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RESEARCH ARTICLE

Vines and MAT-labeled graphs*

Hung Manh Tran [1], Tan Nhat Tran [2,3] and Shuhei Tsujie [4]

[1]Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076;
E-mail: e0511873@u.nus.edu.
[2]Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Fakultät für Mathematik und Physik, Leibniz Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany; E-mail: tan.tran@math.uni-hannover.de (corresponding author).
[3]Department of Mathematics and Statistics, Binghamton University (SUNY), Binghamton, NY 13902-6000, USA;
E-mail: tnhattran@binghamton.edu.
[4]Department of Mathematics, Hokkaido University of Education, Asahikawa, Hokkaido 070-8621, Japan;
E-mail: tsujie.shuhei@a.hokkyodai.ac.jp.

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Abstract

The present paper explores a connection between two concepts arising from different fields of mathematics. The first concept, called vine, is a graphical model for dependent random variables. This concept first appeared in a work of Joe (1994), and the formal definition was given later by Cooke (1997). Vines have nowadays become an active research area whose applications can be found in probability theory and uncertainty analysis. The second concept, called MAT-freeness, is a combinatorial property in the theory of freeness of logarithmic derivation modules of hyperplane arrangements. This concept was first studied by Abe-Barakat-Cuntz-Hoge-Terao (2016), and soon afterwards investigated further by Cuntz-Mücksch (2020).

In the particular case of graphic arrangements, the last two authors (2023) recently proved that the MAT-freeness is completely characterized by the existence of certain edge-labeled graphs, called MAT-labeled graphs. In this paper, we first introduce a poset characterization of a vine. Then we show that, interestingly, there exists an explicit equivalence between the categories of locally regular vines and MAT-labeled graphs. In particular, we obtain an equivalence between the categories of regular vines and MAT-labeled complete graphs.

Several applications will be mentioned to illustrate the interaction between the two concepts. Notably, we give an affirmative answer to a question of Cuntz-Mücksch that MAT-freeness can be characterized by a generalization of the root poset in the case of graphic arrangements.

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1. Introduction

1.1. Motivation

The starting point of our paper is a question of Cuntz-Mücksch [10] (Question 1.3) in the theory of *free hyperplane arrangements*.

Let V be a finite dimensional vector space. A **hyperplane** in V is a 1-codimensional linear subspace of V. Let $\{x_1, \dots, x_\ell\}$ be a basis for the dual space V^*. Any hyperplane in V can be described by a linear equation of the form $a_1x_1 + \cdots + a_\ell x_\ell = 0$ where at least one of the a_i's is nonzero.

A **hyperplane arrangement** $\mathcal{A}$ is a finite set of hyperplanes in V. The **intersection lattice** of $\mathcal{A}$ is the set of all intersections of hyperplanes in $\mathcal{A}$, which is often referred to as the **combinatorics** of $\mathcal{A}$. An arrangement is said to be **free** if its *module of logarithmic derivations* is a free module. For basic definitions and properties of free arrangements, we refer the interested reader to [22, 26]. Freeness is an algebraic property of hyperplane arrangements which has been a major topic of research since the 1970s. A central question in the theory is to study the freeness of an arrangement by combinatorial structures, especially by the intersection lattice of the arrangement.

Among others, *MAT-freeness* is an important concept which was first used by Abe-Barakat-Cuntz-Hoge-Terao [1] to settle the conjecture of Sommers-Tymoczko [24] on the freeness of *ideal subarrangements of Weyl arrangements*. This concept is formally defined later by Cuntz-Mücksch [10], and we will use their definition throughout. For a hyperplane $H \in \mathcal{A}$, define the **restriction** $\mathcal{A}^H$ of $\mathcal{A}$ to H by

$$\mathcal{A}^H := \{K \cap H \mid K \in \mathcal{A} \setminus \{H\}\}.$$

Definition 1.1 (MAT-partition and MAT-free arrangement [10]). Let $\mathcal{A}$ be a nonempty arrangement. An ordered partition (disjoint union of nonempty subsets) $\pi = (\pi_1, \dots, \pi_n)$ of $\mathcal{A}$ is called an **MAT-partition** if the following three conditions hold for every $1 \le k \le n$.

(MP1) The hyperplanes in π_k are linearly independent.
(MP2) There does not exist $H' \in \mathcal{A}_{k-1}$ such that $\bigcap_{H \in \pi_k} H \subseteq H'$, where $\mathcal{A}_{k-1} := \pi_1 \sqcup \cdots \sqcup \pi_{k-1}$ (disjoint union) and $\mathcal{A}_0 := \emptyset$ is the empty arrangement.
(MP3) For each $H \in \pi_k$, $|\mathcal{A}_{k-1}| - |(\mathcal{A}_{k-1} \cup \{H\})^H| = k - 1$.

An arrangement is called **MAT-free** if it is empty or admits an MAT-partition.

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As the name suggests, any MAT-free arrangement is a free arrangement. This follows from the remarkable Multiple Addition Theorem by Abe-Barakat-Cuntz-Hoge-Terao [1, Theorem 3.1] (justifying the abbreviation MAT). MAT-freeness is a helpful combinatorial tool (as it depends only on the intersection lattice) to examine the freeness of arrangements. One of its most famous applications we mentioned earlier is a proof that the ideal subarrangements of Weyl arrangements are free. The MAT-freeness has received increasing attention in recent years; see [2, 3, 9, 20] for some other applications.

Let $V = \mathbb{R}^\ell$ with the standard inner product $(\cdot, \cdot)$. Let Φ be an irreducible (crystallographic) root system in V, with a fixed positive system $\Phi^+ \subseteq \Phi$ and the associated set of simple roots $\Delta := \{\alpha_1, \dots, \alpha_\ell\}$. For $\alpha \in \Phi$, define $H_\alpha := \{x \in V \mid (\alpha, x) = 0\}$. For $\Sigma \subseteq \Phi^+$, the **Weyl subarrangement** $\mathcal{A}_\Sigma$ is defined by $\mathcal{A}_\Sigma := \{H_\alpha \mid \alpha \in \Sigma\}$. In particular, $\mathcal{A}_{\Phi^+}$ is called the **Weyl arrangement**.

We can make Φ^+ into a *poset* (partially ordered set) by defining a partial order $\le$ on Φ^+ as follows: $\beta_1 \le \beta_2$ if $\beta_2 - \beta_1 \in \sum_{i=1}^{\ell} \mathbb{Z}_{\ge 0}\alpha_i$. The poset $(\Phi^+, \le)$ is called the **root poset** of Φ. For an *ideal* $\mathcal{I}$ (Definition 3.4) of the root poset Φ^+, the corresponding Weyl subarrangement $\mathcal{A}_\mathcal{I}$ is called the **ideal subarrangement**.

Theorem 1.2 [1, Theorem 1.1]. *Any ideal subarrangement $\mathcal{A}_\mathcal{I}$ is MAT-free, and hence free.*

The ideal subarrangements form a significant subclass of MAT-free arrangements. However, there are many MAT-free arrangements (or MAT-partitions of a given MAT-free arrangement) that do not arise from ideal subarrangements (Example 7.2). One may wonder if the hyperplanes in an arbitrary MAT-free arrangement satisfy some poset structure similar to the root poset? This question was asked by Cuntz-Mücksch [10] and is the main motivation of our work.

Question 1.3 [10, Problem 47]. *Given an MAT-free arrangement $\mathcal{A}$, can we characterize all possible MAT-partitions of $\mathcal{A}$ by a poset structure generalizing the classical root poset?*

Cuntz-Mücksch's question is difficult in general as the number of different MAT-partitions of a given MAT-free arrangement might be very large. Also, the definition of an MAT-partition itself does not reveal a natural choice of the desirable partial order. In the present paper, we pursue this question along *graphic arrangements*, a well-behaved class of arrangements in which both freeness and MAT-freeness are completely characterized by combinatorial properties of graphs.

Let G be a simple graph (i.e., no loops and no multiple edges) with vertex (or node) set $N_G = \{v_1, \dots, v_\ell\}$ and edge set E_G. The **graphic arrangement** $\mathcal{A}_G$ is an arrangement in an ℓ-dimensional vector space V defined by

$$\mathcal{A}_G := \{x_i - x_j = 0 \mid \{v_i, v_j\} \in E_G\}.$$

A graph is **chordal** if it does not contain an induced *cycle* of length greater than three. A chordal graph is **strongly chordal** if it does not contain a *sun graph* as an induced subgraph (Definition 2.2).

Theorem 1.4 [25], [12, Theorem 3.3]. *The graphic arrangement $\mathcal{A}_G$ is free if and only if G is chordal.*

Theorem 1.5 [27, Theorem 2.10]. *The graphic arrangement $\mathcal{A}_G$ is MAT-free if and only if G is strongly chordal.*

While the definition of an MAT-free arrangement may seem technical at first glance, Theorem 1.5 enables us to view MAT-freeness as a rather natural property. Furthermore, the correspondence between MAT-freeness and strong chordality establishes a nice analog[1] of the classical correspondence between freeness and chordality.

The good thing about graphs is that MAT-partition of a graphic arrangement can be rephrased in terms of a special edge-labeling of graphs, the so-called *MAT-labeling* (Definition 2.4). A graph together with such a labeling is called an **MAT-labeled graph**. To approach Question 1.3 for graphic arrangements, the first question would be how many non-isomorphic MAT-labelings can a (strongly chordal) graph

[1]Many important concepts in the classical theory such as simplicial vertex and perfect elimination ordering of chordal graphs have their analogs in MAT-labeled graphs (see Remark 2.18).

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Table 1. *Correspondence between concepts in vine and graph theories.*

Vine theory	Graph theory
R-vine	MAT-labeled complete graph
LR-vine (= m-vine, or ideal of R-vine)	MAT-labeled graph (= MAT-free graphic arrangement)

have? A computation aided by computer for complete graphs on up to 8 vertices gives us the sequence $1, 1, 1, 2, 6, 40, 560, 17024$. Surprisingly, we found out that this sequence coincides with the number of equivalence classes of *(graphical) regular vines* (or *R-vines*) in dimension up to 8 given in [17, Table 10.4]. This observation is indeed compelling as it leads us to the notion of the *node poset* of a *graphical vine* (Definitions 3.6 and 3.7), which is a perfect candidate for the poset structure we are looking for.

1.2. Main result

In this paper, we first introduce a poset realization of graphical (R-)vines (Definitions 3.12 and 3.14). Our aim is to convert the important terms and properties of graphical vines into the language of posets in which considerable power and development of poset theory would be brought to bear. We also introduce the notion of a *locally regular vine* (or *LR-vine*) (Definition 3.16). Roughly speaking, an LR-vine is a vine that 'locally' looks like an R-vine. It is worth mentioning that any ideal of an R-vine, or *m-vine* (Definition 3.19), gets characterized by an LR-vine (Theorem 6.13).

Having introduced the concepts, we define the category MG of MAT-labeled graphs and the category LRV of LR-vines (Definitions 6.2 and 6.3). Our main result is that these categories are equivalent (Theorem 6.10). In particular, we obtain the equivalence between the category of MAT-labeled complete graphs and the category of R-vines (Corollary 6.11). The correspondences are summarized in Table 1.

To prove the equivalence between MG and LRV, we construct two functors $\Psi\colon \mathsf{MG} \longrightarrow \mathsf{LRV}$ and $\Omega\colon \mathsf{LRV} \longrightarrow \mathsf{MG}$. The former amounts to constructing an LR-vine from a given MAT-labeled graph which will be presented in Definition 4.11 and Theorem 4.13. The proof is direct and largely dependent upon the notion of *MAT-perfect elimination ordering* (Definition 2.16) developed in an earlier work of the last two authors [27]. The argument on the functor Ω is, however, more complicated. We need to show some new properties of LR-vines in §5.1 before giving the construction in Definition 5.16 and Theorem 5.17.

It is known that graphic arrangements are equivalent to Weyl subarrangements when the root system is of type A. It would be interesting to extend our main result to the root systems of other types. However, it is quite challenging since a complete characterization of either MAT-freeness or freeness of Weyl subarrangements is unknown except for type A.

1.3. Applications

From the view point of category theory, the equivalence establishes a strong similarity between the categories and allows many properties and structures to be translated from one to the other. We obtain three main applications from LR-vines to MAT-labeled graphs. First, our main theorem 6.10 gives a new poset characterization of the MAT-free graphic arrangements compared with the characterization by strong chordality in Theorem 1.5. In particular, LR-vine is an answer for Question 1.3 in the case of graphic arrangements (see §7.1.1). We find it interesting that although the class of MAT-free arrangements strictly contains that of ideal subarrangements in general, any MAT-free *graphic* arrangement is characterized by being an ideal of a poset structure. Second, an explicit formula for the number of non-isomorphic MAT-labelings of complete graphs is obtained as it equals the number of equivalence classes of regular vines (see §7.1.2). Third, the notion of *upper truncation* (Definition 4.9) of an LR-vine gives rise to a nontrivial graph operation which produces a new MAT-labeled graph from a given one (see §7.1.3).

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A vine is a graphical tool for representing the joint distribution of random variables. The first construction of a vine was given by Joe [14], and the formal definition was given and refined further by Cooke, Bedford and Kurowicka [7, 5, 15]. Vines have been studied extensively and proved to have various applications in probability theory and related areas. We refer the reader to [17] for a comprehensive handbook of vines. Our main result gives a new appearance and applications of vines in the arrangement theory. In the present paper, we do not pursue the probabilistic or applied aspects of vines (neither does the proof of the main result) but emphasize and develop more on the theoretical aspects. There are several new combinatorial properties of vines presented throughout, and we hope that they will be useful for the future research on vines. For instance, we give an alternative way to associate an m-vine to a strongly chordal graph compared with the work of Zhu-Kurowicka [28] (see §7.2.1), an extension of the notion of *sampling order* [8] from R-vine to LR-vine (see §7.2.2), and a conjectural formula for the number of ideals in a C-vine (see §7.2.3).

2. MAT-labelings of graphs

2.1. Graphs

In this subsection, we recall some basic definitions and notions of graphs. All graphs in this paper are undirected, finite and simple. Let $G = (N_G, E_G)$ be a graph with the set N_G of vertices (or nodes) and the set E_G of edges (unordered pairs of vertices). In this paper, a vertex and a node in a graph are synonyms. The former will be used more often for graphs, while the latter will be used for an element in a poset.

For $S \subseteq N_G$, denote by $G[S] = (S, E_{G[S]})$ where $E_{G[S]} = \{\{u, v\} \in E_G \mid u, v \in S\}$ the **(vertex-) induced subgraph** of S. Denote by $G \setminus S$ the induced subgraph $G[N_G \setminus S]$. In particular, $G \setminus v := G \setminus \{v\}$ when v is a vertex of G. For $F \subseteq E_G$, define the subgraph $G \setminus F := (N_G, E_G \setminus F)$. In particular, $G \setminus e := G \setminus \{e\}$ when e is an edge of G.

A **complete graph** K_n ($n \geq 0$) is a graph with vertex set $N_{K_n} = \{u_1, \ldots, u_n\}$ and edge set

$$E_{K_n} = \{\{u_i, u_j\} \mid 1 \leq i < j \leq n\}.$$

In other words, a complete graph is a graph such that each pair of vertices is an edge. Here, the **order-zero graph** K_0 (i.e., the graph having no vertices is also considered to be a complete graph). A **clique** C of G is a subset of N_G such that the induced subgraph $G[C]$ is a complete graph.

For each $v \in N_G$, the **neighborhood** $\mathrm{Nbd}_G(v)$ of v in G is defined by $\mathrm{Nbd}_G(v) := \{u \in N_G \mid \{u, v\} \in E_G\}$. The **degree** of v in G is defined by $\deg_G(v) := |\mathrm{Nbd}_G(v)|$. A **leaf** is a vertex of degree 1. A vertex is called **simplicial** if its neighborhood is a clique. An ordering $(v_1, \ldots, v_\ell)$ of vertices of a graph G is called a **perfect elimination ordering (PEO)** if v_i is simplicial in the induced subgraph $G[\{v_1, \ldots, v_i\}]$ for each $1 \leq i \leq \ell$.

A **maximal clique** is a clique that it is not a subset of any other clique. A **largest clique** is a clique that has the largest possible number of vertices. Denote by $\mathcal{K}(G)$ the set of all maximal cliques of G. In particular, $|\mathcal{K}(G)| = 0$ or 1 if and only if G is a complete graph. The **clique number** of G, denoted $\omega(G)$, is the number of vertices in a largest clique of G.

A **walk** W in a graph G is a sequence of edges $(e_1, e_2, \ldots, e_p)$ of G for which there is a sequence of vertices $(v_1, v_2, \ldots, v_{p+1})$ of G such that $e_i = \{v_i, v_{i+1}\}$ for $1 \leq i \leq p$. The vertices v_1 and v_{p+1} are said to be **connected** by the walk W, called the **initial** and **final vertices** of W, respectively. The **length** of a walk is the number of edges in the walk (hence, the length of W is $p \geq 0$). Throughout the paper, a walk W is denoted by its **vertex sequence** $W = (v_1, v_2, \ldots, v_{p+1})$. If $W_1 = (v_1, v_2, \ldots, v_{p+1})$ and $W_2 = (v_{p+1}, v_{p+2}, \ldots, v_n)$ are walks, then the **concatenation** of W_1 and W_2 is the walk $(v_1, v_2, \ldots, v_n)$.

A **path** $P = (v_1, v_2, \ldots, v_{p+1})$ is a walk with no repeated vertices except possibly the initial and final vertex. A **subpath** of P is a path of the form $(v_i, v_{i+1}, \ldots, v_j)$ for some $1 \leq i \leq j \leq p + 1$. The following fact is well-known.

Lemma 2.1. *Given two vertices a, b in a graph G, every walk connecting a and b contains a path connecting a and b.*

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A graph G is called **connected** if any two vertices are connected by a walk (hence by a path by Lemma 2.1 above).

A **p-cycle** $C_p = (v_1, v_2, \ldots, v_p)$ for $p \geq 3$ is a path with $v_p = v_1$. The 3-cycle is also called a **triangle**. A **chord** of a cycle is an edge connecting two non-consecutive vertices of the cycle. A **forest** is a graph containing no cycles. A **tree** is a connected forest. In a forest (resp. tree), any two distinct vertices are connected by at most (resp. exactly) one path.

An **n-sun** S_n ($n \geq 3$) is a graph with vertex set $N_{S_n} = \{u_1, \ldots, u_n\} \cup \{v_1, \ldots, v_n\}$ and edge set

$$E_{S_n} = \{\{u_i, u_j\} \mid 1 \leq i < j \leq n\} \cup \{\{v_i, u_j\} \mid 1 \leq i \leq n, j \in \{i, i+1\}\},$$

where we let $u_{n+1} = u_1$.

Definition 2.2 (Strongly chordal graph). A simple graph is **strongly chordal** if it is C_p-free[2] for $p \geq 4$ and S_n-free for $n \geq 3$.

The following property of a strongly chordal graph is a special case of [27, Lemma 5.9].

Lemma 2.3. *Let G be a strongly chordal graph with $|\mathcal{K}(G)| \geq 2$. Then there exist distinct maximal cliques $X_0, Y_0 \in \mathcal{K}(G)$ such that $X_0 \cap Y_0 \supseteq X_0 \cap Y$ for all $Y \in \mathcal{K}(G) \setminus \{X_0\}$.*

2.2. MAT-labeled graphs

In this subsection, we recall some preliminary definitions and facts of MAT-labeled graphs following [27]. An **edge-labeled graph** is pair (G, λ), where G is a simple graph and $\lambda\colon E_G \longrightarrow \mathbb{Z}_{>0}$ is a map, called **(edge-)labeling**. The following definition of an MAT-labeling is equivalent to the original one in [27, Definition 4.2].

Definition 2.4 (MAT-labeling). Let (G, λ) be an edge-labeled graph. For $k \in \mathbb{Z}_{>0}$, let $\pi_k := \lambda^{-1}(k) \subseteq E_G$ denote the set of edges of label k. Define $\pi_{\leq k} := \pi_1 \sqcup \cdots \sqcup \pi_k$ and $\pi_{<1} := \varnothing$. The labeling λ is an **MAT-labeling** if the following two conditions hold for every $k \in \mathbb{Z}_{>0}$.

(ML1) Any edge $e \in \pi_{\leq k}$ does not form a cycle with edges in π_k.
(ML2) Every edge $e \in \pi_k$ forms exactly $k-1$ triangles with edges in $\pi_{<k}$.

Given an edge $e \in \pi_k$, a **conditioning vertex** of e is a vertex that together with the endvertices of e forms two edges both of label $< k$. Condition (ML2) above can be rephrased as every edge e of label k has exactly $k-1$ conditioning vertices.

Definition 2.5 (MAT-labeled (complete) graph). An edge-labeled graph (G, λ) is an **MAT-labeled graph** if λ is an MAT-labeling of G. In particular, an MAT-labeled graph (G, λ) is an **MAT-labeled complete graph** if G is a complete graph.

MAT-partition of a graphic arrangement is nothing but MAT-labeling of the underlying graph [27, Proposition 4.3]. Thus, MAT-free graphic arrangement and MAT-labeled graph are essentially the same object. The following properties of an MAT-labeled graph are deduced thanks to the relation with freeness.

Lemma 2.6 [27, Proposition 4.8]. *Let (G, λ) be an MAT-labeled graph with clique number $\omega(G)$. Then the following statements hold.*

(1) *The largest label of edges in (G, λ) is equal to $\omega(G) - 1$.*
(2) *There exists a bijection between the set of largest cliques of G and π_n, where $n = \omega(G) - 1$ via the relation: For each $e \in \pi_n$, there exists a unique largest clique of G containing the endvertices of e.*

Lemma 2.7 [27, Proposition 4.4]. *If λ is an MAT-labeling of the complete graph K_ℓ, then $|\pi_k| = \ell - k$ for all $1 \leq k \leq \ell - 1$.*

[2]In general, a graph is called H-free if it does not contain H as an induced subgraph. It is not to be confused with 'MAT-free arrangement'.

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Now we present some results on the restrictions of MAT-labelings.

Definition 2.8 (MAT-labeled subgraph). Let (G, λ) be an MAT-labeled graph. An edge-labeled graph (G', λ') is an **MAT-labeled subgraph** of (G, λ), written $(G', \lambda') \leq (G, \lambda)$, if G' is a subgraph of G, $\lambda' = \lambda|_{E_{G'}}$ (i.e., λ' is the restriction of λ on $E_{G'}$), and (G', λ') itself is an MAT-labeled graph.

Lemma 2.9 [27, Lemma 4.7]. *Let (G, λ) be an MAT-labeled graph and let $F_1, F_2 \subseteq E_G$. If $\lambda|_{F_1}$ and $\lambda|_{F_2}$ are MAT-labelings of the subgraphs (N_G, F_1) and (N_G, F_2), respectively, then $\lambda|_{F_1 \cup F_2}$ is an MAT-labeling of $(N_G, F_1 \cup F_2)$.*

Lemma 2.10 [27, Lemma 4.9]. *Let (G, λ) be an MAT-labeled graph. Let X denote the intersection of some maximal cliques of G. Then $(G[X], \lambda|_{E_{G[X]}}) \leq (G, \lambda)$.*

The following is an immediate consequence of Lemmas 2.9 and 2.10 above.

Corollary 2.11. *Let (G, λ) be an MAT-labeled graph. Let $\mathcal{B} \subseteq \mathcal{K}(G)$ be a set of some maximal cliques of G. Let $G_{\mathcal{B}}$ be the subgraph of G with vertex set $N_{G_{\mathcal{B}}} := \cup_{Y \in \mathcal{B}} Y$ and edge set $E_{G_{\mathcal{B}}} := \cup_{Y \in \mathcal{B}} E_{G[Y]}$. Then $(G_{\mathcal{B}}, \lambda|_{E_{G_{\mathcal{B}}}}) \leq (G, \lambda)$.*

Notation 2.12. For simplicity of notation, if $\lambda \colon E_G \longrightarrow \mathbb{Z}_{>0}$ is a labeling and $\{u, v\} \in E_G$ is an edge, we write $\lambda(u, v) := \lambda(\{u, v\})$ for the label of $\{u, v\}$.

The following analogs of simplicial vertex and *perfect elimination ordering* of chordal graphs are important concepts in the study of MAT-labeled graphs.

Definition 2.13 (MAT-simplicial vertex). Given an edge-labeled graph (G, λ), a vertex $v \in N_G$ is **MAT-simplicial** if the following conditions hold.

(MS1) v is a simplicial vertex of G.
(MS2) The edges of G incident on v are labeled by labels from 1 to $\deg_G(v)$ (i.e. $\{\lambda(u, v) \in \mathbb{Z}_{>0} \mid u \in \mathrm{Nbd}_G(v)\} = \{1, 2, \ldots, \deg_G(v)\}$).
(MS3) For any distinct vertices $u_1, u_2 \in \mathrm{Nbd}_G(v)$, $\lambda(u_1, u_2) < \max\{\lambda(u_1, v), \lambda(u_2, v)\}$.

Lemma 2.14 [27, Lemma 5.2]. *If (G, λ) is an MAT-labeled complete graph, then the endvertices of the edge of largest label are MAT-simplicial.*

Lemma 2.15 [27, Proposition 5.3]. *Let (G, λ) be an edge-labeled graph having at least 2 vertices. Suppose that v is an MAT-simplicial vertex of (G, λ). Then λ is an MAT-labeling of G if and only if $\lambda|_{E_{G \setminus v}}$ is an MAT-labeling of $G \setminus v$.*

Definition 2.16 (MAT-PEO). Let (G, λ) be an edge-labeled graph on ℓ vertices. An ordering $(v_1, \ldots, v_\ell)$ of vertices in G is an **MAT-perfect elimination ordering (MAT-PEO)** of (G, λ) if v_i is MAT-simplicial in (G_i, λ_i) for each $1 \leq i \leq \ell$, where $G_i := G[\{v_1, \ldots, v_i\}]$ and $\lambda_i := \lambda|_{E_{G_i}}$.

Theorem 2.17 [27, Theorem 5.5]. *An edge-labeled graph (G, λ) is an MAT-labeled graph if and only if there exists an MAT-PEO of (G, λ).*

Remark 2.18. It is known that a graph is chordal if and only if it has a perfect elimination ordering [13]. Theorem 2.17 is an analog of this classical result for MAT-labeled graphs.

The method of merging regular vines was given in [8, 28]. We have a very similar[3] method for merging MAT-labeled complete graphs.

Lemma 2.19 (Merging MAT-labeled complete graphs [27, Lemma 5.7]). *Let (G_1, λ_1) and (G_2, λ_2) be MAT-labeled complete graphs. Denote the common complete graph $G_1[N_{G_1} \cap N_{G_2}] = G_2[N_{G_1} \cap N_{G_2}]$ by G'. Assume that there exists an MAT-labeling λ' of G' such that $(G', \lambda') \leq (G_1, \lambda_1)$ and $(G', \lambda') \leq (G_2, \lambda_2)$. Let G denote the complete graph with vertex set $N_{G_1} \cup N_{G_2}$. Then there exists an MAT-labeling λ of G such that $(G_1, \lambda_1) \leq (G, \lambda)$ and $(G_2, \lambda_2) \leq (G, \lambda)$.*

[3]These methods are actually equivalent in the sense that they produce the same output as a consequence of our Corollary 6.11.

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Proposition 2.20. *Let (G, λ) be an MAT-labeled graph and K be the complete graph with vertex set N_G. Then there exists an MAT-labeling $\widetilde{\lambda}$ of K such that $(G, \lambda) \leq (K, \widetilde{\lambda})$.*

Proof. We argue by induction on the number $|\mathcal{K}(G)|$ of maximal cliques of G. If $|\mathcal{K}(G)| = 0$ or 1, then the assertion follows since G is a complete graph. Suppose $|\mathcal{K}(G)| \geq 2$.

Note that by Theorem 1.5, G is a strongly chordal graph. By Lemma 2.3, there exist distinct $X_0, Y_0 \in \mathcal{K}(G)$ such that $X_0 \cap Y_0 \supseteq X_0 \cap Y$ for all $Y \in \mathcal{K}(G) \setminus \{X_0\}$. Denote $\mathcal{B} := \mathcal{K}(G) \setminus \{X_0\}$. Let $G_{\mathcal{B}}$ be the subgraph of G with vertex set $N_{G_{\mathcal{B}}} := \cup_{Y \in \mathcal{B}} Y$ and edge set $E_{G_{\mathcal{B}}} := \cup_{Y \in \mathcal{B}} E_{G[Y]}$. Thus, $G_{\mathcal{B}}$ has $|\mathcal{K}(G)| - 1$ maximal cliques. Moreover, $(G_{\mathcal{B}}, \lambda|_{E_{G_{\mathcal{B}}}})$ and $(G[X_0], \lambda|_{E_{G[X_0]}})$ are MAT-labeled graphs by Corollary 2.11. By the induction hypothesis, there exists an MAT-labeling $\lambda_{\mathcal{B}}$ of the complete graph $K_{\mathcal{B}}$ with vertex set $N_{G_{\mathcal{B}}}$ such that $(G_{\mathcal{B}}, \lambda|_{E_{G_{\mathcal{B}}}}) \leq (K_{\mathcal{B}}, \lambda_{\mathcal{B}})$.

Now we are in the setting of Lemma 2.19 with $(G_1, \lambda_1) = (G[X_0], \lambda|_{E_{G[X_0]}})$ and $(G_2, \lambda_2) = (K_{\mathcal{B}}, \lambda_{\mathcal{B}})$. Indeed, first note that

$$N_{G_1} \cap N_{G_2} = X_0 \cap \left(\bigcup_{Y \in \mathcal{B}} Y \right) = \bigcup_{Y \in \mathcal{B}} (X_0 \cap Y) = X_0 \cap Y_0.$$

Hence, $G_1[N_{G_1} \cap N_{G_2}] = G_2[N_{G_1} \cap N_{G_2}] = G[X_0 \cap Y_0]$. Denote this common complete graph by G'. By Lemma 2.10, the restriction $\lambda' := \lambda|_{E_{G'}}$ is an MAT-labeling G'. Hence, $(G', \lambda') \leq (G_1, \lambda_1)$ and $(G', \lambda') \leq (G_2, \lambda_2)$.

Therefore, by Lemma 2.19, there exists an MAT-labeling $\widetilde{\lambda}$ of the complete graph K with vertex set $N_{G_1} \cup N_{G_2} = N_G$ such that $(G_1, \lambda_1) \leq (K, \widetilde{\lambda})$ and $(G_2, \lambda_2) \leq (K, \widetilde{\lambda})$. In particular, $(G, \lambda) \leq (K, \widetilde{\lambda})$. □

The following 'gluing trick' plays an important role in the construction of an MAT-labeling for a given strongly chordal graph in [27, §5.2].

Lemma 2.21 (Gluing MAT-labeled graphs [27, Theorem 5.8]). *Let (G_1, λ_1) and (G_2, λ_2) be MAT-labeled graphs such that $G_1[N_{G_1} \cap N_{G_2}] = G_2[N_{G_1} \cap N_{G_2}]$. Suppose that this common subgraph, denoted G', is a complete graph. Suppose further that there exists an MAT-labeling λ' of G' such that $(G', \lambda') \leq (G_1, \lambda_1)$ and $(G', \lambda') \leq (G_2, \lambda_2)$. Define an edge-labeled graph (G, λ) by $N_G = N_{G_1} \cup N_{G_2}$, $E_G = E_{G_1} \cup E_{G_2}$, $\lambda|_{E_{G_1}} = \lambda_1, \lambda|_{E_{G_2}} = \lambda_2$. Then (G, λ) is an MAT-labeled graph.*

We close this subsection by introducing the notion of *principal cliques* in an MAT-labeled graph. This notion will be useful for the construction of an LR-vine from a given MAT-labeled graph in Definition 4.11.

Lemma 2.22 (Principal clique). *Let (G, λ) be an MAT-labeled graph. Let $e = \{i, j\} \in \pi_k$ be an edge in G of label k and $h_1, \ldots, h_{k-1}$ be the conditioning vertices of e. Then the set $K_e := \{i, j, h_1, \ldots, h_{k-1}\}$ is a clique of G. Moreover, $(G[K_e], \lambda|_{E_{G[K_e]}}) \leq (G, \lambda)$, and all the edges in $G[K_e] \setminus e$ have label $< k$. We call K_e the* ***principal clique generated by*** **e**.

Proof. Let G' denote the graph obtained from G by removing all edges of labels $> k$. By definition, $(G', \lambda') \leq (G, \lambda)$, where $\lambda' := \lambda|_{E_{G'}}$. Since e is an edge of largest label in (G', λ'), by Lemma 2.6, there exists a unique largest clique C of G' containing the endvertices of e. Note also that C does not contain the endvertices of any edge of label k apart from e. Moreover, $(G[C], \lambda|_{E_{G[C]}}) \leq (G', \lambda')$ by Lemma 2.10. Thus, the number of conditioning vertices of e is $k - 1$ in both G and $G[C]$. Hence, $C = K_e$. □

The converse of Lemma 2.22 is also true.

Lemma 2.23. *If C is a clique in an MAT-labeled graph (G, λ) such that $(G[C], \lambda|_{E_{G[C]}}) \leq (G, \lambda)$, then C is a principal clique. In particular, any maximal clique is principal.*

Proof. Let $G' := G[C]$ and $\lambda' := \lambda|_{E_{G[C]}}$. By the assumption, (G', λ') is an MAT-labeled complete graph. Thus, (G', λ') has a unique edge of maximal label by Lemma 2.7. Hence, C is a principal clique in (G', λ') (generated by this unique edge) and hence in (G, λ). The consequence follows directly from Lemma 2.10. □

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Lemma 2.24. *Let (G, λ) be an MAT-labeled graph. Let $e = \{i, j\}$ be an edge in G and K_e the principal clique generated by e. Then both $C_1 := K_e \setminus \{i\}$ and $C_2 := K_e \setminus \{j\}$ are principal cliques in (G, λ). Moreover, if C is a principal clique in (G, λ) such that $C \subsetneq K_e$, then either $C \subseteq C_1$ or $C \subseteq C_2$.*

Proof. Let $G' := G[K_e]$ and $\lambda' := \lambda|_{E_{G[K_e]}}$. By Lemma 2.22, (G', λ') is an MAT-labeled complete graph in which e is the unique edge of largest label. By Lemma 2.14, the endvertices i and j of e are MAT-simplicial. By Lemma 2.15, $(G'[C_1], \lambda'|_{E_{G'[C_1]}})$ and $(G'[C_2], \lambda'|_{E_{G'[C_2]}})$ are MAT-labeled complete graphs. By Lemma 2.23, C_1 and C_2 are principal cliques in (G, λ).

Let C be a principal clique in (G, λ) such that $C \subsetneq K_e$. Then $C = K_f$ for some edge f in $G[K_e]$. Moreover, $\lambda(f) = |C| - 1 < |K_e| - 1 = \lambda(e)$. Thus, e is not an edge of $G[C]$ by Lemma 2.22. Hence, C cannot contain both i and j. It follows that either $C \subseteq C_1$ or $C \subseteq C_2$. □

3. Vines: graphical and poset definitions

3.1. Posets

In this subsection, we recall some basic definitions and notions of posets. All posets $\mathcal{P} = (\mathcal{P}, \leq_{\mathcal{P}})$ in this paper are finite. For a poset $\mathcal{P}$, an element $a \in \mathcal{P}$ is called **maximal** (resp. **minimal**) if there is no other element $b \in \mathcal{P}$ such that $a < b$ (resp. $a > b$). Denote by $\max(\mathcal{P})$ (resp. $\min(\mathcal{P})$) the set of all maximal (resp. minimal) elements in $\mathcal{P}$.

Definition 3.1 (Join). Let $\mathcal{P}$ be a poset and let $x, y \in \mathcal{P}$. An element $v \in \mathcal{P}$ is called the **join** of x and y, denoted $x \vee y$, if the following two conditions are satisfied:

(1) $x \leq v$ and $y \leq v$.
(2) For any $w \in \mathcal{P}$, if $x \leq w$ and $y \leq w$, then $v \leq w$.

The join $x \vee y$ is unique if it exists.

Definition 3.2 (Induced subposet). A poset $(\mathcal{Q}, \leq_{\mathcal{Q}})$ is an **induced subposet** of a poset $(\mathcal{P}, \leq_{\mathcal{P}})$ if $\mathcal{Q} \subseteq \mathcal{P}$ and for any $a, b \in \mathcal{Q}$, it holds that $a \leq_{\mathcal{Q}} b$ if and only if $a \leq_{\mathcal{P}} b$.

For $x, y \in \mathcal{P}$, by y **covers** x, we mean $x < y$ and $x \leq z < y$ implies $x = z$.

Definition 3.3 (Graded poset). A finite poset $\mathcal{P}$ is **graded** if there exists a **rank function** $\mathrm{rk} = \mathrm{rk}_{\mathcal{P}} : \mathcal{P} \longrightarrow \mathbb{Z}_{\geq 0}$ satisfying the following three properties:

(1) For any $x, y \in \mathcal{P}$, if $x < y$, then $\mathrm{rk}(x) < \mathrm{rk}(y)$.
(2) If y covers x, then $\mathrm{rk}(x) = \mathrm{rk}(y) - 1$.
(3) All minimal elements of $\mathcal{P}$ have the same rank. In this paper, we assume[4] $\mathrm{rk}(x) = 1$ for all $x \in \min(\mathcal{P})$.

Equivalently, for every $x \in \mathcal{P}$, all maximal chains among those with x as greatest element have the same length.

The **dimension**[5] $\dim(\mathcal{P})$ of $\mathcal{P}$ is defined as $\dim(\mathcal{P}) := |\min(\mathcal{P})|$. The **rank** $\mathrm{rk}(\mathcal{P})$ of a graded poset $\mathcal{P}$ with rank function rk is defined as

$$\mathrm{rk}(\mathcal{P}) := \max\{\mathrm{rk}(x) \mid x \in \mathcal{P}\}.$$

Definition 3.4 (Ideal, principal ideal). Let $\mathcal{P}$ be a poset. An (order) **ideal** $\mathcal{I}$ of $\mathcal{P}$ is a downward-closed subset (i.e., for every $x \in \mathcal{P}$ and $y \in \mathcal{I}$, $x \leq y$ implies that $x \in \mathcal{I}$). For $a \in \mathcal{P}$, the ideal

$$\mathcal{P}_{\leq a} := \{x \in \mathcal{P} \mid x \leq a\}$$

is called the **principal** ideal of $\mathcal{P}$ generated by a.

[4]A motivation for this assumption is the equivalence between D-vine and root poset of type A (Remark 4.16). The latter is graded by heights of positive roots, and all the minimal elements (simple roots) have rank (height) 1.

[5]The term 'dimension' of a poset may have a different meaning in the other context. The present definition is to make a compatibility for dimensions of a vine (Remark 3.8) and the ambient space of graphic arrangements.

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Definition 3.5 (Poset homomorphism). Let $\mathcal{P}$ and $\mathcal{P}'$ be posets. A **(poset) homomorphism** $\varphi : \mathcal{P} \longrightarrow \mathcal{P}'$ is an order-preserving map, i.e. $x \leq y$ implies $\varphi(x) \leq \varphi(y)$ for all $x, y \in \mathcal{P}$.

We call φ a **join-preserving** homomorphism if for any $x, y \in \mathcal{P}$ such that the join $x \vee y$ exists, then $\varphi(x) \vee \varphi(y)$ exists and $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$.

We call φ an **isomorphism** if φ is bijective and its inverse is a homomorphism. The posets $\mathcal{P}$ and $\mathcal{P}'$ are said to be **isomorphic**, written $\mathcal{P} \simeq \mathcal{P}'$ if there exists an isomorphism $\varphi : \mathcal{P} \longrightarrow \mathcal{P}'$.

When $\mathcal{P} = (\mathcal{P}, \mathrm{rk})$ and $\mathcal{P}' = (\mathcal{P}', \mathrm{rk}')$ are graded posets, a homomorphism $\varphi : \mathcal{P} \longrightarrow \mathcal{P}'$ is called **rank-preserving** if $\mathrm{rk}'(\varphi(x)) = \mathrm{rk}(x)$ for all $x \in \mathcal{P}$.

A rank-preserving homomorphism $\varphi : \mathcal{P} \longrightarrow \mathcal{P}'$ sends a minimal element to a minimal element (i.e., $\varphi(\min(\mathcal{P})) \subseteq \min(\mathcal{P}')$). Any isomorphism between two graded posets is a homomorphism preserving rank and join.

3.2. Vines (graphical definition)

First we recall the graphical definition of a *vine* following [5, Definition 4.1].

Definition 3.6 (Graphical definition of vine). Let $1 \leq n \leq \ell$ be positive integers. A **(graphical) vine** $\mathcal{V}$ on ℓ elements $[\ell] = \{1, \ldots, \ell\}$ (or more generally, on an ℓ-element set called N_1) is an ordered n-tuple $\mathcal{V} = (F_1, F_2, \ldots, F_n)$ such that

(1) F_1 is a forest with nodes $N_1 = [\ell]$ and a set of edges denoted E_1,
(2) for $2 \leq i \leq n$, F_i is a forest with nodes $N_i = E_{i-1}$ and edge set E_i.

We call F_i the **i-th associated forest** of $\mathcal{V}$. A graphical vine is uniquely determined by its associated forests. Denote by $N(\mathcal{V}) = N_1 \cup \cdots \cup N_n$ the set of nodes (of the associated forests) of $\mathcal{V}$. We call the numbers n and ℓ the **rank** and **dimension** of $\mathcal{V}$, respectively.

If node u is an element of node v (i.e., $u \in v$), we say that u is a **child** of v. If v is reachable from u via the membership relation: $u \in u_1 \in \cdots \in v$, we say that u is a **descendant** of v.

Definition 3.7 (Node poset). Let $\mathcal{V}$ be a graphical vine with node set $N(\mathcal{V})$. The **node poset** $\mathcal{P} = \mathcal{P}(\mathcal{V})$ of $\mathcal{V}$ is the poset $(N(\mathcal{V}), \leq)$ defined as follows: For any $u, v \in N(\mathcal{V})$,

$$u \leq v \quad \text{if} \quad u \text{ is a descendant of } v.$$

Remark 3.8. We emphasize that a graphical vine is uniquely determined by its node poset. The terminology 'rank' of a vine has motivation from poset theory. If a vine $\mathcal{V}$ is an ordered n-tuple, then $\mathcal{P} = \mathcal{P}(\mathcal{V})$ is a graded poset with rank function $\mathrm{rk}(v) = i$ for $v \in N_i$ $(1 \leq i \leq n)$. Thus, this number n equals the rank of $\mathcal{P}$. In addition, the dimension of $\mathcal{V}$ equals the number of minimal elements in $\mathcal{P}$, or the dimension of $\mathcal{P}$.

Now we introduce the notion of an *induced subvine*, or more generally, a *subvine* of a vine following [8, §5], [28, §2.2.1].

Definition 3.9 (Subvine, induced subvine). Let $\mathcal{V} = (F_1, F_2, \ldots, F_n)$ be a graphical vine.

1. An ordered p-tuple $\mathcal{V}' = (F'_1, F'_2, \ldots, F'_p)$ for $p \leq n$ is called a **subvine** of $\mathcal{V}$ if F'_i is a subgraph of F_i for each $1 \leq i \leq p$ and $\mathcal{V}'$ itself is a vine.
2. Given a subset $S \subseteq N_1$, define a vine $\mathcal{V}[S] = (F'_1, F'_2, \ldots, F'_p)$ on the set S as follows:
   (1) $F'_1 = F_1[S]$ with the edge set $E'_1 \subseteq E_1 = N_2$,
   (2) for $2 \leq i \leq p$, $F'_i = F_i[E'_{i-1}]$ with the edge set $E'_i \subseteq E_i = N_{i+1}$.
   We call $\mathcal{V}[S]$ the **subvine of $\mathcal{V}$ induced by S**.

Remark 3.10. Any induced subvine is obviously a subvine, but the converse is not necessarily true. For example, let $\mathcal{V} = (F_1, F_2)$ be a vine of dimension 2 with $N_1 = \{1, 2\}$, $N_2 = E_1 = \{\{1, 2\}\}$, $E_2 = \varnothing$. The subvine $\mathcal{V}' = (F'_1)$ defined by $N'_1 = \{1, 2\}$, $E'_1 = \varnothing$ is not an induced subvine of $\mathcal{V}$.

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3.3. Vines (poset definition)

Assumption & Notation 3.11. From now on, unless otherwise stated, we assume that $\mathcal{P}$ is a finite graded poset with a rank function $\mathrm{rk} : \mathcal{P} \longrightarrow \mathbb{Z}_{>0}$. Denote $n := \mathrm{rk}(\mathcal{P})$ and $\ell := \dim(\mathcal{P})$. For $v \in \mathcal{P}$, denote by $\mathcal{E}(v)$ the set of elements covered by v. For $i \geq 0$, define $\mathcal{P}_i := \{v \in \mathcal{P} \mid \mathrm{rk}(v) = i\}$ and $\mathcal{E}(\mathcal{P}_i) := \{\mathcal{E}(v) \mid v \in \mathcal{P}_i\}$. If $\mathcal{P}$ is an ℓ-dimensional poset, we assume $\mathcal{P}_1 = \min(\mathcal{P}) = [\ell]$.

As noted earlier in Remark 3.8, we may think of a graphical vine and its node poset essentially as the same object. It is thus natural to look for a characterization of the node poset of a vine. We give below such a characterization obtained immediately from Definition 3.6.

Definition & Proposition 3.12 (Poset definition of vine). A finite graded poset $\mathcal{P}$ is the node poset of a graphical vine if and only if $\mathcal{P}$ satisfies the following conditions:

(1) Every non-minimal node covers exactly two other nodes.
(2) For each $1 \leq i \leq n = \mathrm{rk}(\mathcal{P})$, the graph $F_i = (N_i, E_i)$ with node set $N_i := \mathcal{P}_i$ and edge set $E_i := \mathcal{E}(\mathcal{P}_{i+1})$ is a forest.

Assumption & Notation 3.13. From now on, unless otherwise stated, by a vine $\mathcal{P}$ we mean a finite graded poset satisfying the two conditions in 3.12. We will also retain the notion i-th associated forest $F_i = (\mathcal{P}_i, \mathcal{E}(\mathcal{P}_{i+1}))$ $(1 \leq i \leq n)$ of $\mathcal{P}$. If v is a node in a vine $\mathcal{P}$ and $\mathcal{E}(v) = \{a, b\}$, we will often abuse notation and write $v = \{a, b\}$. This notation is compatible with the notation of node/edge in the graphical definition of a vine.

The main reason why we choose the poset definition of a vine is because many terms and properties of a (graphical) vine have natural meanings in the language of posets. For example, subvine corresponds to ideal (Lemma 3.18), conditioned set corresponds to join (Lemma 5.1), and m-vine corresponds to LR-vine or local regularity of vine (Theorem 6.13).

Under this consideration, the following poset definition of a *regular vine* is equivalent to the well-known graphical definition of it in the literature (e.g., [5, Definition 4.1]).

Definition 3.14 (R-vine). A vine $\mathcal{P}$ is a **regular vine**, or an *R-vine* for short, if $\mathcal{P}$ satisfies the following conditions:

(1) $\mathrm{rk}(\mathcal{P}) = \dim(\mathcal{P})$ (i.e., $n = \ell$).
(2) Each associated forest $F_i = (\mathcal{P}_i, \mathcal{E}(\mathcal{P}_{i+1}))$ is a tree $(1 \leq i \leq n)$.
(3) **Proximity**: For any distinct nodes $a, b \in \mathcal{P}_i$ for $i \geq 2$, if a and b are covered by a common node, then a and b cover a common node.

Remark 3.15. If $\mathcal{P}$ is an R-vine of rank n, then $|\mathcal{P}_i| = n + 1 - i$ for each $1 \leq i \leq n$. In particular, $|\mathcal{P}| = n(n+1)/2$.

Next, we introduce the notion of a *locally* regular vine.

Definition 3.16 (LR-vine). A vine $\mathcal{P}$ is a **locally regular vine**, or an *LR-vine* for short, if every principal ideal of $\mathcal{P}$ is an R-vine.

Remark 3.17. Intuitively, an LR-vine is a vine that 'locally' looks like an R-vine. In particular, any R-vine is an LR-vine (see Proposition 4.4). Any ideal of a vine (resp. an LR-vine) is itself a vine (resp. an LR-vine).

Lemma 3.18. *Let $\mathcal{V}$ be a graphical vine with the node poset $\mathcal{P}(\mathcal{V})$. A subset $\mathcal{I}$ is an ideal of $\mathcal{P}(\mathcal{V})$ if and only if $\mathcal{I} = \mathcal{P}(\mathcal{V}')$ where $\mathcal{V}'$ is a subvine of $\mathcal{V}$ uniquely determined by $\mathcal{I}$.*

As a result, there is a one-to-one correspondence between the subvines of a graphical vine and the ideals of its node poset.

Proof. Let $\mathcal{V}'$ be a subvine of $\mathcal{V}$. Since $\mathcal{V}'$ itself is a vine, if v is a node in $\mathcal{P}(\mathcal{V}')$, then both children, hence all descendants of v, are also nodes in $\mathcal{P}(\mathcal{V}')$. Hence, $\mathcal{P}(\mathcal{V}')$ is an ideal of $\mathcal{P}(\mathcal{V})$. Conversely, let $\mathcal{I}$

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be an ideal of the vine $\mathcal{P}(\mathcal{V})$. By Remark 3.17, $\mathcal{I}$ itself is a vine. By Remark 3.8, $\mathcal{I}$ uniquely determines a vine $\mathcal{V}'$ which is a subvine of $\mathcal{V}$ and satisfies $\mathcal{I} = \mathcal{P}(\mathcal{V}')$. □

We close this section by recalling the definition of an *m-saturated vine* from [16, Definition 4.2].

Definition 3.19 (M-vine). A vine $\mathcal{P}$ is called an **m-saturated vine**, or an *m-vine* for short, if $\mathcal{P}$ is an ideal of an R-vine.

By Remark 3.17, any m-vine is an LR-vine. We will see in Theorem 6.13 that the converse also holds true.

4. From MAT-labeled graphs to LR-vines

4.1. Some known properties of vines

We begin by defining some statistics on the nodes of a vine. They play an important role in probabilistic applications of vines (e.g., [4, Theorem 3]).

Definition 4.1 (k-fold union, complete union). Let $\mathcal{P}$ be a vine of rank n. For any node $v_i \in \mathcal{P}_i$ $(1 \le i \le n)$ and integer k with $0 \le k \le i-1$, the **k-fold union** of v_i is the subset $U_{v_i}(k) \subseteq \mathcal{P}_{i-k}$ defined by

$$U_{v_i}(k) := \{x \in \mathcal{P}_{i-k} \mid x \le v_i\}.$$

The **complete union** U_{v_i} of $v_i \in \mathcal{P}_i$ is defined as the $(i-1)$-fold union of v_i; that is,

$$U_{v_i} := U_{v_i}(i-1) \subseteq \mathcal{P}_1.$$

Definition 4.2 (Conditioned set, conditioning set). Let $\mathcal{P}$ be a vine of rank n. Let $v_i = \{a, b\} \in \mathcal{P}_i$ $(2 \le i \le n)$ with $a, b \in \mathcal{P}_{i-1}$ (see notation in 3.13). The **conditioning set** D_{v_i} associated with v_i is defined by

$$D_{v_i} := U_a \cap U_b,$$

and the **conditioned set** C_{v_i} associated with v_i is defined by

$$C_{v_i} := U_a \triangle U_b,$$

where $\triangle$ denotes the symmetric difference.

It is easily seen that the nodes of an LR-vine satisfy the proximity condition. The following properties were proved for an R-vine in [5, 15, 16]. The arguments therein apply to a vine satisfying proximity condition as well since we only need the proximity of principal ideals.

Lemma 4.3. *Let $\mathcal{P}$ be a vine of rank n and $v_i \in \mathcal{P}_i$ $(2 \le i \le n)$. Suppose that the proximity condition holds. The following hold:*

(a) $|U_{v_i}(k)| = k+1$ *for* $0 \le k \le i-1$. *In particular,* $|U_{v_i}| = i = \mathrm{rk}(v_i)$.
(b) $|D_{v_i}| = i-2$ *and* $|C_{v_i}| = 2$.

We show below that local regularity and proximity of a vine are actually equivalent.

Proposition 4.4. *A vine $\mathcal{P}$ is locally regular if and only if the proximity condition holds for the nodes of $\mathcal{P}$.*

Proof. It remains to show proximity implies local regularity. Let $v \in \mathcal{P}$. We need to show that the principal ideal $\mathcal{P}_{\le v}$ itself is an R-vine. Write $\mathcal{P}_{\le v} = (T_1, T_2, \ldots, T_p)$, where $p = \mathrm{rk}(v) \le n$. By Lemma 4.3(a), the rank p and dimension $|U_v|$ of $\mathcal{P}_{\le v}$ are equal. Also by Lemma 4.3(a), each forest T_{p-k} $(0 \le k \le p-1)$ has $k+1$ nodes and k edges. Thus, these forests must be trees. Clearly, proximity of a vine is preserved under taking ideals. It follows that $\mathcal{P}_{\le v}$ is an R-vine. □

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Lemma 4.5. *Let $\mathcal{P} = (F_1, \ldots, F_n)$ be an LR-vine and $\mathcal{V}$ be the graphical vine defined by $\mathcal{P}$. Then for every $a \in \mathcal{P}$, the ideal $\mathcal{P}_{\leq a}$ coincides with the node poset of the induced subvine $\mathcal{V}[U_a]$.*

Proof. Write $\mathcal{V}[U_a] = (F_1', \ldots, F_q')$ and $\mathcal{P}_{\leq a} = (T_1, \ldots, T_p)$, where T_i's all are trees. Note that $F_1' = F_1[U_a]$ is a forest with at most $|U_a| - 1$ edges. However, the tree T_1 is a subgraph of F_1 with node set U_a that has exactly $|U_a| - 1$ edges. Hence, $F_1' = T_1$. A repeated application of this argument yields $p = q$ and $F_k' = T_k$ for all $1 \leq k \leq p$. Therefore, $\mathcal{P}_{\leq a}$ is the node poset of $\mathcal{V}[U_a]$. □

Corollary 4.6. *Let $\mathcal{P}$ be an LR-vine and a, b be nodes in $\mathcal{P}$. If $U_a \subseteq U_b$, then $a \leq b$. In particular, if $U_a = U_b$, then $a = b$.*

Proof. Let $\mathcal{V}$ be the graphical vine defined by $\mathcal{P}$. If $U_a \subseteq U_b$, then $\mathcal{V}[U_a]$ is a subvine of $\mathcal{V}[U_b]$. By Lemma 4.5, $\mathcal{P}_{\leq a} \subseteq \mathcal{P}_{\leq b}$. Hence, $a \leq b$. If $U_a = U_b$, then by the first assertion, $a \leq b$ and $b \leq a$. Thus, $a = b$. □

Remark 4.7. By definition, an R-vine has a unique maximal element. Thus, an ideal $\mathcal{I}$ of an LR-vine $\mathcal{P}$ is regular if and only if $\mathcal{I}$ is a principal ideal.

If two posets $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic and $\mathcal{P}$ is an (L)R-vine, then $\mathcal{P}'$ is also an (L)R-vine. The result below enables us to represent a node in an LR-vine by its complete union (see 4.17 for an example).

Proposition 4.8. *Let $\mathcal{P}$ be an LR-vine. Let $\widehat{\mathcal{P}}$ be the poset consisting of the complete unions of the nodes in $\mathcal{P}$ (i.e., $\widehat{\mathcal{P}} = \{U_a \mid a \in \mathcal{P}\}$ with partial order given by set inclusion). Define a map*

$$\eta_{\mathcal{P}} \colon \mathcal{P} \longrightarrow \widehat{\mathcal{P}} \quad \textit{via} \quad a \mapsto U_a.$$

Then $\eta_{\mathcal{P}}$ is a poset isomorphism; hence, $\mathcal{P} \simeq \widehat{\mathcal{P}}$.

Proof. Clearly, $\eta_{\mathcal{P}}$ is a surjective homomorphism. By Corollary 4.6, for any $a, b \in \mathcal{P}$, if $U_a = U_b$, then $a = b$. Thus, $\eta_{\mathcal{P}}$ is injective and hence bijective. Again by Corollary 4.6, for any $a, b \in \mathcal{P}$, $U_a \subseteq U_b$ if and only if $a \leq b$. Thus, the inverse of $\eta_{\mathcal{P}}$ is a poset homomorphism. We conclude that $\eta_{\mathcal{P}}$ is an isomorphism. □

Given a vine, it is important to know which induced subposet is again a vine. This motivates the following notion of *truncation* of a vine [6].

Definition 4.9 (Truncation). Let $(\mathcal{P}, \mathrm{rk})$ be a finite graded poset of rank n and let $1 \leq k \leq n$. The induced subposet $\overline{\mathcal{P}}_{\leq k} := \{x \in \mathcal{P} \mid \mathrm{rk}(x) \leq k\} = \bigcup_{i=1}^{k} \mathcal{P}_i$ with the rank function $\overline{\mathrm{rk}} = \mathrm{rk}$ is called the **k-lower truncation** of $\mathcal{P}$.

Likewise, the induced subposet $\overline{\mathcal{P}}_{\geq k} := \{x \in \mathcal{P} \mid \mathrm{rk}(x) \geq k\} = \bigcup_{i=k}^{n} \mathcal{P}_i$ with the rank function $\overline{\mathrm{rk}}(v) = \mathrm{rk}(v) - k + 1$ for all $v \in \overline{\mathcal{P}}_{\geq k}$ is called the **k-upper truncation** of $\mathcal{P}$.

An induced subposet $\mathcal{Q}$ of $\mathcal{P}$ is called a **lower** (resp. an **upper**) **truncation** if $\mathcal{Q} = (\overline{\mathcal{P}}_{\leq k}, \overline{\mathrm{rk}})$ (resp. $\mathcal{Q} = (\overline{\mathcal{P}}_{\geq k}, \overline{\mathrm{rk}})$) for some k. A truncation $\mathcal{Q}$ of $\mathcal{P}$ is called **proper** if $\mathcal{Q} \neq \mathcal{P}$.

Remark 4.10. Any lower truncation of a vine is an ideal. Hence, by Remark 3.17, any lower truncation of a vine (resp. an LR-vine) is itself a vine (resp. an LR-vine). However, a proper lower truncation of an R-vine of rank > 1 is not an R-vine. See Figure 5 for an example of a lower truncation.

A proper upper truncation of a vine of rank > 1 is not an ideal. However, proximity is preserved under taking either upper or lower truncation. Hence, by Proposition 4.4, any upper truncation of an LR-vine (resp. a vine) is an LR-vine (resp. a vine). Unlike the lower truncation case, any upper truncation of an R-vine is an R-vine by Remark 3.15.

The discussion above indicates that LR-vines are closed under either upper or lower truncation, while R-vines are only closed under upper truncation. We will see in §7.2.2 that these classes are also closed under 'vertical' truncation, or more precisely, *marginalization*.

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4.2. Construct an LR-vine from a given MAT-labeled graph

Definition 4.11. Let (G, λ) be an MAT-labeled graph with $N_G = [\ell]$ and clique number $\omega(G)$. Define a finite graded poset $\mathcal{P} = (\mathcal{P}, \leq_{\mathcal{P}}, \mathrm{rk}_{\mathcal{P}})$ from (G, λ) as follows:

(1) $\mathcal{P}$ consists of the sets $\{i\}$ for $1 \leq i \leq \ell$ and all the principal cliques in (G, λ) (Lemma 2.22).
(2) For $u, v \in \mathcal{P}$, $u \leq_{\mathcal{P}} v$ if u is a subset of v.
(3) $\mathrm{rk}_{\mathcal{P}}(v) = |v|$ for all $v \in \mathcal{P}$.

Remark 4.12. It is easily seen that $\min(\mathcal{P}) = \{\{i\} \mid 1 \leq i \leq \ell\}$. The poset $\mathcal{P}$ is graded by $\mathrm{rk}_{\mathcal{P}}$ because by Lemma 2.24, for every edge $e = \{i, j\}$ in G, the principal clique K_e generated by e covers exactly two principal cliques $K_e \setminus \{i\}$ and $K_e \setminus \{j\}$. Note also that $\mathrm{rk}(\mathcal{P}) = \omega(G)$ by Lemma 2.6.

Theorem 4.13. *The poset $\mathcal{P} = (\mathcal{P}, \leq_{\mathcal{P}}, \mathrm{rk}_{\mathcal{P}})$ from Definition 4.11 is an LR-vine. In particular, if (G, λ) is an MAT-labeled complete graph, then $\mathcal{P}$ is an R-vine.*

Proof. First we prove the first assertion. We argue by induction on the number ℓ of vertices of G. The assertion is clearly true when $\ell = 1$. Suppose $\ell \geq 2$. By Theorem 2.17, there exists an MAT-PEO $(a_1, \ldots, a_\ell)$ of (G, λ). Denote $G' := G \setminus a_\ell$ and $\lambda' := \lambda|_{E_{G'}}$. By Lemma 2.15, (G', λ') is an MAT-labeled graph. Let $\mathcal{P}'$ be the poset defined by (G', λ') using Definition 4.11. By the induction hypothesis, $\mathcal{P}'$ is an LR-vine.

Denote $d := \deg_G(a_\ell)$. If $d = 0$ (i.e., a_ℓ is an isolated vertex), then $\mathcal{P} = \mathcal{P}' \cup \{\{a_\ell\}\}$ is clearly an LR-vine. Suppose $d \geq 1$. Write $\mathrm{Nbd}_G(a_\ell) := \{b_q \mid 1 \leq q \leq d\} \subseteq N_{G'}$ so that $\{a_\ell, b_q\} \in \pi_q$. For $1 \leq q \leq d$, define the following subgraph H_q of G:

$$H_q := G \setminus \{\{a_\ell, b_{q+1}\}, \ldots, \{a_\ell, b_d\}\}.$$

It is not hard to check that a_ℓ is an MAT-simplicial vertex in $(H_q, \lambda|_{E_{H_q}})$ for each $1 \leq q \leq d$. By Lemma 2.15, each $(H_q, \lambda|_{E_{H_q}})$ is an MAT-labeled graph since $G' = H_q \setminus a_\ell$.

For each $1 \leq q \leq d$, let $\mathcal{R}_q$ be the poset defined by $(H_q, \lambda|_{E_{H_q}})$ from Definition 4.11. We may write

$$\mathcal{R}_q = \mathcal{P}' \cup \{v_0, \ldots, v_q\} = \mathcal{P} \setminus \{v_{q+1}, \ldots, v_d\},$$

where $v_0 := \{a_\ell\}$ and each v_q is the principal clique generated by $\{a_\ell, b_q\}$. We claim that for each $1 \leq q \leq d$, the poset $\mathcal{R}_q$ is an LR-vine. In particular, the case $q = d$ whence $H_d = G$ and $\mathcal{R}_d = \mathcal{P}$ yields the first assertion of Theorem 4.13.

We argue by induction on q. The case $q = 1$ is simple. The new non-minimal node $v_1 = \{a_\ell, b_1\}$ covers exactly two nodes $v_0 = \{a_\ell\}$ and $\{b_1\}$. Also, the new minimal node v_0 is only covered by v_1 since $a_\ell \notin v$ for all $v \in \mathcal{P}'$. The first associated forest of $\mathcal{R}_1$ is given by that of $\mathcal{P}'$ with a_ℓ added so that a_ℓ is only connected to b_1. The second associated forest of $\mathcal{R}_1$ is given by that of $\mathcal{P}'$ with an isolated vertex v_1 added. The remaining associated forests of $\mathcal{R}_1$ are the same as those of $\mathcal{P}'$. The proximity condition holds trivially. Hence, $\mathcal{R}_1$ is an LR-vine by Proposition 4.4.

Suppose that the claim is true for some $1 \leq q < d$. First note that $H_{q+1} \setminus \{a_\ell, b_{q+1}\} = H_q$ and $\mathcal{R}_{q+1} \setminus \{v_{q+1}\} = \mathcal{R}_q$. Since a_ℓ is an MAT-simplicial vertex in $(H_q, \lambda|_{E_{H_q}})$, by (MS3), the edges in the complete subgraph $H_q[v_q]$ of H_q induced by the vertices in v_q have label $\leq q-1$ except $\{a_\ell, b_q\} \in \pi_q$. Similarly, any edge of the form $\{b_{q+1}, h\}$ where h is a vertex in $v_q \setminus \{a_\ell\}$ has label $\leq q$. Therefore, $v_q \setminus \{a_\ell\}$ consists of the q conditioning vertices of $\{a_\ell, b_{q+1}\}$ in H_{q+1}. Hence, $v_{q+1} = v_q \cup \{b_{q+1}\}$.

By Lemma 2.24, $u := v_{q+1} \setminus \{a_\ell\}$ is a principal clique (of cardinality $q+1$) in (G', λ') and hence a vertex in $\mathcal{P}'$. Let c be the vertex in u such that c is largest with respect to the MAT-PEO $(a_1, \ldots, a_{\ell-1})$ of (G', λ'). Since the edges in the complete subgraph $G'[u]$ have label $\leq q$, by (MS2), there exists among these edges an edge e incident on c of label q. By the preceding paragraph, this edge e must be $\{b_{q+1}, h^*\}$ for some vertex h^* in $v_q \setminus \{a_\ell\}$. Moreover, such an edge e is unique which is guaranteed by (ML1). Therefore, u is generated by e. In particular, u covers $v_q \setminus \{a_\ell\}$ in $\mathcal{P}'$ by Lemma 2.24. (See Figure 1 for a pictorial illustration of the proof.)

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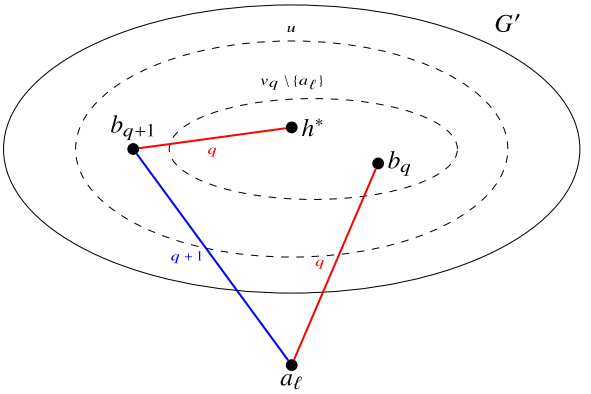


Figure 1. *A pictorial illustration of the proof of Theorem 4.13.*

Now v_{q+1} covers exactly two nodes v_q and u in $\mathcal{R}_{q+1}$. Also, v_q is covered only by v_{q+1} since $a_\ell \notin v$ for all $v \in \mathcal{P}'$. The $(q+1)$-th associated forest of $\mathcal{R}_{q+1}$ is given by that of $\mathcal{R}_q$ with v_q added so that v_q is only connected to u. The $(q+2)$-th associated forest of $\mathcal{R}_{q+1}$ is given by that of $\mathcal{R}_q$ with an isolated vertex v_{q+1} added. The remaining associated forests of $\mathcal{R}_{q+1}$ are the same as those of $\mathcal{R}_q$. This follows that $\mathcal{R}_{q+1}$ is a vine. Furthermore, both $v_q \in \mathcal{R}_q$ and $u \in \mathcal{P}'$ cover $v_q \setminus \{a_\ell\}$. Therefore, the proximity condition holds in $\mathcal{R}_{q+1}$. Hence, $\mathcal{R}_{q+1}$ is an LR-vine by Proposition 4.4.

Finally, we show the second assertion of Theorem 4.13. If (G, λ) is an MAT-labeled complete graph, then $\mathrm{rk}(\mathcal{P}) = \dim(\mathcal{P}) = \ell$. The proofs for the proximity of $\mathcal{P}$ and the fact the associated forests of $\mathcal{P}$ are trees run essentially along the same line as the proof of the first assertion. □

Corollary 4.14. *Given an MAT-labeled graph (G, λ), let $\mathcal{P}$ denote the LR-vine defined by (G, λ) from Definition 4.11. Then a node $v \in \mathcal{P}$ has complete union $U_v = \{\{a\} \mid a \in v\}$. Moreover, if $v = K_e \in \mathcal{P}$ is a non-minimal node where $e = \{i, j\} \in E_G$, then v has conditioned set $C_v = \{\{i\}, \{j\}\}$.*

Proof. Use Remark 4.12 and argue by an induction on $\mathrm{rk}(v)$. □

We close this section by giving an example to illustrate the construction in Definition 4.11 and Theorem 4.13. First we need a definition.

Definition 4.15 (D-vine). An R-vine is called a **D-Vine** if each associated tree has the smallest possible number of vertices of degree 1. Equivalently, each associated tree is a *path graph*.

Remark 4.16. Let Φ be an irreducible root system in $\mathbb{R}^\ell$ with a fixed positive system $\Phi^+ \subseteq \Phi$ and the associated set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Suppose that Φ is of type A_ℓ and the Dynkin diagram of Φ is the path graph $\alpha_1 - \alpha_2 - \cdots - \alpha_\ell$. Then the positive roots of Φ are given by

$$\Phi^+ = \left\{ \sum_{i \le k \le j} \alpha_k \,\middle|\, 1 \le i \le j \le m \right\}.$$

It is not hard to show that the D-vine $\mathcal{P}$ with the first associated tree $1 - 2 - \cdots - \ell$ is isomorphic to the root poset $\mathcal{R}(A_\ell)$ of type A_ℓ under the following isomorphism:

$$\mu \colon \mathcal{P} \longrightarrow \mathcal{R}(A_\ell) \quad \text{via} \quad v \mapsto \sum_{k \in U_v} \alpha_k.$$

Example 4.17. Figure 2 depicts a 4-dimensional D-vine (middle) that can be constructed in three ways. First, it is the node poset of a graphical vine on [4] (left) under the isomorphism in Proposition 4.8. Second, it is the poset defined an MAT-labeled complete graph (right) using Definition 4.11. Third, it

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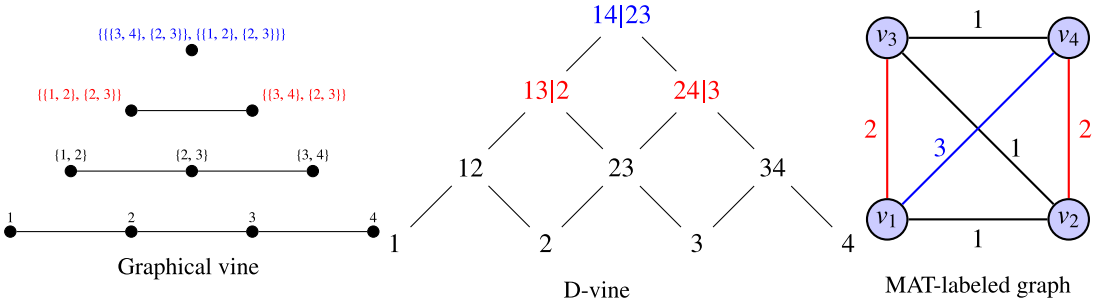


Figure 2. *An MAT-labeled complete graph on* 4 *vertices (right), the D-vine (middle) (= type A root poset) defined by the graph using Definition 4.11, and the corresponding graphical vine (left).*

is the root poset of type A_4 by Remark 4.16. The elements in the poset are written without set symbol for simplicity. The conditioned set of a non-minimal node is given to the left of the '|' sign, while the conditioning set appears on the right. For example, the top node $\{1, 2, 3, 4\}$ (or the largest clique generated by $\{v_1, v_4\}$) is written by 14|23.

Remark 4.18. Max Wakefield let us know an interesting observation that the D-vine is isomorphic to the intersection lattice (with bottom element removed) of the *Pascal arrangement* introduced in [19]. We leave a possible generalization of the main result in [19] to an R-vine for future research.

5. From LR-vines to MAT-labeled graphs

Constructing an MAT-labeled graph from a given LR-vine needs more effort.

5.1. Some new properties of LR-vines

The following lemma provides the key ingredient of our construction.

Lemma 5.1 (Joining path). *Let $\mathcal{P} = (F_1, \dots, F_n)$ be an ℓ-dimensional LR-vine. Let $i, j \in \min(\mathcal{P}) = [\ell]$ be distinct minimal nodes. Let $v \in \mathcal{P}_r$ be a non-minimal node ($2 \le r \le n$). The following are equivalent:*

(1) *$v = i \vee j$ (i.e., v is the join of i and j).*
(2) *$C_v = \{i, j\}$ (i.e., $\{i, j\}$ is the conditioned set of v).*
(3) *There exist r paths (uniquely determined by i and j) $P_k = P_k(i, j) \in E_k$ in the forests F_k ($1 \le k \le r$) satisfying the following three conditions:*
   (a) *$P_1 = (a_{1,1}, a_{1,2}, \dots, a_{1,p_1})$, $p_1 \ge 2$ is a (unique) path connecting nodes $a_{1,1} := i$ and $a_{1,p_1} := j$ in F_1.*
   (b) *For $2 \le k \le r$, $P_k = (a_{k,1}, a_{k,2}, \dots, a_{k,p_k})$, $p_k \ge 1$ is a (unique) path connecting nodes $a_{k,1} := \{a_{k-1,1}, a_{k-1,2}\}$ and $a_{k,p_k} := \{a_{k-1,p_{k-1}-1}, a_{k-1,p_{k-1}}\}$ in F_k.*
   (c) *$P_r = v$.*

In this case, we call the path $P_k(i, j) \in E_k$ ($1 \le k \le r$) the k-joining path *of the pair $\{i, j\}$ (or $P_k(v)$ the k-joining path of v).*

Before giving the proof of Lemma 5.1, we address some remarks.

Remark 5.2. If $\mathcal{P} = (T_1, \dots, T_\ell)$ is an R-vine, then for any distinct $i, j \in [\ell]$, the joining paths of $\{i, j\}$ always exist since T_k is a tree for each $1 \le k \le \ell$ and T_ℓ has only one node.

The implication (2) $\Leftarrow$ (3) was stated in the proof of [11, Lemma 3.9] for R-vines. Unfortunately, the proof was not complete. We give below a complete proof that works for an arbitrary LR-vine.

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Proof of Lemma 5.1. First we prove (2) $\Leftarrow$ (3). By definition, $P_{r-1} = \{a_{r-1,1}, a_{r-1,p_{r-1}}\} = v$. We need to show $C_v = U_{a_{r-1,1}} \triangle U_{a_{r-1,p_{r-1}}} = \{i, j\}$. Since $|C_v| = 2$ by Lemma 4.3(b), it is enough to show that $i \le a_{r-1,1}, i \not\le a_{r-1,p_{r-1}}$ and $j \le a_{r-1,p_{r-1}}, j \not\le a_{r-1,1}$. This follows once we prove that for each $1 \le k \le r$, the following two statements hold:

(S1) $i \le a_{k,1}$ and $i \not\le b$, where b is any node in F_k such that there exists a (unique) path in F_k connecting b and some non-initial node $a_{k,t}$ $(2 \le t \le p_k)$ of P_k but not passing through its initial node $a_{k,1}$.

(S2) $j \le a_{k,p_k}$ and $j \not\le c$, where c is any node in F_k such that there exists a (unique) path in F_k connecting c and some non-final node $a_{k,t}$ $(1 \le t \le p_k - 1)$ of P_k but not passing through its final node a_{k,p_k}.

Since the roles of i and j are symmetric, it suffices to prove Statement (S1). First we need the following crucial property of the paths P_k's.

Claim 5.3. *Fix $1 \le k \le r-1$. Suppose P_k is given by $P_k = (\alpha_1, \alpha_2, \dots, \alpha_p)$ for $p \ge 1$ in terms of its node sequence. Then the node set of P_{k+1} consists of all the edges $\{\alpha_t, \alpha_{t+1}\}$ for $1 \le t \le p-1$ of P_k, and some edges of the form $\{\alpha_{s+1}, \mu\}$ for $1 \le s \le p-2$ and $\mu \in \mathrm{Nbd}_{F_k}(\alpha_{s+1}) \setminus \{\alpha_s, \alpha_{s+2}\}$.*

Proof of Claim 5.3. By definition, the initial and final nodes in P_{k+1} are $\{\alpha_1, \alpha_2\}$ and $\{\alpha_{p-1}, \alpha_p\}$, respectively. The claim is clearly true for $p \le 3$. Suppose $p \ge 4$. By the proximity condition, the length of P_{k+1} is at least two. Again by the proximity, the node adjacent to the initial node in P_{k+1} must have the form either $\{\alpha_1, \beta\}$, where $\beta \in \mathrm{Nbd}_{F_k}(\alpha_1) \setminus \{\alpha_2\}$, or $\{\alpha_2, \gamma\}$, where $\gamma \in \mathrm{Nbd}_{F_k}(\alpha_2) \setminus \{\alpha_1\}$.

The former cannot occur; otherwise, arguing on the proximity yields two different paths in F_k connecting α_1 and α_p; one is P_k passing through α_2, the other passing through some $\beta' \in \mathrm{Nbd}_{F_k}(\alpha_1) \setminus \{\alpha_2\}$. Thus, the latter occurs, and P_{k+1} possibly continues to pass through some node of the same form as $\{\alpha_2, \gamma\}$. The following two conditions hold: The path P_{k+1} must (i) reach the node $\{\alpha_2, \alpha_3\}$, and after that, (ii) does not pass through any further node of this form. If either (i) or (ii) fails, then there are two different paths in F_k connecting α_2 and α_p. Hence, P_{k+1} passes through some node of the form $\{\alpha_3, \delta\}$, where $\delta \in \mathrm{Nbd}_{F_k}(\alpha_3) \setminus \{\alpha_2\}$ until it reaches $\{\alpha_3, \alpha_4\}$ and so on.

A repeated application of the argument above completes the proof of the claim.

$$P_k : \alpha_1 \to \alpha_2 \to \alpha_3 \to \cdots \to \alpha_p,$$
$$P_{k+1} : \{\alpha_1, \alpha_2\} \to \{\alpha_2, \gamma\} \to \cdots \to \{\alpha_2, \alpha_3\} \to \cdots \to \{\alpha_3, \alpha_4\} \to \cdots \to \{\alpha_{p-1}, \alpha_p\}. \qquad \square$$

Now we return to the proof of (S1). The first part is easy since by definition, $a_{1,1} = i$ and $a_{k,1} \le a_{k+1,1}$ for all $1 \le k \le r-1$. We argue the second part by induction on k. The statement is clearly true for $k = 1$. Suppose it is true for any $1 \le k < r$. Let $b = \{b_1, b_2\}$ be an arbitrary node in F_{k+1} such that there exists a path in F_{k+1} connecting b and some non-initial node of P_{k+1} but not passing through its initial node. By the relation of the paths P_k and P_{k+1} proved in Claim 5.3, there exists a path in F_k connecting b_s ($s = 1$ or 2) and some non-initial node of P_k but not passing through its initial node. By the induction hypothesis, we must have $i \not\le b_1$ and $i \not\le b_2$. It follows that $i \not\le b$. This completes the proof of (S1) and hence the proof of (2) $\Leftarrow$ (3).

To prove (2) $\Rightarrow$ (3), the following fact is useful.

Remark 5.4. Suppose $\mathcal{P}$ is an R-vine. By Remark 5.2, for any $1 \le i \ne j \le \ell$, the joining paths of $\{i, j\}$ always exist. Thus, by (2) $\Leftarrow$ (3), there exists a non-minimal node $v \in \mathcal{P}$ such that $C_v = \{i, j\}$. Moreover, by Remark 3.15, the number of non-minimal nodes in $\mathcal{P}$ is equal to $\ell(\ell-1)/2$. Therefore, every pair of distinct elements in $[\ell]$ occurs exactly once as the conditioned set of a non-minimal node.

Now we give the proof of (2) $\Rightarrow$ (3). Write $\mathcal{P}_{\le v} = (T_1, T_2, \dots, T_r)$. By definition, $\mathcal{P}_{\le v}$ is an R-vine. If $C_v = \{i, j\}$, then i and j are nodes in T_1. By Remark 5.2 and (2) $\Leftarrow$ (3), there exist r' joining paths of $\{i, j\}$ and hence a node v' in $\mathcal{P}_{\le v}$ such that $C_{v'} = \{i, j\} = C_v$. Hence, by Remark 5.4, $v = v'$ and $r = r'$.

Next, we show (1) $\Leftarrow$ (3). By Statements (S1) and (S2), $i \le v$ and $j \le v$. Let $u \in \mathcal{P}_s$ for $1 \le s \le n$ be a node such that $i \le u$ and $j \le u$. In particular, i and j are minimal nodes in the R-vine $\mathcal{P}_{\le u}$. By

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Remark 5.2, there exist s joining paths of $\{i, j\}$ in $\mathcal{P}_{\leq u}$. By the uniqueness of the paths in the associated forests F_k's, the joining paths of $\{i, j\}$ in $\mathcal{P}_{\leq u}$ and $\mathcal{P}_{\leq v}$ must be the same. Thus, $r = s$ and $v \leq u$. Hence, v is the join of i and j.

Finally, we show (1) $\Rightarrow$ (3). Suppose $v = i \vee j$. Hence, there exist s' joining paths of $\{i, j\}$ and hence a node u' in $\mathcal{P}_{\leq v}$ such that $i \leq u'$ and $j \leq u'$. By the definition of a join, we must have $v = u'$ and $s' = r$. □

Remark 5.5. The missing piece of the proof of [11, Lemma 3.9] is the fact that $i \not\leq a$ for any non-initial node a in P_k does not automatically imply that $i \not\leq b$ for any non-initial node b in P_{k+1}.

Example 5.6. Let $\mathcal{P}$ be the 4-dimensional D-vine in Figure 2. Let $i = 1$ and $j = 4$ be minimal nodes. The joining paths of 1 and 4 are given by $P_1 = (1, 2, 3, 4)$, $P_2 = (12, 23, 34)$, $P_3 = (13|2, 24|3)$, $P_4 = 14|23$. The join of 1 and 4 is 14|23. The conditioned set of 14|23 is $\{1, 4\}$. These calculations are consistent with Lemma 5.1.

The first two corollaries below are immediate consequences of Lemma 5.1 and Claim 5.3, respectively.

Corollary 5.7. *Let u, v be two non-minimal nodes in an LR-vine $\mathcal{P}$. If $C_u = C_v$, then $u = v$.*

Corollary 5.8. *Let v be a non-minimal node in an LR-vine $\mathcal{P}$. Then all the nodes of the joining paths of v are in $\mathcal{P}_{\leq v}$.*

Corollary 5.9. *Let $\mathcal{P}$ be an LR-vine. Let v be a non-minimal node in $\mathcal{P}$ with conditioned set $C_v = \{i, j\}$ for $i, j \in [\ell]$. If k is in the conditioning set of v, then the joins $u = i \vee k$ and $w = j \vee k$ exist in $\mathcal{P}$. Moreover, $v > u$ and $v > w$.*

Proof. If $k \in D_v$, then k is a minimal node of the R-vine $\mathcal{P}_{\leq v}$. By Remark 5.2, the joining paths of $\{i, k\}$ and $\{k, j\}$ always exist. Thus, by Lemma 5.1, the joins $u = i \vee k$ and $w = j \vee k$ exist in $\mathcal{P}_{\leq v}$ and hence in $\mathcal{P}$. Note that $v \neq u$ and $v \neq w$ since $C_v \neq C_u$ and $C_v \neq C_w$. Thus, $v > u$ and $v > w$. □

We give some further properties of the joining paths of a non-minimal node in an LR-vine.

Lemma 5.10. *Let $\mathcal{P} = (F_1, \ldots, F_n)$ be an LR-vine and let u, v be two distinct non-minimal nodes in $\mathcal{P}$. Suppose that there exists a number k ($1 \leq k \leq \mathrm{rk}(u)$) such that the k-joining path $P_k(u)$ is a proper subpath of the k-joining path $P_k(v)$. Then $P_q(u)$ is a proper subpath of $P_q(v)$ for all $k \leq q \leq \mathrm{rk}(u)$. In particular, $v > u$.*

Proof. Write $P_k(v) = (a_{k,1}, a_{k,2}, \ldots, a_{k,p_k})$, $p_k \geq 2$. Since $P_k(u)$ is a proper subpath of $P_k(v)$, we may write $P_k(u) = (a_{k,s}, \ldots, a_{k,t})$, where $1 \leq s \leq t \leq p_k$ and $(s, t) \neq (1, p_k)$. Note that $P_{k+1}(u)$ is the unique path connecting $\{a_{k,s}, a_{k,s+1}\}$ and $\{a_{k,t-1}, a_{k,t}\}$ in the forest F_{k+1}. Moreover, by Claim 5.3, $P_{k+1}(v)$ passes through $\{a_{k,s}, a_{k,s+1}\}$ and $\{a_{k,t-1}, a_{k,t}\}$. Therefore, $P_{k+1}(u)$ must be a proper subpath of $P_{k+1}(v)$. Applying this argument repeatedly, we may show that $P_q(u)$ is a proper subpath of $P_q(v)$ for all $k \leq q \leq \mathrm{rk}(u)$. In particular, the case $q = \mathrm{rk}(u)$ yields $v > u$. □

Before giving the next property in Lemma 5.13, we need a technical lemma on paths in a forest.

Lemma 5.11. *Let F be a forest. Let $i_1, i_2, \ldots, i_m$ for $m \geq 3$ be mutually distinct nodes in F. For each $1 \leq s \leq m$, suppose that there exists a (unique) path $P_{s,s+1}$ in F connecting i_s and i_{s+1}. Here, we take the index modulo m. Denote by $e'_{s,s+1}$ and $e''_{s,s+1}$ the edges in $P_{s,s+1}$ incident on i_s and i_{s+1}, respectively. Suppose that there exists $1 \leq t \leq m$ such that $e''_{t,t+1} \neq e'_{t+1,t+2}$. Then among the paths $P_{s,s+1}$'s for $s \notin \{t, t+1\}$ there exist two paths $P_{a,a+1}$ and $P_{b,b+1}$ (not necessarily distinct) both of length ≥ 2 containing $e''_{t,t+1}$ and $e'_{t+1,t+2}$, respectively.*

Proof. Let T denote the subgraph of F induced by the vertices of the paths $P_{s,s+1}$'s for all $1 \leq s \leq m$. Note that T is a connected subgraph of F; hence, T is a tree.

If $m = 3$, then the path $P_{t,t+2}$ of length ≥ 2 contains both $e''_{t,t+1}$ and $e'_{t+1,t+2}$. Suppose $m > 3$. By the assumption, the concatenation of $P_{t,t+1}$ and $P_{t+1,t+2}$, denoted P, is the unique path in T connecting i_t and i_{t+2}. Moreover, there exists a walk in T connecting i_t and i_{t+2} whose edge set is the union of the edge

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sets of $P_{t+2,t+3}, P_{t+3,t+4}, \ldots, P_{t-1,t}$. By Lemma 2.1, this walk must contain the path P. In particular, the edge $e''_{t,t+1}$ (resp. $e'_{t+1,t+2}$) is contained in a path $P_{a,a+1}$ (resp. $P_{b,b+1}$) for some $a, b \notin \{t, t+1\}$. Clearly, both $P_{a,a+1}$ and $P_{b,b+1}$ have lengths ≥ 2. □

Notation 5.12. Let $\mathcal{P}$ be an LR-vine. In what follows, for two distinct minimal nodes $i, j \in \min(\mathcal{P})$, if the join $i \vee j$ exists in $\mathcal{P}$, we denote $v_{i,j} := i \vee j \in \mathcal{P}$. Sometimes, two minimal nodes in $\mathcal{P}$ are denoted by i_s, i_t, in which case, we write $v_{s,t} := v_{i_s,i_t}$. Note that by Lemma 5.1, the nodes $v_{i,j}$'s are mutually distinct (i.e., if $\{i, j\} \neq \{i', j'\}$, then $v_{i,j} \neq v_{i',j'}$).

Lemma 5.13. *Let $\mathcal{P} = (F_1, \ldots, F_n)$ be an LR-vine of rank n. Let $i_1, i_2, \ldots, i_m \in \min(\mathcal{P})$ for $m \geq 3$ be mutually distinct minimal nodes in $\mathcal{P}$. Suppose that the join $v_{s,s+1}$ exists in $\mathcal{P}$ for each $1 \leq s \leq m$. Here again, we take the index modulo m. Then there exist a node i_t and a join $v_{a,a+1}$ for $a, t \in [m]$ such that $a \notin \{t-1, t\}$ and i_t belongs to the conditioning set of $v_{a,a+1}$.*

Proof. By Lemma 5.1, for each $1 \leq s \leq m$, there exist the k-joining paths $P_k(v_{s,s+1})$'s of $v_{s,s+1}$ for $1 \leq k \leq \mathrm{rk}(v_{s,s+1})$. In particular, $P_1(v_{s,s+1})$ is the unique path in F_1 connecting i_s and i_{s+1}. For each $1 \leq s \leq m$, denote by $e'_{s,s+1}$ and $e''_{s,s+1}$ the edges in $P_1(v_{s,s+1})$ incident on i_s and i_{s+1}, respectively. If there exists $t \in [m]$ such that $e''_{t,t+1} \neq e'_{t+1,t+2}$, then by Lemma 5.11, $e''_{t,t+1}$ is an edge of a path $P_1(v_{a,a+1})$ of length ≥ 2 for some $a \in [m] \setminus \{t, t+1\}$. By Claim 5.3, $e''_{t,t+1}$ is a node in $P_2(v_{a,a+1})$. By Corollary 5.8, $i_{t+1} < e''_{t,t+1} \leq v_{a,a+1}$. Hence, i_{t+1} belongs to the conditioning set of $v_{a,a+1}$.

Now consider the case $e''_{s-1,s} = e'_{s,s+1}$ for each $1 \leq s \leq m$. Denote this common edge by j_s. Note that these edges become nodes in F_2. By definition, $P_2(v_{s,s+1})$ is the path in F_2 connecting j_s and j_{s+1} for each $1 \leq s \leq m$. Denote by $f'_{s,s+1}$ and $f''_{s,s+1}$ the edges in $P_2(v_{s,s+1})$ incident on j_s and j_{s+1}, respectively. By a similar argument as in the preceding paragraph with the aid of Lemma 5.11, we only need to consider the case $f''_{s-1,s} = f'_{s,s+1}$ for all $1 \leq s \leq m$.

A repeated application of this argument leads us to the situation that for every k, there exist mutually distinct nodes $h_1, h_2, \ldots, h_m$ in F_k such that the k-joining path $P_k(v_{s,s+1})$ connects h_s and h_{s+1} for each $1 \leq s \leq m$. Furthermore, if $g'_{s,s+1}$ and $g''_{s,s+1}$ are the edges in $P_k(v_{s,s+1})$ incident on h_s and h_{s+1}, respectively, then $g''_{s-1,s} = g'_{s,s+1}$ for each $1 \leq s \leq m$.

However, let $q \in [m]$ such that $\mathrm{rk}(v_{q,q+1}) = \min\{\mathrm{rk}(v_{s,s+1}) \mid 1 \leq s \leq m\}$ and let $k = \mathrm{rk}(v_{q,q+1}) - 1$. Then the path $P_k(v_{q,q+1})$ has length 1 (or simply an edge in F_k). The situation in the paragraph above implies that $P_k(v_{q,q+1})$ is a proper subpath of $P_k(v_{q-1,q})$. By Lemma 5.10, $i_{q+1} < P_k(v_{q,q+1}) \leq v_{q-1,q}$. Hence, i_{q+1} belongs to the conditioning set of $v_{q-1,q}$. □

We have a stronger statement when $m = 3$ in Lemma 5.13. First we address a remark.

Remark 5.14. Let F be a forest. Let i_1, i_2 and i_3 be three mutually distinct nodes in F. Suppose there exist the paths $P_{1,2}, P_{2,3}, P_{3,1}$ in F connecting i_1 and i_2, i_2 and i_3, i_3 and i_1, respectively. Let T denote the subgraph (tree) of F induced by the vertices of the paths $P_{1,2}, P_{2,3}, P_{3,1}$. One may show that there does not exist a path among $P_{1,2}$, $P_{2,3}$ and $P_{3,1}$ being the concatenation of the other two paths if and only if i_1, i_2, i_3 are leaves in T. In particular, it is not the case if one of the paths has length 1.

Lemma 5.15. *Let $\mathcal{P} = (F_1, \ldots, F_n)$ be an LR-vine of rank n. Let $a_1, b_1, c_1 \in \min(\mathcal{P})$ be mutually distinct minimal nodes in $\mathcal{P}$. Suppose that the joins $u = a_1 \vee b_1$, $v = b_1 \vee c_1$ and $w = a_1 \vee c_1$ exist in $\mathcal{P}$. Then there exists a number $k (1 \leq k \leq n)$ such that one of the three k-joining paths $P_k(u)$, $P_k(v)$ and $P_k(w)$ is the concatenation of the other two paths.*

As a consequence, there exists a node among three nodes u, v and w strictly greater than the other two.

Proof. The proof of the first assertion is similar to the proof of Lemma 5.13 with the use of Remark 5.14 in place of Lemma 5.11. The second assertion follows from the first and Lemma 5.10. □

5.2. Construct an MAT-labeled graph from a given LR-vine

Definition 5.16. Let $\mathcal{P}$ be an LR-vine of dimension ℓ and rank n. Define an edge-labeled graph (G, λ) from $\mathcal{P}$ as follows:

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(1) The vertex set N_G is given by the set of minimal nodes (i.e., $N_G := \min(\mathcal{P}) = [\ell]$).
(2) The edge set E_G is given by the conditioned sets of non-minimal nodes; that is,

$$\begin{aligned} E_G &:= \{C_v \mid v \in \mathcal{P} \setminus \min(\mathcal{P})\} \\ &= \{\{i, j\} \subseteq [\ell] \mid i \neq j \text{ and the join} v_{i,j} = i \vee j \text{ exists in } \mathcal{P}\}. \end{aligned}$$

(The second expression of E_G above follows from Lemma 5.1.)
(3) The labeling $\lambda : E_G \longrightarrow \mathbb{Z}_{>0}$ is defined by

$$\lambda(i, j) := \mathrm{rk}_{\mathcal{P}}(v_{i,j}) - 1.$$

Theorem 5.17. *The edge-labeled graph (G, λ) from Definition 5.16 is an MAT-labeled graph. In particular, if $\mathcal{P}$ is an R-vine, then (G, λ) is an MAT-labeled complete graph.*

Proof. The first assertion follows from Lemmas 5.18 and 5.19 below. The second assertion follows from the first and Remark 5.4. □

Lemma 5.18. *(G, λ) satisfies (ML1).*

Proof. Suppose to the contrary that there exist $1 \leq k \leq n-1$ and an m-cycle C_m for $m \geq 3$ with edges $\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_m, i_1\}$ such that $\lambda(i_1, i_2) < k$ and $\lambda(i_s, i_{s+1}) = k$ for $2 \leq s \leq m$. Here, we take the index modulo m. We choose the smallest such m.

If $m = 3$, then by Lemma 5.15, there exists a node among the nodes $v_{1,2}$, $v_{2,3}$, $v_{3,1}$ strictly greater than the other two. This is a contradiction since there are two nodes of the same rank $k+1$, while the remaining node has rank $< k+1$. We may assume $m > 3$.

By Lemma 5.13, there exist a minimal node i_t and a join $v_{a,a+1} = i_a \vee i_{a+1}$ in $\mathcal{P}$ for $a, t \in [m]$ such that $t \notin \{a, a+1\}$ and i_t belongs to the conditioning set of $v_{a,a+1}$. By Corollary 5.9, the joins $v_{a,t}$, $v_{t,a+1}$ exist in $\mathcal{P}$, and both are strictly smaller than $v_{a,a+1}$. Hence, the edges $\{i_a, i_t\}$, $\{i_t, i_{a+1}\}$ exist in (G, λ) and both have labels $< k$. Therefore, there exists a cycle in (G, λ) of length $< m$ with one edge of label $< k$ and the other edges of the same label k. This contradicts the minimality of m. Thus, for every $k \in \mathbb{Z}_{>0}$, an edge $e \in \pi_{<k}$ does not form a cycle with edges in π_k.

Now suppose (G, λ) contains an m-cycle C_m for $m \geq 3$ with all edges of the same label k for some $1 \leq k \leq n-1$. By Lemma 5.15, we may assume $m > 3$. By a similar argument as in the preceding paragraph with the aid of Lemma 5.13, the cycle C_m has a chord of label $< k$. This contradicts the conclusion of the preceding paragraph. □

Lemma 5.19. *Fix $1 \leq k \leq n-1$ and let $\{i, j\} \in \pi_k$. Then the conditioning set of $v_{i,j} = i \vee j$ coincides with the set of conditioning vertices of $\{i, j\}$. In particular, (G, λ) satisfies (ML2).*

Proof. Note that $v_{i,j} \in \mathcal{P}_{k+1}$. By Lemma 4.3(b), the conditioning set $D_{v_{i,j}}$ of $v_{i,j}$ contains $k-1$ elements. We may write it as $D_{v_{i,j}} = \{h_t \mid 1 \leq t \leq k-1\}$. By Corollary 5.9, the joins v_{i,h_t}, $v_{h_t,j}$ for $1 \leq h_t \leq k-1$ exist in $\mathcal{P}$, and all are strictly smaller than $v_{i,j}$. In particular, the edges $\{i, h_t\}$, $\{h_t, j\}$ have label $< k$ for all h_t. This implies that $D_{v_{i,j}}$ is contained in the set of conditioning vertices of $\{i, j\}$.

Now let $h \in [\ell] \setminus U_{v_{i,j}}$ (i.e., $h \notin v_{i,j}$). We show that if the joins $v_{i,h}$ and $v_{h,j}$ both exist, then either $\{i, h\}$ or $\{h, j\}$ has label $> k$. It cannot happen that $v_{i,h} < v_{i,j}$ or $v_{h,j} < v_{i,j}$; otherwise, $h \leq v_{i,j}$. Hence, by Lemma 5.15, either $v_{i,h} > v_{i,j}$ or $v_{h,j} > v_{i,j}$.

Thus, the conditioning set of $v_{i,j}$ coincides with the set of conditioning vertices of $\{i, j\} \in \pi_k$. Since this is true for every k, we conclude that (G, λ) satisfies (ML2). □

6. Equivalences of categories

For basic definitions and notations of category theory, we refer the reader to [18, Chapter 1]. In this section, we will define the categories of MAT-labeled graphs and LR-vines, and construct an explicit equivalence between these two categories.

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6.1. Equivalence of MAT-labeled graphs and LR-vines

Definition 6.1 (Label-preserving homomorphism). Let (G,λ) and (G',λ') be edge-labeled graphs. A **label-preserving homomorphism** from (G,λ) to (G',λ'), written $\sigma\colon (G,\lambda) \longrightarrow (G',\lambda')$, is a map $\sigma\colon N_G \longrightarrow N_{G'}$ such that for all $u,v \in N_G$, $\{u,v\} \in E_G$ implies $\{\sigma(u),\sigma(v)\} \in E_{G'}$ and $\lambda(u,v) = \lambda'(\sigma(u),\sigma(v))$.

We call σ an **isomorphism** if σ is bijective and its inverse is a label-preserving homomorphism. The edge-labeled graphs (G,λ) and (G',λ') are said to be **isomorphic**, written $(G,\lambda) \simeq (G',\lambda')$ if there exists an isomorphism $\sigma\colon (G,\lambda) \longrightarrow (G',\lambda')$. If $(G,\lambda) \simeq (G,\lambda')$, we say that two labelings λ and λ' are the same (or isomorphic).

If $(G,\lambda) \simeq (G',\lambda')$ and (G,λ) is an MAT-labeled graph, then (G',λ') is also an MAT-labeled graph.

Definition 6.2 (Category of MAT-labeled (complete) graphs). The **category** MG **of MAT-labeled graphs** is the category whose objects are the MAT-labeled graphs and whose morphisms are the label-preserving homomorphisms. The **category** MCG **of MAT-labeled complete graphs** is a full subcategory of MG whose objects are the MAT-labeled complete graphs.

Recall the definition of rank and join-preserving homomorphisms of graded posets from Definition 3.5.

Definition 6.3 (Category of (L)R-vines). The **category** LRV **of LR-vines** is the category whose objects are the LR-vines and whose morphisms are the homomorphisms preserving rank and join. The **category** RV **of R-vines** is a full subcategory of LRV whose objects are the R-vines.

First we need some lemmas.

Lemma 6.4. *Let $\varphi\colon \mathcal{P} \longrightarrow \mathcal{P}'$ be a rank-preserving homomorphism of LR-vines. Suppose φ preserves join of minimal pairs (i.e., if $x,y \in \min(\mathcal{P})$ such that the join $x \vee y$ exists, then $\varphi(x) \vee \varphi(y)$ exists and $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$). Then φ induces an isomorphism $\mathcal{P}_{\leq v} \simeq \mathcal{P}'_{\leq \varphi(v)}$ for every $v \in \mathcal{P}$.*

Proof. Clearly, φ induces a homomorphism $\varphi|_{\mathcal{P}_{\leq v}}\colon \mathcal{P}_{\leq v} \longrightarrow \mathcal{P}'_{\leq \varphi(v)}$. In particular, $\varphi(U_v) \subseteq U_{\varphi(v)}$. Note that for any distinct $i,j \in U_v$, the join $i \vee j$ exists in the R-vine $\mathcal{P}_{\leq v}$ by Remark 5.4. Since φ preserves rank, $\mathrm{rk}'(\varphi(i \vee j)) = \mathrm{rk}(i \vee j) > \mathrm{rk}(i)$. Since φ preserves join of minimal pairs, $\varphi(i) \vee \varphi(j)$ exists and $\varphi(i \vee j) = \varphi(i) \vee \varphi(j)$. In particular, $\varphi(i) \neq \varphi(j)$. Hence, the elements in $\varphi(U_v)$ are pairwise distinct. Thus, $\varphi(U_v) = U_{\varphi(v)}$ since $|\varphi(U_v)| = |U_v| = \mathrm{rk}(v) = \mathrm{rk}'(\varphi(v)) = |U_{\varphi(v)}|$.

Let $a,a' \in \mathcal{P}_{\leq v}$ be such that $\varphi(a) = \varphi(a')$. We may write $a = i \vee j$ and $a' = i' \vee j'$ for minimal nodes $i \neq j$, $i' \neq j'$. Thus, $\varphi(i) \vee \varphi(j) = \varphi(i') \vee \varphi(j')$. Again by Remark 5.4, $\{\varphi(i),\varphi(j)\} = \{\varphi(i'),\varphi(j')\}$. Since the elements in $\varphi(U_v)$ are pairwise distinct, $\{i,j\} = \{i',j'\}$. Hence, $a = a'$. This implies that $\varphi|_{\mathcal{P}_{\leq v}}$ is injective. Moreover, $|\mathcal{P}_{\leq v}| = \left|\mathcal{P}'_{\leq \varphi(v)}\right|$ by Remark 3.15. Hence, $\varphi|_{\mathcal{P}_{\leq v}}$ is bijective.

Now let $a,b \in \mathcal{P}_{\leq v}$ be such that $\varphi(a) \leq \varphi(b)$. Therefore, $U_{\varphi(a)} \subseteq U_{\varphi(b)}$; hence, $\varphi(U_a) \subseteq \varphi(U_b)$. It follows that $U_a \subseteq U_b$. By Corollary 4.6, $a \leq b$. Thus, the inverse of $\varphi|_{\mathcal{P}_{\leq v}}$ is a homomorphism. We conclude that $\varphi|_{\mathcal{P}_{\leq v}}$ is an isomorphism. □

Lemma 6.5. *Let $\varphi\colon \mathcal{P} \longrightarrow \mathcal{P}'$ be a rank-preserving homomorphism of LR-vines such that φ preserves join of minimal pairs (Lemma 6.4). Then φ is join-preserving.*

Proof. Let $a,b \in \mathcal{P}$ and suppose the join $a \vee b$ exists in $\mathcal{P}$. Write $v = a \vee b$. Note that $\varphi(a),\varphi(b) \in \mathcal{P}'_{\leq \varphi(v)}$. By Lemma 6.4, $\varphi|_{\mathcal{P}_{\leq v}}\colon \mathcal{P}_{\leq v} \longrightarrow \mathcal{P}'_{\leq \varphi(v)}$ is a poset isomorphism. It follows that $\varphi|_{\mathcal{P}_{\leq v}}$ is join-preserving. Therefore, $\varphi(a) \vee \varphi(b)$ exists in $\mathcal{P}'_{\leq \varphi(v)}$ and hence in $\mathcal{P}'$ and $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$. Thus, φ is join-preserving. □

Lemma 6.6. *Let $\mathcal{P}$ and $\mathcal{P}'$ be LR-vines and suppose there is a homomorphism $\varphi\colon \mathcal{P} \longrightarrow \mathcal{P}'$ preserving rank and join. Let (G,λ) (resp. (G',λ')) denote the MAT-labeled graph defined by $\mathcal{P}$ (resp. $\mathcal{P}'$) from Definition 5.16 and Theorem 5.17. Define a map $\Omega(\varphi)\colon N_G \longrightarrow N_{G'}$ by sending each node i in G to a node $\varphi(i)$ in G' for $1 \leq i \leq \ell$. Then $\Omega(\varphi)$ is a label-preserving homomorphism from (G,λ) to (G',λ').*

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Proof. Let $\{i, j\} \in E_G$ be an edge in G for distinct nodes $i, j \in N_G$. Then the join $i \vee j$ exists in $\mathcal{P}$. Since φ is join-preserving, $\varphi(i) \vee \varphi(j)$ exists and $\varphi(i \vee j) = \varphi(i) \vee \varphi(j)$. Since φ is rank-preserving, $\varphi(i) \neq \varphi(j)$. Therefore, $\{\varphi(i), \varphi(j)\} \in E_{G'}$. Furthermore,

$$\lambda'(\varphi(i), \varphi(j)) = \mathrm{rk}'(\varphi(i) \vee \varphi(j)) - 1 = \mathrm{rk}(i \vee j) - 1 = \lambda(i, j).$$

Thus, $\Omega(\varphi)$ is a label-preserving homomorphism. □

Lemma 6.7. *Let (G, λ) and (G', λ') be MAT-labeled graphs and suppose there is a label-preserving homomorphism $\sigma\colon (G, \lambda) \longrightarrow (G', \lambda')$. Let $\mathcal{P}$ (resp. $\mathcal{P}'$) denote the LR-vine defined by (G, λ) (resp. (G', λ')) from Definition 4.11 and Theorem 4.13. Define a map $\Psi(\sigma)\colon \mathcal{P} \longrightarrow \mathcal{P}'$ by sending $v \in \mathcal{P}$ to $\{\sigma(a) \mid a \in v\} \in \mathcal{P}'$. Then $\Psi(\sigma)$ is a homomorphism preserving rank and join between $\mathcal{P}$ and $\mathcal{P}'$.*

Proof. Let $\varphi = \Psi(\sigma)$. Note that since φ is label-preserving, if $v = K_e \in \mathcal{P}$ where K_e is a principal clique in (G, λ) for $e \in E_G$, then $\sigma(K_e) = \{\sigma(a) \mid a \in K_e\} = K_{\sigma(e)}$ is a principal clique in (G', λ'). Hence, φ is indeed well-defined. It is also easily seen that φ is a rank-preserving homomorphism since $\lambda(e) = |K_e| - 1$. Let $\{i\}, \{j\} \in \mathcal{P}$ be distinct minimal nodes in $\mathcal{P}$ such that $\{i\} \vee \{j\}$ exists in $\mathcal{P}$. By Corollary 4.14, we may write $K_e = \{i\} \vee \{j\} \in \mathcal{P}$ for $e = \{i, j\} \in E_G$. Also by Corollary 4.14, $K_{\sigma(e)} = \{\sigma(i)\} \vee \{\sigma(j)\} \in \mathcal{P}'$ since $\sigma(e) = \{\sigma(i), \sigma(j)\} \in E_{G'}$. Therefore, $\varphi(\{i\} \vee \{j\}) = \varphi(\{i\}) \vee \varphi(\{j\})$. Thus, φ preserves join of minimal pairs. By Lemma 6.5, φ is join-preserving. □

The following two lemmas give a construction of functors between MG and LRV. The proofs are routine.

Lemma 6.8. *Define a mapping $\Omega\colon$ LRV $\longrightarrow$ MG by associating*

(1) *each object $\mathcal{P}$ in* LRV *to an object $\Omega(\mathcal{P}) = (G, \lambda)$ in* MG*, where (G, λ) is the MAT-labeled graph defined by $\mathcal{P}$ from Definition 5.16 and Theorem 5.17, and*
(2) *each morphism $\varphi\colon \mathcal{P} \longrightarrow \mathcal{P}'$ in* LRV *to a morphism $\Omega(\varphi)\colon \Omega(\mathcal{P}) \longrightarrow \Omega(\mathcal{P}')$ in* MG*, where $\Omega(\varphi)$ is the label-preserving homomorphism from Lemma 6.6.*

Then Ω is a functor from LRV *to* MG*.*

Lemma 6.9. *Define a mapping $\Psi\colon$ MG $\longrightarrow$ LRV by associating*

(1) *each object (G, λ) in* MG *to an object $\Psi(G, \lambda) = \mathcal{P}$ in* LRV*, where $\mathcal{P}$ is the LR-vine defined by (G, λ) from Definition 4.11 and Theorem 4.13, and*
(2) *each morphism $\sigma\colon (G, \lambda) \longrightarrow (G', \lambda')$ in* MG *to a morphism $\Psi(\sigma)\colon \Psi(G, \lambda) \longrightarrow \Psi(G', \lambda')$ in* LRV*, where $\Psi(\sigma)$ is the homomorphism preserving rank and join from Lemma 6.7.*

Then Ψ is a functor from MG *to* LRV*.*

We are now ready to prove the main result of the paper.

Theorem 6.10. *The composite functor $\Psi\Omega$ (resp. $\Omega\Psi$) is naturally isomorphic to the identity functor 1_{LRV} (resp. 1_{MG}). As a result, the categories* MG *and* LRV *are equivalent.*

Proof. For every LR-vine $\mathcal{P}$, recall from Proposition 4.8 the LR-vine $\widehat{\mathcal{P}} = \{U_v \mid v \in \mathcal{P}\}$ and the isomorphism

$$\eta_{\mathcal{P}}\colon \mathcal{P} \longrightarrow \widehat{\mathcal{P}} \quad \text{via} \quad v \mapsto U_v.$$

By Lemmas 6.9 and 6.8, the functor $\Psi\Omega\colon$ LRV $\longrightarrow$ LRV assigns

(1) each LR-vine $\mathcal{P}$ to the LR-vine $\Psi\Omega(\mathcal{P}) = \widehat{\mathcal{P}}$, and
(2) each morphism $\varphi\colon \mathcal{P} \longrightarrow \mathcal{P}'$ in LRV to a morphism $\Psi\Omega(\varphi)\colon \widehat{\mathcal{P}} \longrightarrow \widehat{\mathcal{P}'}$ in LRV defined by sending $U_v \in \widehat{\mathcal{P}}$ to $\varphi(U_v) = U_{\varphi(v)} \in \widehat{\mathcal{P}'}$ for every $v \in \mathcal{P}$. The equality holds by Lemma 6.4.

Similarly, the functor $\Omega\Psi\colon$ MG $\longrightarrow$ MG assigns

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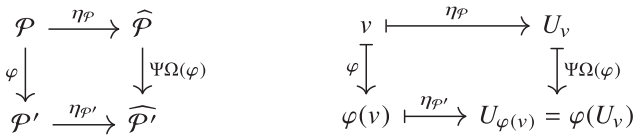


Figure 3. *Commutative diagram that shows* $\Psi\Omega \simeq 1_{\mathsf{LRV}}$.

(1') each MAT-labeled graph (G, λ) to an MAT-labeled graph $\Omega\Psi(G, \lambda) = (\widehat{G}, \widehat{\lambda})$, where $N_{\widehat{G}} = \{\{i\} \mid i \in N_G\}$, $E_{\widehat{G}} = \{\{\{i\}, \{j\}\} \mid \{i, j\} \in E_G\}$ and $\widehat{\lambda}(\{i\}, \{j\}) = \lambda(i, j)$ (see also Corollary 4.14), and

(2') each morphism $\sigma\colon (G, \lambda) \longrightarrow (G', \lambda')$ in MG to a morphism $\Omega\Psi(\sigma)\colon (\widehat{G}, \widehat{\lambda}) \longrightarrow (\widehat{G'}, \widehat{\lambda'})$ in MG defined by sending $\{i\} \in N_{\widehat{G}}$ to $\{\sigma(i)\} \in N_{\widehat{G'}}$ for every $i \in N_G$.

First we prove $\Psi\Omega \simeq 1_{\mathsf{LRV}}$ (i.e., $\Psi\Omega$ is naturally isomorphic to 1_{LRV}). For every morphism $\varphi\colon \mathcal{P} \longrightarrow \mathcal{P}'$ in LRV, we have a commutative diagram in Figure 3.

This follows that $\eta\colon 1_{\mathsf{LRV}} \longrightarrow \Psi\Omega$ with component $\eta_{\mathcal{P}}$ at $\mathcal{P}$ is a natural isomorphism. Thus, $\Psi\Omega$ is naturally isomorphic to 1_{LRV}.

The proof for $\Omega\Psi \simeq 1_{\mathsf{MG}}$ is similar and easier. For every object (G, λ) in MG, the following map is an isomorphism:

$$\epsilon_{(G,\lambda)}\colon (G, \lambda) \longrightarrow (\widehat{G}, \widehat{\lambda}) \quad \text{via} \quad N_G \ni i \mapsto \{i\} \in N_{\widehat{G}}.$$

Furthermore, $\Omega\Psi$ is naturally isomorphic to 1_{MG} via the natural isomorphism $\epsilon\colon 1_{\mathsf{MG}} \longrightarrow \Omega\Psi$ with component $\epsilon_{(G,\lambda)}$ at (G, λ). □

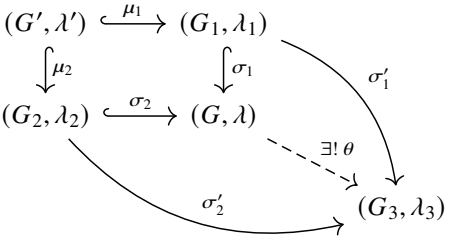
The following corollary is straightforward from Theorem 6.10.

Corollary 6.11. *The restriction* $\Psi|_{\mathsf{MCG}}$ *(resp.* $\Omega|_{\mathsf{RV}}$*) is a functor from* MCG *(resp.* RV*) to* RV *(resp.* MCG*). Furthermore, the composite functor* $\Psi|_{\mathsf{MCG}}\Omega|_{\mathsf{RV}}$ *(resp.* $\Omega|_{\mathsf{RV}}\Psi|_{\mathsf{MCG}}$*) is naturally isomorphic to the identity functor* 1_{RV} *(resp.* 1_{MCG}*). As a result, the categories* MCG *and* RV *are equivalent.*

Another interesting consequence is the existence of a pushout in the category MG and hence in LRV owing to the gluing method in Lemma 2.21.

Corollary 6.12. *Suppose we are in the situation of Lemma 2.21. Denote by* $\mu_1\colon (G', \lambda') \hookrightarrow (G_1, \lambda_1)$, $\mu_2\colon (G', \lambda') \hookrightarrow (G_2, \lambda_2)$, $\sigma_1\colon (G_1, \lambda_1) \hookrightarrow (G, \lambda)$ *and* $\sigma_2\colon (G_2, \lambda_2) \hookrightarrow (G, \lambda)$ *the embeddings (in particular, morphisms in* MG*) of the MAT-labeled graphs. Then* $((G, \lambda), \sigma_1, \sigma_2)$ *is a pushout of* μ_1 *and* μ_2.

Proof. It is easily seen that $\sigma_1\mu_1 = \sigma_2\mu_2$. Hence, the square diagram commutes. Now given another triple $((G_3, \lambda_3), \sigma_1', \sigma_2')$ with $\sigma_1'\mu_1 = \sigma_2'\mu_2$, let $\theta\colon (G, \lambda) \longrightarrow (G_3, \lambda_3)$ be defined by $\theta(u) = \sigma_1'(u)$ for $u \in N_{G_1}$, $\theta(u) = \sigma_2'(u)$ for $u \in N_{G_2}$. Thus, $\theta\sigma_1 = \sigma_1'$ and $\theta\sigma_2 = \sigma_2'$. It is also easy to see that θ is the unique morphism making the diagram commute.



We conclude that $((G, \lambda), \sigma_1, \sigma_2)$ is a pushout of μ_1 and μ_2. □

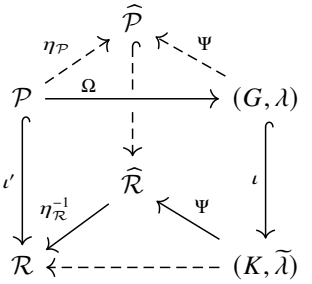
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6.2. Equivalence of LR-vines and m-vines

Theorem 6.13. *Let $\mathcal{P}$ be a vine. The following are equivalent:*

(1) *$\mathcal{P}$ is an m-vine (i.e., by definition, $\mathcal{P}$ is an ideal of an R-vine).*
(2) *$\mathcal{P}$ satisfies the proximity condition.*
(3) *$\mathcal{P}$ is an LR-vine.*

Proof. (2) $\Leftrightarrow$ (3) is shown in Proposition 4.4. (1) $\Rightarrow$ (3) is straightforward from Remark 3.17. It remains to show (1) $\Leftarrow$ (3). The proof is based on the following diagram:



First, given an LR-vine $\mathcal{P}$, let $(G,\lambda) = \Omega(\mathcal{P})$ be the MAT-labeled graph associated to $\mathcal{P}$ via the functor Ω from Lemma 6.8. Next, let $(K,\widetilde{\lambda})$ be the MAT-labeled complete graph such that $(G,\lambda) \leq (K,\widetilde{\lambda})$ from Proposition 2.20. In particular, there exists an embedding $\iota\colon (G,\lambda) \hookrightarrow (K,\widetilde{\lambda})$. Then let $\Psi|_{\mathsf{MCG}}(K,\widetilde{\lambda}) = \widehat{\mathcal{R}}$ be the R-vine associated to $(K,\widetilde{\lambda})$ via the functor $\Psi|_{\mathsf{MCG}}$ from Corollary 6.11. Finally, let $\mathcal{R}$ be the R-vine isomorphic to $\widehat{\mathcal{R}}$ via the (inverse of) poset isomorphism $\eta_{\mathcal{R}}$ from Proposition 4.8.

(1) $\Leftarrow$ (3) is proved once we show that $\mathcal{P}$ is an ideal of $\mathcal{R}$. Indeed, by the construction, $\mathcal{P}$ is an induced subposet of $\mathcal{R}$. (One may see this via the sequence $\mathcal{P} \longrightarrow \widehat{\mathcal{P}} \hookrightarrow \widehat{\mathcal{R}} \longrightarrow \mathcal{R}$ in the diagram.) Note also that $\mathcal{P}$ and $\mathcal{R}$ have the same set of minimal nodes. Let $w, v \in \mathcal{R}$ be two nodes with $w \leq v$ and $v \in \mathcal{P}$. We need to show that $w \in \mathcal{P}$. The assertion follows easily if either w or v is a minimal node. We may assume that both w and v are not minimal. Since $C_w \subseteq U_w \subseteq U_v$ in $\mathcal{R}$, and U_v in $\mathcal{P}$ is the same as U_v in $\mathcal{R}$, we have $C_w \subseteq U_v$ in $\mathcal{P}$. By Remark 5.4, there exists a non-minimal node w' in the R-vine $\mathcal{P}_{\leq v}$ such that $C_w = C_{w'}$. Applying Corollary 5.7 for two non-minimal nodes w, w' in $\mathcal{R}$, we have $w = w'$. Hence, w is an element in $\mathcal{P}$, as desired. □

7. Applications

7.1. From LR-vines to MAT-labeled graphs

7.1.1. A poset characterization of MAT-free graphic arrangements

The most important application of our main result (Theorem 6.10) is an affirmative answer for the question of Cuntz-Mücksch (Question 1.3) in the case of graphic arrangements: MAT-free graphic arrangements have a poset characterization by LR-vines. (Note that LR-vines generalize the root poset of type A by Remark 4.16.) We give below two examples to illustrate the correspondence. First we need a definition.

Definition 7.1 (C-vine). An R-vine is called a **C-Vine** if each associated tree has the largest possible number of vertices of degree 1. Equivalently, each associated tree is a *star graph*.

D-vine and C-vine can be regarded as the 'extreme' cases of R-vines.

Example 7.2. In dimension 4, there are exactly two non-isomorphic R-vine structures: D-vine and C-vine. Likewise, there are exactly two non-isomorphic MAT-labeled complete graphs on 4 vertices. Figure 4 depicts a C-vine on [4] (top right), the associated forests (top left), the associated MAT-labeled complete graph (bottom right) via the functor Ω from Lemma 6.8, and the MAT-partition (bottom left)

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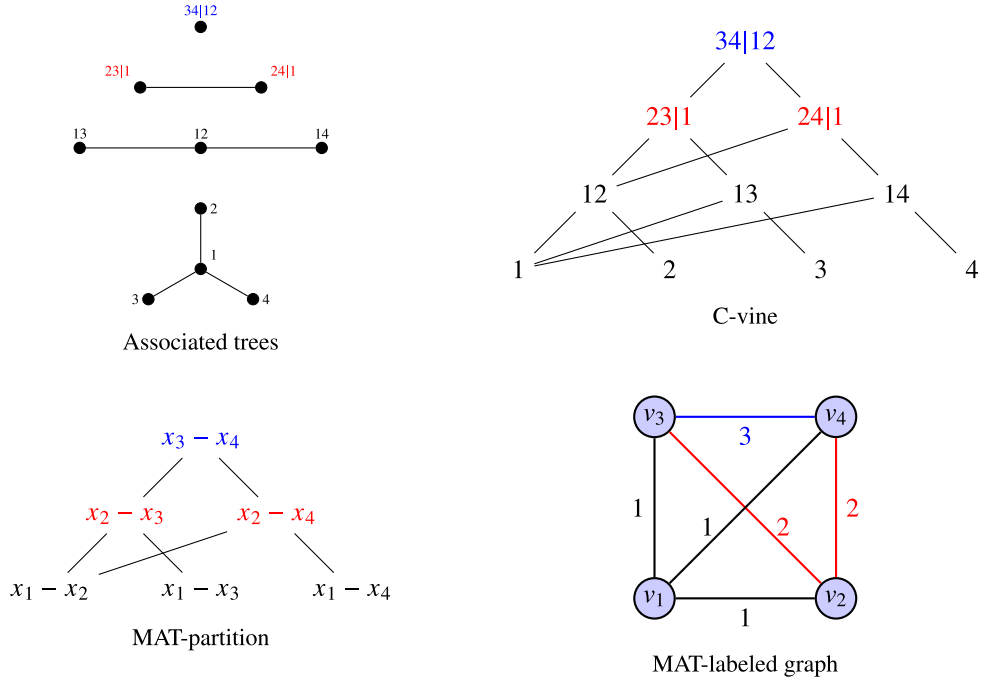


Figure 4. *C-vine on* 4 *elements, the associated trees, MAT-labeled complete graph and MAT-partition from Example 7.2.*

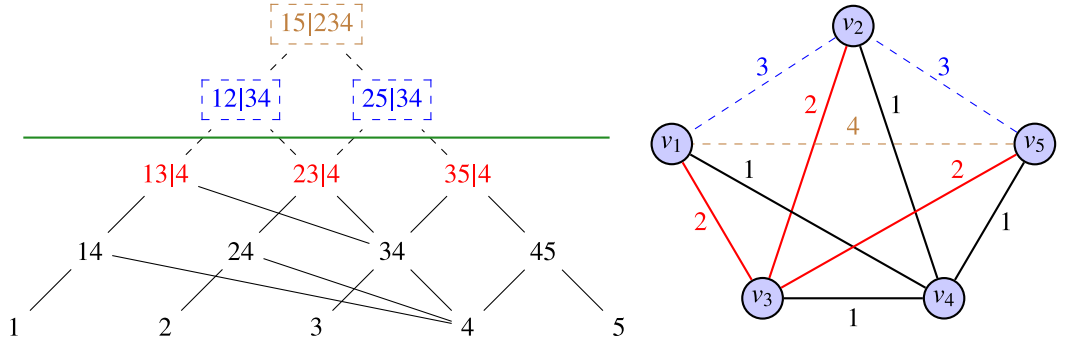


Figure 5. *An MAT-labeled graph on* 5 *vertices (right) and the LR-vine $\mathcal{P} = \Psi(G, \lambda)$ (left) from Lemma 6.9 with its* 3*-lower truncation.*

of the corresponding graphic arrangement. The C-vine in dimension ≥ 4 is not an ideal of any D-vine; hence, the corresponding MAT-partition is not obtained from an ideal of the type A root poset.

Example 7.3. Figure 5 depicts on the right an MAT-labeled graph (G, λ) on 5 vertices and an MAT-labeled complete graph $(K, \widetilde{\lambda})$ such that $(G, \lambda) \leq (K, \widetilde{\lambda})$. The complementary edges are shown in dashed lines. The graphs (G, λ) and $(K, \widetilde{\lambda})$ correspond (via the functor Ψ from Lemma 6.9) to the LR-vine $\mathcal{P}$ and R-vine $\mathcal{R}$ on the left, respectively. In this case, $\mathcal{P}$ is the 3-lower truncation of $\mathcal{R}$.

7.1.2. Number of non-isomorphic MAT-labelings of complete graphs

The number of equivalence classes of R-vines in dimension ℓ is given in [17, §10.3]. By our Corollary 6.11, this number is equal to the number of non-isomorphic MAT-labelings of the complete graph on ℓ vertices. We immediately have the following:

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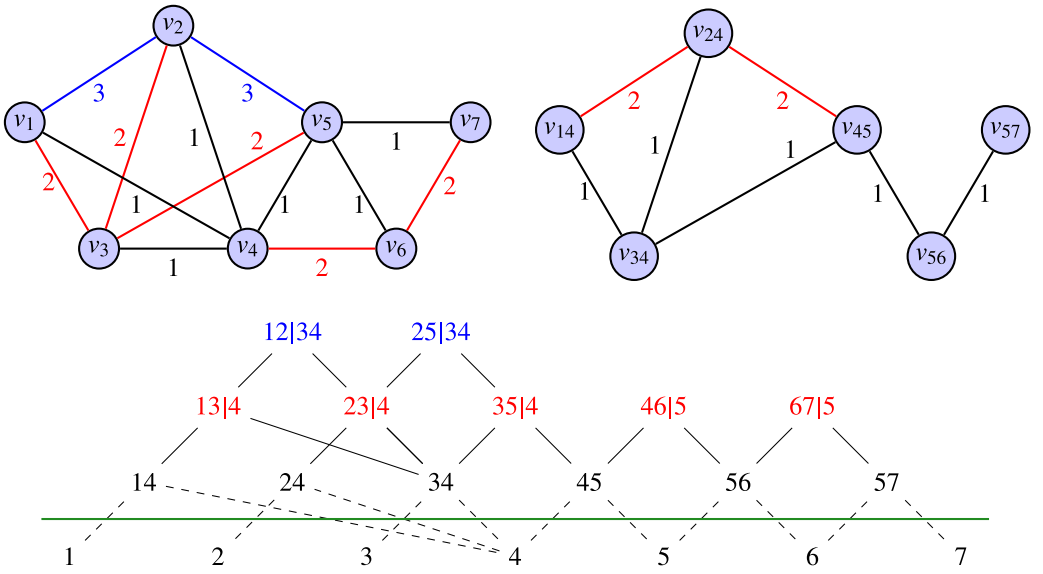


Figure 6. *An MAT-labeled graph on 7 vertices (top left), the LR-vine $\mathcal{P} = \Psi(G, \lambda)$ (bottom) from Lemma 6.9 and its 1-upper truncation $\overline{\mathcal{P}}_{\geq 1}$, and the MAT-labeled graph $\Omega(\overline{\mathcal{P}}_{\geq 1})$ (top right) from Lemma 6.8.*

Theorem 7.4. *The number E_ℓ of non-isomorphic MAT-labelings of the complete graph K_ℓ for $\ell \geq 1$ is given by $E_1 = E_2 = E_3 = 1$ and $E_\ell = (A_\ell + B_\ell)/2$ for $\ell \geq 4$, where*

$$A_\ell = 2^{(\ell-2)(\ell-3)/2}, \quad B_\ell = \sum_{k=1}^{\lfloor \ell/2 \rfloor - 1} A_\ell c_k 2^{-k+\sum_{i=0}^{k-1}(\ell-4-2i)},$$

and $c_k = 1$ for all k except $c_{\lfloor \ell/2 \rfloor - 1} = 2$.

The first 8 elements of the sequence (E_ℓ) are $1, 1, 1, 2, 6, 40, 560, 17024$ mentioned in §1.1. In particular, $E_4 = 2$ and these MAT-labelings are given in Figures 2 and 4.

7.1.3. Upper truncation of MAT-labeled graphs

In Remark 4.10, we discussed two ways to obtain a new LR-vine from a given LR-vine by upper or lower truncation. From our main result 6.10, the lower truncation simply corresponds to deleting the edges of high label in the associated MAT-labeled graph (Figure 5). The upper truncation, however, gives rise to a nontrivial graph operation which produces an MAT-labeled graph from a given one. We shall not give an explicit formulation of the operation but instead illustrate it by an example in Figure 6.

In terms of hyperplane arrangement, any upper truncation of an MAT-free graphic arrangement is again MAT-free. This fact is not true in general. For example, the Weyl subarrangement defined by the 1-upper truncation of the root poset of type B_3 is not free, and hence not MAT-free. It would be interesting to find for which MAT-free arrangement or for which upper truncation of a given MAT-free arrangement this property holds true.

7.2. From MAT-labeled graphs to LR-vines

7.2.1. Strongly chordal graphs and m-vines

Given a strongly chordal graph G, Zhu-Kurowicka [28, §3.4] showed that there exists an m-vine, equivalently, an LR-vine $\mathcal{P}$ (by our Theorem 6.13) such that the principal ideals generated by the maximal elements of $\mathcal{P}$ are in one-to-one correspondence with the maximal cliques of G. Their method

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is based on the existence of a *strong clique tree* of G. We give below a different construction of such an LR-vine thanks to the equivalence between the LR-vines and MAT-labeled graphs from Theorem 6.10.

Theorem 7.5. *Given a strongly chordal graph G, there exists an LR-vine $\mathcal{P}$ such that the principal ideals generated by the maximal elements of $\mathcal{P}$ are in one-to-one correspondence with the maximal cliques of G.*

Proof. By Theorem 1.5, there exists an MAT-labeling λ of G. The construction of such a λ can be found in [27, Theorem 5.12] based on the notion of *clique intersection poset* of G first appeared in [21]. Let $\mathcal{P} = \Psi(G, \lambda)$ (Lemma 6.9). Theorem 7.5 is proved once we prove that the set $\max(\mathcal{P})$ of maximal elements of $\mathcal{P}$ coincides with the set $\mathcal{K}(G)$ of maximal cliques of G. By Lemma 2.23, any maximal clique C in G is principal and hence an element in $\mathcal{P}$. Moreover, $C \in \max(\mathcal{P})$. Otherwise, there exists a clique $C' \in \max(\mathcal{P})$ such that $C \subsetneq C'$. This contradicts the maximality of C. Hence, $\mathcal{K}(G) \subseteq \max(\mathcal{P})$. The reserve inclusion is proved similarly. Thus, $\mathcal{P}$ is a desired LR-vine. □

7.2.2. Marginalization and sampling order

The notions of *marginalization* and *sampling order* of an R-vine were introduced in [8, §3]. We give below an extension of these notions to an LR-vine.

Definition 7.6 (Marginalization). Let $(\mathcal{P}, \leq_{\mathcal{P}}, \mathrm{rk}_{\mathcal{P}})$ be an LR-vine. Let $v \in \min(\mathcal{P})$ be a minimal node. The **marginalization** $(\mathcal{P} \parallel v, \leq_{\mathcal{P}})$ of $\mathcal{P}$ by v is the induced subposet of $\mathcal{P}$ obtained by removing v and the nodes whose conditioned sets contain v; that is,

$$\mathcal{P} \parallel v := \mathcal{P} \setminus (\{v\} \cup \{x \in \mathcal{P} \mid v \in C_x\}).$$

Let $\mathcal{Q}$ be an induced subposet of a finite graded poset $(\mathcal{P}, \leq_{\mathcal{P}}, \mathrm{rk}_{\mathcal{P}})$. We say that $\mathcal{Q}$ is **graded by** $\mathrm{rk}_{\mathcal{P}}$ if the restriction $\mathrm{rk}_{\mathcal{P}}|_{\mathcal{Q}}$ is a rank function on $\mathcal{Q}$. The marginalization of $\mathcal{P}$ by a minimal node is not necessarily graded by $\mathrm{rk}_{\mathcal{P}}$ in general. For example, let $\mathcal{P}$ be the C-vine with 4 minimal elements $1, 2, 3, 4$ in Figure 4. Then the marginalization $\mathcal{Q} := (\mathcal{P} \parallel 2, \leq_{\mathcal{P}})$ is not graded by $\mathrm{rk}_{\mathcal{P}}$ since $\mathrm{rk}_{\mathcal{P}}(34|12) = 4$ while $34|12$ cannot have rank 4 in $\mathcal{Q}$.

Definition 7.7 (Sampling order). Let $(\mathcal{P}, \leq_{\mathcal{P}}, \mathrm{rk}_{\mathcal{P}})$ be an ℓ-dimensional (L)R-vine. An ordering $(v_1, \ldots, v_\ell)$ of minimal nodes in $\mathcal{P}$ is a **sampling order** if the marginalization $\mathcal{P}_i := \mathcal{P}_{i+1} \parallel v_{i+1}$ is an (L)R-vine graded by $\mathrm{rk}_{\mathcal{P}}$ for each $1 \leq i \leq \ell - 1$. Here, we let $\mathcal{P} = \mathcal{P}_\ell$.

It is shown in [8, Theorem 5.1] that an R-vine always has a sampling order. Now we generalize this result to LR-vines.

Theorem 7.8. *Let $\mathcal{P}$ be an ℓ-dimensional LR-vine and let $(G, \lambda) = \Omega(\mathcal{P})$ (Lemma 6.8). If $(v_1, \ldots, v_\ell)$ is an MAT-PEO of (G, λ), then $(v_1, \ldots, v_\ell)$ is a sampling order of $\mathcal{P}$. As a consequence, an LR-vine always has a sampling order.*

Proof. The assertion is clearly true when $\ell \leq 1$. Suppose that $\ell \geq 2$ and $(v_1, \ldots, v_\ell)$ is an MAT-PEO of (G, λ). By Lemma 2.15, (G_i, λ_i) is an MAT-labeled graph for each $1 \leq i \leq \ell - 1$, where $G_i := G[\{v_1, \ldots, v_i\}]$ and $\lambda_i := \lambda|_{E_{G_i}}$. By the proof of Theorem 4.13, for each i, $\Psi(G_i, \lambda_i) = \widehat{\mathcal{P}_i}$ where $\mathcal{P}_i := \mathcal{P}_{i+1} \parallel v_{i+1}$, $\mathcal{P}_\ell := \mathcal{P}$ and $\widehat{\mathcal{P}_i}$ is the poset isomorphic to $\mathcal{P}_i$ in Proposition 4.8. Thus, each $\mathcal{P}_i$ is an LR-vine graded by $\mathrm{rk}_{\mathcal{P}}$. Hence, $(v_1, \ldots, v_\ell)$ is a sampling order of $\mathcal{P}$. The consequence is straightforward since an MAT-labeled graph always has an MAT-PEO by Theorem 2.17. □

Remark 7.9. The converse of the main assertion in Theorem 7.8 is not true in general. Namely, a sampling order of $\mathcal{P}$ is not necessarily an MAT-PEO of (G, λ). The reason is that even if the marginalization $\mathcal{P} \parallel v$ for some $v \in \min(\mathcal{P})$ is an LR-vine graded by $\mathrm{rk}_{\mathcal{P}}$, the node v is not necessarily an MAT-simplicial vertex of (G, λ). For example, let $\mathcal{P}$ be the 3-lower truncation of the C-vine in Figure 4 (see Figure 7 below). The associated MAT-labeled graph $(G, \lambda) = \Omega(\mathcal{P})$ is the graph obtained from the corresponding complete graph by removing the edge $\{v_3, v_4\}$. Then the marginalization $\mathcal{P} \parallel 2$

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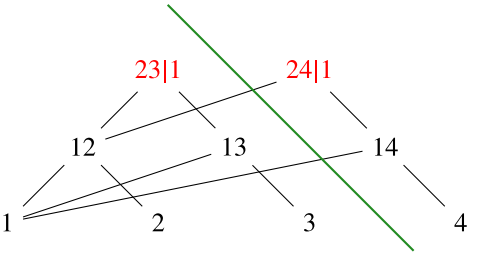


Figure 7. *An LR-vine and its marginalization by the node* 4 *visualized as 'vertical' truncation.*

is an LR-vine graded by $\mathrm{rk}_{\mathcal{P}}$, but the vertex v_2 is not (MAT-)simplicial in (G, λ) since its neighborhood does not form a clique. In addition, $(1, 3, 4, 2)$ is a sampling order of $\mathcal{P}$, but (v_1, v_3, v_4, v_2) is not an MAT-PEO of (G, λ).

While upper or lower truncation of an LR-vine can be visualized as 'horizontal' truncation, the marginalization is a 'vertical' truncation. Figure 7 depicts the 3-lower truncation $\mathcal{P}$ of the C-vine in Figure 4 and the marginalization $\mathcal{P} \parallel 4$. We may continue marginalizing and get the sampling order $(1, 2, 3, 4)$ of $\mathcal{P}$. Furthermore, (v_1, v_2, v_3, v_4) is an MAT-PEO of the MAT-labeled graph associated to $\mathcal{P}$.

7.2.3. Number of ideals in a D-vine

Given an arbitrary R-vine $\mathcal{P}$, a natural question is to find the number of m-vines or ideals in $\mathcal{P}$. This question is still open in general (see, for example, [28, §4]). However, in the particular case of D-vine, we have an explicit answer owing to a classical result of Shi [23] in the theory of hyperplane arrangements.

Theorem 7.10. *The number of ideals in the D-vine of dimension* $\ell - 1$ *for* $\ell \geq 2$ *is given by the* ℓ*-th Catalan number* $\mathrm{Cat}_\ell = \frac{1}{\ell+1}\binom{2\ell}{\ell} = \prod_{k=2}^{\ell} \frac{\ell+k}{k}$.

Proof. By Remark 4.16, the D-vine of dimension $\ell - 1$ is isomorphic to the type $A_{\ell-1}$ root poset. It is known that the number of ideals in the latter is given by the ℓ-th Catalan number Cat_ℓ [23, Theorem 1.4]. □

Some calculation on C-vines suggests us the following conjecture:

Conjecture 7.11. *The number of ideals in the C-vine of rank* $\ell \geq 1$ *is given by* $\sum_{i=0}^{\ell+1}((i+1)^{\ell+1-i} - i^{\ell+1-i})$*, the* $(\ell + 1)$*-th number in the OEIS sequence A047970.*

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Competing interest. The authors have no competing interest to declare.

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