Expressions for rational approximations to square roots of integers using Pell's equation

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Introduction

There are recursive expressions (see [1]) for sequentially generating the integer solutions to Pell's equation: $p^2 - Dq^2 = 1$, where D is any positive non-square integer. With known positive integer solution p_1 and q_1 we can compute, using these recursive expressions, p_n and q_n for all $n > 1$. See Table in [2] for a list of smallest integer, or *fundamental*, solutions p_1 and q_1 for $D \leqslant 128$. These (p_n, q_n) pairs also form *rational approximations* to $\sqrt{D}$ that, as noted in [3, Chapter 3], match with convergents ($C_n = p_n/q_n$) of the Regular Continued Fractions (RCF, continued fractions with the numerator of all fractions equal to 1) for $\sqrt{D}$. These recursive expressions, being linear, help us derive new expressions for computing p_n and q_n directly (as opposed to sequentially) for any n in terms of just p_1, q_1, and D. In Section 1, we use the eigenvalues of the 2×2 matrix that is associated with these recursive expressions and compute the nth power of these eigenvalues for deriving new direct expressions. In this process, we also prove formally that $C_n \to \sqrt{D}$ as $n \to \infty$. In Section 2, based on a well-known historical observation attributed to Brahmagupta [4], we identify a class of integers that we name after him: *Brahmagupta integers* (BI). These are formed by (i) $D = N^2 \pm 1$, (ii) $D = N^2 \pm 2$ and (iii) $D = N^2 \pm 4$, where $N > 0$ takes all the integer values such that $D > 1$. These BIs use, instead of fundamental solutions, easily obtainable initial solutions (i) $p_1 = \lfloor \sqrt{D} \rfloor$ and $q_1 = 1$ and (ii) $p_1 = (\lfloor \sqrt{D} \rfloor + 1)$ and $q_1 = 1$ in their recursive expressions for p_n and q_n. We also prove formally the applicability of these revised recursive relations to BIs. In Section 3, we briefly illustrate the use of all these expressions with a few simple examples. Two of these examples lead us to a conjecture that shows the limitations of the revised recursive expressions.

1. *Direct expressions for rational approximation to* $\sqrt{D}$

The following Pell's Equation always has integer solutions (see [3]):

$$p^2 - Dq^2 = 1. \tag{1}$$

Knowing the fundamental solution (or any integer solution) p_1 and q_1 to (1) (see [2]), it is possible to compute the higher order ($n > 1$) solutions that form the rational approximations to $\sqrt{D}$ using the following expressions given in [1]:

$$p_{n+1} = p_1 p_n + D.q_1 q_n \tag{2}$$

and

$$q_{n+1} = q_1 p_n + p_1 q_n. \tag{3}$$

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Starting with the above two *recursive* expressions, we develop new expressions that help compute *directly* any specific (say, nth) rational approximation (i.e. p_n and q_n) to $\sqrt{D}$ in terms of just D, p_1 and q_1. This is, in effect, a generalisation of an approach that was used for $\sqrt{2}$ in [5]. To establish the viability of this approach, we also show in Theorem 1, stated below, that the resulting ratios p_n/q_n, formed using their *direct expressions*, converge to $\sqrt{D}$ as $n \to \infty$.

Theorem 1: Given the recursive relations in (2) and (3) pertaining to $\sqrt{D}$ along with known initial solutions p_1 and q_1: (i) the nth convergent $C_n\left(= p_n/q_n\right) \to \sqrt{D}$ as $n \to \infty$ and (ii) the following expressions for p_n and q_n in terms of just D, p_1 and q_1 are valid. The expressions for odd-numbered subscripts, for $n > 1$, are:

$$p_n = \sum_{k=0}^{\frac{1}{2}(n-1)} \binom{n}{2k} D^k p_1^{(n-2k)} q_1^{2k}; \; q_n = \sum_{k=0}^{\frac{1}{2}(n-1)} \binom{n}{2k+1} D^k p_1^{(n-2k-1)} q_1^{(2k+1)} \tag{4}$$

and the expressions for the even-numbered subscripts, for $n > 1$, are:

$$p_n = \sum_{k=0}^{\frac{1}{2}n} \binom{n}{2k} D^k p_1^{(n-2k)} q_1^{2k}; \; q_n = \sum_{k=0}^{\frac{1}{2}(n-2)} \binom{n}{2k+1} D^k p_1^{(n-2k-1)} q_1^{(2k+1)}. \tag{5}$$

Proof: Let p and q be two elements of a general vector $X = \begin{bmatrix} p & q \end{bmatrix}^T$ and $X_1 = \begin{bmatrix} p_1 & q_1 \end{bmatrix}^T$. Let A be a 2×2 matrix with the following four elements: $a_{11} = p_1$, $a_{12} = Dq_1$, $a_{21} = q_1$ and $a_{22} = p_1$. Hence, from (2) and (3) we note that

$$X_2 = AX_1 \tag{6}$$

and that, for all $n > 1$,

$$X_n = \begin{bmatrix} p_n & q_n \end{bmatrix}^T = A^{n-1}X_1. \tag{7}$$

Using λ for the eigenvalue, the characteristic equation of the matrix A is $(p_1 - \lambda)^2 - Dq_1^2 = 0$ and the two eigenvalues are $\lambda_1 = \left(p_1 - q_1\sqrt{D}\right)$ and $\lambda_2 = \left(p_1 + q_1\sqrt{D}\right)$. Since these eigenvalues are distinct, matrix A is diagonalisable. We now express the matrix A^{n-1} in terms of the powers of these eigenvalues and use them in (7) to evaluate X_n. Skipping the details, with I_2 as the 2×2 identity matrix, we state below the matrix equation for A^{n-1}:

$$A^{n-1} = \left\{\frac{\lambda_2(\lambda_1)^{n-1} - \lambda_1(\lambda_2)^{n-1}}{\lambda_2 - \lambda_1}\right\} I_2 + \left\{\frac{(\lambda_2)^{n-1} - (\lambda_1)^{n-1}}{\lambda_2 - \lambda_1}\right\} A.$$

Simplifying this and substituting it into (7), we get the following expressions for the two elements of X_n:

$$p_n = \frac{\lambda_2^n + \lambda_1^n}{2}; \qquad q_n = \frac{\lambda_2^n - \lambda_1^n}{2\sqrt{D}}.$$

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Factoring out p_1 and letting $y_1 = q_1/p_1$ in both λ_1 and λ_2, the two elements of X_n become

$$p_n = p_1^n \frac{(1 + y_1\sqrt{D})^n + (1 - y_1\sqrt{D})^n}{2} \tag{8}$$

and

$$q_n = p_1^n \frac{(1 + y_1\sqrt{D})^n - (1 - y_1\sqrt{D})^n}{2\sqrt{D}}. \tag{9}$$

From (8) and (9), the n th convergent is

$$C_n = \frac{p_n}{q_n} = \frac{(1 + y_1\sqrt{D})^n + (1 - y_1\sqrt{D})^n}{(1 + y_1\sqrt{D})^n - (1 - y_1\sqrt{D})^n}\sqrt{D}. \tag{10}$$

Result 1: In (10) we note that $(1 + y_1\sqrt{D}) > 1$ and $|(1 - y_1\sqrt{D})| < 1$. Hence, as $n \to \infty$, $(1 - y_1\sqrt{D}) \to 0$ and $C_n \to \sqrt{D}$.

Also

$$(1 + \sqrt{D}y_1)^n = \binom{n}{0}(\sqrt{D}y_1)^0 + \binom{n}{1}(\sqrt{D}y_1)^1 + \ldots + \binom{n}{n-1}(\sqrt{D}y_1)^{n-1} + \binom{n}{n}(\sqrt{D}y_1)^n \tag{11}$$

and

$$(1 - \sqrt{D}y_1)^n = \binom{n}{0}(-\sqrt{D}y_1)^0 + \binom{n}{1}(-\sqrt{D}y_1)^1 + \ldots + \binom{n}{n-1}(-\sqrt{D}y_1)^{n-1} + \binom{n}{n}(-\sqrt{D}y_1)^n. \tag{12}$$

When the above expansions are used in (8) and (9), the terms with $\sqrt{D}$ cancel out, (i) for p_n, while adding the two expansions and (ii) for q_n, after subtracting, with the $\sqrt{D}$ in the denominator. The number of elements in both the expressions for p_n and q_n vary depending on the subscript n being odd or even. Hence, after replacing y_1 with q_1/p_1, we formulate separate direct expressions for the odd and even subscripts of both p_n and q_n for $n > 1$ as stated in (4) and (5).

With known p_1 and q_1 (such as the fundamental solution, see [2]), the expressions in (4) and (5) can be used to find any n th solution *directly* that forms a rational approximation to $\sqrt{D}$. In Section 3, we illustrate these expressions with examples. Before that, in Section 2, we identify a class of integers that use simpler initial values such as $p_1 = \lfloor\sqrt{D}\rfloor$ and $q_1 = 1$. Most of these values are not solutions to $p^2 - Dq^2 - 1$ initially but seem to lead to such solutions in a few iterations.

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2. *Brahmagupta integers: expressions for rational approximations to their square roots*

As an alternative to the fundamental solutions, we examine two obvious initial solutions: (i) $p_1 = \lfloor\sqrt{D}\rfloor = t$ with $q_1 = 1$ and (ii) $p_1 = u = (t + 1)$ with $q_1 = 1$, where $t^2 < D < u^2$. In order to establish the class of integers which can use such simpler initial solutions in a set of recursive expressions for generating the rational approximations to their square roots, we make use of a clue from the 7th Century (see [4]).

Using a pair (p, q) – a trial solution – that satisfies $p^2 - Dq^2 = k$, where $k = -1, 2, -2, 4$ or -4, Brahmagupta was able to find a solution for the Pell's equation of interest (i.e., with $k = 1$) using the *composition method*.

This suggests a possibility that, in order to find a class of integers that benefit from the two sets of initial solutions proposed above, one could let the above initial solutions satisfy a set of Pell's equations that includes additionally $-1, \pm2$ and ±4:

$$p^2 - Dq^2 = \pm C, \text{ where } C = 1, 2 \text{ or } 4. \tag{13}$$

With $t = \lfloor\sqrt{D}\rfloor$ and $u = (t + 1)$, which are the nearest integers to $\sqrt{D}$, the following two sets of starting approximations: (i) $p_1 = t$ and $q_1 = 1$ and (ii) $p_1 = u$ and $q_1 = 1$ satisfy respectively:

$$p^2 - Dq^2 = -g, \text{ where } g = D - t^2 \tag{14}$$

and

$$p^2 - Dq^2 = h, \text{ where } h = u^2 - D. \tag{15}$$

By limiting the values of g and h to 1, 2, or 4, (14) and (15) become subsets of (13) thus ensuring that the chosen initial solutions comply with the observation in [4]. This leads us to a special class of integers $D = (t^2 + g)$ and $D = (u^2 - h)$ with $g = 1, 2$, and 4 and $h = 1, 2$, and 4 that could benefit from such initial solutions. We present these *Brahmagupta integers* (BI) in Table 1. Here, t and u take on values 1, 2, 3, … etc., such that any resulting BI is always greater than 1. There are two rows for $g = 4$ and $h = 4$ to distinguish between odd and even values of t and u (i.e. either t, $u = 2m + 1$ or $t, u = 2m$, where m is an integer greater than 0) . The BIs in the last row are all four times those in the first row; note also that 2, 3, 5 and 8 are repeated.

g	*Expression for BI*	*BI example*	h	*Expression for BI*	*BI example*
1	$t^2 + 1$	2, 5, 10, …	1	$u^2 - 1$	3, 8, 15, …
2	$t^2 + 2$	3, 6, 11, …	2	$u^2 - 2$	2, 7, 14, …
4	$(2m + 1)^2 + 2^2$	13, 29, 53, …	4	$(2m + 1)^2 - 2^2$	5, 21, 45, …
4	$(2m)^2 + 2^2$	8, 20, 40, …	4	$(2m)^2 - 2^2$	12, 32, 60, …

TABLE 1: Brahmagupta integers: Class of D with $g, h = 1, 2$ or 4

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The two sets of initial solutions, except for the case with $h = 1$, differ from the fundamental solutions. There is a need to revise the recursive expressions in (2) and (3) when D is confined to be a BI. This revision is carried out with the understanding that the resulting revised expressions (i) are applicable only to BIs and (ii) offer solutions to an expanded set of Pell's equations $p^2 - Dq^2 = \pm C$, where $C = 1$, 2 or 4, rather than just $p^2 - Dq^2 = 1$. With the two sets of starting approximations to $\sqrt{D}$: (i) $p_1 = \lfloor \sqrt{D} \rfloor = t$ with $q_1 = 1$ and (ii) $p_1 = u = (t + 1)$ with $q_1 = 1$, (2) and (3) become (with D restricted to be a BI):

(i) for $p_1 = t$ and $q_1 = 1$,

$$p_{n+1} = tp_n + Dq_n \tag{16}$$

$$\text{and } q_{n+1} = p_n + tq_n \tag{17}$$

(ii) for $p_1 = u$ and $q_1 = 1$,

$$p_{n+1} = up_n + Dq_n \tag{18}$$

$$\text{and } q_{n+1} = p_n + uq_n. \tag{19}$$

We discuss the applicability of these expressions in the form of two Theorems relating to BIs.

Theorem 2: For BIs formed (see Table 1) by

(i) $D = t^2 + 1, t > 1$

and (ii) $D = (2m + 1)^2 + 4, m > 0,$

p and q are integer solutions to $p^2 - Dq^2 = -1$.

Proof:

(i) For $D = t^2 + 1$, the corresponding initial pair $p_1 = t$ and $q_1 = 1$ itself is a solution.

(ii) For $D = (2m + 1)^2 + 4$, we let $t = 2m + 1$ where m is any integer, so that t is odd.

Using $p_1 = t, q_1 = 1$ and $D = t^2 + 4$ in (16) and (17), we get, in two iterations, $p_3 = 4(t^3 + 3t)$ and $q_3 = 4(t^2 + 1)$. Since t is odd, both $(t^3 + 3t)$ and $(t^2 + 1)$ are even. Dividing p_3 and q_3 by their common factor 8, we get $p_3 = \frac{1}{2}(t^3 + 3t)$ and $q_3 = \frac{1}{2}(t^2 + 1)$. With $p = p_3, q = q_3$ and $D = t^2 + 4$, the left-hand side of the Pell's equation becomes

$$p^2 - Dq^2 = \left(\frac{t^3 + 3t}{2}\right)^2 - (t^2 + 4)\left(\frac{t^2 + 1}{2}\right)^2.$$

Upon simplification, we find $p^2 - Dq^2 = -1$.

Theorem 3: By design, the two sets of recursive expressions given in (16) to (19), with D as a BI, solve at least one of the Pell's equations $p^2 - Dq^2 = \pm C$, where $C = 1$, 2 or 4. Irrespective of the value of C $(-1, \pm 2, \pm 4)$ produced by the initial pair (p_1, q_1), these expressions offer integer solutions to $p^2 - Dq^2 = 1$ subsequently (in no more than five steps).

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Proof: There are five different situations here: three use expressions given in (16) and (17) and two use expressions given in (18) and (19). ($D = u^2 - 1$ is excluded.)

(a) $D = t^2 + 1$

Here t is any integer greater than 1. Using $p_1 = t$, $q_1 = 1$ and $D = t^2 + 1$ in (16) and (17), we find $p_2 = (2t^2 + 1)$ and $q_2 = 2t$. With $p = p_2$, $q = q_2$ and $D = t^2 + 1$, the left-hand side of the Pell's equation becomes

$$p^2 - Dq^2 = (2t^2 + 1)^2 - (t^2 + 1)(2t)^2) = 1.$$

(b) $D = t^2 + 2$

Using $p_1 = t$, $q_1 = 1$ and $D = t^2 + 2$ in (16) and (17), we find $p_2 = 2t^2 + 2$ and $q_2 = 2t$. Dividing both p_2 and q_2 by 2, we find $p_2 = t^2 + 1$ and $q_2 = t$. With $p = p_2$, $q = q_2$ and $D = t^2 + 2$, the left-hand side of the Pell's equation becomes

$$p^2 - Dq^2 = (t^2 + 1)^2 - (t^2 + 2)t^2 = 1.$$

(c) $D = t^2 + 4$

Here $t > 1$ is an odd integer. Using $p_3 = \frac{1}{2}(t^3 + 3t)$ and $q_3 = \frac{1}{2}(t^2 + 1)$, a result under proof of Theorem 2, and $D = t^2 + 4$ in (16) and (17) we find, in the third step, $p_6 = \frac{1}{2}(t^6 + 6t^4 + 9t^2 + 2)$ and $q_6 = \frac{1}{2}(t^5 + 4t^3 + 3t)$. (Since the numerators of p_6 and q_6 are even, we divide the results by 2.) With $p = p_6$, $q = q_6$ and $D = t^2 + 4$, the left-hand side of the Pell's equation becomes

$$p^2 - Dq^2 = \left(\frac{t^6 + 6t^4 + 9t^2 + 2}{2}\right)^2 - (t^2 + 4)\left(\frac{t^5 + 4t^3 + 3t}{2}\right)^2.$$

Upon simplification, we find $p^2 - Dq^2 = 1$.

(d) $D = u^2 - 2$

Using $p_1 = u$, $q_1 = 1$ and $D = u^2 - 2$ in (18) and (19), we find $p_2 = (2u^2 - 2)$ and $q_2 = 2u$. Dividing p_2 and q_2 by 2, we get $p_2 = (u^2 - 1)$ and $q_2 = u$. With $p = p_2$, $q = q_2$ and $D = u^2 - 2$, the left-hand side of the Pell's equation becomes

$$p^2 - Dq^2 = (u^2 - 1)^2 - (u^2 - 2)u^2 = 1.$$

(e) $D = u^2 - 4$

Here u is an odd integer. Using $p_1 = u$, $q_1 = 1$ and $D = u^2 - 4$ in (18) and (19), we find $p_3 = u(u^2 - 3)$ and $q_3 = (u^2 - 1)$. Since u is odd, both p_3 and q_3 are even. Dividing p_3 and q_3 by 2, we get $p_3 = \frac{1}{2}u(u^3 - 3)$ and $q_3 = \frac{1}{2}(u^2 - 1)$. With $p = p_3$, $q = q_3$ and $D = u^2 - 4$, the left-hand side of the Pell's equation becomes

$$p^2 - Dq^2 = \left(\frac{u^3 - 3u}{2}\right)^2 - (u^2 - 4)\left(\frac{u^2 - 1}{2}\right)^2.$$

Upon simplification, we find $p^2 - Dq^2 = 1$.

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For cases (a), (b) and (d) above, we obtained the solutions to the Pell's equation $p^2 - Dq^2 = 1$ in the very next step. For case (c) it took five steps and for case (e) it took two steps. In the next Section we illustrate the use of all the expressions given in (4) and (5) and (16) to (19) using suitable examples.

3. *Illustrations: rational approximations to $\sqrt{D}$ using direct and recursive expressions*

As an illustration of the use of direct expressions in (4) and (5) which work for any non-square integer, we consider the first non-BI i.e. $D = 19$. The fundamental (first) solution for the Pell's equation [2] $p^2 - 19q^2 = 1$ is $q_1 = 39$ and $p_1 = 170$. For $D = 19$, the direct expressions for (q_n, p_n) for odd and even values of n, respectively, are:

$$p_n = \sum_{k=0}^{\frac{1}{2}(n-1)} \binom{n}{2k} 19^k 170^{(n-2k)} 39^{2k};\ q_n = \sum_{k=0}^{\frac{1}{2}(n-1)} \binom{n}{2k+1} 19^k 170^{(n-2k-1)} 39^{(2k+1)};$$

$$p_n = \sum_{k=0}^{\frac{1}{2}n} \binom{n}{2k} 19^k 170^{(n-2k)} 39^{2k};\ q_n = \sum_{k=0}^{\frac{1}{2}(n-2)} \binom{n}{2k+1} 19^k 170^{(n-2k-1)} 39^{(2k+1)}.$$

Using the above, it is possible to compute directly, for any specific n, the corresponding convergents of RCF to $\sqrt{19}$. To verify the above, we manually computed just the values for $n = 2$ (for even) and $n = 3$ (for odd). The two rational approximations are: (i) $p_2 = 57799$ and $q_2 = 13260$; and (ii) $p_3 = 19651490$ and $q_3 = 4508361$. For a BI, we consider 83 (with $t = p_1 = 9$ and $q_1 = 1$). The manually computed values for $n = 4$ are $p_4 = 13447$ and $q_4 = 1476$; and, for $n = 5$, after dividing the computed p_5 and q_5 by their common factor 4, are $p_5 = 243531$ and $q_5 = 26731$. All these rational approximations match with the respective CF values found in [6]. (For 83, the fundamental solution is $q_1 = 9$ and $p_1 = 82$; using these, $p_2 = 13447$ and $q_2 = 1476$.) Since $q_1 = 1$ for BIs, it is somewhat easier to compute p_n and q_n using (4) and (5).

Example 1: $g = 1: \sqrt{5}$ Example 2: $h = 1: \sqrt{15}$

n	p	q	p/q	$p^2 - 5q^2$		n	p	q	p/q	$p^2 - 15q^2$
1	2	1	2	−1		1	4	1	4	1
2	9	4	2.25	1		2	31	8	3.875	1
3	38	17	2.235294118	−1		3	244	63	3.873015873	1
4	161	72	2.236111111	1		4	1921	496	3.872983871	1

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Example 3: $g = 2 : \sqrt{83}$

n	p	q	p/q	$p^2 - 83q^2$
1	9	1	9	−2
2	82	9	9.111111111	1
3	1485	163	9.110429448	−2
4	13447	1476	9.110433604	1

Example 4: $h = 2 : \sqrt{23}$

n	p	q	p/q	$p^2 - 23q^2$
1	5	1	5	2
2	24	5	4.8	1
3	235	49	4.795918367	2
4	1151	240	4.795833333	1

Example 5: $g = 4 : \sqrt{13}$

n	p	q	p/q	$p^2 - 13q^2$
1	3	1	3	−4
2	11	3	3.666666667	4
3	18	5	3.6	−1
4	119	33	3.606060606	4
5	393	109	3.605504587	−4
6	649	180	3.605555556	1

Example 6: $h = 4 : \sqrt{21}$

n	p	q	p/q	$p^2 - 21q^2$
1	5	1	5	4
2	23	5	4.6	4
3	55	12	4.583333333	1
4	527	115	4.582608696	4
5	2525	551	4.582577132	4
6	6049	1320	4.582575758	1

TABLE 2: Sample rational approximations of square roots of Brahmagupta integers

In Table 2, we illustrate the application of the revised sets of recursive expressions (that are intended only for BIs). The values $D = 5$ (for $g = 1$), $D = 83$ (for $g = 2$) and $D = 13$ (for $g = 4$) serve as examples for (16) and (17). The values $D = 15$ (for $h = 1$), $D = 23$ (for $h = 2$) and $D = 21$ (for $h = 4$) serve as examples for (18) and (19). For both $\sqrt{83}$ and $\sqrt{23}$, we divide the computed p and q values by $2^{\mathrm{mod}(\frac{1}{2}(n-1))}$ and for both $\sqrt{13}$ and $\sqrt{21}$, we divide the computed p and q values by $2^{\mathrm{mod}(\frac{1}{3}(n-1))}$. The resulting rational approximations to the respective square roots of these BIs match with the respective numerators and denominators, which can be found by searching for convergents in [6]. To verify the assertion that, for non-BIs, these revised sets of recursive expressions will not produce results that match with the correct convergents, we used expressions in (16) and (17) for $D = 19$ (with $g = 3$) and expressions in (18) and (19) for $D = 61$ (with $h = 3$). These results, shown in Table 3, lead us to Conjecture 1 stated below.

Conjecture 1: For non-BIs, expressions in (16) to (19) will not generate (i) the correct rational approximations to their respective square roots and (ii) integer solutions for $p^2 - Dq^2 = 1$.

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Example 7: $g = 3 : \sqrt{19}$ Example 8: $h = 3 : \sqrt{61}$

n	p	q	p/q	$p^2 - 19q^2$		n	p	q	p/q	$p^2 - 61q^2$
1	4	1	4	−3		1	8	1	8	3
2	35	8	4.375	9		2	125	16	7.8125	9
3	292	67	4.358209	−27		3	1976	253	7.810277	27
4	2441	560	4.3589286	81		4	31241	4000	7.81025	81

TABLE 3: Sample rational approximations of square roots of non-Brahmagupta integers

4. *Conclusion*

Since the recursive expressions for solving the Pell's equations are linear, it was easy to formulate the matrix that captured the behaviour of rational approximations. The approach used here is a generalisation of the results presented in [5] for $\sqrt{2}$. The formulations presented here help (i) to prove that the convergence of the ratio p_n/q_n to $\sqrt{D}$ as $n \to \infty$ and (ii) to develop expressions for computing directly any specific (say n th) pair of rational approximations using only the integer D and any known (e.g. the fundamental) solution. This mechanism for computing directly any specific set of rational approximation p and q presented in Section 1 appears to be a new result and its potentials are yet to be explored. Further, the odd convergents (p_n/q_n) which are computed using (4) monotonically increase and converge to $\sqrt{D}$; and the even convergents (p_n/q_n) which are computed using (5) monotonically decrease and converge to $\sqrt{D}$. We identified in Section 2 a set of integers that we named after Brahmagupta. The BIs are formed by (i) $D = N^2 \pm 1$, (ii) $D = N^2 \pm 2$ and (iii) $D = N^2 \pm 4$ where the integer $N > 0$ such that $D > 1$. The two sets of revised recursive expressions given in (16) to (19) were formulated for finding the rational approximations of the square root of BIs, which use obvious initial solution ($p_1 = \lfloor\sqrt{D}\rfloor$ or $p_1 = \lfloor\sqrt{D}\rfloor + 1$, $q_1 = 1$). We chose a few examples for illustration such as 83 and 23 (in lieu of 92) in light of their historical association with Brahmagupta, who is believed to have found integer solutions for equations such as $p^2 - 83q^2 = 1$ and $p^2 - 92q^2 = 1$ using a trial solution and the composition method.

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