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RESPONDING TO THE INFLATION TAX

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We adopt mechanism design to study the effects of inflation on output, trade, and capital accumulation. Our theory captures multiple channels for individuals to respond to inflation: search intensity, market participation, and substitution between money and a higher return asset. We characterize constrained efficient allocations and show inflation has nonmonotonic effects on the frequency of trades (extensive margin) and the total quantity traded (intensive margin). The model features monetary superneutrality for low inflation rates, nonlinearities in trading frequencies, and substitution of money for capital for higher inflation rates. While these effects are difficult to capture in previous models, we show how they are intimately related by all being features of an optimal trading mechanism.

Keywords: Inflation, Search Intensity, Money and Capital, Mechanism Design

1. INTRODUCTION

This paper develops a monetary model with multiple channels for inflation to affect aggregate activity, trade, and welfare. Since inflation is a tax on cash transactions, the conventional wisdom suggests that individuals shift away from cash-intensive activities as inflation rises [Cooley and Hansen (1989), Lucas (2000)]. In this paper, we emphasize additional consequences of inflation important for the functioning of monetary economies. These include the efforts taken by individuals engaging in market activities to economize on cash holdings, the accumulation of real assets or capital goods that can substitute for money as a means of payment, and the exchange patterns society adopts. The objective is to analyze how inflation affects these channels and how they interact with one another. As we highlight in detail below, key to our approach is the use of mechanism design to determine the terms

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of trade, which allows us to study how inflation induces changes in the trading arrangements implemented by society.

Our framework builds on the New Monetarist model of Lagos and Wright (2005) where frictions like limited commitment and lack of record keeping make money or other liquid assets essential for trade. Since inflation directly affects the opportunity cost of holding money, output always falls with inflation absent other margins for individuals to respond. To incorporate additional consequences of inflation, we modify the Lagos and Wright (2005) model of monetary exchange to include search intensity, capital accumulation, and a pricing mechanism that respond endogenously to inflation. For instance, in episodes of high inflation, individuals try to reduce the time of carrying money by increasing their trading frequency.[1] We endogenize search effort as a way to model this effect. Moreover, while money is a primary liquid asset in most low inflation economies, societies in times of high inflation tend to use other assets for transactions, such as capital goods. To capture this substitution effect, we follow Lagos and Rocheteau (2008) and introduce capital goods that can serve both as a means of production and a means of payment. While stylized, capital in our model captures two salient features of real assets: first, overaccumulation of such assets is inefficient, and second, such assets can serve as payment (or collateral) to facilitate trade.[2]

A key ingredient of our analysis is the use of mechanism design to determine the terms of trade in pairwise meetings. Following Hu et al. (2009), we consider socially optimal allocations that are individually rational and immune to pairwise defections. As in Hu and Rocheteau (2013), our economy features coexistence of money and higher return capital as a feature of the optimal trading mechanism. Relative to those papers, we obtain new insights on the nontrivial interaction between individual search decisions and portfolio choices. Moreover, a novelty of this approach is that we can study how inflation induces changes in society's trading arrangements; these changes may be innocuous, i.e., for low inflation rates, or more dramatic, i.e., for high inflation rates.[3]

Our results provide a picture of the consequences of inflation for allocations and trading patterns for different levels of inflation. In particular, the model generates three qualitatively different regimes distinguished by whether inflation is at a low, intermediate, or high level. Which regime arises depends on endogenous thresholds for inflation, which affects individual trading decisions and hence dictates how welfare-relevant variables interact. We characterize constrained efficient allocations in each regime and describe how they respond to anticipated changes in inflation.

In the low inflation regime, there is monetary superneutrality: output, search effort, and the capital stock remain at their efficient levels, irrespective of changes in inflation. In the intermediate inflation regime, search effort can increase with inflation even though the buyer's real balances fall. Here, the optimal mechanism dictates the buyer to have more surplus as inflation rises, which induces buyers to search harder. Inflation therefore leads to a higher trading frequency but lower

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output per trade. Under a weak sufficient condition, capital remains at the first-best level even though search intensity increases. We also provide examples where trade in the decentralized market (DM) increases with inflation, while search efforts are inefficiently high. Our finding that low inflation is costless and becomes socially harmful only with higher inflation is broadly consistent with the nonlinear relationship between inflation and output documented empirically, e.g., by Bullard and Keating (1995). However, in these two regimes, capital accumulation is unaffected by inflation.

In the high-inflation regime, agents overaccumulate capital that gradually crowds out money as the main medium of exchange. As inflation tends to infinity, search effort can remain inefficiently high, and the economy never collapses to autarky. These outcomes contrast with that of a pure currency economy, such as in Lagos and Rocheteau (2005) where the economy approaches autarky and buyers stop searching as inflation becomes high enough.[4] These results suggest capital overaccumulation is a symptom only of high inflation, while Tobin effects are small or absent for moderate inflation. Our finding that individuals first try to get rid of their real balances before substituting to another asset is also consistent with the responses of high inflation documented historically.

The paper proceeds as follows. Section 1.1 discusses related literature. Section 2 presents the baseline environment, and Section 3 describes the trading mechanism. Section 4 characterizes implementable allocations and the effects of inflation on output, search effort, capital accumulation, and welfare. We conclude in Section 5. All proofs are in Appendix.

1.1. Related Literature

In contrast with previous studies on inflation and endogenous search efforts that take the pricing mechanism as given, we use mechanism design following Hu et al. (2009) and Rocheteau (2012).[5] While in Hu et al. (2009), the buyer's surplus is indeterminate when inflation is sufficiently low, the optimal search effort pins down the buyer's surplus in our model. With fixed search intensity, the optimal mechanism always gives buyers the full surplus, which no longer holds in our model. In our model, the externality from buyers' search decisions is the key driving force for search intensity to rise with inflation. When inflation is in an intermediate range, this hot potato effect arises for all parameters but appears only for a small parameter set under suboptimal mechanisms.[6]

Our coexistence result is similar to the main results in Hu and Rocheteau (2013). However, different from that paper, our main focus is on the interaction between search intensity and the endogenous choice of media of exchange. While in pure-currency economies [such as those considered in Lagos and Rocheteau (2005) and Liu et al. (2011)], agents stop exerting search effort as inflation rate tends infinity, we find the presence of capital creates a range of parameters where search intensity remains inefficiently high even as inflation gets arbitrarily high.[7]

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2. ENVIRONMENT

Time is discrete and continues forever. The economy is populated by a continuum of infinitely lived agents, divided into a set of buyers, denoted by **B**, and a set of sellers, denoted by **S**. Each date has two stages: the first has pairwise meetings in a DM and the second has centralized meetings (CM). Time starts in the CM of period zero, and all agents discount across periods according to $\beta = (1 + r)^{-1} \in (0, 1)$. There is a single perishable good produced in each stage, where the CM good is the numéraire. In the CM, all agents can produce and wish to consume. Agents' labels as buyers and sellers depend on their roles in the DM where only sellers are able to produce and only buyers wish to consume.

The numéraire can be transformed into a capital good one for one. Capital goods accumulated at the end of period t are used by sellers at the beginning of the CM of period $t + 1$ to produce numéraire according to the technology $F(k)$, where F is twice continuously differentiable, strictly increasing, strictly concave, and satisfies the Inada conditions $F'(0) = \infty$ and $F'(\infty) = 0$.[8] We also assume $F'(k)k$ is strictly increasing, strictly concave in k, and has range $\mathbf{R}_+$. Capital goods depreciate fully after one period, and the rental (or purchase) price of capital in terms of numéraire at period t is denoted R_t.[9] The assumption of full depreciation is with no loss in generality. For instance, we could have assumed a production technology $f(k)$ and depreciation rate $\delta \in (0, 1)$, and then define $F(k)$ as $F(k) = f(k) + (1 - \delta)k$, which will give us exactly the same analytical results.

There is also an intrinsically useless, perfectly divisible, and storable asset called money. The quantity of money at the end of the period-t CM is M_t, and the relative price of money in terms of numéraire is denoted as ϕ_t. The money supply evolves according to $M_{t+1} = \gamma M_t$, where γ is the gross growth rate of the money supply. New money is injected if $\gamma > 1$, or withdrawn if $\gamma < 1$, by lump sum transfers or taxes in the beginning of the CM to buyers.[10] The lack of record-keeping and private information over individual trading histories rules out unsecured credit, which gives a role for money or capital to serve as means of payment. In addition, individual asset holdings are common knowledge in a match. We assume that sellers do not carry real balances or capital across periods.[11] We also assume that the portfolios of both agents are common knowledge in a match.[12]

Agents are matched pairwise in the DM. The measure of sellers and buyers is normalized each to 1. We assume the seller's search intensity is exogenously given, but buyers can choose their search intensity. At the beginning of the DM, each buyer $b \in \mathbf{B}$ chooses search intensity, $e_b \in [0, 1]$. The average search intensity of buyers is $\bar{e}$, defined as

$$\bar{e} = \int_{b \in \mathbf{B}} e_b db.$$

A buyer exerting effort e to search in the DM incurs cost $\psi(e)$. For all $e \in [0, 1)$, $\psi(e) \in [0, \infty)$ is twice continuously differentiable, strictly increasing, strictly

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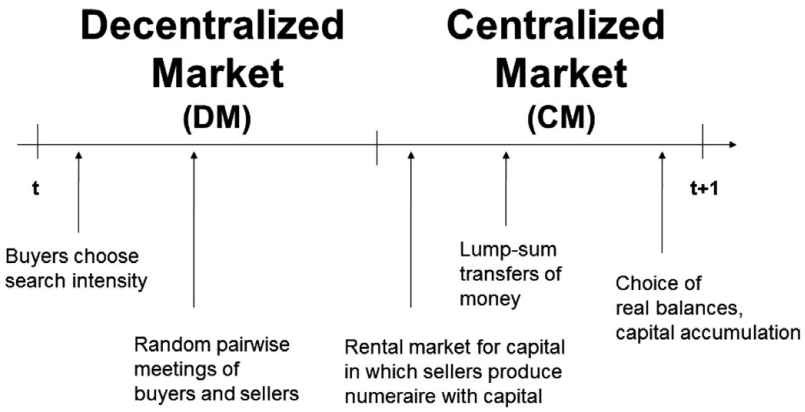


FIGURE 1. Timing of representative period.

convex, and satisfies Inada conditions $\psi(0) = \psi'(0) = 0$, $\lim_{e\to 1} \psi(e) = \infty$, and $\lim_{e\to 1} \psi'(e) = \infty$. Figure 1 summarizes the timing of a representative period.

Given $\bar{e}$, the number of DM matches is determined by a constant-returns-to-scale matching function that depends on market tightness, defined as $\theta \equiv 1/\bar{e} \in [1, \infty]$, or the ratio of sellers to the effective buyers searching. A high θ implies a thick market for buyers and a thin one for sellers. Given θ, the meeting probability for an individual buyer with search intensity e is $e\alpha(\theta)$, while the meeting probability of a seller is $\alpha(\theta)/\theta$. The function $\alpha(\theta)$ satisfies $\alpha(\theta) \in [0, 1]$ for any $\theta \geq 1$ and is twice continuously differentiable, strictly increasing, strictly concave for $\theta \in [1, \infty)$, and satisfies Inada conditions $\lim_{\theta\to\infty} \alpha(\theta) = 1$, $\lim_{\theta\to 1} \alpha(\theta) = 0$, $\lim_{\theta\to 1} \alpha'(\theta) \geq 1$, and $\lim_{\theta\to 1} \alpha(\theta)/\theta = 1$.

The buyer's instantaneous utility function is

$$U^b(q, e, x) = u(q) - \psi(e) + x, \tag{1}$$

where q is DM consumption, x is the utility of consuming $x \in \mathbf{R}$ units of numéraire ($x < 0$ is interpreted as production), and e is the buyer's search effort.[13] We assume $u(0) = 0$, $u'(0) = \infty$, $u'(q) > 0$, and $u''(q) < 0$ for $q > 0$. Similarly, the sellers' instantaneous utility function is

$$U^s(q, x) = -c(q) + x, \tag{2}$$

where q is the DM production and x is defined as before. We assume $c(0) = c'(0) = 0$, $c'(q) > 0$, and $c''(q) \geq 0$. Furthermore, we let $c(q) = u(q)$ for some $q > 0$ and denote by q^* the solution to $u'(q^*) = c'(q^*)$.

3. IMPLEMENTATION

We study outcomes that can be implemented with a mechanism designer's proposal. A *proposal* consists of (i) a sequence of functions in bilateral matches,

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$o_t : \mathbf{R}_+^2 \to \mathbf{R}_+^3$, each of which maps the buyer's portfolio, (z_t, k_t), into a proposed trade, $(q_t, d_{z,t}, d_{k,t}) \in \mathbf{R}_+ \times [0, z_t] \times [0, k_t]$, where q_t is the DM output produced by the seller and consumed by the buyer, $d_{z,t}$ is the transfer of real balances, and $d_{k,t}$ is the transfer of capital from the buyer to the seller; (ii) an initial distribution of money, μ; (iii) a sequence of prices for money, $\{\phi_t\}_{t=1}^{\infty}$, and a sequence of rental prices for capital, $\{R_t\}_{t=1}^{\infty}$, both in terms of numéraire; (iv) a sequence of search intensities for buyers, $\{e_t\}_{t=1}^{\infty}$.

The trading procedure in the DM is given by the following game. Given the buyer's portfolio holdings and the proposed trade, both the buyer and the seller simultaneously respond with *yes* or *no*: if both say *yes*, the proposed trade is carried out; otherwise, there is no trade. Since both agents can turn down the proposed trade, it ensures that trades are individually rational. We also require the proposed trade to be in the pairwise core.[14] Agents in the CM trade competitively against the proposed prices, which is consistent with the pairwise core requirement in the DM due to the equivalence between the core for the CM and competitive equilibria.

We denote s_b as the strategy of buyer $b \in \mathbf{B}$, whose component at date t consists of three parts for any of his private trading history h^t at the beginning of period t: (i) $s_b^{h^t,0}(z,k) = e \in \mathbf{R}_+$ that maps the buyer's portfolio, (z,k), into his search intensity, e, at the beginning of the DM; (ii) $s_b^{h^t,1}(z,k) \in \{yes, no\}$ that, contingent on being matched in the DM, maps the buyer's portfolio (z,k) to his *yes* or *no* response in the DM; (iii) $s_b^{h^t,2}(z,k,a_b,a_s) \in \mathbf{R}_+^2$ that maps the buyer's original portfolio, (z,k), and the buyer's and seller's choices whether to accept the trade, $a_b, a_s \in \{yes, no\}$, to his final real balances and capital holdings after the CM. The strategy of a seller $s \in \mathbf{S}$ at the beginning of period t, given his private history h^t, consists of a function, $s_s^{h^t,1}(z,k) \in \{yes, no\}$, that represent the seller's response to trade contingent on the buyer's portfolio.

DEFINITION 1. *Equilibrium is a list,* $\langle (s_b : b \in \mathbf{B}), (s_s : s \in \mathbf{S}), \mu, \{o_t, \phi_t, R_t, e_t\}_{t=1}^{\infty} \rangle$*, composed of a strategy for each agent and a proposal such that* (i) *each strategy is sequentially rational given other agents' strategies and* (ii) *the centralized market clears at every date.*

In the following, we focus on stationary proposals where real balances are constant over time and equilibria where (i) agents follow symmetric and stationary strategies, (ii) agents respond with *yes* in all DM meetings, and (iii) the initial distribution of money is uniform across buyers. In such an equilibrium, $\phi_t = \gamma \phi_{t+1}$ for all t; hence, we can discuss real balances only and leave out ϕ_t from a proposal. Moreover, the proposed DM trades, $o_t(z_t, k_t)$, are the same across time periods and can be written as $o(z,k) = [q(z,k), d_z(z,k), d_k(z,k)]$.

The outcome of interest is a list, $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$, where (q^p, d_z^p, d_k^p) are the terms of trade in the DM, e^p is the buyer's search intensity, and (z^p, k^p) are the portfolio holdings of those buyers. Such an outcome, $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$, is said to be *implementable* if it is the equilibrium outcome associated with a proposal

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$\{o, R, e\}$. Given proposals and a rental price for capital, we let $\mathcal{CO}(z, k; R)$ denote the set of allocations in the pairwise core for each (z, k).

Given o, θ, and R, let $V^b(z, k)$ and $W^b(z, k)$ denote the continuation values for a buyer holding portfolio (z, k) upon entering the DM and CM, respectively. Similarly, let $W^s(z, k)$ denote the continuation value for a seller holding (z, k). A buyer in the CM solves

$$W^b(z, k) = \max_{x, \hat{z} \geq 0, \hat{k} \geq 0} \left[x + \beta V^b(\hat{z}, \hat{k}) \right]$$

$$\text{s.t. } x + \gamma \hat{z} + \hat{k} = z + Rk + T,$$

where $\hat{z}$ and $\hat{k}$ are the real balances and capital taken into the next DM, and $T = (M_{t+1} - M_t)\phi_t$ is the lump sum transfer of money. In a stationary equilibrium, $\gamma = \frac{\phi_t}{\phi_{t+1}} = \frac{M_{t+1}}{M_t}$. Hence, in order to hold $\hat{z}$ real balances in the next period, the buyer must accumulate $\gamma \hat{z}$ real balances this period. Substituting $x = z + Rk + T - \gamma \hat{z} - \hat{k}$ from the budget constraint, a buyer's CM value function is

$$W^b(z, k) = z + Rk + T + \max_{\hat{z} \geq 0, \hat{k} \geq 0} \left[-\gamma \hat{z} - \hat{k} + \beta V^b(\hat{z}, \hat{k}) \right]. \tag{3}$$

Due to linear preferences in the CM, the buyer's value function is linear in total wealth, $W^b(z, k) = z + Rk + W^b(0, 0)$, and the maximizing choice of $\hat{z}$ and $\hat{k}$ is independent of the buyer's current wealth.

The value function of a buyer with portfolio (z, k) upon entering the DM is

$$V^b(z, k) = \max_{e \in [0,1]} \left\{ -\psi(e) + e\alpha(\theta) \left\{ u\left[q(z, k)\right] + W^b\left[z - d_z(z, k), k - d_k(z, k)\right] \right\} \right.$$
$$\left. + \left[1 - e\alpha(\theta)\right] W^b(z, k) \right\}. \tag{4}$$

According to (4), a buyer searching with intensity e meets a seller with probability $e\alpha(\theta)$, consumes $q(z, k)$, and transfers to the seller $d_z(z, k)$ real balances and $d_k(z, k)$ units of capital. The buyer therefore enters the CM with $z - d_z(z, k)$ real balances and $k - d_k(z, k)$ units of capital. With probability $1 - e\alpha(\theta)$, a buyer is unmatched so there is no trade. From the linearity of W^b, (4) simplifies to

$$V^b(z, k) = \max_{e \in [0,1]} \left\{ -\psi(e) + e\alpha(\theta) \left\{ u\left[q(z, k)\right] - d_z(z, k) - Rd_k(z, k) \right\} \right.$$
$$\left. + W^b(z, k) \right\}. \tag{5}$$

For each portfolio (z, k), we let $e(z, k)$ denote the optimal search intensity that solves (5). Since ψ is strictly convex, $e(z, k)$ is uniquely defined. In addition, $(z, k) = (0, 0)$ implies $e(z, k) = 0$. Since $\theta = 1/e^p$ in equilibrium, the buyer's search intensity, $e^p = e(z^p, k^p)$, solves

$$-\psi'(e^p) + \alpha(1/e^p) \left[u(q^p) - d_z^p - Rd_k^p \right] = 0. \tag{6}$$

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Substituting $V^b(z,k)$ from (5) into (3), using the linearity of $W^b(z,k)$, and omitting constant terms, the buyer's portfolio problem in the CM is

$$\max_{(z,k)} \{-iz - (1+r-R)k - \psi[e(z,k)] + e(z,k)\alpha(\theta)\{u[q(z,k)] - d_z(z,k) - Rd_k(z,k)\}\}, \tag{7}$$

where $i = \frac{\gamma-\beta}{\beta}$ is the cost of holding money and $1+r-R$ is the cost of holding capital, which is the difference between the gross rate of time preference and the rental price of capital. Since holding the equilibrium portfolio, (z^p, k^p), is better than $(0,0)$ in equilibrium, we must have

$$-iz^p - (1+r-R)k^p - \psi(e^p) + e^p\alpha(1/e^p)[u(q^p) - d_z^p - Rd_k^p] \geq 0. \tag{8}$$

Similarly, the Bellman equation for a seller in the CM is

$$W^s(z,k) = z + Rk + \max_{\hat{k}\geq 0}\left[F(\hat{k}) - R\hat{k}\right]. \tag{9}$$

According to (9), the seller's choice of capital solves $F'(\hat{k}) = R$. From market clearing in the CM, $k^p = \hat{k}$. Consequently, the equilibrium capital stock, k^p, solves

$$F'(k^p) = R \leq 1 + r. \tag{10}$$

According to (10), the equilibrium capital stock equates the marginal product of capital, $F'(k^p)$, with the rental rate, R. It is also necessary that $R \leq 1+r$. If $R > 1+r$, buyers hold an infinite amount of capital, but perfect competition implies $F'(\infty) = 0 < 1+r < R$, a contradiction. Using (9), the seller responds with *yes* to the proposed trade (q^p, d_z^p, d_k^p) only if

$$-c(q^p) + d_z^p + Rd_k^p \geq 0. \tag{11}$$

The above analysis implies that (6), (8), (10), and (11) are necessary conditions to implement $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$. In addition, we also impose the pairwise core requirement. Given buyer's portfolio and a rental price, the pairwise core, $\mathcal{CO}(z^p, k^p; R)$, is defined as the set of feasible allocations, $(q, d_z, d_k) \in \mathbf{R}_+ \times [0, z^p] \times [0, k^p]$, such that no alternative feasible allocations would make both parties in the match strictly better off, taking the continuation value as given. A characterization of the pairwise core in a related setting can be found in Hu and Rocheteau (2013)'s Supplementary Appendix B. The following proposition shows that these necessary conditions and the pairwise core requirement are also sufficient.

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PROPOSITION 1. *An outcome,* $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$*, is implementable if and only if*

$$-iz^p - [1 + r - F'(k^p)]k^p + e^p\alpha(1/e^p)[u(q^p) - d_z^p - F'(k^p)d_k^p] - \psi(e^p) \geq 0, \tag{12}$$

$$d_z^p \leq z^p,\ d_k^p \leq k^p, \tag{13}$$

$$\psi'(e^p) = \alpha(1/e^p)[u(q^p) - d_z^p - F'(k^p)d_k^p], \tag{14}$$

$$-c(q^p) + d_z^p + F'(k^p)d_k^p \geq 0, \tag{15}$$

$$F'(k^p) \leq 1 + r, \tag{16}$$

and $(q^p, d_z^p, d_k^p) \in \mathcal{CO}(z^p, k^p; R)$.

The proof of Proposition 1 is constructive as we explicitly provide the proposed trades to implement the candidate outcome. In contrast with the implementability result in Hu and Rocheteau (2013), here we have to worry about the implementation of search efforts determined by the buyer's trade surplus through (14). As a result, the buyer's surplus affects both their portfolio choice and search decision. This generates a new tradeoff we discuss in detail below.

4. OPTIMAL ALLOCATION

We now describe implementable outcomes that are socially optimal. Given an outcome, social welfare is defined as the sum of buyers' and sellers' lifetime discounted expected utilities:

$$\begin{aligned}&\mathcal{W}(q^p, d_z^p, d_k^p, z^p, k^p, e^p)\\ &= -k^p + \lim_{T\to\infty}\sum_{t=1}^{T}\beta^t\left\{e^p\alpha\left(\frac{1}{e^p}\right)[u(q^p) - c(q^p)] - \psi(e^p) + [F(k^p) - k^p]\right\}\\ &= \frac{1}{r}\left\{e^p\alpha\left(\frac{1}{e^p}\right)[u(q^p) - c(q^p)] - \psi(e^p) + [F(k^p) - (1+r)k^p]\right\}.\end{aligned} \tag{17}$$

The first term after the first equality is the utility cost incurred by agents in the initial CM to accumulate k^p; the second term captures the utility flows in subsequent periods and consists of the sum of expected surpluses in pairwise meetings, $e^p\alpha(1/e^p)[u(q^p) - c(q^p)]$, net of the search cost, $\psi(e^p)$, and CM output net of the depreciated capital stock, $F(k^p) - k^p$.

DEFINITION 2. *An outcome,* $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$*, is constrained efficient if it maximizes (17) subject to (12)–(16) and the pairwise core requirement.*

As a benchmark, we begin with the unconstrained problem that maximizes social welfare (17) without constraints (12)–(16). The solution to this unconstrained

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problem, which we call the *first-best allocation*, is given by $q^p = q^*$, $k^p = k^*$, and $e^p = e^*$ that solve

$$u'(q^*) = c'(q^*), \tag{18}$$

$$F'(k^*) = 1 + r, \tag{19}$$

$$\left[\alpha(1/e^*) - \alpha'(1/e^*)/e^*\right][u(q^*) - c(q^*)] = \psi'(e^*). \tag{20}$$

Since (q^*, k^*, e^*) are uniquely determined and (17) is concave in q and e (but is not jointly concave), these necessary conditions are also sufficient. The first-best level of output, q^*, maximizes the match surplus between a buyer and seller, and the first-best level of capital, k^*, ensures that the marginal product of capital compensates for the opportunity cost of holding capital. The first-best level of search intensity, e^*, is derived from the first-order condition on the objective function with respect to e, but taking $q^p = q^*$. Accordingly, the marginal cost of searching, $\psi'(e^*)$, is equal to the corresponding social marginal contribution of searching, $[\alpha(1/e^*) - \alpha'(1/e^*)/e^*]$, times the surplus generated in each trade, $u(q^*) - c(q^*)$.

Next we consider an economy where capital is the only liquid asset. Imposing $z = 0$, an outcome is denoted by (q, d_k, k, e).

LEMMA 1. *Suppose $z = 0$. A constrained-efficient outcome, (q^c, d_k^c, k^c, e^c), exists, and the first best is implementable if and only if*

$$(1 + r)k^* \geq u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}.$$

When the first best is not implementable, $d_k^c = k^c > k^*$. Maximal social welfare given by

$$\mathcal{W}^c = \frac{1}{r}\left\{e^c\alpha(1/e^c)[u(q^c) - c(q^c)] - \psi(e^c) + F(k^c) - (1 + r)k^c\right\}$$

is strictly greater than what is achievable with the additional constraint $k = k^*$, denoted as $\mathcal{W}^0$.

Lemma 1 implies the first best is implementable without money when k^* is sufficiently large. In that case, the aggregate capital stock is sufficiently abundant to allow buyers to finance consumption of the first best. Since the first best is implementable with $z = 0$, money is not essential.

When instead k^* is insufficient to meet the economy's liquidity needs, the optimal mechanism features an overaccumulation of capital where $k^c > k^*$. In addition, quantities traded in the DM are inefficiently low ($q^c < q^*$). With a shortage of liquidity, society faces a trade-off between two inefficiencies, as highlighted by Hu and Rocheteau (2013): (i) the shortage of capital for liquidity purposes, and (ii) the overaccumulation of capital for productive purposes. Lemma 2 then shows that whenever the first best is not implementable, overaccumulation of capital is

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TABLE 1. Constrained efficient outcomes with capital alone

	First-best	$\sigma = 0.3$	First-best	$\sigma = 0.7$
Output q	1	0.29	1	0.32
Search effort e	0.22	0.18	0.34	0.41
Capital k	0.17	0.32	0.17	0.37

socially optimal in order to mitigate the shortage of liquidity. Note that since it is always feasible to set $z = 0$, $\mathcal{W}^c$ gives a lower bound on welfare when both money and capital are present.

We provide numerical examples in Table 1 for the economy with capital alone. We assume $u(q) = \frac{(q+b)^{1-\sigma}-b^{1-\sigma}}{1-\sigma}$, $c(q) = \frac{q^\kappa}{\kappa}$, $\psi(e) = c\left(\frac{e}{1-e}\right)^\rho$, $\alpha(\theta) = 1 - \exp(1-\theta)$ where $\theta = 1/e$, $F(k) = Ak^a + (1-\delta)k$, and set $b = 0.0001$, $c = 0.4$, $\rho = 2$, $\kappa = 1$, $r = 0.02$, $a = 0.3$, $A = 0.8$, $\delta = 0.8$, and consider two values for σ, 0.3 and 0.7. In both cases, the first best is not implementable and there is overaccumulation of capital. However, when $\sigma = 0.3$, equilibrium search intensity is inefficiently low; for $\sigma = 0.7$, search intensity is inefficiently high.

Here, we present our main proposition, which summarizes the effects of inflation on implementable allocations when there is a shortage of capital. To simplify notation, we call a tuple $[q^p(i), z^p(i), k^p(i), e^p(i)]$ a constrained-efficient outcome under nominal interest rate i if there exists $(d_z^p, d_k^p) \leq [z^p(i), k^p(i)]$ such that $[q^p(i), d_z^p, d_k^p, z^p(i), k^p(i), e^p(i)]$ maximizes social welfare, (17), subject to (12)–(16) and the pairwise core requirement.

PROPOSITION 2. *Suppose* $(1+r)k^* < u(q^*) - \psi'(e^*)/\alpha(1/e^*)$. *For any* $i \geq 0$, *a constrained efficient outcome,* $[q^p(i), z^p(i), k^p(i), e^p(i)]$, *exists, and satisfies the following.*

1. *Let* $i^* = \frac{e^*\psi'(e^*)-\psi(e^*)}{u(q^*)-(1+r)k^*-\psi'(e^*)/\alpha(1/e^*)} > 0$. *For all* $i \in [0, i^*]$, *the constrained-efficient outcome,* $[q^p(i), z^p(i), k^p(i), e^p(i)]$, *is unique [except for* $z^p(i)$*] , and satisfies* $q^p(i) = q^*$, $z^p(i) > 0$, $k^p(i) = k^*$, *and* $e^p(i) = e^*$.
2. *Suppose* $1 + r + F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^*}$. *There exists* $\bar{i}' > i^*$ *such that for all* $i \in (i^*, \bar{i}']$, *the unique constrained-efficient outcome,* $[q^p(i), z^p(i), k^p(i), e^p(i)]$, *satisfies* $q^p(i) < q^*$, $k^p = k^*$, $z^p(i) > 0$, *and* $e^p(i) > e^*$. *Moreover,* $e^p(i)$ *is strictly increasing in* $i \in [i^*, \bar{i}']$.
3. *There exists* $\hat{i}$ *such that, for each* $i > \hat{i}$, *and for each constrained-efficient outcome, we have* $k^p(i) > k^*$. *Moreover,* $z^p(i) \to 0$ *as* $i \to \infty$ *but maximum welfare converges to* $\mathcal{W}^c > \mathcal{W}^0$.

Proposition 2 assumes $(1+r)k^* < u(q^*) - \psi'(e^*)/\alpha(1/e^*)$ to allow a role for money. While the proof of Proposition 2 (1) only requires verifying constraints (12)–(16), the proof of Proposition 2 (2) is nonstandard since the constraint set is not convex and the objective function is not globally concave. Instead, we employ the Implicit Function Theorem to find a solution to the first-order conditions and use continuity to establish that the solution is also a global maximizer. While

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we cannot give an explicit expression for the upper bound on the inflation rate below which search intensity increases, we later provide examples to quantify this threshold.

According to Proposition 2 (1), the highest nominal interest rate for implementing the first best is strictly positive: $i^* > 0$. Hence, the Friedman rule, defined as $i = 0$, is sufficient but not necessary to achieve maximal welfare.[15] For all $i \in [0, i^*]$, money is superneutral and all welfare-relevant variables are at their first-best levels. In contrast to Hu et al. (2009), a difference here is the first best cannot be implemented by giving all the surplus to buyers with the equilibrium amount of real balances. If this were the case, then under the first-best level of output, search intensity would be given by (14) with $q = q^*, d_z + F'(k)d_k = c(q^*)$, and $k = k^*$, and hence equal to $\hat{e}$ given by

$$\psi'(\hat{e})/\alpha(1/\hat{e}) = [u(q^*) - c(q^*)]. \tag{21}$$

But due to search externalities, $\hat{e} > e^*$. By (20),

$$\begin{aligned}\psi'(e^*)/\alpha(1/e^*) &< \psi'(e^*)/[\alpha(1/e^*) - \alpha'(1/e^*)/e^*] = [u(q^*) - c(q^*)] \\ &= \psi'(\hat{e})/\alpha(1/\hat{e}).\end{aligned}$$

To discourage buyers from oversearching, the optimal mechanism gives buyers a fraction of the surplus, while the seller's participation constraint, (15), is not binding at the optimum.

For intermediate inflation rates, Proposition 2 (2) gives a sufficient condition for inflation to have no effects on the capital stock even though output is inefficiently low and search intensity is inefficiently high. For instance, when $F(k) = Ak^a$, the sufficient condition holds if a is not too large or i^* is relatively small. We remark here that money is essential for a range of nominal interest rates above i^*, even without imposing the sufficient condition, $1 + r + F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^*}$ [see Claim 2 in the proof of Proposition 2 (2)]. As in Hu and Rocheteau (2013), there is rate-of-return dominance whenever both money and capital are used.

While this result resembles the hot potato effect of inflation, the mechanism in our model differs from the conventional rationale. The standard explanation is that higher inflation itself induces buyers to search harder in order to get rid of their money holdings. However, this reasoning implicitly assumes a cash-in-advance constraint without which buyers may not hold cash in the first place. Instead in our setting, the optimal mechanism dictates buyers to have more surplus as inflation rises above i^*, thereby inducing buyers to search harder.

The intuition for why both the buyer's surplus and search effort increases with inflation can be seen from Figures 2 and 3, which depict the implementable set under $k = k^*$ for $i = i^*$ and a slightly higher i', denoted by $A^m(i^*)$ and $A^m(i')$, respectively. In Figure 2, the first-best allocation, (q^*, e^*), lies on the boundary of the lower curve, which corresponds to the buyer's participation constraint, (12), being binding, but lies strictly below the upper curve, which corresponds

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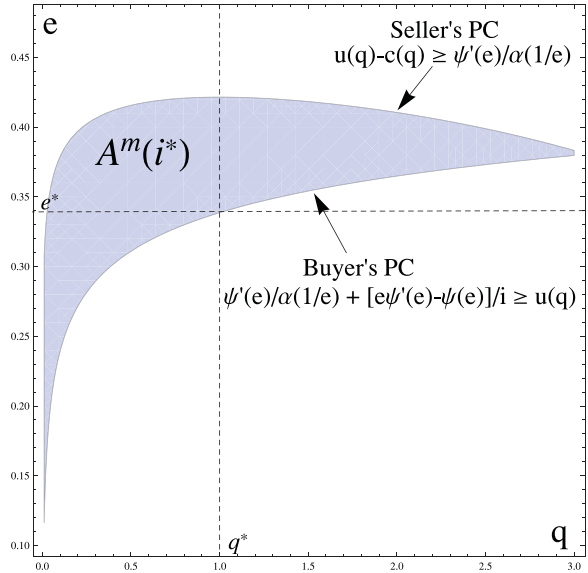


FIGURE 2. Implementable set, $\mathcal{A}^m(i^*)$.

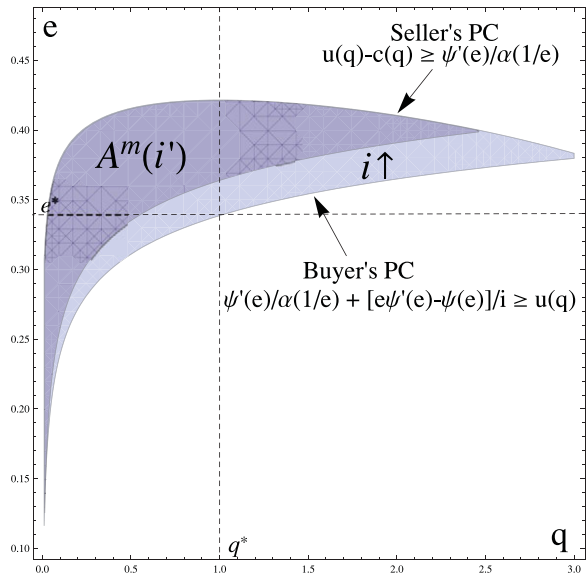


FIGURE 3. Implementable set shrinks As $i \uparrow$.

to the seller's participation constraint, (15), being slack. Figure 3 shows that as the nominal interest rate increases from i^* to $i' > i^*$, the buyer's constraint shifts upward, while the seller's constraint is not affected. As the objective function, (17),

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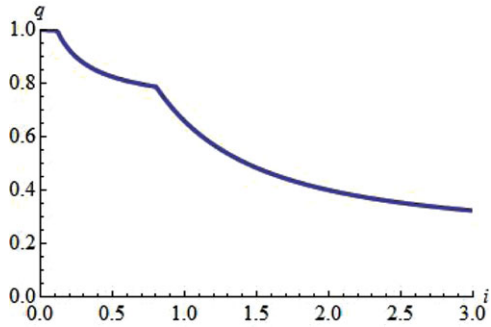


FIGURE 4. Output per match.

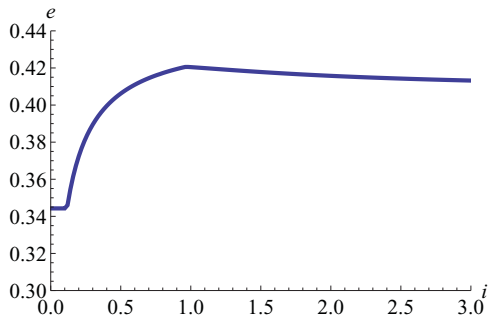


FIGURE 5. Search intensity.

is locally concave, the constrained-efficient level of search intensity increases with inflation. Since output higher than q^* would violate the pairwise core requirement, output falls with inflation.

In the high-inflation regime, the economy with both money and capital has several novel features. Proposition 2 (3) shows capital overaccumulation is bound to occur as inflation rises, even without the sufficient condition in part (2). In turn, the monetary sector eventually collapses. In contrast to the pure currency economy studied in the literature, the economy never collapses into autarky since capital can always be used as a medium of exchange. In addition, search intensity can remain inefficiently high even at high-inflation rates. Indeed, as welfare converges to the level where only capital is the medium of exchange, $\mathcal{W}^c$, Proposition 2 (3) suggests search intensity also converges to its level without money, which may be higher or lower than e^*.

Figures 4–9 are examples illustrating Proposition 2. We assume the same functional forms as before with $b = 0.0001$, $c = 0.4$, $\rho = 2$, $\kappa = 1$, $r = 0.02$, $a = 0.3$, $A = 0.8$, $\delta = 0.8$, and $\sigma = 0.7$. Figure 5 shows search intensity remains inefficiently high and approaches the value reported in Table 1 for the economy with capital alone, and hence the rise in search intensity persists even for high

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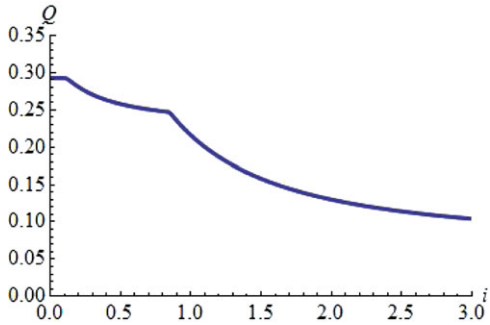


FIGURE 6. Aggregate output.

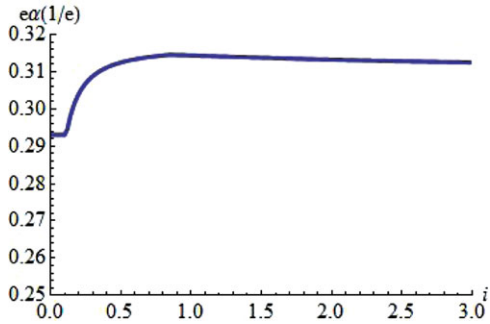


FIGURE 7. Matching probability.

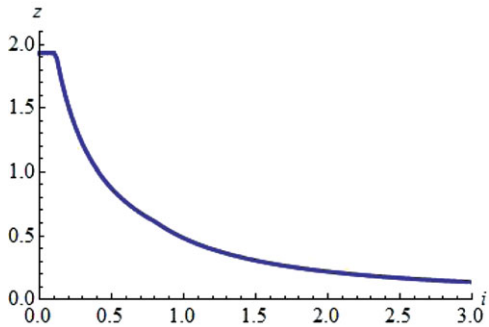


FIGURE 8. Real balances.

inflation rates. For high inflation rates, there is a substitution effect where the optimal mechanism prescribes buyers to substitute money for capital as inflation increases. This can be seen in Figures 8 and 9. This Tobin effect turns out to be an optimal way of responding to inflation as doing so allows agents to maintain consumption in the DM even as inflation gets very high. The fact that capital

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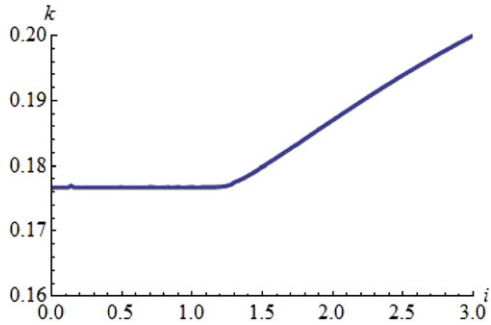


FIGURE 9. Capital.

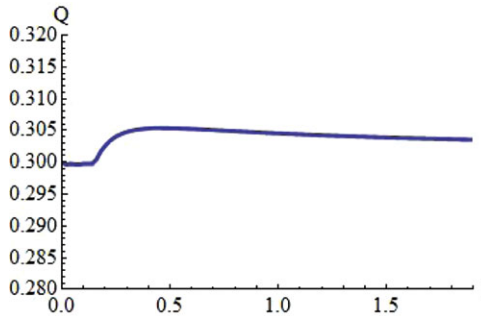


FIGURE 10. Aggregate output.

increases with inflation comes from our broader notion of capital as an asset that competes with money as media of exchange. Indeed, since agents can use both money and capital as means of payment, but incur a real cost for overaccumulating of capital, search intensity can still rise with inflation even as inflation gets very high.

Finally, we provide examples where DM aggregate output can rise with inflation. DM aggregate output is defined as the total quantity of goods traded or total production in the DM:

$$\mathcal{Q} \equiv \underbrace{e\alpha(1/e)}_{\text{matching prob.}} \times \underbrace{q}_{\text{output per trade}} . \tag{22}$$

Figure 10 plots DM aggregate output with $\kappa = 5$. When $i \in (i^*, \bar{i}']$, our model has two counteracting effects: Search intensity, and hence the frequency of trades as in Figure 11 $e\alpha(1/e)$, increases with inflation while DM quantity traded per match, q, falls. In our examples, the responsiveness of DM output to inflation is decreasing in κ, i.e., output is less responsive to inflation when $c(q)$ is more convex. Hence, when κ is relatively large, it is possible for the total quantity traded,

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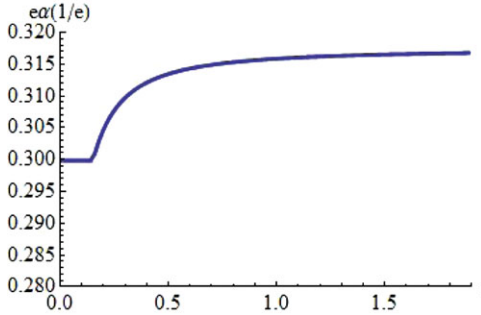


FIGURE 11. Matching probability.

$e\alpha(1/e)q$, to go up with inflation. This nonmonotonicity result on search efforts and quantity traded contrasts with many previous studies that study endogenous search decisions under suboptimal trading mechanisms. In particular, Lagos and Rocheteau (2005) show under Nash bargaining, the buyer's search effort falls monotonically with inflation. As this trading protocol is held constant for different inflation rates, both the buyer's real balances and surplus fall with inflation, and hence search efforts fall as well.

5. CONCLUDING REMARKS

In this paper, we adopt mechanism design to revisit some classical issues in monetary economics, namely the long-run effects of inflation on output, search efforts, and capital accumulation as well as the social costs of inflation. We develop a tractable monetary model featuring costly search efforts to endogenize the frequency of trade, capital accumulation to endogenize the choice of a means of payment, and an endogenous trading mechanism that adjusts with the inflation tax.

On the normative side, our results suggest that zero inflation rate is not necessarily the optimal policy under the optimal trading mechanism and a strictly positive rate can also be optimal, and we show that this can be the case with endogenous search intensity as well as with other means of payments.

On the positive side, the model is able to replicate several qualitative patterns emphasized in empirical macrostudies and historical anecdotes, including monetary superneutrality for a range of low-inflation rates, nonlinearities in trading frequencies and aggregate output, and substitution of money for capital for high inflation rates. While we acknowledge that certain aspects of our findings have appeared separately in previous studies, we show how they are intimately related by all being features of an optimal trading mechanism. The fact that changes in inflation can have severe consequences on economic exchange and social interactions has also been emphasized by economic historians [Bresciani-Turroni (1931), Heymann and Leijohnhufvud (1995), O'Dougherty (2002)]. Here, we remark on a few caveats to our analysis and posit some directions for future research.

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5.1. Optimal Trading Mechanism

In our framework, the economy's trading mechanism evolves to the optimal mechanism as the inflation rate changes. The inflation rate itself however is taken as exogenous, and our focus is to study the consequences of changes in inflation. Importantly, however, we do endogenize society's trading mechanism and obtain very different results from previous studies, most of which treat the trading mechanism as a primitive. Indeed, we show that under the optimal mechanism, the hot potato effect and substitution between money and capital are both optimal ways of responding to the inflation tax. Although it is unlikely for societies to change trading patterns for small changes in inflation, it seems plausible that societies would adjust trading mechanisms for large and persistent changes and our results are qualitatively in line with historical episodes of such changes.

5.2. Other Substitutes for Domestic Currency

Our model assumes that capital goods are the only alternative means of payments to money. However, capital goods in the model can be interpreted more broadly to include other real assets that may provide a hedge against inflation. This includes the use of assets not only for immediate settlement, but also as collateral [Caballero (2006)].[16] An example is the use of home equity as collateral to finance future consumption. Moreover, individuals often resort to using foreign currencies for transactions during periods of high or hyperinflation [Calvo and Vegh (2002)]. While our current framework cannot fully accommodate for the circulation of foreign currencies, an extension of our model to multiple countries and currencies is a fruitful topic for future research. Such a model could then determine how the presence of foreign currencies affects the consequences of inflation on international trade and welfare, as in Zhang (2014).

5.3. Sellers' Search Intensity

While we assume that only buyers choose their search intensity, we can extend the analysis to allow sellers to choose as well. This would add an additional constraint in Proposition 1, through a first order condition for sellers' search effort. This implies that we always have to give sellers *some* trade surpluses to encourage their search decisions. Under appropriate conditions, Proposition 2 (1) still holds, but with a modified expression for i^*. We also conjecture that if the cost of search is relatively small for sellers, Proposition 2 (2) and (3) will still hold and leave a more comprehensive analysis for future work.

NOTES

1. See e.g. descriptions by Guttmann and Meehan (1975) for the 1920s hyperinflation in Germany, Heymann and Leijohnhufvud (1995), and O'Dougherty (2002) for the 1990s high inflations in Latin America.

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2. In this sense, our notion of capital is rather broad; it includes all durable commodities or intermediate goods that can serve as direct means-of-payments or collateral. See Bresciani-Turroni (1931, Chap. 5), for evidence on the effects of overaccumulation of such goods in German hyperinflation from 1914 to 1923.

3. As emphasized by Casella and Feinstein (1990), inflation not only affects how individuals economize on their real balances, it also changes the economy's trading patterns: "[H]istorians emphasize hyperinflation's disruptive impact on individuals and on their socioeconomic relationships. Previously stable trading connections were severed, transactions patterns were altered, and normally well-functioning markets collapsed."

4. In our working paper Hu and Zhang (2015), we consider a pure currency economy and obtain this same result under the optimal trading mechanism.

5. This includes Li (1994) that studies a model with indivisible money and Lagos and Rocheteau (2005) in a model with divisible money.

6. Another way to generate the hot potato effect is to have periodic access to the centralized market, as in Ennis (2009), or to introduce preference shocks, as in Ennis (2008) and Nosal (2011). Liu et al. (2011) focus instead on buyers' participation decisions and show inflation decreases the number of buyers and hence the frequency of trades. In our working paper Hu and Zhang (2015), we consider an extension with endogenous entry by buyers and obtain a similar result as Liu et al. (2011) under an optimal mechanism.

7. Alternative formalizations of capital accumulation in monetary search models include Shi (1999), Aruoba and Wright (2003), and Aruoba et al. (2011). These papers focus on the use of capital for production and rule out its role as a means of payment, a key focus of the present paper.

8. Our model can be reinterpreted as one where buyers are endowed with one unit of labor in the CM and enjoy no utility from leisure, where the CM technology is a constant-return-to-scale neoclassical production function, $G(K, L)$. Our analysis would be unchanged if we replace $F(k)$ with $G(k, 1)$. Although it would complicate the presentation, allowing for elastic labor supply in $G(K, L)$ would not alter our main results. In addition, our main results go through with a linear production technology for capital, e.g., $F(k) = Ak$ where $A > 0$.

9. Who operates the technology, F, is irrelevant for our analysis provided the residual profits, $F(k) - kF'(k)$, are not pledgable in the DM due to lack of commitment.

10. Since we focus on the effects of inflation, money growth is not chosen optimally and is taken as given in the mechanism design problem. To model deflation, the government is assumed to have enough coercive power to collect and enforce taxes in the CM, but has no coercive power in the DM.

11. Lemma 3 in Hu and Rocheteau (2013) shows that, in a similar environment without endogenous search intensity, this assumption is with no loss in generality, as far as optimal allocation is concerned. It is straightforward to modify their proof to show the same result in our environment with search intensity.

12. Our results will go through if the portfolio is private information, as in Hu et al. (2009), but, since this informational friction is not the focus here, we make this assumption to simply the exposition.

13. For tractability, the model requires either the utility of consuming or the cost of producing the CM good is linear. Here, we assume both CM consumption and production is linear though it would be straightforward to generalize to quasilinear preferences.

14. The pairwise core requirement can be implemented directly with a trading mechanism that adds a renegotiation stage as in Hu et al. (2009), following the *yes* responses from both agents. The renegotiation stage will work as follows. An agent will be chosen at random to make an alternative offer to the one made by the mechanism. The other agent will then have the opportunity to choose between the two offers.

15. This finding differs from the typical result in monetary models that rely on exogenously given trading mechanisms such as pairwise bargaining. There, the Friedman rule is typically necessary for efficiency, at least with regard to the amount of output traded in a match. With endogenous participation or entry however, the Friedman rule need not be optimal. See also Rocheteau and Wright (2005) and Berentsen et al. (2007) for a related discussion.

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16. The role of assets as collateral also appears in Kiyotaki and Moore (2008) where assets do not change hands along the equilibrium path. This would entail DM trades using secured credit with capital playing the role of collateral. Then, in the CM, debtors would settle obligations in numéraire. In our setup, capital goods are transferred between individuals and there is finality in each DM trade.

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APPENDIX

PROOF OF PROPOSITION 1

We proved the necessity of constraints (12)–(16) in the main text. Here, we prove their sufficiency. Let $(q^p, d_z^p, d_k^p, z^p, k^p, e^p)$ be an outcome that satisfies (12)–(16) and the pairwise core requirement. Consider the following trading mechanism with $R = F'(k^p)$:

1. If $(z, k) \geq (z^p, k^p)$, then

$$o(z,k) \in \max_{q,d_z,d_k} \{d_z + Rd_k - c(q)\} \quad \textbf{(A.1)}$$
$$\text{s.t. } u(q) - d_z - Rd_k \geq u(q^p) - d_z^p - Rd_k^p,$$
$$q \geq 0,\ d_z \in [0,z],\ d_k \in [0,k].$$

2. Otherwise,

$$o(z,k) \in \max_{q,d_z,d_k} \{d_z + Rd_k - c(q)\} \quad \textbf{(A.2)}$$
$$\text{s.t. } u(q) - d_z - Rd_k \geq 0,$$
$$q \geq 0,\ d_z \in [0,z],\ d_k \in [0,k].$$

Solutions to the maximization problems (A.1) and (A.2) exist and are denoted by $o(z,k) = [q(z,k), d_z(z,k), d_z(z,k)]$. Each solution has a unique $q(z,k)$. Although d_z and d_k may not be uniquely determined, we select the solution such that $d_z(z,k) = z$ if it exists and $d_k(z,k) = 0$ otherwise for any $(z,k) \neq (z^p, k^p)$. Indeed, in the Supplemental Material, Section 1, we show that the total wealth transfer is in fact uniquely determined.

To show that (q^p, z^p, k^p) is a solution to (A.1) for $(z,k) = (z^p, k^p)$, notice that (A.1) is the dual problem that defines the core of a pairwise meeting. Because $(q^p, z^p, k^p) \in \mathcal{CO}(z^p, k^p; R)$, it is also a solution to (A.1). This gives us a well-defined mechanism, o.

Now we show that the following strategy profile, (s_b^*, s_s^*), form a simple equilibrium: for all t and for all h^t, $(s_b^*)^{h^t,0}(z,k) = e^p$ if $(z,k) \geq (z^p, k^p)$, $(s_b^*)^{h^t,0}(z,k) = 0$ otherwise; for all portfolios (z,k), $(s_b^*)^{h^t,1}(z,k) = yes$ for all portfolios (z,k), and $(s_s^*)^{h^t,1}(z,k) = yes$; for all portfolios (z,k) and all responses (a_b, a_s), $(s_b^*)^{h^t,2}(z,k,a_b,a_s) = (z^p, k^p)$. In words, irrespective of their portfolios when entering the CM, buyers exit the CM with their proposed portfolios, (z^p, k^p). The effort choice is e^p if the buyer holds no less than the proposed portfolio in both assets; it is zero otherwise. In the DM they always say *yes* to the proposals. We show that s_b^* and s_s^* are optimal strategies following any history, given that all other agents follow (s_b^*, s_s^*).

Conditions (12) and (15), as well as the constraints in (A.1) and (A.2), ensure that both buyers and sellers are willing to respond with *yes* to the mechanism, both on and off equilibrium paths.

Now, by (A.1) and (A.2), the buyer's surplus is given by

$$u\left[q(z,k)\right] - d_z(z,k) - Rd_k(z,k) = u(q^p) - d_z^p - Rd_k^p \text{ if } (z,k) \geq (z^p,k^p); \quad \textbf{(A.3)}$$
$$u\left[q(z,k)\right] - d_z(z,k) - Rd_k(z,k) = 0 \text{ otherwise.}$$

As a result, because e^p satisfies (14) and $R = F'(k^p)$, it follows that $e(z,k) = e^p$ if $(z,k) \geq (z^p, k^p)$ and $e(z,k) = 0$ otherwise. Now consider the problem (7). By (A.3), any

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choice (z, k) with $(z, k) \geq (z^p, k^p)$ are strictly dominated by (z^p, k^p) and other choices are dominated by $(0, 0)$, but (z^p, k^p) is better than $(0, 0)$ by (12). This implies that (z^p, k^p) is the unique solution to the problem (7). ■

PROOF OF LEMMA 1

From Proposition 1, an outcome (q, d_k, k, e) is implementable if and only if

$$-[1 + r - F'(k)]k + e\alpha(1/e)[u(q) - F'(k)d_k] \geq \psi(e), \tag{A.4}$$

$$-c(q) + F'(k)d_k \geq 0, \tag{A.5}$$

$$1 + r \geq F'(k), \tag{A.6}$$

$$\psi'(e) = \alpha(1/e)[u(q) - F'(k)d_k], \tag{A.7}$$

and $(q, d_k) \in \mathcal{CO}(0, k; R)$ with $R = F'(k)$.

(1) Suppose $(1+r)k^* \geq u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$. We show the first-best allocation, (q^*, d_k^*, k^*, e^*), is implementable, where

$$d_k^* = \frac{1}{1+r}\left[u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}\right].$$

Because $F'(k^*) = 1 + r$, (A.4) and (A.6) are satisfied. Note that $d_k^* \leq k^*$ because $(1+r)k^* \geq u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$ and (A.7) is satisfied by construction. Finally, (A.5) holds if and only if $u(q^*) - \psi'(e^*)/\alpha(1/e^*) \geq c(q^*)$, that is, $\alpha(1/e^*)[u(q^*) - c(q^*)] \geq \psi'(e^*)$. But, by (20) and the fact that $\alpha'(\theta) > 0$ for all θ, $\alpha(1/e^*)[u(q^*) - c(q^*)] = [\alpha'(1/e^*)/e^*][u(q^*) - c(q^*)] + \psi'(e^*) > \psi'(e^*)$.

(2) Suppose $(1 + r)k^* < u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$. Here, we show $k^0 > k^*$ and hence the first best is not implementable and that $\mathcal{W}^c > \mathcal{W}^0$.

First, we show $\mathcal{W}^0 > 0$. Consider the outcome $(\bar{q}, \bar{d}_k, k^*, \bar{e})$ given as follows: $\bar{q} = u^{-1}[(1 + r)k^*] > 0$, $\bar{e}$ solves

$$[\alpha(1/e) - \alpha'(1/e)/e][u(\bar{q}) - c(\bar{q})] = \psi'(e),$$

$\bar{d}_k = u(\bar{q}) - \psi'(\bar{e})/\alpha(1/\bar{e}) > c(\bar{q})$. The outcome is implementable and is associated with positive welfare.

Second, we show a constrained-efficient outcome (under the additional constraint $z = 0$), (q^c, d_k^c, k^c, e^c), exists. Note first that any outcome (q, d_k, k, e) with $q > q^*$ is strictly dominated by another outcome with $q' \leq q^*$; the proof follows exactly the same arguments as in the proof of Lemma 1. Second, any outcome (q, d_k, k, e) with $d_k < k$ is strictly dominated as well. If $k > k^*$, then we can decrease k and obtain higher welfare. Otherwise, assume that $k = k^*$ and consider two cases: (i) $q < q^*$. Then, consider another outcome (q', d_k', k, e) such that $q < q' < q^*$ and that $u(q') - F'(k)d_k' = u(q) - F'(k)d_k$. So buyer surplus is unchanged; the seller constraint is satisfied [note that $u(q') - c(q') > u(q) - c(q)$]:

$$F'(k)d_k' - c(q') = u(q') - c(q') - u(q) + F'(k)d_k > -c(q) + F'(k)d_k \geq 0.$$

So (q', d_k', k, e) is implementable but has strictly higher welfare. (ii) $q = q^*$ and $k = k^*$. Then, because $(1 + r)k^* < u(q^*) - \frac{\psi'(e^*)}{\alpha(1/e^*)}$ and because (q, d_k, k, e) satisfies

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(A.7), we have

$$\psi'(e)/\alpha(1/e) = [u(q^*)-(1+r)d_k] \geq [u(q^*)-(1+r)k^*] > \psi'(e^*)/\alpha(1/e^*), \quad \textbf{(A.8)}$$

and hence $e > e^*$. So lowering e will increase welfare. Consider (q, d_k', k, e') with $d_k < d_k' < k = k^*$ and that

$$\frac{\psi'(e')}{\alpha(1/e')} = [u(q^*) - F'(k^*)d_k'].$$

So $e' \in (e^*, e)$. Then, (q, d_k', k, e') is implementable but has strictly higher welfare. Thus, we may only consider outcomes with $k = d_k$, and $q \leq q^*$. This implies that $k \leq \hat{k}$ that is given by

$$F'(\hat{k})\hat{k} = u(q^*). \quad \textbf{(A.9)}$$

Therefore, we may consider outcomes of the form (q, k, k, e) that satisfies (A.4)–(A.7) and $q \in [0, q^*]$, $k \in [0, \hat{k}]$. Thus, we have a maximization problem of a continuous objective function with a compact feasible set, which admits a maximum.

Now we show that, in any constrained-efficient outcome, (q^c, d_k^c, k^c, e^c), $k^c > k^*$. Suppose, by contradiction, that $k^c = k^*$. Consider two cases.

(a) $q^c < q^*$. We have shown that $d_k^c = k^*$. Note that because $k^c = k^*$ and because of (A.7), (A.4) holds with strict inequality. Let $k' > k^*$ be sufficiently close to k^* such that, by setting q' to satisfy $u(q') - F'(k')k' = u(q^c) - F'(k^*)k^*$, we have

$$q^c < q' < q^* \text{ and } e^c\alpha(1/e^c)[u'(q') - c'(q')]\frac{g'(k')}{u'(q^c)} > [1 + r - F'(k')],$$

where $g(k) = F'(k)k$, a concave function by assumption, and that (A.4) holds for $q = q'$, $e = e^c$, and $F'(k)d_k = F'(k')k'$. Note that the second requirement to define k' can be satisfied because the right side is zero at k^* but the left side is bounded away from zero. Because $u'(q')-c'(q') > u(q^c)-c(q^c)$, it follows that $F'(k')k' > c(q')$. Thus, (q', k', k', e^c) is implementable but the welfare difference is

$$e^c\alpha(1/e^c)[u(q')-c(q')-u(q^c)+c(q^c)]-\{[(1+r)k'-F(k')]-[(1+r)k^*-F(k^*)]\}$$

$$> e^c\alpha(1/e^c)[u'(q') - c'(q')]\frac{g'(k')}{u'(q^c)}(k' - k^*) - [(1 + r) - F'(k')](k' - k^*) > 0.$$

(b) $q^c = q^*$, and hence, by (A.8), $e^c > e^*$. Let $k' > k^*$ be sufficiently close to k^* such that, by setting e' to satisfy $u(q^*) - F'(k')k' = \psi'(e')/\alpha(1/e')$, we have

$$e^c > e' > e^* \text{ and } l'(e')\frac{g'(k')}{\max_{e\in[e',e^0]} j'(e)} > [1 + r - F'(k')],$$

where $j(e) = \psi'(e)/\alpha(1/e)$ and $l(e) = e\alpha(1/e)h(q^*) - \psi(e)$ [note that $j(e)$ is strictly increasing]. Again, the second requirement that defines e' above can be satisfied because $1+r-F'(k^*) = 0$ but the left-hand side is bounded away from zero. (q^*, k', k', e') is implementable but the welfare difference is

$$[l(e') - l(e^c)] - \{[(1 + r)k' - F(k')] - [(1 + r)k^* - F(k^*)]\}$$

$$> l'(e')\frac{g'(k')}{\max_{e\in[e',e^c]} j'(e)}(k' - k^*) - [(1 + r) - F'(k')](k' - k^*) > 0.$$

Therefore, we have $k^c > k^*$. ■

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PROOF OF PROPOSITION 2

(1) It is straightforward to show that the first-best allocation given by (18)–(20) is unique using standard arguments. Now we show that for all $i \in [0, i^*]$, the outcome $(q^*, d_z^*, k^*, z^*, k^*, e^*)$ with

$$d_z^* = u(q^*) - (1+r)k^* - \frac{\psi'(e^*)}{\alpha(1/e^*)} > 0$$

satisfies constraints (12)–(16). Clearly, $F'(k^*) = 1 + r$ implies (16) is satisfied. Note that (14) holds by construction. Plugging this into (12), it is straightforward to verify that it holds if and only if $i \leq i^*$ by definition of i^*. Note that (15) holds if and only if $u(q^*) - \psi'(e^*)/\alpha(1/e^*) \geq c(q^*)$, that is, $\alpha(1/e^*)[u(q^*) - c(q^*)] \geq \psi'(e^*)$. But, by (20) and the fact that $\alpha'(\theta) > 0$ for all θ,

$$\alpha(1/e^*)[u(q^*) - c(q^*)] = \alpha'(1/e^*)/e^*[u(q^*) - c(q^*)] + \psi'(e^*) > \psi'(e^*).$$

(2) First, we show that when $i > i^*$, any outcome (q, d_z, d_k, z, k, e) with $d_z < z$ or $d_k < k$ is strictly dominated. Note that any outcome with $q > q^*$ is strictly dominated by another with $q' \leq q^*$. The case with $d_k < k$ follows the same arguments as those in the proof of Lemma 2. Consider the case with $d_z < z$ and $d_k = k$. If $k > k^*$, then we may decrease k and d_k and increase d_z to keep the buyer surplus unchanged, and by doing so we keep the constraints but increase the welfare. So assume that $k = k^*$. If $q < q^*$, then we may increase both q and d_z to keep the buyer surplus unchanged, and by doing so we keep the constraints but increase the welfare. So assume that $k = k^*$ and $q = q^*$. Then, by (12) and (14),

$$\frac{e\psi'(e) - \psi(e)}{i} \geq z > d_z = u(q^*) - \psi'(e)/\alpha(1/e) - (1+r)k^*,$$

and hence

$$\frac{e\psi'(e) - \psi(e)}{u(q^*) - \psi'(e)/\alpha(1/e) - (1+r)k^*} > i > i^*,$$

which implies that $e > e^*$. Thus, we may increase d_z and decrease e to keep (14) intact, and by doing so increase welfare.

Thus, we may only consider outcomes with $d_k = k$, $d_z = z$, and with $q \leq q^*$. Because $q \leq q^*$, to satisfy (12) it must be the case that $F'(k)k \leq u(q^*)$, that is, $k \leq \hat{k}$, which is given by (A.9). Thus, we may restrict attention to outcomes, (q, z, k, e), that satisfy

$$-iz - [1 + r - F'(k)]k + e\alpha(1/e)[u(q) - z - F'(k)k] \geq \psi(e), \quad \textbf{(A.10)}$$

$$-c(q) + z + F'(k)k \geq 0, \quad \textbf{(A.11)}$$

$$1 + r \geq F'(k), \quad \textbf{(A.12)}$$

$$\psi'(e) = \alpha(1/e)[u(q) - z - F'(k)k]. \quad \textbf{(A.13)}$$

Note that because $d_z = z$ and $d_k = k$, $(q, d_z, d_k) \in \mathcal{CO}(z, k; R)$.

Given these preliminary observations, we follow the same logic as the proof of Proposition 2, and prove the result by two claims. ■

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Claim 1. (i) When $i = i^*$, the seller's participation constraint, (A.11), holds with strict inequality at the optimum, (q^*, z^*, k^*, e^*); (ii) for all $i > i^*$, the buyer's participation constraint, (A.10), binds, and $q^p < q^*$ at the optimum.

Given that (A.10) and (A.13) bind, we can solve for z and q as a function of (k, e, i):

$$z(k, e, i) = \frac{1}{i}\{e\psi'(e) - \psi(e) - [1 + r - F'(k)]k\},$$

$$q(k, e, i) = f\left\{g(e, i) + \frac{[-(1 + r) + (1 + i)F'(k)]k}{i}\right\},$$

where

$$g(e, i) = \frac{1}{i}[e\psi'(e) - \psi(e)] + \frac{\psi'(e)}{\alpha(1/e)} \text{ and } f(x) = u^{-1}(x).$$

The objective function can be written as

$$G(k, e, i) = e\alpha(1/e)\{u[q(k, e, i)] - c[q(k, e, i)]\} - \psi(e) + F(k) - (1 + r)k. \quad \textbf{(A.14)}$$

Claim 2. There is $\bar{i}'$ such that for all $i \in [i^*, \bar{i}']$, there is a unique maximizer, $[k^p(i), e^p(i)]$, to

$$\max_{k \in [k^*, \bar{k}], e \in [0,1]} G(k, e, i),$$

with $z^p(i) = z[k^p(i), e^p(i), i] > 0$. Moreover, $[q^p(i), z^p(i), k^p(i), e^p(i)]$ is the unique constrained-efficient outcome $\{q^p(i) = q[k^p(i), e^p(i), i]\}$, and, if $1 + r + F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^*}$, then $\frac{d}{de}e^p(i) > 0$ and $k^p(i) = k^*$.

The result follows directly from Claim 2. Now we prove the two claims.

PROOF OF CLAIM 1

(i) We have shown it in (1).

(ii) To show that (A.10) binds for all $i > i^*$, we consider two cases:

(a) At the optimum, $k^p > k^*$. Suppose, by contradiction, that (A.10) does not bind. Let (z', k') be such that $k^* \le k' < k^p$ but $z' + F'(k')k' = z^p + F'(k^p)k^p$, and, by continuity, the tuple (q^p, z', k', e^p) also satisfies (A.10). Because $k' < k^p$, this leads to an increase in the welfare, a contradiction.

(b) At the optimum, $k^p = k^*$. Consider the Lagrangian associated with (A.10)–(A.13), $q \ge 0$, $z \ge 0$, and $e \ge 0$:

$$\begin{aligned}\mathcal{L}(q, z, k, e; \lambda, \mu, \xi, \eta) = {} & e\alpha(1/e)[u(q) - c(q)] + [F(k) - (1 + r)k] - \psi(e) \\ & + \lambda\{-iz - [(1 + r) - F'(k)]k + e\alpha(1/e)[u(q) \\ & \quad - z - F'(k)k] - \psi(e)\} \\ & + \mu\{[F'(k)k + z - c(q)]\} + \xi[(1 + r) - F'(k)] \\ & + \eta\{\psi'(e) - \alpha(1/e)[u(q) - z - F'(k)k]\},\end{aligned}$$

where $\lambda \ge 0$, $\mu \ge 0$, $\xi \ge 0$, and η are the Lagrange multipliers associated with (A.10), (A.11), (A.12), and (A.13). From the Kuhn–Tucker theorem, the following

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are the first-order necessary conditions with respect to q, z, e (with $k^p = k^*$):

$$e^p\alpha(1/e^p)[u'(q^p) - c'(q^p)] + \lambda e^p\alpha(1/e^p)u'(q^p) - \mu c'(q^p) - \eta\alpha(1/e^p)u'(q^p) = 0, \tag{A.15}$$

$$\lambda[i + e^p\alpha(1/e^p)] = \mu + \eta\alpha(1/e^p), \tag{A.16}$$

$$[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^p) - c(q^p)] - \psi'(e^p) + \lambda\{[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^p) - z^p - (1+r)k^*] - \psi'(e^p)\} = -\eta\{\psi''(e^p) + [\alpha'(1/e^p)/(e^p)^2][u(q^p) - z^p - (1+r)k^*]\}. \tag{A.17}$$

In addition, (A.10) and (A.11) are not binding only if $\lambda = 0$ and $\mu = 0$, respectively.

Here, we show that (A.10) binds at the optimum for all $i > i^*$. Suppose, by contradiction, that (A.10) does not bind and hence $\lambda = 0$. It also implies that $q^p > 0$ and $e^p > 0$. Then, from (A.11), $q^p > 0$, $e^p > 0$, and $k^p = k^*$, we have $z^p > 0$. Combining (A.15) and (A.16) yields

$$\frac{u'(q^p)}{c'(q^p)} = \frac{e^p\alpha(1/e^p) + \mu}{e^p\alpha(1/e^p) + \mu - \lambda i}. \tag{A.18}$$

From (A.18), $q^p = q^*$, and hence, from (A.16) and $\lambda = 0$ we have $-\alpha(1/e^p)\eta = \mu$. If $\mu = 0$, then from (A.17), $e^p = e^*$, a contradiction. If $\mu > 0$, then (A.11) is binding and hence $d_z^p + (1+r)k^* = c(q^*)$. By (A.16), $\mu = -\eta\alpha(1/e^p) > 0$. But then, by (A.17), this implies $[\alpha(1/e^p) - \alpha'(1/e^p)/e^p][u(q^*) - c(q^*)] - \psi'(e^p) > 0$, and hence $e^p < e^*$. However, by (A.13), $\psi'(e^p) = \alpha(1/e^p)[u(q^*) - c(q^*)]$, and, because $c(q^*) < d_z^* + (1+r)k^*$, this implies that $e^p > e^*$. This leads to a contradiction. Hence, $\lambda > 0$ and so (A.10) is binding. Moreover, because $\lambda > 0$, (A.18) implies that $u'(q^p) > c'(q^p)$ and hence $q^p < q^*$. ■

PROOF OF CLAIM 2

First note that

$$\frac{\partial G(k^*, e^*, i^*)}{\partial e} = \left[\alpha(1/e^*) - \frac{\alpha'(1/e^*)}{e^*}\right]\left[u(q^*) - c(q^*)\right] - \psi'(e^*) = 0, \tag{A.19}$$

$$\frac{\partial G(k^*, e^*, i^*)}{\partial k} = e^*\alpha(1/e^*)\left[u'(q^*) - c'(q^*)\right]\frac{\partial q(k^*, e^*, i^*)}{\partial k} + F'(k^*) - (1+r) = 0.$$

Now we show that

$$\frac{\partial^2 G(k^*, e^*, i^*)}{\partial k^2} < 0, \quad \frac{\partial^2 G(k^*, e^*, i^*)}{\partial e^2} < 0, \quad \frac{\partial^2 G(k^*, e^*, i^*)}{\partial k^2}\frac{\partial^2 G(k^*, e^*, i^*)}{\partial e^2} - \frac{\partial^2 G(k^*, e^*, i^*)}{\partial k \partial e} > 0. \tag{A.20}$$

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The second partial derivatives are

$$\frac{\partial^2 G(k^*, e^*, i^*)}{\partial e^2}$$

$$= \alpha''(1/e^*)/(e^*)^3[u(q^*) - c(q^*)] + e^*\alpha(1/e^*)\left[\frac{\partial}{\partial e}q(k^*, e^*, i^*)\right]^2 [u''(q^*) - c''(q^*)]$$

$$- \psi''(e^*) < 0,$$

$$\frac{\partial^2 G(k^*, e^*, i^*)}{\partial k^2} = e^*\alpha(1/e^*)\left[\frac{\partial}{\partial k}q(k^*, e^*, i^*)\right]^2 [u''(q^*) - c''(q^*)] + F''(k^*) < 0,$$

$$\frac{\partial^2 G(k^*, e^*, i^*)}{\partial k \partial e} = e^*\alpha(1/e^*)\left[\frac{\partial}{\partial k}q(k^*, e^*, i^*)\frac{\partial}{\partial e}q(k^*, e^*, i^*)\right][u''(q^*) - c''(q^*)].$$

Hence,

$$\frac{\partial^2 G(k^*, e^*, i^*)}{\partial e^2}\frac{\partial^2 G(k^*, e^*, i^*)}{\partial k^2}$$

$$> e^*\alpha(1/e^*)\left[\frac{\partial}{\partial e}q(k^*, e^*, i^*)\right]^2 [u''(q^*) - c''(q^*)]e^*\alpha(1/e^*)$$

$$\times\left[\frac{\partial}{\partial k}q(k^*, e^*, i^*)\right]^2 [u''(q^*) - c''(q^*)]$$

$$= \left[\frac{\partial^2 G(k^*, e^*, i^*)}{\partial k \partial e}\right]^2.$$

Because of (A.19) and (A.20), and by the IFT, there is an open neighborhood $O = (k_0, k_1)\times(e_0, e_1)\times(i_0, i_1)$ around (e^*, i^*) and a continuously differentiable implicit function $(k_0^p, e_0^p) : (i_0, i_1) \to (k_0, k_1) \times (e_0, e_1)$ such that for all $i \in [i^*, i_1)$, $[k^p(i), e_0^p(i)]$ is the unique $(k, e) \in (k_0, k_1) \times (e_0, e_1)$ such that

$$\frac{\partial}{\partial e}G[k_0^p(i), e_0^p(i), i] = 0 \text{ and } \frac{\partial}{\partial k}G[k^p(i), e_0^p(i), i] = 0,$$

and another continuously differentiable implicit function $e^p : (i_0, i_1) \to (e_0, e_1)$ such that for all $i \in [i^*, i_1)$, $e_1^p(i)$ is the unique $e \in (e_0, e_1)$ such that

$$\frac{\partial}{\partial e}G[k^*, e_1^p(i), i] = 0,$$

and that $G(\cdot, \cdot, i)$ is strictly concave over O. Now, define $[k^p(i), e^p(i)]$ as

$$[k^p(i), e^p(i)] = \begin{cases} [k_0^p(i), e_0^p(i)] \text{ if } k_0^p(i) \geq k^* \\ [k^*, e_1^p(i)] \text{ otherwise.} \end{cases}$$

Because $G(\cdot, \cdot, i)$ is strictly concave over O, by the Kuhn–Tucker conditions, $[k^p(i), e^p(i)]$ is a local maximizer; using the same arguments as those in Proposition 2, we can show that $[k^p(i), e^p(i)]$ is the global maximizer as well, at least for some interval $[i^*, i_2]$ with

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$i_2 \in (i^*, i_1]$ and, using similar arguments there about seller participation constraint, one can show

$$(q^p, z^p, k^p, e^p) = \{q[k^p(i), e_0^p(i), i], z[k^p(i), e_0^p(i), i], k^p(i), e_0^p(i)\}$$

is the unique constrained-efficient outcome for $i \in [i^*, i_2]$. Note that, by continuity, $e^p(i) > 0$ and $k^p(i)$ is close to k^* at least locally and hence $z^p > 0$.

Now we show that if $1 + r + F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^*}$, then $k^p(i) = k^*$ and $e^p(i)$ is increasing. For all $i \in [i^*, i_2]$, let $q(i) = q[k^*, e_1^p(i), i]$,

$$\frac{\partial}{\partial k} G[k^*, e_1^p(i), i] = e_1^p(i)\alpha[1/e_1^p(i)]\{u'[q(i)] - c'[q(i)]\} f'\{u[q(i)]\}$$
$$\times \left\{[1 + r + F''(k^*)k^*] + F''(k^*)k^*/i\right\}.$$

Because $1 + r + F''(k^*)k^* < \frac{-F''(k^*)k^*}{i^*}$, there exists $i_3 \leq i_2$ such that for all $i \in [i^*, i_3]$, $1 + r + F''(k^*)k^* + \frac{F''(k^*)k^*}{i} \leq 0$, and hence, for all such i's, $\frac{\partial}{\partial k} G[k^*, e_1^p(i), i] \leq 0$. Recall that $G(\cdot, \cdot, i)$ is strictly concave over O. Because

$$\frac{\partial}{\partial e} G[k^*, e_1^p(i), i] = 0 \text{ and } \frac{\partial}{\partial k} G[k^*, e_1^p(i), i] \leq 0,$$

for all $i \in [i^*, i_2]$, it follows that $e^p(i) = e_1^p(i)$ for all $i \in [i^*, i_3]$ and hence the constrained efficient outcome has $k^p = k^*$.

Finally, by IFT again, $e^p(i)$ is continuously differentiable and for all $i \in (i^*, i_3]$,

$$(e^p)'(i) = -\frac{\partial^2}{\partial e \partial i} G[k^*, e^p(i), i] / \frac{\partial^2}{\partial e^2} G[k^*, e^p(i), i].$$

We have shown that $\frac{\partial^2}{\partial e^2} G(k^*, e^*, i^*) < 0$. Now,

$$\frac{\partial^2}{\partial e \partial i} G(k^*, e^*, i^*) = e^* \alpha(1/e^*)[u''(q^*) - c''(q^*)]\{f'[u(q^*)]\}^2 g_e(e^*, i^*) g_i(e^*, i^*) > 0,$$

because

$$g_e(e^*, i^*) = \frac{e^* \psi''(e^*)}{i^*} + \frac{\psi''(e^*)\alpha(1/e^*) + \psi'(e^*)\alpha'(1/e^*)/(e^*)^2}{\alpha(1/e^*)^2}$$
$$> 0, \; g_i(e^*, i^*) = \frac{e^* \psi'(e^*) - \psi(e^*)}{(i^*)^2} < 0.$$

So $\frac{d}{di} e^p(i^*) > 0$ and, by continuity, there is $\bar{i}' \in (i^*, i_3]$ such that $\frac{d}{di} e^p(i) > 0$ for all $i \in [i^*, \bar{i}']$. ■

(3) Recall from Lemma 1 that for any i and in any constrained-efficient outcome w.r.t. i, $e^p(i) \leq \hat{e}$. Note that the arguments there are not affected by the presence of capital. Moreover, by (A.10), we have

$$z^p(i) \leq e^p(i)\alpha[1/e^p(i)]\{u[q^p(i)] - c[q^p(i)]\}/i \leq \hat{e}\alpha(1/\hat{e})[u(q^*) - c(q^*)]/i.$$

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Again, we prove the result by two claims below. Claim 3 show that $\mathcal{W}(i)$, the welfare associated with a constrained-efficient outcome under i, is arbitrarily close to $\mathcal{W}^k$ as i goes to infinity.

Claim 3. For any $\varepsilon > 0$, there exists i_ε for which $i > i_\varepsilon$ implies $\mathcal{W}(i) \leq \mathcal{W}^k + \varepsilon$.

Because it is always feasible to set $z = 0$ and hence $\mathcal{W}(i) \geq \mathcal{W}^k$ for all i, the result that $\lim_{i\to\infty} \mathcal{W}(i) = \mathcal{W}^k$ follows immediately from Claim 3. By Lemma 2, if we impose the additional constraints $z = 0$ and $k = k^*$, then the resulting maximum welfare, denoted $\mathcal{W}^0$, is strictly less than $\mathcal{W}^k$, and hence $[\mathcal{W}^k - \mathcal{W}^0]/2 > 0$. The following claim shows that, if we impose $k = k^*$, then, for i sufficiently large, the maximum achievable welfare is less than $\mathcal{W}^k - [\mathcal{W}^k - \mathcal{W}^0]/2$. ■

Claim 4. Define $\mathcal{W}^0(i)$ to be the maximum welfare achievable by outcomes satisfying $k = k^*$, together with constraints (A.10)–(A.13). There exists $\hat{i}$ such that for all $i > \hat{i}$, $\mathcal{W}^0(i) < \mathcal{W}^k - [\mathcal{W}^k - \mathcal{W}^0]/2$.

Claim 4 implies that for all $i > \hat{i}$, $k^p(i) > k^*$, for otherwise $\mathcal{W}(i) = \mathcal{W}^0(i) < \mathcal{W}^k$, a contradiction. Now we prove the two claims.

PROOF OF CLAIM 3

First note that in any constrained-efficient outcome, $q^p(i) \leq q^*$ and $k^p(i) \leq \hat{k}$. For each i, define $\tilde{k}(i)$ by the capital stock that satisfies

$$F'[\tilde{k}(i)]\tilde{k}(i) - F'(\hat{k})\hat{k} = \frac{\hat{e}\alpha(1/\hat{e})[u(q^*) - c(q^*)]}{i}.$$

Because the function $F'(k)k$ is strictly increasing in k with range $\mathbf{R}_+$, $\tilde{k}(i)$ is well defined and is a decreasing function of i. Moreover, as $i \to \infty$, $\tilde{k}(i)$ converges to $\hat{k}$.

Let $S(k) = F'(k)k$. Given $\epsilon > 0$, let i_ϵ be so large that $i > i_\epsilon$ implies

$$\{1 + r - F'[\tilde{k}(i)]\}[\tilde{k}(i) - \hat{k}] \leq \epsilon, \ S'[\tilde{k}(i)](1 + i) \geq 1 + r. \tag{A.21}$$

Note that i_ϵ is well-defined because $\tilde{k}(i)$ converges to $\hat{k}$ and S' is a decreasing function.

Now we show that if $i > i_\epsilon$, then $\mathcal{W}(i) \leq \mathcal{W}^k + \epsilon$. Fix some $i > i_\epsilon$, and a constrained-efficient outcome, $[q^p(i), d_z^p(i), d_k^p(i), z^p(i), k^p(i), e^p(i)]$. Consider an alternative outcome

$$(q', d_z', d_k', z', k', e') = [q^p(i), 0, d_k', 0, k', e^p(i)],$$

where k' and d_k' are such that

$$F'(k')k' - F'[k^p(i)]k^p(i) = z^p(i) \leq \frac{\hat{e}\alpha(1/\hat{e})[u(q^*) - c(q^*)]}{i}, \tag{A.22}$$

$$F'(k')d_k' = F'[k^p(i)]d_k^p(i) + d_z^p(i). \tag{A.23}$$

Note that $k' \leq \tilde{k}(i)$. Now we show that the outcome $(q', d_z', d_k', z', k', e')$ satisfies incentive compatibility constraints (A.10)–(A.13) and has welfare equal to $\mathcal{W}' \geq \mathcal{W}(i) - \epsilon$. Note that, by definition, $\mathcal{W}' \leq \mathcal{W}^k$ and hence this implies that $\mathcal{W}^k \geq \mathcal{W}(i) - \epsilon$.

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First consider the buyer's participation constraint, (A.10). Because the original outcome satisfies (A.10), it suffices to show that

$$-iz^p(i) - \{1 + r - F'[k^p(i)]\}k^p(i) \leq -[1 + r - F'(k')]k',$$

which holds if and only if

$$(1 + r)[k' - k^p(i)] - z^p(i) \leq iz^p(i) \Leftrightarrow (1 + r)[k' - k^p(i)] \leq (1 + i)z^p(i) \Leftrightarrow \frac{z^p(i)}{k' - k^p(i)} \geq \frac{1 + r}{1 + i}.$$

By definition of k', $z^p(i) = F'(k')k' - F'[k^p(i)]k^p(i)$ and hence [note that $k' \leq \tilde{k}(i)$], by (A.21),

$$\frac{z^p(i)}{k' - k^p(i)} = \frac{F'(k')k' - F'[k^p(i)]k^p(i)}{k' - k^p(i)} \geq S'(k') \geq S'[\tilde{k}(i)] \geq \frac{1 + r}{1 + i}.$$

In addition, because $k' \geq k^p(i)$ and because of (A.23), the alternative outcome satisfies (A.11)–(A.13).

Here, we show that $\mathcal{W}' \geq \mathcal{W}(i) - \epsilon$. First note that

$$\{F[k^p(i)] - (1 + r)k^p(i)\} - [F(k') - (1 + r)k'] \leq [F'(k') - (1 + r)][k^p(i) - k'] = [1 + r - F'(k')][k' - k^p(i)].$$

Then, note that, in terms of variables relevant to the welfare, the alternative outcome differ from the original outcome only in the capital stock, and hence the difference in welfare, $\mathcal{W}' - \mathcal{W}(i)$, can be written as

$$\mathcal{W}' - \mathcal{W}(i) = -(\{F[k^p(i)] - (1 + r)k^p(i)\} - [F'(k') - (1 + r)k']) \geq -[1 + r - F'(k')][k' - k^p(i)] \geq -\{1 + r - F'[\tilde{k}(i)]\}[\tilde{k}(i) - \hat{k}] \geq -\epsilon.$$

The second last inequality follows from the fact that $k' - k^p(i) = z^p(i) \geq \tilde{k}(i) - \hat{k}$ and the fact that the function $S(k) = F'(k)k$ is concave in k, and the last inequality follows from (A.21). Hence, $\mathcal{W}' \geq \mathcal{W}(i) - \epsilon$. ■

PROOF OF CLAIM 4

We show that for any $\epsilon > 0$, there exists i'_ϵ such that $\mathcal{W}^0(i) < \mathcal{W}^0 + \epsilon$ for all $i > i'_\epsilon$. The claim follows immediately.

Because $\mathcal{W}^0(i) > 0$ (as it is always feasible to set $k = k^*$, q such that $c(q) = (1 + r)k^*$, and e that solves $\psi'(e)/\alpha(1/e) = [u(q) - (1+r)k^*] > 0$) for all i, we can find a lower bound $\underline{q}$ and $\underline{e}$ such that for any outcome $[q^0(i), d_z^0(i), d_k^0(i), z^0(i), k^0(i), e^0(i)]$ that achieves the maximum welfare under the constraints (A.10)–(A.13) plus $k = k^*$, we have $q^0(i) > \underline{q}$ and $e^0(i) > \underline{e}$ for all i. Because $i > i^*$, at the optimum we must have $d_z^0(i) = z^0(i)$ and $d_k^0(i) = k^*$. Moreover, it follows that we can choose $u(\underline{q})$ to be strictly greater than $(1 + r)k^*$, for otherwise the buyer will have arbitrarily small surplus and hence the search intensity will be arbitrarily small as well.

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Now, the welfare, as a function of (q, k, e), is continuous and hence is uniformly continuous in $[\underline{q}, q^*] \times \{k^*\} \times [\underline{e}, \hat{e}]$. Thus, there exists $\delta > 0$ such that if $\|(q, e) - [q^0(i), e^0(i)]\| < \delta$, then the welfare associated with (q, k^*, e), differs from the welfare $\mathcal{W}^0(i)$ by less than ϵ for all i.

Let $l(e) = \psi'(e)/\alpha(1/e)$. Then, $l'(e) > 0$ for all $e \in [\underline{e}, \hat{e}]$ and hence $A = \min_{e\in[\underline{e},\hat{e}]} l'(e) > 0$. Let i'_ϵ be so large that if $i > i'_\epsilon$,

$$\max\{2, 1 + u'(\underline{q})/A\}\frac{[u(q^*) - c(q^*)]}{c'(\underline{q}/2)i} < \min\{\underline{q}/2, \delta/2, \underline{q} - u^{-1}[(1 + r)k^*]\}. \quad \textbf{(A.24)}$$

Fix $i > i'_\epsilon$ and an outcome $[q^0(i), z^0(i), k^*, e^0(i)]$ that achieves $\mathcal{W}^0(i)$. We construct an alternative outcome, $(q', 0, k^*, e')$ such that $\|[q', e') - [q^0(i), e^0(i)]\| < \delta$ and satisfies (A.10)–(A.13). Then, the welfare associated with the alternative outcome, denoted by $\mathcal{W}'$, is within ϵ of $\mathcal{W}^0(i)$, but $\mathcal{W}' \le \mathcal{W}^0$.

The outcome $(q', 0, k^*, e')$ is given by

$$c(q') = c[q^0(i)] - z^0(i) \ge 0 \text{ and } \frac{\psi'(e')}{\alpha(1/e')} = u(q') - (1 + r)k^*.$$

Because $z^0(i) \le [u(q^*) - c(q^*)]/i$, it follows from (A.24) that $q' \ge \underline{q}/2$ and that $u(q') \ge (1 + r)k^*$. Moreover, because $[q^0(i), z^0(i), k^*, e^0(i)]$ satisfies (A.11),

$$-c(q') + (1 + r)k^* = -c[q^0(i)] + z^0(i) + (1 + r)k^* \ge 0,$$

and hence $(q', 0, k^*, e')$ satisfies (A.11) as well. Note that it also satisfies (A.10) and (A.13) by construction.

Thus, we have

$$0 \le c[q^0(i)] - c(q') \le z^0(i) \le \frac{\hat{e}\alpha(1/\hat{e})[u(q^*) - c(q^*)]}{i},$$

and so, by (A.24),

$$|q^o(i) - q'| \le |c[q^0(i)] - c(q')|/c'(\underline{q}/2) \le \delta/2.$$

By (A.13),

$$\begin{aligned}|l[e^0(i)] - l(e')| &= |\psi'[e^0(i)]/\alpha[e^0(i)] - \psi'(e')/\alpha(1/e')| = |u[q^0(i)] - u(q') - z^0(i)| \\ &\le u'(\underline{q}/2)[q^0(i) - q'] + z^0(i),\end{aligned}$$

and so, by (A.24),

$$|e' - e^0(i)| \le (1/A)\{u'(\underline{q}/2)[q^0(i) - q'] + z^0(i)\} < \delta/2.$$

Thus, we have $\|(q', e') - [q^0(i), e^0(i)]\| < \delta$, and hence $\mathcal{W}^0 \ge \mathcal{W}' > \mathcal{W}^0(i) - \epsilon$. Finally, take $\hat{i} = i'_{[\mathcal{W}^k - \mathcal{W}^0]/2}$. ■

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