \sup\left\{\tau \geq 0 : \limsup_{n\to\infty} \frac{1}{n^\tau} N\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right) = \infty\right\}.\end{aligned}$$

(2) If $\mathcal{U}$ is a generating open cover, that is, $\lim_{n\to\infty} \operatorname{diam}(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}) = 0$, then

$$\overline{D}(X, T) = \overline{D}(T, \mathcal{U}).$$

Similarly, we can define entropy dimension for measurable dynamical systems. By a measurable dynamical system (MDS, for short), we mean a quadruple $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a probability space and T is a measurable transformation on X that preserves μ. Let $(X, \mathcal{B}, \mu, T)$ be an MDS and let $\mathcal{P}$ be a finite measurable partition of X. An increasing sequence of positive integers $S = \{s_1 < s_2 < \cdots\}$ is called an *entropy generating sequence* of $\mathcal{P}$ (with respect to μ) if

$$\liminf_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=1}^{n} T^{-s_i}\mathcal{P}\right) > 0.$$

Then the *(upper) entropy dimension* of $\mathcal{P}$ is defined by

$$\overline{D}_\mu(T, \mathcal{P}) = \sup\{\overline{D}(S) : S \text{ is an entropy generating sequence of } \mathcal{P} \text{ (w.r.t } \mu)\}.$$

The (upper) entropy dimension of the MDS $(X, \mathcal{B}, \mu, T)$ is defined by

$$\overline{D}_\mu(X, T) = \sup_{\mathcal{P}} \overline{D}_\mu(T, \mathcal{P}),$$

where the supremum is taken over all the finite measurable partitions $\mathcal{P}$ of X. It is shown in [10] that $\overline{D}_\mu(X, T)$ equals $\overline{D}_\mu(T, \mathcal{P})$ when $\mathcal{P}$ is a generating partition.

2.2. *DWRS.* Let $\mathbf{T} = \mathbb{R}/\mathbb{Z}$ be the unit circle and let $\alpha \in (0, 1)$ be an irrational number. Let $(\mathbf{T}, \alpha)$ be the rotation by α on the unit circle. For any $\theta \in \mathbf{T}$, we define a bi-infinite $\{-1, 1\}$ sequence $z = z(\theta) = (\ldots, z_{-1}, z_0, z_1 \ldots)$ by

$$z_n = \begin{cases} 1 & \text{if } \theta + n\alpha \bmod 1 \in \left[0, \frac{1}{2}\right), \\ -1 & \text{if } \theta + n\alpha \bmod 1 \in \left[\frac{1}{2}, 1\right). \end{cases}$$

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Let $Z = cl\{\sigma^n z(0) : n \in \mathbb{Z}\}$, where σ is the left-shift map. Then (Z, σ) forms a TDS which is called a Sturmian flow of type $(\alpha, \frac{1}{2})$ (see [21, (2.53)]). Since α and $\frac{1}{2}$ are rationally independent, (Z, σ) is minimal and uniquely ergodic. Moreover, (Z, σ) is an almost one-to-one extension of the irrational rotation $(\mathbf{T}, \alpha)$. Let φ be the factor map between (Z, σ) and $(\mathbf{T}, \alpha)$. It is known that $\varphi^{-1}(\theta)$ has exactly two points if $\theta \in Orb(0) \cup Orb(\frac{1}{2})$ and that $\varphi^{-1}(\theta) = z(\theta)$ otherwise.

Let $\mathcal{B}_{\mathbf{T}}$ and $\mathcal{B}_Z$ be the Borel σ-algebra of $\mathbf{T}$ and Z, respectively. Let $\mathbf{m}$ and μ be the unique ergodic invariant probability measures of $(\mathbf{T}, \alpha)$ and (Z, σ), respectively. Then $\mathbf{m}$ is the Haar–Lebesgue measure of $\mathbf{T}$ and, moreover, from the measure-theoretic point of view, the MDSs $(\mathbf{T}, \mathcal{B}_{\mathbf{T}}, \mathbf{m}, \alpha)$ and $(Z, \mathcal{B}_Z, \mu, \sigma)$ are conjugate.

Sometimes we will not distinguish between $\mathbf{T}$ and Z in this paper.

Let (Y, S) be an invertible TDS. We define the map $T : Z \times Y \to Z \times Y$ by

$$T(z, y) = (\sigma z, S^{z_0} y),$$

for $z = (\ldots, z_{-1} z_0 z_1 \cdots) \in Z$. That is, for any point $(z, y) \in Z \times Y$, if z_0 is 1, we then push y forward to Sy; otherwise we pull y backward to $S^{-1}y$. Hence we have constructed a skew-product system $(Z \times Y, T)$. For the measure-theoretic case, for an invertible MDS $(Y, \mathcal{D}, \nu, S)$, this skew-product system can also be viewed as a deterministic walk along the orbits of $(Y, \mathcal{D}, \nu, S)$.

For simplicity, we let $\mathfrak{Y}$ denote either the invertible TDS (Y, S) or the invertible MDS $(Y, \mathcal{D}, \nu, S)$. We denote by $\mathfrak{X} = \mathfrak{X}_{\alpha,\mathfrak{Y}}$ either the TDS $(Z \times Y, T)$ or the MDS $(Z \times Y, \mathcal{B}_Z \otimes \mathcal{D}, \mu \times \nu, T)$ and call it a deterministic walk with rotation α along $\mathfrak{Y}$. In particular, when $\mathfrak{Y}$ is a Bernoulli system (or a positive entropy system), we call $\mathfrak{X}$ a DWRS.

Remark 2.2. Since $(\mathbf{T}, \mathcal{B}_{\mathbf{T}}, \mathbf{m}, \alpha)$ and $(Z, \mathcal{B}_Z, \mu, \sigma)$ are conjugate, sometimes $\mathfrak{X}$ is also viewed as $(\mathbf{T} \times Y, \mathcal{B}_{\mathbf{T}} \otimes \mathcal{D}, \mathbf{m} \times \nu, T)$. In this case, T also denotes the corresponding skew-product map on $\mathbf{T} \times Y$.

It is easy to see that the following propositions hold and we omit the proofs.

Proposition 2.3. $\mathfrak{X}$ is minimal (respectively, transitive or ergodic) if and only if $\mathfrak{Y}$ is.

Proposition 2.4. For every $\alpha \in \mathbf{T}$ and any system $\mathfrak{Y}$, the entropy of $\mathfrak{X}$ is zero.

Let $z = (\ldots, z_{-1} z_0 z_1 \cdots) \in Z$. For any $m, n \in \mathbb{N}$ with $m \le n$, denote

$$\omega(z, m, n) = \sum_{i=m}^{n} z_i, \ \ \omega(z, 1, n) = \omega(z, n).$$

For $\theta \in \mathbf{T}$, denote

$$\omega(\theta, m, n) = \omega(z(\theta), m, n), \ \ \omega(\theta, n) = \omega(z(\theta), n) = \omega(\theta, 1, n)$$

and

$$\omega(n) = \omega(0, n) = \omega(z(0), n).$$

If $\varphi(z)$ does not belong to the orbits of 0 and $\frac{1}{2}$, then $\omega(z, k) = \omega(\varphi(z), k)$. Otherwise, $|\omega(z, k) - \omega(\varphi(z), k)| \le 2$.

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Denote

$$M(\theta, n) = \max_{1\le \ell \le n} \omega(\theta, \ell), \quad m(\theta, n) = \min_{1\le \ell \le n} \omega(\theta, \ell)$$

and

$$M(z, n) = \max_{1\le \ell \le n} \omega(z, \ell), \quad m(z, n) = \min_{1\le \ell \le n} \omega(z, \ell).$$

If $\varphi(z)$ does not belong to the orbits of 0 and $\frac{1}{2}$, then $M(z, n) = M(\varphi(z), n)$ and $m(z, n) = m(\varphi(z), n)$. Otherwise, $|M(z, n) - M(\varphi(z), n)| \le 2$ and $|m(z, n) - m(\varphi(z), n)| \le 2$.

Let $\theta \in \mathbf{T}$ and $z = z(\theta) = (\ldots, z_{-1}z_0z_1 \cdots) \in Z$. Recall that $T(z, y) = (\sigma z, S^{z_0} y)$. Hence, for $i \ge 0$,

$$\begin{aligned} T^i(z, y) &= (\sigma^i z, S^{\omega(z,0,i-1)} y), \\ T^{-i}(z, y) &= (\sigma^{-i} z, S^{-\omega(\sigma^{-i} z,0,i-1)} y), \end{aligned} \tag{2.1}$$

where $\omega(z, 0, -1) := 0$.

2.3. *The best approximation and continued fraction expansion of irrationals.* Fix an irrational number $\alpha \in (0, 1)$. The growth of $\omega(n)$ depends on the recurrent properties of α. Here we collect some basic facts on the continued fraction expansion of irrationals (see Khinchin's classic book [17] for details).

For a real number β, let $\|\beta\| = \min\{\{\beta\}, 1 - \{\beta\}\}$, where $\{\beta\}$ denotes the fractional part of β.

Let $[a_0; a_1, a_2, \ldots]$ be the continued fraction expansion of α and let $p_n/q_n = [a_0; a_1, a_2, \ldots, a_n]$ be the continued fraction approximation of α. Then

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \cdots < \alpha < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1}$$

and

$$q_n = a_n q_{n-1} + q_{n-2}, \tag{2.2}$$

where $q_{-1} = 0$ and $q_{-2} = 1$. Since α is chosen from $(0, 1)$, we have that $a_0 = 0$ and $q_0 = 1$. Moreover, p_n and q_n have the following properties:

$$\|q_n\alpha\| = \begin{cases} \{q_n\alpha\} & \text{if } n = 2k, \\ 1 - \{q_n\alpha\} & \text{if } n = 2k + 1; \end{cases} \tag{2.3}$$

$$\|q_{n-1}\alpha\| < \|k\alpha\| \quad \text{for all } k < q_n, k \ne q_{n-1}; \tag{2.4}$$

$$\frac{1}{q_n(q_n + q_{n+1})} \le (-1)^n \left(\alpha - \frac{p_n}{q_n}\right) \le \frac{1}{q_n q_{n+1}}. \tag{2.5}$$

3. *Some recurrence properties of* $(\mathbf{T}, \alpha)$

Let α be a fixed irrational in $[0, 1)$. In this section, we will discuss the growth of $\omega(\theta, n)$ for $\theta \in \mathbf{T}$, which will be related to the complexity of our DWRS models. First, we study

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the growth of $\omega(n)$. For convenience, we will identify $\mathbf{T}$ with the interval $[0, 1)$. When we talk about some point $\ell\alpha$, it should be understood as $\ell\alpha$ mod 1 due to the context. And when we talk about some interval, it should be understood as an arc of $\mathbf{T}$. For each $n \in \mathbb{N}$, let $Q_n = \{k\alpha : 1 \le k \le q_n\} \subset \mathbf{T}$.

The following two propositions will show that the points in Q_n have 'almost equi-distribution' properties on $\mathbf{T}$ with respect to any translation of the partition $\{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$.

Proposition 3.1. Let p_n/q_n be the continued fraction approximation of α. Then

$$\omega(q_n) = \begin{cases} 0 & \text{if } q_n = 2r, \\ 1 & \text{if } q_n = 2r+1 \text{ and } n = 2k, \\ -1 & \text{if } q_n = 2r+1 \text{ and } n = 2k+1. \end{cases}$$

Proof. We will just prove the case when $n = 2k$ since a parallel argument holds for $n = 2k + 1$.

We note that $\|q_n\alpha\| = \{q_n\alpha\}$.

For $1 \le \ell \le q_n - 1$, if $\ell\alpha \in (0, \frac{1}{2})$, then $-\ell\alpha \in (\frac{1}{2}, 1)$. Hence $(q_n - \ell)\alpha \in (\frac{1}{2} + \|q_n\alpha\|, 1 + \|q_n\alpha\|)$. Since no element in Q_n belongs to $[0, \|q_n\alpha\|)$, we know that $(q_n - \ell)\alpha \in (\frac{1}{2} + \|q_n\alpha\|, 1)$. Likewise, if $\ell\alpha \in (\frac{1}{2} + \|q_n\alpha\|, 1)$, then $(q_n - \ell)\alpha \in (0, \frac{1}{2})$. Moreover, $\ell\alpha \in (\frac{1}{2}, \frac{1}{2} + \|q_n\alpha\|)$ if and only if $(q_n - \ell)\alpha \in (\frac{1}{2}, \frac{1}{2} + \|q_n\alpha\|)$.

If $q_n = 2r$, then $r\alpha = \frac{1}{2} + \frac{1}{2}\|q_n\alpha\| \in (\frac{1}{2}, \frac{1}{2} + \|q_n\alpha\|)$. Note that $q_n\alpha \in [0, \frac{1}{2})$, and hence $\omega(q_n) = 0$. If $q_n = 2r + 1$, then no element of the set Q_n belongs to $(\frac{1}{2}, \frac{1}{2} + \|q_n\alpha\|)$ since, otherwise, there will be two elements of Q_n belonging to this interval, which contradicts the fact that p_n/q_n is the best approximation of α. Hence $\omega(q_n) = 1$. □

Proposition 3.2. For $\theta \in \mathbf{T}$, the following hold.

(1) If $\omega(q_n) = 0$, then $\omega(\theta, q_n) = 0$ or ± 2.
(2) If $\omega(q_n) = \pm 1$, then $\omega(\theta, q_n) = \pm 1$ or ± 3.
(3) If $q_n = 2r + 1$ and $kq_n < q_{n+1}$, then $|\omega(kq_n)| \le k$.

Proof. (1) Suppose $\omega(q_n) = 0$. If $\theta = 0$ or $\frac{1}{2}$, then, obviously, $\omega(\theta, q_n) = 0$. Now we treat the case for $0 < \theta < \frac{1}{2}$; the case for $\frac{1}{2} < \theta < 1$ is similar. Then

$$\begin{aligned} \omega(\theta, q_n) &= \#\big((\theta + Q_n) \cap \big[0, \tfrac{1}{2}\big)\big) - \#\big((\theta + Q_n) \cap \big[\tfrac{1}{2}, 1\big)\big) \\ &= \#\big(Q_n \cap \big([-\theta, 0) \cup \big[0, \tfrac{1}{2} - \theta\big)\big)\big) - \#\big(Q_n \cap \big(\big[\tfrac{1}{2} - \theta, \tfrac{1}{2}\big) \cup \big[\tfrac{1}{2}, -\theta\big)\big)\big) \\ &= \#(Q_n \cap [-\theta, 0)) + \#\big(Q_n \cap \big[0, \tfrac{1}{2}\big)\big) - \#\big(Q_n \cap \big[\tfrac{1}{2} - \theta, \tfrac{1}{2}\big)\big) \\ &\quad - \#\big(Q_n \cap \big[\tfrac{1}{2} - \theta, \tfrac{1}{2}\big)\big) - \#\big(Q_n \cap \big[\tfrac{1}{2}, 1\big)\big) + \#(Q_n \cap [-\theta, 0)). \end{aligned}$$

Since $\omega(q_n) = \#(Q_n \cap [0, \frac{1}{2})) - \#(Q_n \cap [\frac{1}{2}, 1)) = 0$,

$$\omega(\theta, q_n) = 2\big(\#(Q_n \cap [-\theta, 0)) - \#\big(Q_n \cap \big[\tfrac{1}{2} - \theta, \tfrac{1}{2}\big)\big)\big). \tag{3.1}$$

From Proposition 3.1, $\omega(q_n) = 0$ if and only if $q_n = 2r$ for some r. As before, we will just consider this for the case when $n = 2k$. Note that, in this case, $\|q_n\alpha\| = \{q_n\alpha\}$ and $r\alpha = \frac{1}{2} + \frac{1}{2}\|q_n\alpha\|$. Hence $Q_n \cap [\frac{1}{2} - \frac{1}{2}\|q_n\alpha\|, \frac{1}{2} + \frac{1}{2}\|q_n\alpha\|) = \emptyset$.

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For $0 < \ell \le r$,

$$\begin{aligned} \ell\alpha \in [-\theta, 0) &\Leftrightarrow (\ell + r)\alpha \in [\tfrac{1}{2} - \theta + \tfrac{1}{2}\|q_n\alpha\|, \tfrac{1}{2} + \tfrac{1}{2}\|q_n\alpha\|) \\ &\Leftrightarrow (\ell + r)\alpha \in \big[\tfrac{1}{2} - \theta + \tfrac{1}{2}\|q_n\alpha\|, \tfrac{1}{2}\big). \end{aligned} \tag{3.2}$$

For $r < \ell \le q_n$,

$$\begin{aligned} \ell\alpha \in [-\theta, 0) &\Leftrightarrow (\ell - r)\alpha \in \big[\tfrac{1}{2} - \theta - \tfrac{1}{2}\|q_n\alpha\|, \tfrac{1}{2} - \tfrac{1}{2}\|q_n\alpha\|\big) \\ &\Leftrightarrow (\ell - r)\alpha \in \big[\tfrac{1}{2} - \theta - \tfrac{1}{2}\|q_n\alpha\|, \tfrac{1}{2}\big). \end{aligned} \tag{3.3}$$

Notice that $[\frac{1}{2} - \theta - \frac{1}{2}\|q_n\alpha\|, \frac{1}{2} - \theta + \frac{1}{2}\|q_n\alpha\|)$ contains at most one point of Q_n (since the length of this interval is $\|q_n\alpha\|$). Together with (3.1), (3.2) and (3.3), we conclude that

$$\omega(\theta, q_n) = \begin{cases} 2 & \text{if } (\ell - r)\alpha \in \big[\frac{1}{2} - \theta - \frac{1}{2}\|q_n\alpha\|, \frac{1}{2} - \theta\big) \text{ for some } r < \ell \le q_n, \\ -2 & \text{if } (\ell + r)\alpha \in \big[\frac{1}{2} - \theta, \frac{1}{2} - \theta + \frac{1}{2}\|q_n\alpha\|\big) \text{ for some } 0 < \ell \le r, \\ 0 & \text{otherwise.} \end{cases}$$

(2) We just consider the case for $\omega(q_n) = 1$; the case for $\omega(q_n) = -1$ is similar.

By Proposition 3.1, if $\omega(q_n) = 1$, then $n = 2k$ and $q_n = 2r + 1$.

It is clear that $\omega(\theta, q_n) = 1$ if $\theta = 0$ and $\omega(\theta, q_n) = -1$ if $\theta = \frac{1}{2}$. Now assume that $0 < \theta < \frac{1}{2}$. Similarly to (3.1),

$$\omega(\theta, q_n) = 2\big(\#(Q_n \cap [-\theta, 0)) - \#\big(Q_n \cap \big[\tfrac{1}{2} - \theta, \tfrac{1}{2}\big)\big)\big) + 1. \tag{3.4}$$

Assume that $\frac{1}{2} \in (s_1\alpha, s_2\alpha)$, where $s_1\alpha, s_2\alpha \in Q_n$ and $(s_1\alpha, s_2\alpha) \cap Q_n = \emptyset$. If $s_2 \le q_n - q_{n-1}$, then $(s_2 + q_{n-1})\alpha \in Q_n$ and no element of Q_n can lie between $(s_2 + q_{n-1})\alpha$ and $s_2\alpha$. Hence $s_1 = s_2 + q_{n-1}$ and $|s_2\alpha - s_1\alpha| = \|q_{n-1}\alpha\|$. If $s_2 > q_n - q_{n-1}$, then $(s_2 + q_{n-1} - q_n)\alpha \in Q_n$ and no element of Q_n can lie between $(s_2 + q_{n-1} - q_n)\alpha$ and $s_2\alpha$. Hence $s_1 = s_2 + q_{n-1} - q_n$ and $|s_2\alpha - s_1\alpha| = \|q_{n-1}\alpha\| + \|q_n\alpha\|$. So $(-\theta + s_1\alpha, -\theta + s_2\alpha)$ contains at most two elements in Q_n.

For $0 < \ell \le q_n - s_2$, since $Q_n \cap [\frac{1}{2}, s_2\alpha) = \emptyset$,

$$\begin{aligned} \ell\alpha \in [-\theta, 0) &\Leftrightarrow (\ell + s_2)\alpha \in [-\theta + s_2\alpha, s_2\alpha) \\ &\Leftrightarrow (\ell + s_2)\alpha \in [-\theta + s_2\alpha, \tfrac{1}{2}). \end{aligned} \tag{3.5}$$

For $q_n - s_2 < \ell \le q_n$, since no element of Q_n lies between $s_2\alpha - \|q_n\alpha\|$ and $\frac{1}{2}$,

$$\begin{aligned} \ell\alpha \in [-\theta, 0) &\Leftrightarrow (\ell + s_2 - q_n)\alpha \in [-\theta + s_2\alpha - \|q_n\alpha\|, s_2\alpha - \|q_n\alpha\|) \\ &\Leftrightarrow (\ell + s_2 - q_n)\alpha \in [-\theta + s_2\alpha - \|q_n\alpha\|, \tfrac{1}{2}). \end{aligned} \tag{3.6}$$

In the proof of Proposition 3.1 (the last paragraph), we have shown that Q_n does not intersect the interval $(\frac{1}{2}, \frac{1}{2} + \|q_n\alpha\|)$ when $q_n = 2r + 1$. So $s_2\alpha - \|q_n\alpha\| \ge \frac{1}{2}$. Noticing that $[-\theta + \frac{1}{2}, -\theta + s_2\alpha)$ contains at most two elements of Q_n and that $[-\theta + \frac{1}{2}, -\theta + s_2\alpha - \|q_n\alpha\|)$ contains at most one element of Q_n, together with (3.4), (3.5) and (3.6), we conclude that $\omega(\theta, q_n) = \pm 1$ or -3.

If $\frac{1}{2} < \theta < 1$, then $\omega(\theta, q_n) = -\omega(\theta - \frac{1}{2}, q_n)$, and hence $\omega(\theta, q_n) = \pm 1$ or 3.

(3) If $q_n = 2r + 1$ and $n = 2t$, that is, $\omega(q_n) = 1$, then $\|q_n\alpha\| = \{q_n\alpha\}$. We now order the points in Q_n on $\mathbf{T}$ by $0 < q_n\alpha < \cdots < s_1\alpha < \frac{1}{2} < s_2\alpha < \cdots < q_{n-1}\alpha < 1$. Since

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$kq_n < q_{n+1}$, the points in $\{\ell\alpha\}_{\ell=1}^{kq_n}$ should be ordered as

$$\begin{aligned}
0 < q_n\alpha < 2q_n\alpha < \cdots < kq_n\alpha \\
\cdots \\
< s_1\alpha < (s_1+q_n)\alpha < \cdots < (s_1+(k-1)q_n)\alpha \\
< s_2\alpha < (s_2+q_n)\alpha < \cdots < (s_2+(k-1)q_n)\alpha \\
\cdots \\
< q_{n-1}\alpha < (q_{n-1}+q_n)\alpha < \cdots < (q_{n-1}+(k-1)q_n)\alpha < 1.
\end{aligned}$$

Owing to the place where $\frac{1}{2}$ lies, obviously, $|\omega(kq_n)| \le k$.

Similarly, when $q_n = 2r+1$ and $n = 2t+1$, we also have $|\omega(kq_n)| \le k$. □

The following proposition gives estimates for $m(\theta, q_n)$ and $M(\theta, q_n)$.

Proposition 3.3. Let $\theta \in \mathbf{T}$. Then we have the following estimation.

(1) For $q_n < t \le q_{n+1}$, $M(\theta, t) \le M(\theta, q_n) + 3\lceil t/q_n \rceil$.
(2) For $q_n < t \le q_{n+1}$, $m(\theta, t) \ge m(\theta, q_n) - 3\lceil t/q_n \rceil$.
(3) $\max\{M(0, q_{n+1}), |m(0, q_{n+1})|\} \ge |\omega(q_n)| \lfloor q_{n+1}/6q_n \rfloor$.

Proof. For any $0 < \ell \le t$, we write $\ell = \ell_0 + kq_n$, where $0 \le \ell_0 < q_n$ and $0 \le k \le \lfloor t/q_n \rfloor$. Then

$$\begin{aligned}
\omega(\theta, \ell) &= \omega(\theta, \ell_0) + \omega(\theta, \ell_0+1, \ell_0+q_n) + \cdots + \omega(\theta, \ell_0+1+(k-1)q_n, \ell_0+kq_n) \\
&= \omega(\theta, \ell_0) + \omega(\theta+\ell_0\alpha, q_n) + \cdots + \omega(\theta+(\ell_0+(k-1)q_n)\alpha, q_n).
\end{aligned}$$

Hence, by Proposition 3.2,

$$m(\theta, q_n) - 3k \le \omega(\theta, \ell) \le M(\theta, q_n) + 3k.$$

This proves (1) and (2).

Now we prove (3). Obviously, (3) is true if $\omega(q_n) = 0$. Without loss of generality, we assume that $\omega(q_n) = 1$.

Suppose $s_1\alpha < \frac{1}{2} < s_2\alpha$, where $0 < s_1, s_2 < q_n$ and no element in Q_n lies between $s_1\alpha$ and $s_2\alpha$. From the property of p_n/q_n (inequality (2.4)), $|s_1 - s_2| \le q_{n-1}$ and $\|s_1\alpha - s_2\alpha\| \ge \|q_{n-1}\alpha\|$. If $\frac{1}{2} - \{s_1\alpha\} < \|q_n\alpha\|$, then

$$\omega(0, (k-1)q_n+1, kq_n) = \begin{cases} 1 & \text{if } k = 1, \\ -1 & \text{if } k = 2, 3, \ldots, \left\lfloor \dfrac{\|q_{n-1}\alpha\|}{\|q_n\alpha\|} \right\rfloor. \end{cases} \tag{3.7}$$

If $\frac{1}{2} - \{s_1\alpha\} \ge \|q_n\alpha\|$, then

$$\omega(0, (k-1)q_n+1, kq_n) = \begin{cases} 1 & \text{if } k = 1, 2, \ldots, \left\lfloor \dfrac{\frac{1}{2} - \{s_1\alpha\}}{\|q_n\alpha\|} \right\rfloor, \\ -1 & \text{if } k = \left\lfloor \dfrac{\frac{1}{2} - \{s_1\alpha\}}{\|q_n\alpha\|} \right\rfloor + 1, \ldots, \left\lfloor \dfrac{\|q_{n-1}\alpha\|}{\|q_n\alpha\|} \right\rfloor. \end{cases} \tag{3.8}$$

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We also note that, by inequality (2.5),

$$\frac{\|q_{n-1}\alpha\|}{\|q_n\alpha\|} \geq \frac{1}{q_{n-1}(q_{n-1}+q_n)} q_n q_{n+1} \geq \left\lfloor \frac{q_{n+1}}{q_n} \right\rfloor \frac{q_n^2}{q_{n-1}(q_{n-1}+q_n)}$$
$$\geq \frac{1}{2}\left\lfloor \frac{q_{n+1}}{q_n} \right\rfloor \geq \left\lfloor \frac{q_{n+1}}{2q_n} \right\rfloor.$$

This implies that

$$\left\lfloor \frac{q_{n+1}}{6q_n} \right\rfloor \|q_n\alpha\| \leq \frac{1}{3}\|q_{n-1}\alpha\|. \tag{3.9}$$

We assume that $q_{n+1} \geq 6q_n$ (otherwise (3) already holds).

If $\frac{1}{2} - s_1\alpha > \frac{1}{3}\|q_{n-1}\alpha\| (\geq \|q_n\alpha\|)$, then $\lfloor (\frac{1}{2} - \{s_1\alpha\})/\|q_n\alpha\| \rfloor \geq \lfloor (q_{n+1})/6q_n \rfloor$. By (3.8), $\omega(kq_n) = k$ for any $k = 1, 2, \ldots, \lfloor (\frac{1}{2} - \{s_1\alpha\})/\|q_n\alpha\| \rfloor$. So, in this case, $M(0, q_{n+1}) \geq |\omega(q_n)| \lfloor q_{n+1}/6q_n \rfloor$.

If $\|q_n\alpha\| \leq \frac{1}{2} - s_1\alpha \leq \frac{1}{3}\|q_{n-1}\alpha\|$, then, by (3.8), $\omega(kq_n) = 2\lfloor (\frac{1}{2} - \{s_1\alpha\})/\|q_n\alpha\| \rfloor - k$ for any $\lfloor (\frac{1}{2} - \{s_1\alpha\})/\|q_n\alpha\| \rfloor < k \leq \lfloor \|q_{n-1}\alpha\|/\|q_n\alpha\| \rfloor$. So, in this case,

$$m(0, q_{n+1}) \leq \omega\left(\left\lfloor \frac{\|q_{n-1}\alpha\|}{\|q_n\alpha\|} \right\rfloor q_n \right) = 2\left\lfloor \frac{\frac{1}{2} - \{s_1\alpha\}}{\|q_n\alpha\|} \right\rfloor - \left\lfloor \frac{\|q_{n-1}\alpha\|}{\|q_n\alpha\|} \right\rfloor$$
$$\leq -\left\lfloor \frac{q_{n+1}}{6q_n} \right\rfloor.$$

If $\frac{1}{2} - s_1\alpha \leq \frac{1}{3}\|q_{n-1}\alpha\|$ and $\frac{1}{2} - \{s_1\alpha\} < \|q_n\alpha\|$, then, by (3.7), it obviously holds that

$$m(0, q_{n+1}) \leq \omega\left(\left\lfloor \frac{\|q_{n-1}\alpha\|}{\|q_n\alpha\|} \right\rfloor q_n \right) = -\left\lfloor \frac{\|q_{n-1}\alpha\|}{\|q_n\alpha\|} \right\rfloor + 2$$
$$\leq -\left\lfloor \frac{q_{n+1}}{6q_n} \right\rfloor.$$

□

Denote

$$M(k) = \max_{z \in Z} M(z, k) \text{ and } m(k) = \min_{z \in Z} m(z, k).$$

Proposition 3.4. For $q_n < k \leq q_{n+1}$,

$$M(k) \leq q_1 + 3\left\lceil \frac{k}{q_n} \right\rceil + 3\sum_{j=2}^{n} \lceil \frac{q_j}{q_{j-1}} \rceil + 2$$

and

$$m(k) \geq -q_1 - 3\left\lceil \frac{k}{q_n} \right\rceil - 3\sum_{j=2}^{n} \lceil \frac{q_j}{q_{j-1}} \rceil - 2.$$

Proof. By (1) of Proposition 3.3, for $q_n < k \leq q_{n+1}$,

$$M(\theta, k) \leq M(\theta, q_n) + 3\left\lceil \frac{k}{q_n} \right\rceil \leq \cdots$$

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$$\le M(\theta, q_1) + 3\left\lceil \frac{k}{q_n} \right\rceil + 3\sum_{j=2}^{n} \left\lceil \frac{q_j}{q_{j-1}} \right\rceil$$
$$\le q_1 + 3\left\lceil \frac{k}{q_n} \right\rceil + 3\sum_{j=2}^{n} \left\lceil \frac{q_j}{q_{j-1}} \right\rceil.$$

Hence

$$M(k) = \max_{z\in Z} M(z,k) \le \max_{\theta\in\mathbf{T}} M(\theta,k) + 2$$
$$\le q_1 + 3\left\lceil \frac{k}{q_n} \right\rceil + 3\sum_{j=2}^{n} \left\lceil \frac{q_j}{q_{j-1}} \right\rceil + 2.$$

The proof is similar for $m(k)$. □

4. *Topological entropy dimension*

Let $\mathfrak{X} = \mathfrak{X}_{\alpha,\mathfrak{Y}}$ be the DWRS with rotation α along $\mathfrak{Y}$, where $\mathfrak{Y}$ is an invertible TDS with positive entropy. In this section, we will show that, for any $\tau \in [0, 1)$, there exists α such that $\mathfrak{X}$ has topological entropy dimension τ.

Let $\tau \in [0, 1)$ be given. For any positive integers $\{t_1, t_2, \ldots, t_k\}$, we define an irrational number $\alpha(t_1, t_2, \ldots, t_k) = [0; a_1, a_2, \ldots] \in [0, 1)$ as follows.

$$a_n = t_n \text{ for } 1 \le n \le k; \begin{cases} a_{k+1} = a_{k+2} = 2 & \text{if } q_k q_{k-1} \text{ is odd,} \\ a_{k+1} = 3, a_{k+2} = 2 & \text{if } q_{k-1} \text{ is even and } q_k \text{ is odd,} \\ a_{k+1} = a_{k+2} = 3 & \text{if } q_k \text{ is even and } q_{k-1} \text{ is odd;} \end{cases} \tag{4.1}$$

$$a_{n+1} = 2\lfloor q_n^{\frac{\tau}{1-\tau}} \rfloor \text{ for } n > k+2.$$

By (2.2), $q_{n+1} = a_{n+1}q_n + q_{n-1}$ is odd for every $n \ge k$. We remark here that if q_n is even, then q_{n-1} must be odd (otherwise, $q_{n-2}, \ldots, q_1, q_0$ are all even, which contradicts the fact that $q_0 = 1$).

Denote by E_τ the collection of such irrational numbers $\alpha(t_1, t_2, \ldots, t_k)$. Then E_τ is dense in $[0, 1)$.

Let $\alpha \in E_\tau$ be fixed. By (2.2), a simple estimation shows that, for any $c > 0$,

$$\frac{q_{n+1}}{q_n} \sim a_{n+1} \sim 2q_n^{\tau/(1-\tau)} \gg n^c, \tag{4.2}$$

when n is sufficiently large.

Let $l_n = \lfloor (q_{n+1})/12q_n \rfloor$. By (2.5),

$$\frac{\|q_{n-1}\alpha\|}{\|q_n\alpha\|} \ge \frac{1/(q_{n-1}+q_n)}{1/(q_{n+1})} \ge \frac{q_{n+1}}{2q_n}.$$

Hence

$$l_n \le \left\lfloor \frac{\|q_{n-1}\alpha\|}{6\|q_n\alpha\|} \right\rfloor. \tag{4.3}$$

Moreover, by (4.2), there exists n_0 such that $l_n > 0$ and $\sum_{i=n_0}^{n-1} l_i < l_n$ whenever $n \ge n_0$.

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Now we set

$$F = \bigcup_{n=n_0}^{\infty} \{q_n, 2q_n, \ldots, l_n q_n\} = \{s_1 < s_2 < \cdots\}. \tag{4.4}$$

LEMMA 4.1.

$$\overline{D}(F) \geq \tau.$$

Proof. From the definition of F,

$$\begin{aligned}
\overline{D}(F) &= \limsup_{n\to\infty} \frac{\log n}{\log s_n} \geq \lim_{m\to\infty} \frac{\log \sum_{i=n_0}^{m} l_i}{\log(l_m q_m)} \\
&\geq \lim_{m\to\infty} \frac{\log l_m}{\log(l_m q_m)} = \lim_{m\to\infty} \frac{\log\lfloor q_{m+1}/12q_m\rfloor}{\log(\lfloor q_{m+1}/12q_m\rfloor q_m)} \\
&= \lim_{m\to\infty} \frac{\log((q_{m+1})/q_m)}{\log q_{m+1}} = \frac{\tau/(1-\tau)}{\tau/(1-\tau)+1} \text{ (by (4.2))} \\
&= \tau.
\end{aligned}$$

□

The following lemma will be used while computing the entropy dimension.

LEMMA 4.2. *When n is large enough,*

$$\mathbf{m}\left\{\theta \in \mathbf{T} : \omega(\theta, iq_n) = i, 1 \leq i \leq \left\lfloor \frac{\|q_{n-1}\alpha\|}{6\|q_n\alpha\|} \right\rfloor\right\} > \frac{1}{8}.$$

Proof. We only consider the case when $n = 2t$; the case $n = 2t+1$ is similar. By Proposition 3.1, $\omega(q_n) = 1$. Also, we note that $\{q_n\alpha\} = \|q_n\alpha\|$.

Order the points in Q_n on $\mathbf{T}$ by $0 < q_n\alpha < \cdots < s_1\alpha < \frac{1}{2} < s_2\alpha < \cdots < q_{n-1}\alpha < 1$. Then $\|s_2\alpha - s_1\alpha\| \geq \|q_{n-1}\alpha\|$. We note that, from (4.2), when n is sufficiently large, $\|q_{n-1}\alpha\| \gg \|q_n\alpha\|$.

Case 1. $\|\frac{1}{2} - s_1\alpha\| \geq \frac{1}{2}\|q_{n-1}\alpha\|$.

CLAIM 4.3. *For any $\ell\alpha \in Q_n$ and $0 \leq \theta < \|\frac{1}{2} - s_1\alpha\| - \|q_n\alpha\|$, $\omega(\ell\alpha + \theta, q_n) = \omega(q_n) = 1$.*

Proof of Claim 4.3. Notice that

$$\ell\alpha + \theta + Q_n = (\theta + \ell\alpha + \{\alpha, 2\alpha, \ldots, (q_n - \ell)\alpha\}) \bigsqcup (\theta + \|q_n\alpha\| + \{\alpha, 2\alpha, \ldots, \ell\alpha\}).$$

Comparing elements in $\ell\alpha + \theta + Q_n$ with elements in Q_n, we see that the first part of the right-hand side of the above equality is a translation by θ of a subset of Q_n and the second part is a translation by $\theta + \|q_n\alpha\|$ of a subset of Q_n. Since the distance of $s_1\alpha$ and $\frac{1}{2}$ is larger than $\theta + \|q_n\alpha\|$, we have that $\omega(\ell\alpha + \theta, q_n) = \omega(q_n) = 1$. □

For any $\ell\alpha \in Q_n$, $0 \leq \theta < \frac{1}{4}\|q_{n-1}\alpha\|$ and $0 \leq i \leq \lfloor \|q_{n-1}\alpha\|/4\|q_n\alpha\|\rfloor - 1$, it is easy to see that $\theta + i\|q_n\alpha\| + \|q_n\alpha\| < \|\frac{1}{2} - s_1\alpha\|$. Hence, for $1 \leq i \leq \lfloor \|q_{n-1}\alpha\|/4\|q_n\alpha\|\rfloor - 1$, by the claim,

$$\begin{aligned}
\omega(\ell\alpha + \theta, iq_n) = \omega(\ell\alpha + \theta, q_n) &+ \omega(\ell\alpha + \theta + \|q_n\alpha\|, q_n) \\
&+ \cdots + \omega(\ell\alpha + \theta + (i-1)\|q_n\alpha\|, q_n) = i.
\end{aligned}$$

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By (2.5),

$$\|q_{n-1}\alpha\| = q_{n-1}(-1)^{n-1}\left(\alpha - \frac{p_{n-1}}{q_{n-1}}\right) \geq \frac{1}{q_n + q_{n-1}} > \frac{1}{2q_n}.$$

Let n be large enough to make $\lfloor(\|q_{n-1}\alpha\|)/6\|q_n\alpha\|\rfloor \leq \lfloor\|q_{n-1}\alpha\|/4\|q_n\alpha\|\rfloor - 1$. Since

$$\left\{\theta \in \mathbf{T} : \omega(\theta, iq_n) = i, 1 \leq i \leq \left\lfloor \frac{\|q_{n-1}\alpha\|}{6\|q_n\alpha\|} \right\rfloor\right\} \supset \bigsqcup_{\ell=1}^{q_n}\left(\ell\alpha + \left[0, \frac{1}{4}\|q_{n-1}\alpha\|\right)\right),$$

we have that

$$\mathbf{m}\left\{\theta \in \mathbf{T} : \omega(\theta, iq_n) = i, 1 \leq i \leq \left\lfloor \frac{\|q_{n-1}\alpha\|}{6\|q_n\alpha\|} \right\rfloor\right\} \geq q_n\frac{1}{4}\|q_{n-1}\alpha\| > \frac{1}{8}.$$

Case 2. $\|\frac{1}{2} - s_1\alpha\| < \frac{1}{2}\|q_{n-1}\alpha\|$.

CLAIM 4.4. *For any* $\ell\alpha \in Q_n$ *and* $\|\frac{1}{2} - s_1\alpha\| < \theta \leq \|q_{n-1}\alpha\| - \|q_n\alpha\|$, $\omega(\ell\alpha + \theta, q_n) = -1$.

Proof of Claim 4.4. As in the proof of Claim 4.3,

$$\ell\alpha + \theta + Q_n = \big(\theta + \ell\alpha + \{\alpha, 2\alpha, \ldots, (q_n - \ell)\alpha\}\big) \bigsqcup \big(\theta + \|q_n\alpha\| + \{\alpha, 2\alpha, \ldots, \ell\alpha\}\big).$$

Then elements in $\ell\alpha + \theta + Q_n$ are obtained by translating elements in Q_n by θ or $\theta + \|q_n\alpha\|$. We see that only one element $s_1\alpha$ is moved from the upper semicircle to the lower semicircle. Hence $\omega(\ell\alpha + \theta, q_n) = -1$. □

Similarly to Case 1, when n is large enough,

$$\mathbf{m}\left\{\theta \in \mathbf{T} : \omega(\theta, iq_n) = i, 1 \leq i \leq \left\lfloor \frac{\|q_{n-1}\alpha\|}{6\|q_n\alpha\|} \right\rfloor\right\} \geq q_n\frac{1}{4}\|q_{n-1}\alpha\| > \frac{1}{8}.$$

THEOREM 4.5. *Let* $\mathfrak{X} = \mathfrak{X}_{\alpha,\mathfrak{Y}}$ *be the DWRS with rotation* α *along* $\mathfrak{Y}$*, where* α *is defined by* (4.1) *and* $\mathfrak{Y}$ *is a TDS with positive entropy. Then the topological entropy dimension of* $\mathfrak{X}$ *is* τ.

Proof. Suppose that $\mathfrak{Y} = (Y, S)$ and $\mathfrak{X} = (Z \times Y, T)$ as defined in §2. Let $\mathcal{U}$ be a finite open cover of Z and let $\mathcal{V}$ be a finite open cover of Y. For $q_n \leq k \leq q_{n+1}$ and $i = 1, \ldots, k$, we now partition Z due to the range of $\omega(z, i)$ for $z \in Z$. Note that $m(i) \leq \omega(z, i) \leq M(i)$. Let $\widetilde{U}_{i,\ell} = \{z \in Z : \omega(z, i) = \ell\}$, where $m(i) \leq \ell \leq M(i)$. Then $\widetilde{\mathcal{U}}_i = \{\widetilde{U}_{i,m(i)}, \widetilde{U}_{i,m(i)+1}, \ldots, \widetilde{U}_{i,M(i)}\}$ is an open cover of Z. Note that $z \in \sigma^{i+1}U_{i,\ell}$ if and only if $\omega(\sigma^{-i}z, 0, i-1) = \omega(\sigma^{-(i+1)}z, i) = \ell$. Hence

$$\begin{aligned}\bigvee_{i=1}^{k} T^{-i}(\mathcal{U} \times \mathcal{V}) &\prec \bigvee_{i=1}^{k} T^{-i}((\mathcal{U} \vee \sigma^{i+1}\widetilde{\mathcal{U}}_i) \times \mathcal{V}) \\ &\prec \left(\bigvee_{i=1}^{k} \sigma^{-i}(\mathcal{U} \vee \sigma^{i+1}\widetilde{\mathcal{U}}_i)\right) \times \left(\bigvee_{j=m(k)}^{M(k)} S^{-j}\mathcal{V}\right).\end{aligned}$$

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So

$$\begin{aligned}
&\log N\Big(\bigvee_{i=1}^{k} T^{-i}(\mathcal{U}\times\mathcal{V})\Big)\\
&\le \log N\Big(\Big(\bigvee_{i=1}^{k}\sigma^{-i}(\mathcal{U}\vee\sigma^{i+1}\widetilde{\mathcal{U}}_i)\Big)\times\Big(\bigvee_{j=m(k)}^{M(k)} S^{-j}\mathcal{V}\Big)\Big)\\
&\le \log N\Big(\bigvee_{i=1}^{k}\sigma^{-i}\mathcal{U}\Big)+\log N\Big(\bigvee_{i=1}^{k}\sigma(\widetilde{\mathcal{U}}_i)\Big)+\log N\Big(\bigvee_{j=m(k)}^{M(k)} S^{-j}\mathcal{V}\Big)\\
&\le \log N\Big(\bigvee_{i=1}^{k}\sigma^{-i}\mathcal{U}\Big)+\log N\Big(\bigvee_{i=1}^{k}\sigma(\widetilde{\mathcal{U}}_i)\Big)+(M(k)-m(k)+1)\log N(\mathcal{V}).
\end{aligned}$$

For the Sturmian system (Z,σ), its sequence entropy and entropy dimension are both zero. Noticing that $\bigvee_{i=1}^{k}\sigma(\widetilde{\mathcal{U}}_i)\prec\{\text{the cover formed by the blocks of length } k+1\}$, we have that, for any $\epsilon>0$,

$$\limsup_{k\to\infty}\frac{1}{k^{\tau+\epsilon}}\Big(\log N\Big(\bigvee_{i=1}^{k}\sigma^{-i}\mathcal{U}\Big)+\log N\Big(\bigvee_{i=1}^{k}\sigma(\widetilde{\mathcal{U}}_i)\Big)\Big)=0.$$

For each k, there exists a unique $n(k)$ such that $q_{n(k)}<k\le q_{n(k)+1}$. By Proposition 3.4,

$$\begin{aligned}
&\limsup_{k\to\infty}\frac{1}{k^{\tau+\epsilon}}(M(k)-m(k)+1)\\
&\le\limsup_{k\to\infty}\frac{1}{k^{\tau+\epsilon}}\Big(6\Big\lceil\frac{k}{q_{n(k)}}\Big\rceil+6\sum_{j=2}^{n}(k)\frac{q_j}{q_{j-1}}+2q_1+5\Big)\\
&\le\limsup_{k\to\infty}\frac{6(k/q_{n(k)})}{k^{\tau+\epsilon}}+\limsup_{k\to\infty}\frac{6n(k)(q_{n(k)}/q_{n(k)-1})}{k^{\tau+\epsilon}}\\
&\le 6\Big(\limsup_{k\to\infty}\frac{q_{n(k)+1}^{1-\tau}}{q_{n(k)}k^{\epsilon}}+\limsup_{k\to\infty}\frac{q_{n(k)}^{1-\tau}n(k)}{q_{n(k)-1}k^{\epsilon}}\Big)=0.
\end{aligned}$$

Hence, by (1) of Remark 2.1, $\overline{D}(X,T)\le\tau$.

If $\tau=0$, $\overline{D}(X,T)=0$ has already been proved. In the following, we will show that $\overline{D}(X,T)\ge\tau$ for $\tau>0$.

Since $h_{\mathrm{top}}(Y,S)>0$, there exist two non-empty disjoint closed sets A_1, A_2 of Y, such that $h_{\mathrm{top}}(S,\{A_1^c,A_2^c\})>0$ (c.f. [5]). Let $\widetilde{\mathcal{U}}=\{Z\times A_1^c, Z\times A_2^c\}$, which is an open cover of X.

By Lemma 4.2, when n is large enough, we can take any small closed interval from $\{\theta\in\mathbf{T}:\omega(\theta,iq_n)=i,\ 1\le i\le\lfloor(\|q_{n-1}\alpha\|)/6\|q_n\alpha\|\rfloor\}$ and denote it by I. Together with (4.3), for any fixed $1\le i\le l_n$ and any point $z\in\varphi^{-1}(I)$, $\omega(z,iq_n)=i$. By (2.1), for $j=1,2$,

$$T^{-iq_n}(\varphi^{-1}((iq_n+1)\alpha+I)\times A_j)=\varphi^{-1}(\alpha+I)\times S^{-i}A_j.$$

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Hence, for any $1 \le \ell \le l_n$,

$$\begin{aligned}
&N\Big(\bigvee_{i=1}^{\ell} T^{-iq_n}\{Z \times A_1^c, Z \times A_2^c\}\Big)\\
&\quad \ge N\Big(\bigvee_{i=1}^{\ell} T^{-iq_n}\{(\varphi^{-1}((iq_n+1)\alpha + I) \times A_1)^c, (\varphi^{-1}((iq_n+1)\alpha + I) \times A_2)^c\}\Big)\\
&\quad = N\Big(\bigvee_{i=1}^{\ell}\{(\varphi^{-1}(\alpha + I) \times S^{-i}A_1)^c, (\varphi^{-1}(\alpha + I) \times S^{-i}A_2)^c\}\Big)\\
&\quad = N\Big(\bigvee_{i=1}^{\ell}\{\varphi^{-1}(\alpha + I)^c \times Y \cup \varphi^{-1}(\alpha + I) \times S^{-i}A_1^c,\\
&\qquad \varphi^{-1}(\alpha + I)^c \times Y \cup \varphi^{-1}(\alpha + I) \times S^{-i}A_2^c\}\Big)\\
&\quad = N\Big(\bigvee_{i=1}^{\ell} S^{-i}\{A_1^c, A_2^c\}\Big).
\end{aligned}$$

Now let $F = \{s_1 < s_2 < \cdots\}$, as defined in (4.4). For any k, there exists $n(k)$ such that $q_{n(k)} \le s_k < q_{n(k)+1}$. Assume that $s_k = \ell q_{n(k)}$. Then $k = \sum_{i=n_0}^{n(k)-1} l_i + \ell$ and $\max\{\sum_{i=n_0}^{n(k)-1} l_i, \ell\} \ge \frac{1}{2}k$.

Hence, along the sequence F,

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{1}{k} N\Big(\bigvee_{i=1}^{k} T^{-s_i}\{Z \times A_1^c, Z \times A_2^c\}\Big)\\
&\quad \ge \liminf_{k\to\infty} \frac{1}{k} \max\Big\{N\Big(\bigvee_{i=1}^{l_{n(k)-1}} T^{-iq_{n(k)-1}}\{Z \times A_1^c, Z \times A_2^c\}\Big),\\
&\qquad N\Big(\bigvee_{i=1}^{\ell} T^{-iq_{n(k)}}\{Z \times A_1^c, Z \times A_2^c\}\Big)\Big\}\\
&\quad \ge \liminf_{k\to\infty} \frac{1}{k} \max\Big\{N\Big(\bigvee_{i=1}^{l_{n(k)-1}} S^{-i}\{A_1^c, A_2^c\}\Big), N\Big(\bigvee_{i=1}^{\ell} S^{-i}\{A_1^c, A_2^c\}\Big)\Big\}\\
&\quad = \liminf_{k\to\infty} \max\Big\{\frac{l_{n(k)-1}}{k} \frac{N\big(\bigvee_{i=1}^{l_{n(k)-1}} S^{-i}\{A_1^c, A_2^c\}\big)}{l_{n(k)-1}}, \frac{\ell}{k} \frac{N\big(\bigvee_{i=1}^{\ell} S^{-i}\{A_1^c, A_2^c\}\big)}{\ell}\Big\}\\
&\quad \ge \tfrac{1}{4} h_{\mathrm{top}}(S, \{A_1^c, A_2^c\}) > 0.
\end{aligned}$$

So F is an entropy generating sequence of $\tilde{\mathcal{U}}$. Noticing that $\overline{D}(F) = \tau$, we can deduce that $\overline{D}(X, T) \ge \tau$. Hence $\overline{D}(X, T) = \tau$. □

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5. *Metric entropy dimension*

Let $\mathfrak{X} = \mathfrak{X}_{\alpha,\mathfrak{Y}}$ be the DWRS with rotation α along $\mathfrak{Y}$, where $\alpha \in E_\tau$ is defined as in §4 and $\mathfrak{Y}$ is an MDS with positive metric entropy. In this section, we will show that the metric entropy dimension of $\mathfrak{X}$ also equals τ.

By Remark 2.2, in this section, we assume that $\mathfrak{X} = (X, \mathcal{B}, \mu, T) = (\mathbf{T} \times Y, \mathcal{B}_{\mathbf{T}} \otimes \mathcal{D}, \mathbf{m} \times \nu, T)$.

Let us consider a measurable partition of X, say, $\mathcal{P} = \{\mathbf{T} \times A, \mathbf{T} \times A^c\}$, where $\{A, A^c\}$ is a measurable partition of Y with $h_\nu(S, \{A, A^c\}) > 0$.

Proposition 5.1. Let $F = \{s_1 < s_2 < \cdots\}$ be defined by (4.4). Then F is an entropy generating sequence of $\mathcal{P}$. Hence, by Lemma 4.1,

$$\overline{D}_\mu(T, \mathcal{P}) \geq \tau.$$

Proof. For $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$, a collection of measurable subsets of X (which does not need to be a partition), we still denote $H_\mu(\mathcal{A}) = \sum_{i=1}^k -\mu(A_i) \log \mu(A_i)$.

Let $W = A$ or A^c and denote $I_i = \{\theta \in \mathbf{T} : \omega(\theta, iq_n) = i\}$ and $I = \bigcap_{i=1}^\ell I_i$. Then, by (2.1),

$$\begin{aligned} T^{-iq_n}(\mathbf{T} \times W) &= \bigcup_{\theta \in \mathbf{T}} (\theta - iq_n\alpha, S^{-\omega(\theta - iq_n\alpha, 0, iq_n - 1)} W) \\ &\supset \{\theta \in \mathbf{T} : \omega(\theta, 0, iq_n - 1) = i\} \times S^{-i} W \\ &= (I_i + \alpha) \times S^{-i} W. \end{aligned}$$

So

$$\begin{aligned} H_\mu\Big(\bigvee_{i=1}^\ell T^{-iq_n}\mathcal{P}\Big) &\geq H_\mu\Big(\bigvee_{i=1}^\ell (I_i + \alpha) \times S^{-i}\{A, A^c\}\Big) \\ &= H_\mu\Big(\bigcap_{i=1}^\ell (I_i + \alpha) \times \bigvee_{i=1}^\ell S^{-i}\{A, A^c\}\Big) \\ &= -\mathbf{m}(I) \log \mathbf{m}(I) + \mathbf{m}(I) H_\nu\Big(\bigvee_{i=1}^\ell S^{-i}\{A, A^c\}\Big) \\ &\geq \frac{1}{8} H_\nu\Big(\bigvee_{i=1}^\ell S^{-i}\{A, A^c\}\Big). \end{aligned}$$

For any k, there exists $n(k)$ such that $q_{n(k)} \leq s_k < q_{n(k)+1}$. Assume that $s_k = \ell q_{n(k)}$. Then $k = \sum_{i=1}^{n(k)-1} l_i + \ell$ and $\max\{\sum_{i=n_0}^{n(k)-1} l_i, \ell\} \geq \frac{1}{2}k$.

Hence, along the sequence F,

$$\begin{aligned} &\liminf_{k\to\infty} \frac{1}{k} H_\mu\Big(\bigvee_{i=1}^k T^{-s_i}\mathcal{P}\Big) \\ &\quad \geq \liminf_{k\to\infty} \frac{1}{k} \max\left\{H_\mu\Big(\bigvee_{i=1}^{l_{n(k)-1}} T^{-iq_{n(k)-1}}\mathcal{P}\Big), H_\mu\Big(\bigvee_{i=1}^\ell T^{-iq_{n(k)}}\mathcal{P}\Big)\right\} \end{aligned}$$

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$$\geq \liminf_{k\to\infty} \frac{1}{k} \max\left\{\frac{1}{8}H_\nu\Big(\bigvee_{i=1}^{l_{n(k)-1}} S^{-i}\{A, A^c\}\Big), \frac{1}{8}H_\nu\Big(\bigvee_{i=1}^{\ell} S^{-i}\{A, A^c\}\Big)\right\}$$

$$= \liminf_{k\to\infty} \max\left\{\frac{l_{n(k)-1}}{8k}\frac{H_\nu(\bigvee_{i=1}^{l_{n(k)-1}} S^{-i}\{A, A^c\})}{l_{n(k)-1}}, \frac{\ell}{8k}\frac{H_\nu(\bigvee_{i=1}^{\ell} S^{-i}\{A, A^c\})}{\ell}\right\}$$

$$\geq \frac{1}{32}h_\nu(S, \{A, A^c\}) > 0.$$

So F is an entropy generating sequence of $\mathcal{P}$. □

Since the metric entropy dimension is bounded from above by the topological entropy dimension, together with Theorem 4.5 and Proposition 5.1, we have the following theorem.

THEOREM 5.2. *The metric entropy dimension of* $\mathfrak{X}$ *equals* τ.

Remark 5.3. (1) Take an increasing sequence $\{\tau_n\}$ such that $\lim_{n\to\infty} \tau_n = 1$. In the construction of the irrational number $\alpha = \alpha(t_1, t_2, \ldots, t_k) = [0; a_1, a_2, \ldots]$ (see (4.1)), when $n > k + 2$, modify a_{n+1} by $a_{n+1} = 2\lfloor q_n^{\tau_n/(1-\tau_n)}\rfloor$. By a proof similar to those of Lemma 4.1 and Proposition 5.1, we can show that the metric entropy dimension of $\mathfrak{X}$ equals 1 for such α. Hence the topological entropy dimension is also 1.

(2) For $\tau \in [0, 1)$, since E_τ is dense in $[0, 1)$, by Theorem 4.5 and Theorem 5.2, the collection of irrational α such that the system $\mathfrak{X}$ has entropy dimension τ is dense.

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