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Resonance between Cantor sets

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Abstract. Let C_a be the central Cantor set obtained by removing a central interval of length $1-2a$ from the unit interval, and then continuing this process inductively on each of the remaining two intervals. We prove that if $\log b/\log a$ is irrational, then

$$\dim(C_a + C_b) = \min(\dim(C_a) + \dim(C_b),\, 1),$$

where dim is Hausdorff dimension. More generally, given two self-similar sets K, K' in $\mathbb{R}$ and a scaling parameter $s > 0$, if the dimension of the arithmetic sum $K + sK'$ is strictly smaller than $\dim(K) + \dim(K') \le 1$ ('geometric resonance'), then there exists $r < 1$ such that all contraction ratios of the similitudes defining K and K' are powers of r ('algebraic resonance'). Our method also yields a new result on the projections of planar self-similar sets generated by an iterated function system that includes a scaled irrational rotation.

1. *Introduction and statement of results*

In many situations in dynamical systems and geometric measure theory, one encounters results which are valid for almost every member of a parametrized family. In general, those results are sharp in the sense that the set of exceptions may have positive, or even full, dimension. But when the constructions are done in a dynamical or geometrically regular way, there is often some countable set of parameters, usually arising from an algebraic condition, where the result fails, and it is natural to conjecture that these are all of the parameters for which the result fails; see, for example, [**6–8, 18, 20**]. However, there are very few cases where the set of exceptions has been explicitly determined. In this paper we shall find the exact set of parameters for which a certain kind of resonance between two self-similar sets occurs. This set is countable and given by a simple, natural algebraic relation.

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Let C_a be the central Cantor set obtained by removing a middle interval of length $1-2a$ from the unit interval, and then continuing this process inductively on each of the remaining intervals. Explicitly,

$$C_a = \left\{ (1-a) \sum_{i=0}^{\infty} \omega_i a^i : \omega_i \in \{0, 1\} \right\}.$$

It is well known that $\dim(C_a) = \log 2/\log(1/a)$.

Our main result is the following.

THEOREM 1. *If* $\log b/\log a$ *is irrational, then*

$$\dim(C_a + C_b) = \min(\dim(C_a) + \dim(C_b), 1). \tag{1}$$

When $\log b/\log a$ is rational and $\dim(C_a) + \dim(C_b) \leq 1$, the equality (1) does not hold; this is well known, and we sketch a proof after (4) below.

An *iterated function system* (or i.f.s. for short) on a complete metric space $\mathbb{X}$ is a finite collection of maps $\{f_1, \ldots, f_n\}$ from $\mathbb{X}$ to itself, such that all the f_j are contractions (i.e. have Lipschitz constant strictly less than 1). The *attractor* of this i.f.s. is the unique non-empty compact set E such that

$$E = \bigcup_{i=1}^{n} f_i(E).$$

In particular, C_a is the attractor of the i.f.s. $\{ax, ax + (1-a)\}$ in $\mathbb{R}$. When $\mathbb{X} = \mathbb{R}^n$ and the maps are similitudes, we say that E is a *self-similar* set. See [**4**] for background on iterated function systems and self-similar sets.

Theorem 1 generalizes to arbitrary self-similar sets in $\mathbb{R}$.

THEOREM 2. *Let* K *(respectively* K'*) be the attractor of the i.f.s.* $\{r_i x + t_i\}_{i=1}^{n}$ *(respectively* $\{r'_i x + t'_i\}_{i=1}^{n'}$*). If there exist* j, j' *such that* $\log(|r_j|)/\log(|r'_{j'}|)$ *is irrational, then*

$$\dim(K + K') = \min(\dim(K) + \dim(K'), 1).$$

Next, we define two notions of *resonance* and restate Theorem 2 using these notions. Let $\{r_i x + t_i\}_{i=1}^{n}$ and $\{r'_i x + t'_i\}_{i=1}^{n'}$ be two i.f.s. consisting of similitudes, with attractors K and K', respectively. We say that these two i.f.s. are *algebraically resonant* if for all $j \leq n$ and $j' \leq n'$, the ratio $\log|r_j|/\log|r'_{j'}|$ is rational. We say that the two i.f.s. are *geometrically resonant* if there is some $s > 0$ such that

$$\dim(K + sK') < \dim(K) + \dim(K') \leq 1.$$

Theorem 2 implies that two geometrically resonant iterated function systems must also be algebraically resonant. Below we will state a converse under a mild separation condition; see Theorem 4.

The paper is structured as follows. In the remainder of this section we discuss some background related to Theorem 1. Section 2 contains a discrete version of the Marstrand projection theorem, which will be needed for the main proofs. The proof of Theorem 1 is in §3, while §4 contains the proofs of the additional results stated at the end of this section. We finish the paper with some remarks and open questions in §5.

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1.1. *Background.* The study of the arithmetic sums of Cantor sets is a classical topic in real analysis, motivated in part by its rich applications to Diophantine approximation, the study of homoclinic bifurcations in dynamical systems, and other topics. The basic problem is to compute or estimate the size of the arithmetic sum $K + K'$ in terms of the sizes of K and K'. 'Size' can mean different things, but in this paper we focus on Hausdorff dimension.

Let K, K' be two compact subsets of the real line. Observe that

$$K + K' \quad \text{is congruent to} \quad \sqrt{2}P_{\pi/4}(K \times K'),$$

where P_θ denotes the orthogonal projection onto the line $\{(t\cos\theta, t\sin\theta)\}_{t\in\mathbb{R}}$ making angle θ with the x-axis. Let $\gamma = \dim(K) + \dim(K')$. By Marstrand's projection theorem (see [**12**, Ch. 9]) we have

$$\gamma \le 1 \implies \dim(P_\theta(K \times K')) = \gamma \text{ for almost every } \theta \in [0, \pi),$$
$$\gamma > 1 \implies \mathcal{L}(P_\theta(K \times K')) > 0 \text{ for almost every } \theta \in [0, \pi),$$

where $\mathcal{L}$ denotes Lebesgue measure. Unfortunately, Marstrand's theorem gives no information about specific values of θ, and therefore cannot be directly applied to obtain information about sums of Cantor sets. It does, however, support the heuristic principle that, 'typically',

$$\dim(K + K') = \min(\dim(K) + \dim(K'), 1).$$

Note that our definition of geometrical resonance for the iterated function systems that generate K and K' (in the case where the sum of the dimensions is at most one) requires at least one exception to the projection theorem for the set $K \times K'$. We now discuss some earlier results that motivated our work.

The following result is due to Peres and Solomyak [**19**, Theorem 1.1], and generalizes an earlier result of Solomyak [**23**].

THEOREM 3. (Peres and Solomyak) *Let*

$$\gamma(a) = \dim(C_a) = \log(2)/\log(1/a).$$

Given a fixed compact set $K \subset \mathbb{R}$, the following holds for almost every $a \in (0, 1/2)$:

$$\gamma(a) + \dim(K) \le 1 \implies \dim(C_a + K) = \gamma(a) + \dim(K); \tag{2}$$
$$\gamma(a) + \dim(K) > 1 \implies \mathcal{L}(C_a + K) > 0. \tag{3}$$

In this theorem, the Cantor sets C_a can be replaced by more general homogeneous self-similar sets (see [**19**] for details). If the ratio of $\log a$ and $\log b$ is rational, then

$$\dim(C_a + C_b) < \dim(C_a) + \dim(C_b). \tag{4}$$

This is folklore and can be seen as follows. If $\log a/\log b$ is rational, then there are $0 < r < 1$ and positive integers m, n such that $a = r^m$, $b = r^n$. By iterating n times the construction of C_a and m times the construction of C_b, we can decompose the product $C_a \times C_b$ into a union of cylinder sets whose convex hulls are squares. Moreover, two of these squares are $[0, r^{mn}] \times [1 - r^{mn}, 1]$ and $[1 - r^{mn}, 1] \times [0, r^{mn}]$. The projections of

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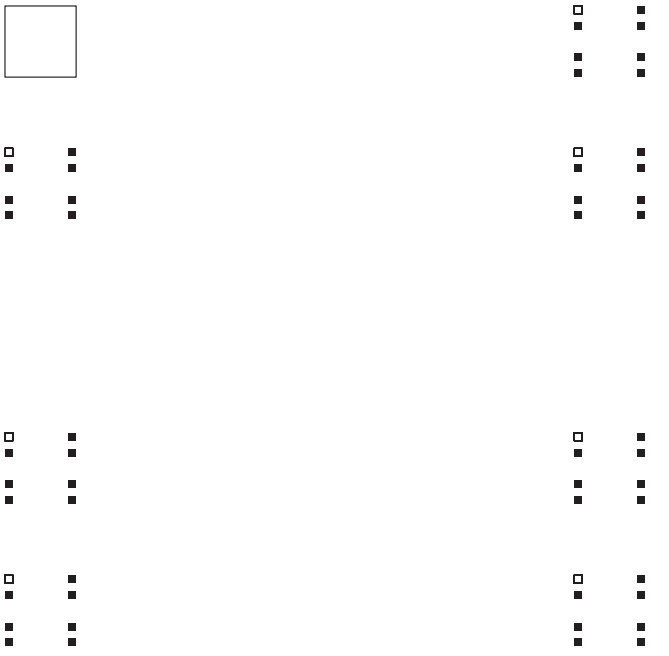


FIGURE 1. The structure of $C_{1/9} \times C_{1/3}$: for each white square, there is a black square of the same level and with the same projection onto the line $\{(t, t)\}_{t\in\mathbb{R}}$. This results in a dimension drop for the projection, which is affinely equivalent to the arithmetic sum $C_{1/9} + C_{1/3}$.

these squares on the line $\{(t, t)\}_{t\in\mathbb{R}}$ coincide. Thus, if we delete one of them from the construction of $C_a \times C_b$, and delete the corresponding square at every level, we obtain a self-similar set $K \subset C_a \times C_b$ with $P_{\pi/4}(K) = P_{\pi/4}(C_a \times C_b)$ and $\dim(P_{\pi/4}(K)) \leq \dim(K) < \dim(C_a \times C_b)$. See Figure 1 for an illustration of this idea.

It follows that one cannot expect (2) to hold for *all* a, even when K is also a central Cantor set. In [**17**] (as part of a more general framework) the dimension of the set of exceptions in (2) was estimated, but it was unknown until now whether the set of exceptions (for $K = C_b$) is countable. Eroğlu [**3**] proved that if $\dim(C_a) + \dim(C_b) \leq 1$, then the Hausdorff measure of the sum set $C_a + C_b$ in the dimension $\dim(C_a) + \dim(C_b)$ is zero.

In a different direction, Moreira proved a result on the dimension of the sum of dynamically defined, nonlinear Cantor sets. This result was stated in [**14**], but the proof sketched there is incorrect; see [**22**] for a correct proof based on Moreira's ideas.

We remark that Moreira's theorem has explicit checkable conditions, but it applies to nonlinear constructions; on the other hand, the result of Peres and Solomyak applies to (affine) self-similar sets, but is an almost-everywhere type of result and gives no information about specific cases.

Theorem 1 fills this gap by determining the exact set of exceptions to (2) in the case where K is also a central Cantor set. In fact, our result holds for sums of general self-similar sets; see Theorem 2 below.

Our research was also inspired by some conjectures of H. Furstenberg (personal communication), who, in the 1980s, asked about the validity of Theorem 1 in the particular

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case where a^{-1} and b^{-1} are integers which are not powers of a common integer. Other conjectures of Furstenberg that are similar in spirit remain open. For example, let S be the one-dimensional Sierpiński gasket, i.e.

$$S = \left\{ \sum_{i=1}^{\infty} z_i 3^{-i} : z_i \in \{(0,0), (1,0), (0,1)\} \right\}.$$

Furstenberg conjectured that $P_\theta(S)$ has dimension one but measure zero for all θ for which $\tan(\theta)$ is irrational. The part of the conjecture concerning measure was proved by Kenyon [**10**] and generalized by Lagarias and Wang in [**11**], but the dimension part is still open. Some related conjectures and results can be found in [**6**].

Palis [**16**] conjectured that, generically, if $K + K'$ has positive Lebesgue measure, then it has non-empty interior. This conjecture motivated much of the research on this topic, and it was eventually answered positively in [**15**]. Although our work is not directly related to Palis's conjecture (indeed, we focus on the case where $\dim(K) + \dim(K') < 1$ and, in particular, $K + K'$ has zero measure), it can be seen as a continuation of the same line of research. We remark that Palis's conjecture restricted to self-similar sets is still open, i.e. it is not known whether sums of self-similar sets generically are either of zero measure or have non-empty interior.

The topological structure of $C_a + C_b$ when $\dim(C_a) + \dim(C_b) > 1$ was investigated in [**13**] and [**1**]; the condition $\log b / \log a \notin \mathbb{Q}$ also arises naturally in this context.

1.2. *Further results.* The irrationality condition in Theorem 2 is equivalent to $\{\log |r_i|\} \cup \{\log |r_i'|\}$ not being an arithmetic set (a subset of $\mathbb{R}$ is arithmetic if it is contained in some lattice $\alpha\mathbb{N}$). Our next result shows that, under the assumption that the Hausdorff dimension equals the so-called similarity dimension, Theorem 2 is sharp. We remark that this assumption is weaker than the well-known open set condition; see e.g. [**4**].

THEOREM 4. *Let K (respectively K') be the attractor of the i.f.s. $\{r_i x + t_i\}_{i=1}^n$ (respectively $\{r_i' x + t_i'\}_{i=1}^{n'}$), where $0 < r_i, r_i' < 1$, and assume that the Hausdorff dimension of K (respectively K') is given by the only $\beta > 0$ (respectively $\beta' > 0$) satisfying $\sum_{i=1}^n r_i^\beta = 1$ (respectively $\sum_{i=1}^{n'} (r_i')^{\beta'} = 1$). Suppose that the set $\{\log |r_i|\} \cup \{\log |r_i'|\}$ is arithmetic. Then there exists some $s > 0$ such that*

$$\dim(K + sK') < \dim(K) + \dim(K'). \tag{5}$$

Using self-similarity, it is easy to show that if there is one s such that (5) holds, then there are infinitely many. On the other hand, according to Marstrand's projection theorem, the set of parameters s for which a dimension drop (5) occurs has zero length. Furstenberg conjectured that this set is in fact countable, at least in the case where K and K' are central Cantor sets; however, our methods do not seem to yield progress on this problem.

A modification of the proofs of Theorems 1 and 2 yields a new result on the projections of planar self-similar sets generated by an iterated function system that includes a scaled irrational rotation.

THEOREM 5. *Let $\{A_i x + d_i\}_{i=1}^n$ be a self-similar i.f.s. in $\mathbb{R}^2$ with attractor E. Suppose that each linear map A_i is written as $\zeta_i R_{\theta_i} O_i$, where $|\zeta_i| < 1$, the map R_{θ_i} is the rotation*

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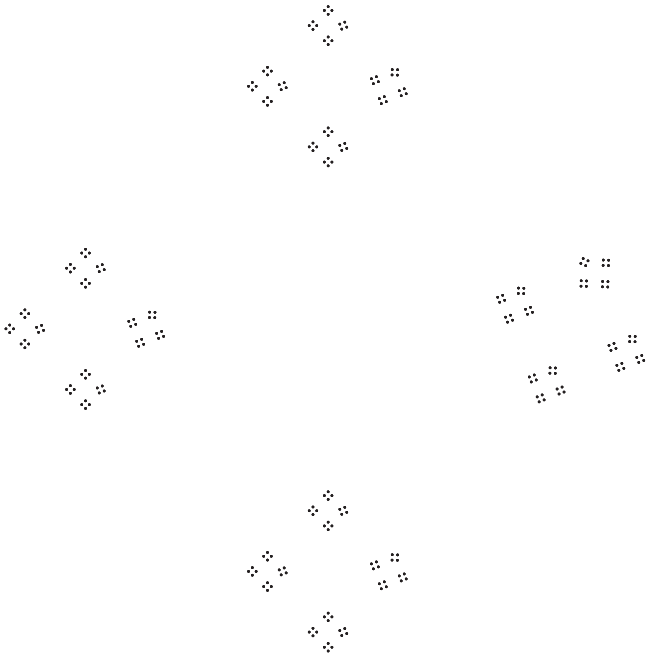


FIGURE 2. The attractor of an i.f.s. to which Theorem 5 applies. The dimension of this self-similar set is one. Theorem 5 says that all orthogonal projections are one-dimensional as well.

by angle θ_i and O_i is either the identity or a reflection about the x-axis. Suppose that the group G generated by $\{R_{\theta_i} O_i\}_{i=1}^n$ is such that the set $\{\theta : R_\theta \in G\}$ is dense in $[0, \pi)$. Then

$$\dim(P_\xi(E)) = \min(\dim(E), 1) \quad \textit{for all } \xi \in [0, \pi).$$

Figure 2 depicts one of the self-similar sets to which Theorem 5 applies. We remark that Eroğlu [**2**] proved that, under the assumptions of Theorem 5, if the i.f.s. satisfies the open set condition and $\gamma = \dim(E) \leq 1$, then all projections $P_\xi(E)$ have zero γ-dimensional Hausdorff measure. Thus, in this case, the projections are smaller than the original set E in measure, but not in dimension.

As part of the proofs of Theorems 2 and 5, we will need the following proposition, which as far as we know has not appeared previously in the literature.

PROPOSITION 6. *Let $\{A_i x + d_i\}_{i=1}^n$ be a self-similar i.f.s. on $\mathbb{R}^p$, where p is either 1 or 2, that satisfies the open set condition; let E be its attractor.*

Then, for all $\varepsilon > 0$, there exists an i.f.s. of the form $\{Ax + z_i\}_{i=1}^N$ with attractor H, such that $H \subset E$ and

$$\dim(H) \geq \dim(E) - \varepsilon.$$

2. *A discrete Marstrand projection theorem*

We prove a discrete version of Marstrand's theorem on projections. Similar results were obtained by Moreira (personal communication, see [**22**]) and by Rams, in the context of intersection numbers; see [**21**] and references therein.

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PROPOSITION 7. *Given constants* $A > 1$, $A_1, A_2 > 0$ *and* $0 < \gamma < 1$, *there exists a constant* δ *such that the following holds.*

Fix $\rho > 0$. *Let* $\mathcal{Q}$ *be a collection of disjoint closed convex subsets of the unit disk such that each element contains a disk of radius* $A^{-1}\rho$ *and is contained in a disk of radius* $A\rho$. *Suppose that* $\mathcal{Q}$ *has cardinality at least* $\rho^{-\gamma}/A_1$, *yet any disk of radius* $\ell \in (\rho, 1)$ *intersects at most* $A_2(\ell/\rho)^\gamma$ *elements of* $\mathcal{Q}$. *Then for any* $\varepsilon > 0$, *there exists a set* $J \subset [0, \pi]$ *with the following properties:*

(1) $\mathcal{L}([0, \pi]\backslash J) \le \varepsilon$;

(2) *if* $\theta \in J$, *then there exists a subcollection* $\mathcal{Q}_1$ *of* $\mathcal{Q}$ *of cardinality at least* $\varepsilon\delta|\mathcal{Q}|$ *such that the orthogonal projections of the sets in* $\mathcal{Q}_1$ *onto a line with direction* θ *are all disjoint and* ρ*-separated;*

(3) J *is a finite union of open intervals.*

Proof. In the course of the proof, c denotes a universal constant, and the A_i denote constants which depend only on A, A_1, A_2 and γ. Let E be the union of all elements of $\mathcal{Q}$, and let μ be normalized Lebesgue measure on E. Consider the Riesz energy

$$I_1(\mu) = \iint |z - w|^{-1}\, d\mu(z)\, d\mu(w).$$

We claim that

$$I_1(\mu) \le A_4\rho^{\gamma-1}. \tag{6}$$

This is standard, but we provide a proof below for completeness.

Assuming (6), we apply [**12**, Theorem 9.9] with $n = 2$ and $m = 1$ to obtain

$$\int_0^\pi |P_\theta E|^{-1}\, d\theta \le cI_1(\mu) \le A_5\rho^{\gamma-1},$$

where each $P_\theta E$ is the projection of $E = \mathrm{Supp}(\mu)$ onto a line making angle θ with the x-axis, and $|P_\theta E|$ is its length. Therefore, letting

$$J_* = \{\theta : |P_\theta E| > \varepsilon A_5^{-1}\rho^{1-\gamma}\},$$

we have $\mathcal{L}([0, \pi]\backslash J_*) \le \varepsilon$.

Let $J = J(\delta)$ be the set of all $\theta \in [0, \pi)$ with the following property: there exists a subcollection $\mathcal{Q}_1 = \mathcal{Q}_1(\theta)$ of $\mathcal{Q}$, of cardinality at least $\varepsilon\delta|\mathcal{Q}|$, such that the orthogonal projections of the sets in $\mathcal{Q}_1$ onto a line with direction θ are all disjoint and ρ-separated.

Owing to the convexity of the elements of $\mathcal{Q}$, the set $J = J(\delta)$ is a finite union of open intervals. We claim that $J_* \subset J(\delta)$ for $\delta = (5A + 5)^{-1}A_5^{-1}$. To prove this, consider $\theta \in J_*$. By the definition of J_*, we can find in $P_\theta E$ at least $\delta\varepsilon\rho^{-\gamma}$ points x_j that are $3(A + 1)\rho$-separated. Choose for each of these points x_j one element in $\mathcal{Q}$ with projection that contains x_j. This yields a family $\mathcal{Q}_1 \subset \mathcal{Q}$, of cardinality at least $\delta\varepsilon\rho^{-\gamma}$, such that the projections $\{P_\theta(Q) : Q \in \mathcal{Q}_1\}$ are at least $(A + 1)\rho$-separated, as desired.

Proof of (6). We will need the following properties of μ:

(i) $A_2^{-1}A^{-4}\rho^\gamma \le \mu(Q) \le A_1A^4\rho^\gamma$ for any element Q of $\mathcal{Q}$;

(ii) $\mu(D) \le A_7\ell^\gamma$ for any disk D of radius $\ell \in [\rho, 1]$;

(iii) $\mu \le A_8\rho^{\gamma-2}\mathcal{L}_2$, where $\mathcal{L}_2$ denotes two-dimensional Lebesgue measure.

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To begin, observe that if Q_1, Q_2 are elements of $\mathcal{Q}$, then $\mu(Q_1)/\mu(Q_2) \le A^4$. Since $\mathcal{Q}$ has between $\rho^{-\gamma}/A_1$ and $A_2\rho^{-\gamma}$ elements, (i) follows. If D is a disk of radius ℓ, where $\rho \le \ell \le 1$, then D intersects at most $A_2(\ell/\rho)^\gamma$ elements of $\mathcal{Q}$; hence (ii) follows from (i), with $A_7 = A_1A^4A_2$. Next, observe that

$$\mathcal{L}_2(E) \ge (A_1^{-1}\rho^{-\gamma})(A^{-2}\rho^2) = A_1^{-1}A^{-2}\rho^{2-\gamma},$$

and from this we deduce that for any Borel set $B \subset \mathbb{R}^2$,

$$\mu(B) = \mu(B \cap E) = \mathcal{L}_2(E)^{-1}\mathcal{L}_2(B \cap E) \le A_1A^2\rho^{\gamma-2}\mathcal{L}_2(B),$$

which shows that (iii) holds with $A_8 = A_1A^2$.

Now fix $w \in E$. Using (iii) we estimate

$$\int_{|z-w|\le\rho} |z-w|^{-1}\, d\mu(z) \le A_8\rho^{\gamma-2} \int_{|u|\le\rho} |u|^{-1}\, du = 2\pi A_8\rho^{\gamma-1}. \tag{7}$$

On the other hand, for $j \ge 1$ we can apply (ii) to obtain

$$\mu(\{z : |z-w| \le e^j\rho\}) \le A_7(e^j\rho)^\gamma,$$

whence we deduce

$$\int_{e^{j-1}\rho<|z-w|\le e^j\rho} |z-w|^{-1}\, d\mu(z) \le A_7(e^j\rho)^\gamma e^{1-j}\rho^{-1} = eA_7(e^j\rho)^{\gamma-1}. \tag{8}$$

Adding (7) and the sum of (8) over all $j \in \mathbb{N}$, we get

$$\int |z-w|^{-1}\, d\mu(z) \le A_4\rho^{\gamma-1}.$$

Finally, integrating over w yields (6), and the proof is complete. □

3. *Proof of Theorem 1*

We first discuss the main ideas of the proof of Theorem 1; full details follow below. Recall that Marstrand's theorem says that

$$\dim(C_a + e^tC_b) = \min(\dim(C_a) + \dim(C_b),\, 1)$$

for almost every $t \in \mathbb{R}$ (for reasons which will become apparent later, it will be useful to work in a logarithmic scale). Proposition 7 gives a discrete version of this: we decompose $C_a \times C_b$ into cylinder rectangles of size approximately $\rho \times \rho$. For $t \in \mathbb{R}$, consider the projection mapping Π_{e^t}. It will follow from Proposition 7 that there is a set F_0 of 'good' values, whose complement has very small measure, such that for $t \in F_0$ a large number of rectangles have disjoint Π_{e^t} projections.

Because of the homogeneity of C_a, C_b, all these cylinders are translates of each other, and they are pairwise disjoint. The idea now is to decompose these first cylinders into subcylinders of size approximately $\rho^2 \times \rho^2$ and consider the set of 'good' scales F_1 associated with them; because the cylinders are a (anisotropic) rescaling of $C_a \times C_b$, F_1 is of the form $F_0 + \alpha_1$ for some $\alpha_1 \in \mathbb{R}$. We continue this process inductively to obtain sets of good scales F_n after n steps; we have $F_{n+1} = F_n + \alpha_n$ for some sequence $\{\alpha_n\}$.

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Notice that there is some latitude in choosing the sequence $\{\alpha_n\}$; the main idea is to exploit this freedom and pick the α_n in such a way that $t \mapsto t + \alpha_n$ is essentially an irrational rotation on the circle. By construction, the sets F_i are robust: they can be taken to be a finite union of intervals. Then Weyl's equidistribution principle implies that for any specific angle θ, there are many cylinders at many levels (more precisely, at a set of levels of well-defined density close to one) whose images under Π_{e^t} are disjoint. From here, a standard application of the mass distribution principle gives the desired lower bound on the dimension.

We remark that the idea of using the discrete version of Marstrand's theorem and then iterating the process on each cylinder is due to Moreira. However, while he uses a recurrence result which requires a nonlinearity assumption, we go in a different direction by using Weyl's equidistribution principle. Furstenberg used Weyl's theorem in a related setting in [**6**].

We now proceed to the details of the proof. We will establish a slightly more general version which will be needed to prove Theorem 2.

THEOREM 8. *Let K be the attractor of the i.f.s. $\{rx + t_i\}_{i=1}^{n}$, and let K' be the attractor of the i.f.s. $\{r'x + t_i'\}_{i=1}^{n'}$.*

Let I and I' be the convex hulls of K and K', respectively. Assume that the intervals $rI + t_i$ are pairwise disjoint, as are the intervals $r'I' + t_i'$.

Assume further that $\log(|r|)/\log(|r'|)$ *is irrational.*

Then

$$\dim(K + K') = \min(\dim(K) + \dim(K'), 1). \tag{9}$$

The central Cantor sets C_a satisfy the hypotheses of Theorem 8, hence Theorem 1 is an immediate consequence of Theorem 8.

Proof of Theorem 8.

Notation and remarks. The upper estimate

$$\dim(K + K') \leq \min(\dim(K) + \dim(K'), 1).$$

is well known, so we only need to prove that the opposite inequality also holds.

If $\dim(K) + \dim(K') \geq 1$, then for any $\varepsilon > 0$ we can, by iterating the original i.f.s. and throwing away some maps, find a subset $\widetilde{K}$ of K such that the hypotheses of the theorem still hold for the pair $(\widetilde{K}, K')$, and

$$1 - \varepsilon < \dim(\widetilde{K}) + \dim(K') < 1.$$

Thus we can assume without loss of generality that

$$\gamma := \dim(K) + \dim(K') < 1.$$

Dilating both K and K' by the same factor and translating either K or K' will affect neither the hypotheses nor the result; hence we can also assume that $I = [0, 1]$ and $I' = [0, e^{\tau}]$ for some $\tau \in \mathbb{R}$. Therefore it is enough to show that

$$\dim(K + e^{\tau}K') \geq \dim(K) + \dim(K') - \eta$$

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for all $\eta > 0$, under the additional assumption that the unit interval is the convex hull of both K and K'. We will fix τ for the rest of the proof.

Let Σ^* be the family of all finite words $u = (u_1, \ldots, u_k)$ with $u_i \in \{1, \ldots, n\}$; we define $(\Sigma')^*$ analogously. As is customary, $|u|$ will denote the length of u. Concatenation of words will be denoted by juxtaposition; 1^j denotes the word consisting of j consecutive ones.

Let $f_i(x) = rx + t_i$ and $f_i'(x) = r'x + t_i'$. For $u \in \Sigma^*$, $u' \in (\Sigma')^*$ we shall write

$$f_u = f_{u_1} \ldots f_{u_k}, \quad f'_{u'} = f'_{u'_1} \ldots f_{u'_{k'}},$$

where $|u| = k$, $|u'| = k'$. Also let $I(u) = f_u(I)$ and $I'(u') = f'_{u'}(I)$. Finally, write $Q(u, u') = I(u) \times I'(u')$. We emphasize that f' here is not a derivative.

Recall that P_θ is the orthogonal projection onto a line making angle θ with the x-axis. The projection mappings $(x, y) \to x + ty$ will be denoted by Π_t. A trivial but useful observation is that $P_\theta(\Lambda)$ and $\Pi_{\tan\theta}(\Lambda)$ are the same up to affine equivalence.

Application of the discrete version of Marstrand's theorem. In the course of the proof, c will denote a constant which depends only on K and K'; its value may be different in each line.

Given $k \geq 0$, consider the families

$$\mathcal{Q}_k = \{Q(u, u') : |u| = k, |u'| = k'\},$$
$$\widetilde{\mathcal{Q}}_k = \{Q(u, u') : |u| = k, |u'| = k' + 1\},$$

where k' is the largest integer such that $r^k < (r')^{k'}$. Note that rectangles in $\mathcal{Q}_k$ have size $r^k \times M_k r^k$, where $1 < M_k < 1/r'$, and rectangles in $\widetilde{\mathcal{Q}}_k$ have size $r^k \times M_k r' r^k$.

Observe that $|\mathcal{Q}_j| > cr^{-j\gamma}$. This is well known; see, for example, [**5**, Proposition 7.4] and the remarks preceding it. Moreover, if $j < k$, then any $Q \in \mathcal{Q}_j$ contains at most $cr^{(j-k)\gamma}$ elements of $\mathcal{Q}_k$. Therefore if S is any square of side r^j, then S intersects a uniformly bounded number of elements of $\mathcal{Q}_j$, and combined with the previous remark we deduce that

$$|\{Q \in \mathcal{Q}_k : Q \subset S\}| \leq cr^{(j-k)\gamma}.$$

It follows from these observations that $\mathcal{Q}_k$ satisfies the conditions of Proposition 7 for some A, A_1, A_2 that depend only on K, K' and which, in particular, are independent of k.

Now pick some large integer m and some small $\varepsilon > 0$, and apply Proposition 7 to $\mathcal{Q}_m$ to obtain a set J of 'good' angles. Note that because every rectangle of $\widetilde{\mathcal{Q}}_m$ is contained in a rectangle of $\mathcal{Q}_m$, property (2) of J in Proposition 7 is valid for $\widetilde{\mathcal{Q}}_m$ as well.

Write $\alpha = \log(M_m)$, $\beta = \log(1/r')$. A crucial fact is that α/β is irrational; this follows easily from the irrationality assumption. In particular, the map

$$\mathbf{R}(x) = x + \alpha (\mathrm{mod}\ \beta) \tag{10}$$

is uniquely ergodic on $[0, \beta)$ endowed with normalized Lebesgue measure. We observe that $\mathbf{R}^k(0) = \log(M_{mk})$; this fact will be useful later.

We will show that

$$\dim(K + e^\tau K') \geq \gamma - \eta,$$

where η depends on m and ε, and can be made arbitrarily small by letting $m \to \infty$, $\varepsilon \to 0$.

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If $\theta \in [0, \pi)$, then the orthogonal projection P_θ can be identified with the map $\Pi_s(x, y) = x + sy$, where $s = \tan(\theta)$. Since the map $\theta \to \log(\tan(\theta))$ is a diffeomorphism on any compact subset of the set of directions with positive and finite slope, it follows from Proposition 7 that there exists $\widetilde{F} \subset [\tau, \tau + \beta)$ and constants L, δ_1 (independent of m and ε) such that:

(i) $\mathcal{L}([\tau, \tau + \beta) \backslash \widetilde{F}) \leq L\varepsilon$;

(ii) if $t \in \widetilde{F}$, then there exist $\mathcal{D} = \mathcal{D}(t) \subset \mathcal{Q}_m$ and $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}(t) \subset \widetilde{\mathcal{Q}}_m$ such that $|\mathcal{D}| > \delta_1 \varepsilon r^{-m\gamma}$ and the family

$$\{\Pi_{e^t}(Q) : Q \in \mathcal{D}\}$$

is r^m-separated, with analogous assertions holding for $\widetilde{D}$;

(iii) $\widetilde{F}$ is a union of finitely many open intervals.

Finally, let $F = \widetilde{F} - \tau$.

Inductive construction. We will now use properties (i) and (ii) to inductively construct a tree $\mathcal{R}$ with vertices labeled by cylinder rectangles. The set of rectangles of level j will be denoted by $\mathcal{R}_j$. We will prove that the tree has the following properties:

(A) if Q is the parent of Q', then $Q' \subset Q$;

(B) if $Q \in \mathcal{R}_j$, then Q has size $r^{mj} \times \exp(\mathbf{R}^j(0))r^{mj}$;

(C) the family $\{\Pi_{e^\tau}(Q) : Q \in \mathcal{R}_j\}$ is disjoint and r^{jm}-separated;

(D) all rectangles of level j have the same number of children C_j and, moreover,

$$\mathbf{R}^j(0) \notin F \implies C_j = 1,$$
$$\mathbf{R}^j(0) \in F \implies C_j \geq \delta_1 \varepsilon r^{-m\gamma};$$

(E) each $Q \in \mathcal{R}_j$ is of the form $Q(v, v')$ for some $v \in \Sigma^*$, $v' \in (\Sigma')^*$.

Let the root of the tree be the unit square. Define $\mathcal{R}_1$ as follows: if $\tau \in \widetilde{F}$, then $\mathcal{R}_1 = \mathcal{D}(t)$; otherwise, $\mathcal{R}_1 = \{\widehat{Q}\}$ where $\widehat{Q}$ is any element of $\mathcal{Q}_m$ (e.g. $\widehat{Q} = Q(1^m, 1^{m'})$). Property (B) follows from the definition of α. Properties (A), (C), (D) and (E) are also clear.

Now assume that $\mathcal{R}_j$ has been defined such that (A)–(E) are satisfied. We will consider two cases:

(I) $\mathbf{R}^j(0) + \alpha < \beta$;

(II) $\mathbf{R}^j(0) + \alpha > \beta$.

Let $\sigma = 0$ if (I) holds, and $\sigma = 1$ otherwise. Observe that

$$\mathbf{R}^{j+1}(0) = \mathbf{R}^j(0) + \alpha - \sigma\beta. \tag{11}$$

If P and Q are rectangles, define their product $Q \cdot P$ as $f_Q f_P(Q_0)$, where Q_0 is the unit square and f_P and f_Q are the orientation-preserving linear maps mapping Q_0 onto P and Q, respectively. Now, given $Q \in \mathcal{R}_j$, we will define its set of offspring $C(Q)$ as follows. If $\mathbf{R}^j(0) \in F$, let

$$C(Q) = \begin{cases} \{Q \cdot P : P \in \mathcal{D}(\mathbf{R}^j(0) + \tau)\} & \text{if case (I) holds,} \\ \{Q \cdot P : P \in \widetilde{\mathcal{D}}(\mathbf{R}^j(0) + \tau)\} & \text{if case (II) holds.} \end{cases}$$

Otherwise, if $\mathbf{R}^j(0) \notin F$, then let $C(Q) = Q \cdot Q(1^m, 1^{m'+\sigma})$.

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Property (A) is clear. Property (B) follows from (11) since each element of $\mathcal{R}_{j+1}$ has size

$$r^{mj} r^m \times r^{mj} \exp(\mathbf{R}^j(0)) M_m(r')^\sigma r^m = r^{m(j+1)} \times \exp(\mathbf{R}^j(0) + \alpha - \sigma\beta) r^{m(j+1)}.$$

If $\mathbf{R}^j(0) \notin F$, then property (C) is trivial, so we will assume that $\mathbf{R}^j(0) \in F$ or, in other words, $\mathbf{R}^j(0) + \tau \in \widetilde{F}$. It is enough to show that if S_1, S_2 are children of the same $Q \in \mathcal{R}_j$, then their projections are $r^{m(j+1)}$-separated. Moreover, by translating Q if needed, we can assume that $Q = r^{mj}(I \times \exp(\mathbf{R}^j(0))I)$ where I is the unit interval. Let $S_i = Q \cdot P_i$, where $P_i = J_i \times J_i'$. We have

$$\begin{aligned}
\Pi_{e^\tau}(S_i) &= \Pi_{e^\tau}(Q \cdot P_i) \\
&= r^{jm} \Pi_{e^\tau}((I \times \exp(\mathbf{R}^j(0))I) \cdot (J_i \times J_i')) \\
&= r^{jm}(J_i \times \exp(\mathbf{R}^j(0) + \tau) J_i') \\
&= r^{jm} \Pi_{\exp(\mathbf{R}^j(0)+\tau)} P_i.
\end{aligned}$$

By property (ii) of $\widetilde{F}$, the family $\{\Pi_{\exp(\mathbf{R}^j(0)+\tau)} P_i\}$ is r^m-separated; it follows that the family $\{\Pi_{e^\tau}(S_i)\}$ is $r^{(j+1)m}$-separated, as desired.

Property (D) follows immediately from property (ii) above. Finally, property (E) is a direct consequence of the construction.

Application of Weyl's equidistribution principle. We recall that $\widetilde{F}$, and hence F, is a finite union of open intervals. Therefore we can apply Weyl's equidistribution principle to get

$$\lim_{j\to\infty} \frac{1}{j} |\{i < j : \mathbf{R}^i(0) \in F\}| = \frac{\mathcal{L}(F \cap [0, \beta))}{\mathcal{L}([0, \beta))} \geq 1 - (L/\beta)\varepsilon. \tag{12}$$

Let

$$E_\tau = \bigcap_{j=1}^{\infty} \bigcup_{Q \in \mathcal{R}_j} \Pi_{e^\tau}(Q).$$

By property (A) this set is a countable intersection of nested non-empty compact sets, so E_τ is compact and non-empty. It follows from property (E) that $E_\tau \subset K + e^\tau K'$.

We will estimate the dimension of E_τ in a standard way, by means of a Frostman measure. Let μ_τ be the probability measure which assigns the same mass $|\mathcal{R}_j|^{-1}$ to all intervals $\Pi_{e^\tau}(Q)$, for $Q \in \mathcal{R}_j$. This measure is well defined thanks to properties (C) and (D), and is supported on E_τ by definition.

Let $x \in E_\tau$. Because of (C), the interval $(x - r^{jm}, x + r^{jm})$ intersects exactly one interval $\Pi_{e^\tau}(Q)$ with $Q \in \mathcal{R}_j$, so we have

$$\mu_\tau(x - r^{jm}, x + r^{jm}) \leq \mu_\tau(\Pi_{e^\tau}(Q)) = |\mathcal{R}_j|^{-1}.$$

On the other hand, using (D) once again we get that

$$\log |\mathcal{R}_j| \geq |\{i < j : \mathbf{R}^i(0) \in F\}| \ \log(\delta_1 \varepsilon r^{-m\gamma}).$$

Hence we deduce from (12) that if j is large enough, then

$$\log \mu_\tau(x - r^{jm}, x + r^{jm}) \leq j(1 - (2L/\beta)\varepsilon) \log(\delta_1^{-1} \varepsilon^{-1} r^{m\gamma}).$$

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Thus, from the mass distribution principle (see e.g. [**5**, Proposition 2.3]), we conclude that

$$\dim(K + e^{\tau} K') \geq \dim(E_\tau) \geq \frac{(1 - (2L/\beta)\varepsilon)\log(\delta_1^{-1}\varepsilon^{-1}r^{m\gamma})}{m \log r}.$$

The right-hand side can be made arbitrarily close to γ by letting $m \to \infty$, $\varepsilon \to 0$. Therefore the Hausdorff dimension of $K + e^{\tau} K'$ must be at least γ, and this completes the proof. □

4. *Remaining proofs*

Proof of Theorem 2. Let K and K' be as in the statement of the theorem. We can assume without loss of generality that $\log(|r_1|)/\log(|r_1'|) \notin \mathbb{Q}$.

Fix $\varepsilon > 0$. Pick δ small enough so that for any covering $\{I_j\}$ of K by intervals of diameter at most 3δ, we have

$$\sum_j |I_j|^{\dim(K)-\varepsilon} > 1.$$

Let I be the convex hull of K. Consider a covering of K by cylinder intervals $f_u(I)$ of diameter between $r_*\delta$ and δ, where $r_* = \min_i |r_i|$. Pick a maximal pairwise disjoint collection $\{f_{u_j}(I)\}_{j=1}^N$ among these cylinders. By maximality, if I_j is the interval with the same center as $f_{u_j}(I)$ and length $|f_{u_j}(I)| + 2\delta$, then $\{I_j\}_{j=1}^N$ is a covering of K. Then we have

$$\sum_{j=1}^{N} (3\delta)^{\dim(K)-\varepsilon} \geq \sum_{j=1}^{N} |I_j|^{\dim(K)-\varepsilon} > 1.$$

It follows that $N \geq c\delta^{\varepsilon-\dim(K)}$, where $c = 3^{\varepsilon-\dim(K)}$. Let $\widetilde{K}$ be the attractor of the i.f.s. $\{f_{u_j}\}_{j=1}^N$. We have that $\widetilde{K} \subset K$ and

$$\dim(\widetilde{K}) \geq \frac{\log N}{|\log(r_*\delta)|} \geq \frac{\log c + (\dim(K) - \varepsilon)|\log \delta|}{|\log(r_*)| + |\log \delta|}.$$

Hence, by taking δ very small, we can ensure that $\dim(\widetilde{K}) > \dim(K) - 2\varepsilon$. Moreover, all the intervals $f_{u_j}(I)$ are disjoint. In the same way, we obtain an appropriate set $\widetilde{K}' \subset K'$. We can therefore assume, without loss of generality, that the families

$$\{r_i I + t_i\}_{i=1}^n, \quad \{r_i' I' + t_i'\}_{i=1}^{n'}$$

are disjoint for some intervals I and I' (one can take these intervals to be the convex hulls of K and K'). At this point the irrationality condition may fail to hold.

We now apply Proposition 6 to K and K', to obtain iterated function systems $\{g_i(x)\}_{i=1}^N$ and $\{g_i'(x)\}_{i=1}^{N'}$ with attractors H and H', where

$$g_i(x) = \rho x + z_i, \quad g_i'(x) = \rho' x + z_i',$$

the dimensions of H, H' are very close to those of K, K', and $H \subset K$, $H' \subset K'$. If $\log|\rho|/\log|\rho'|$ is irrational, we can apply Theorem 8 to $H + H'$, and we are done. If not, then consider the new i.f.s.

$$\{f_1 \circ g_i(x)\}_{i=1}^N,$$

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where $f_1(x) = r_1 x + t_1$. The dimension of the attractor $\widetilde{H} \subset K$ is given by

$$\frac{\log N}{-(\log|\rho| + \log|r_1|)},$$

which, by picking ρ small enough, can be made arbitrarily close to the dimension of H (and hence to the dimension of K). Define $\widetilde{H}'$ analogously. Since we are assuming that $\log(|\rho|)/\log(|\rho'|) \in \mathbb{Q}$ and that $\log(|r_1|)/\log(|r_1'|) \notin \mathbb{Q}$, it follows that $\widetilde{H}$, $\widetilde{H}'$ satisfy the irrationality hypothesis and we can apply Theorem 8 to $\widetilde{H} + \widetilde{H}'$. This completes the proof. □

Proof of Theorem 4. By assumption there exist $\xi > 0$ and integers $\{a_i\}$, $\{b_i\}$ such that $r_i = \xi^{a_i}$ and $r_i' = \xi^{b_i}$.

We first consider the case in which all the a_i are equal to some a and all the b_i are equal to some b. In this case, $K + K'$ is also self-similar, and the proof is much easier; notice that algebraically resonant central Cantor sets fall into this case. By iterating the first i.f.s. b times and the second i.f.s. a times, and replacing ξ by ξ^{ab}, we can assume that $a = b = 1$. In this case, the attractors are, up to affine equivalence, explicitly given by

$$K = \left\{ \sum_{i=1}^{\infty} d_i \xi^i : d_i \in \{t_1, \ldots, t_n\} \right\}, \quad K' = \left\{ \sum_{i=1}^{\infty} d_i \xi^i : d_i \in \{t_1', \ldots, t_{n'}'\} \right\}.$$

Moreover, $\dim(K) = \log(n)/\log(1/\xi)$ and $\dim(K') = \log(n')/\log(1/\xi)$. Therefore, letting $D = \{t_1, \ldots, t_n\}$ and $D' = \{t_1', \ldots, t_{n'}'\}$, we have

$$K + sK' = \left\{ \sum_{i=1}^{\infty} d_i \xi^i : d_i \in D + sD' \right\},$$

whence

$$\dim(K + sK') \leq \frac{\log|D + sD'|}{\log(1/\xi)}.$$

One can clearly take s so that $|D + sD'| < |D||D'|$; for example, let $s = (t_n - t_1)/(t_{n'}' - t_1')$. For any such s, a dimension drop occurs.

We now consider the general case. We will use the same notation as in the proof of Theorem 8. We claim that by iterating the constructions and reordering, we can assume that $a_1 = a_2 = b_1 = b_2$. Indeed, one can replace f_i $(i = 1, 2)$ by the collection of maps $\{f_i f_u : |u| = a_{3-i} b_1 b_2 - 1\}$, and f_i' $(i = 1, 2)$ by $\{f_i' f_u' : |u| = b_{3-i} a_1 a_2 - 1\}$. This operation does not change the attractors K and K'. But now $f_1 \circ f_1^{a_2 b_1 b_2 - 1}$ and $f_2 \circ f_2^{a_1 b_1 b_2 - 1}$ are maps in the first i.f.s. (they are obtained by taking $i = 1$ and u a word consisting of all 1s, and then $i = 2$ and u a word consisting of all 2s, respectively), and they both have similarity ratio $\xi^{a_1 a_2 b_1 b_2}$. Likewise, there are two maps in the second i.f.s. with this similarity ratio; this proves the claim.

Denote the common value of a_1, a_2, b_1, b_2 by ℓ. By a further translation and dilation we can also assume that the convex hull of both K and K' is the unit interval.

After these simplifications, notice that $Q(1, 1)$ and $Q(2, 2)$ are squares of the same size, and we can choose s so that $\Pi_s(Q(1, 1)) = \Pi_s(Q(2, 2))$. From our assumption that all the maps are orientation-preserving and by self-similarity, it follows that

$$\Pi_s(f_1(K) \times f_1'(K')) = \Pi_s(f_2(K) \times f_2'(K')).$$

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More generally, if for some u, u' we have $r_u = r'_{u'}$, then

$$\Pi_s(f_{(u1)}(K) \times f'_{(u'1)}(K')) = \Pi_s(f_{(u2)}(K) \times f'_{(u'2)}(K')). \tag{13}$$

Let a be the g.c.d. of $\{a_i\}$ and b the g.c.d. of $\{b_i\}$. Also write $A = \max\{a_i\}$, $B = \max\{b_i\}$. By [**9**, Theorem 1.4.1], there is a smallest integer M_0 with the following property: all multiples of ab that are greater than or equal to M_0 can be represented as a linear combination of $\{a_i\}$ with positive integral coefficients, and can also be represented as a linear combination of $\{b_i\}$ with positive integral coefficients. Pick some $M \geq M_0 + \max(A, B)$ which is a multiple of ab.

Let $\Sigma = \{1, \ldots, n\}^{\mathbb{N}}$. Given $\omega \in \Sigma$, let

$$S_j(\omega) = \sum_{i=1}^{j} a_{\omega_i},$$
$$n_k(\omega) = \min\{j : S_j(\omega) \geq kM\},$$
$$L_k(\omega) = S_{n_k}(\omega) - kM.$$

All these definitions also apply to finite words of the appropriate length. Recall that the Hausdorff dimension of K is given by the only $\beta > 0$ such that

$$\sum_{i=1}^{n} \xi^{\beta a_i} = 1.$$

Endow Σ with the Bernoulli measure μ for the probability vector $(\xi^{\beta a_i})_{i=1}^n$. Define Σ', S'_j, n'_k, L'_k, β', μ' analogously.

Notice that L_k can take the values $0, a, \ldots, (A/a - 1)a$. Because of the way M was defined, there is a finite sequence $\{j_m\}$ such that

$$(k+1)M = S_{n_k}(\omega) + \sum a_{j_m}.$$

Since μ is Bernoulli, this shows that there exist $\{p_i\}$, independent of k, such that

$$\mu(S_{n_{k+1}} = (k+1)M | L_k = ia) = p_i \quad \text{for } i = 0, 1, \ldots A/a - 1, \tag{14}$$

and analogously for μ'. Let $\nu = \mu \times \mu'$; we will treat ν like a probability distribution according to which pairs of sequences are drawn at random. Consider the events

$$\Omega_k = \{S_{n_k} = S'_{n'_k} = kM\},$$
$$\Xi_{k,i} = \{\omega(n_k + 1) = i, \omega'(n'_k + 1) = i\} \quad (i = 1, 2),$$
$$\Theta_{k,i} = \Omega_k \cap \Xi_{k,i}.$$

Letting $p = \min(\{p_i\}, \{p'_i\})$, it follows from (14) and independence that

$$\nu(\Omega_k) \geq p^2, \quad \nu(\Xi_{k,i}) = \xi^{\ell(\beta+\beta')}, \quad \nu(\Theta_{k,i}) \geq \xi^{\ell(\beta+\beta')} p^2.$$

Therefore

$$\begin{aligned}
\nu(\Theta_{k+1,2} \cap \Theta_{k,2}^c) &> \nu(\Theta_{k+1,2} \cap \Theta_{k,1}) \\
&= \nu((\Theta_{k+1,2} | S_{n_k+1} = S'_{n'_{k'}+1} = kM + \ell) \cap \Theta_{k,1}) \\
&\geq (\xi^{\ell(\beta+\beta')} p^2)^2 =: q,
\end{aligned}$$

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where in the last inequality we have used independence again. In particular, we have shown that $\nu(\Theta_{k+1,2}^c|\Theta_{k,2}^c) < 1-q$, and from this we deduce that

$$\nu\left(\bigcap_{j=1}^{k}\Theta_{j,2}^c\right) < (1-q)^k. \tag{15}$$

We now come back to the geometric picture. Consider the family

$$\mathcal{C}_k = \{u \in \Sigma^* : S_{|u|-1}(u) < kM \le S_{|u|}(u)\}.$$

Notice that $\{f_u(K) : u \in \mathcal{C}_k\}$ is a covering of K; define $\mathcal{C}_k'$ analogously. Let us call a pair $(u, u') \in \mathcal{C}_k \times \mathcal{C}_k'$ *redundant* if $(u, u') \in \Theta_{j,2}$ for some $j < k$; otherwise, we call (u, u') *essential*.

We claim that

$$\mathcal{E}_k = \{\Pi_s(Q(u, u')) : (u, u') \in \mathcal{C}_k \times \mathcal{C}_k' \text{ is essential}\}$$

is a covering of $\Pi_s(K \times K')$. Indeed, if (u, u') is redundant, then it can be decomposed as $(v_1 2 v_2, v_1' 2 v_2')$ where $r_{v_1} = r_{v_1'}' = \xi^{jM}$. Hence it follows from (13) that

$$\Pi_s(K_u \times K_{u'}') \subset \Pi_s(K_{(v_1 1)} \times K_{(v_1' 1)}'),$$

and if we delete (u, u') from $\mathcal{C}_k \times \mathcal{C}_k'$, we still get a covering of the projection.

Notice that each interval in $\mathcal{E}_k$ has length bounded by $(s+1)\xi^{kM}$. Any essential pair (u, u') has ν-probability at least $\xi^{A\beta+B\beta'}\xi^{k(\beta+\beta')M}$ of occurring. On the other hand, the probability of a pair in $\mathcal{C}_k \times \mathcal{C}_k'$ being essential is, by (15), no more than $(1-q)^{k-1}$. It follows that there are at most

$$\frac{1}{(1-q)\xi^{A\beta+B\beta'}}((\xi^{-(\beta+\beta')M})(1-q))^k$$

essential pairs. As $k \to \infty$, we get coverings of $K + e^s K'$ by intervals of arbitrarily small diameter. In the limit we can ignore the factors which do not depend on k, and we conclude that

$$\dim(K + e^s K') \le \frac{(\beta+\beta')M\log(1/\xi) + \log(1-q)}{M\log(1/\xi)} < \beta + \beta',$$

as desired. □

Proof of Theorem 5. By proceeding as in the proof of Theorem 2, we can assume that E is the attractor of an i.f.s. of the form $\{\zeta R_\theta x + d_i\}_{i=1}^n$, where θ/π is irrational, and the open set condition is satisfied. Write $f_i(x) = \zeta R_\theta x + d_i$.

We can assume without loss of generality that $f_i(B) \subset B$ for all i, where B is the unit ball. Fix some small $\varepsilon > 0$ and some large m. We can apply Proposition 7 to the family $\mathcal{Q} = \{f_u(B) : |u| = m\}$, with $\rho = \zeta^m$, $\gamma = \dim(E)$, and some A, A_1, A_2 which depend only on E. In this way, we obtain a set of 'good' angles J satisfying properties (1)–(3).

Let $\mathbf{R}$ be rotation by $m\theta$ on the circle. Because of the irrationality assumption, this is a uniquely ergodic transformation.

Fix some $\xi \in [0, \pi)$; we will use an inductive construction to show that

$$\dim(P_\xi(E)) \ge \dim(E) - \eta,$$

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where η depends on m and ε and can be made arbitrarily small. Our construction will yield a tree $\mathcal{R}$ with vertices labelled by disks, such that if $\mathcal{R}_j$ is the set of vertices at level j, then the following conditions hold:

(A) if D is the parent of D', then $D' \subset D$;

(B) if $D \in \mathcal{R}_j$, then $D = f_u(B)$ for some word u of length mj and, in particular, D has radius ζ^{jm};

(C) the family $\{P_\xi(D) : D \in \mathcal{R}_j\}$ is disjoint and ζ^{jm}-separated;

(D) all vertices of level j have the same number of children C_j and, moreover,

$$\mathbf{R}^j(\xi) \notin J \implies C_j = 1,$$
$$\mathbf{R}^j(\xi) \in J \implies C_j \geq \varepsilon\delta\zeta^{-m\gamma}.$$

Let B be the root of the tree, i.e. the only vertex of level 0. Now assume that a vertex D of level j has been defined; by property (E) we have $D = f_u(B)$, where u is a word of length jm. We consider two cases: if $\mathbf{R}^j(\xi) \notin J$, then D has just one child, $f_{u1^m}(B)$; and if $\mathbf{R}^j(\xi) \in J$, then Proposition 7 guarantees that the family $\mathcal{Q}$ has a subcollection $\{f_{v_i}(B)\}_{i=1}^M$ such that all the projections $P_{\mathbf{R}^j(\xi)}(f_{v_i}(B))$ are ζ^m-separated, and $M \geq \varepsilon\delta\zeta^{-m\gamma}$. We define the set of offspring of D to be $\{f_u \circ f_{v_i}(B)\}_{i=1}^M$.

Properties (A), (B) and (D) are immediate from the construction. To establish (C), it is enough to show that the projections of the offspring of a given $D \in \mathcal{R}_j$ are $\zeta^{(j+1)m}$-separated. Let $D = f_u(B)$; since $|u| = jm$, it follows that

$$f_u(x) = \zeta^{jm} R_{\mathbf{R}^j(0)}(x) + d_u$$

for some translation d_u. Using this, we get

$$\begin{aligned} P_\xi(f_u \circ f_{v_i}(B)) &= P_\xi(\zeta^{jm} R_{\mathbf{R}^j(0)} f_{v_i}(B) + d_u) \\ &= \zeta^{jm} P_{\mathbf{R}^j(\xi)}(f_{v_i}(B)) + P_\xi(d_u), \end{aligned}$$

which shows that (C) holds as well.

Now let

$$E_\xi = \bigcap_{j=1}^{\infty} \cup\{D : D \in \mathcal{R}_j\}.$$

By properties (A) and (B), this set is well defined and contained in $P_\xi(E)$. Let μ_ξ be the probability measure which assigns equal mass to all intervals $P_\xi(D)$ for $D \in \mathcal{R}_j$. This is well defined because of properties (C) and (D); moreover, μ_ξ is supported on E_ξ. We estimate the dimension of E_ξ by using μ_ξ as a Frostman measure, and applying Weyl's equidistribution principle to bound from above the μ_ξ-measure of projections $P_\xi(D)$. This is done exactly as in the proof of Theorem 2, so we skip the details. In the end we obtain the estimate

$$\dim(P_\xi(E)) \geq \frac{(1-\varepsilon)(\log(\varepsilon\delta) + m\gamma\log(1/\zeta))}{m\log(1/\zeta)}.$$

The right-hand side can be made arbitrarily close to γ, and this completes the proof. □

Proof of Proposition 6. We will present the proof in the case $p = 2$; the case $p = 1$ is analogous but simpler. The linear maps A_i can be decomposed as

$$A_i(x) = r_i R_{\theta_i} O_i(x),$$

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where $|r_i| < 1$, R_{θ_i} is rotation by angle θ_i, and O_i is either the identity or a reflection. Without loss of generality we can assume that E does not contain reflections, i.e. all the O_i are the identity. Indeed, if this were not the case, say O_1 is a reflection, then we can iterate the i.f.s. a large number of times and compose each of the resulting maps which contains a reflection with A_1; this gives us a new self-similar set without reflections, which is contained in E and with dimension arbitrarily close to the dimension of E.

Now let γ be the only real number such that $\sum_i r_i^\gamma = 1$. Since E satisfies the open set condition, this is also the Hausdorff dimension of E.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Consider the random walk which starts at 0 and moves from x to $x + e_i$ with probability r_i^γ. Let $\mathbb{X}_k$ be the position of this random walk after k steps. The mean of $\mathbb{X}_k$ is given by

$$\mathbb{E}\mathbb{X}_k = \sum_{i=1}^{n} k r_i^\gamma e_i.$$

Let

$$v = \sum_{i=1}^{n} (\lceil k r_i^\gamma \rceil) e_i,$$

and notice that we have $|v - \mathbb{E}\mathbb{X}_k| \le \sqrt{n}$. It follows from [**24**, Ch. II, Proposition P9] that

$$P(\mathbb{X}_k = v) \ge c(\sqrt{k})^{1-n} \tag{16}$$

for some $c > 0$ independent of k. Each of the paths which end in v has probability

$$\prod_{i=1}^{n} r_i^{\gamma v_i} \le \prod_{i=1}^{n} r_i^{\gamma k r_i^\gamma}.$$

If N_k is the number of such paths, it follows from (16) that

$$N_k \ge c(\sqrt{k})^{1-n} \prod_{i=1}^{n} r_i^{-k\gamma r_i^\gamma}. \tag{17}$$

Let

$$\rho = \prod_{i=1}^{n} r_i^{v_i} \ge \prod_{i=1}^{n} r_i \prod_{i=1}^{n} r_i^{k r_i^\gamma}. \tag{18}$$

We can identify paths on the lattice with compositions of maps in the original i.f.s. Notice that all paths ending in v are associated with a map of the form $Ax + z_i$, where the similarity ratio of A is ρ. The attractor H of the i.f.s. generated by such maps is contained in E, and its dimension τ satisfies

$$N_k \rho^\tau = 1 \implies \tau = \frac{\log(N_k)}{\log(1/\rho)}. \tag{19}$$

From (17), (18) and (19) we obtain

$$\tau \ge \frac{\log c + (1-n)\log\sqrt{k} + k\gamma \sum_{i=1}^{n} r_i^\gamma \log(1/r_i)}{\sum_{i=1}^{n} \log(1/r_i) + k \sum_{i=1}^{n} r_i^\gamma \log(1/r_i)}.$$

Since the rightmost terms dominate in both numerator and denominator, it follows that τ can be made arbitrarily close to γ by taking k large enough, completing the proof. □

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5. *Concluding remarks and open questions*

(1) *Sums of graph-directed attractors*. Theorem 2 can be further generalized. The sets K and K' can be attractors of more general graph-directed systems. In order to see this, one can either observe that the proof extends to this more general situation with minor variations, or just use the fact that attractors of graph-directed systems contain self-similar sets of arbitrarily close dimension.

(2) *Resonance as an equivalence relation*. Let $\mathcal{I}$ denote the set of all self-similar i.f.s. satisfying the open set condition and such that the attractor has dimension at most $1/2$. Algebraic resonance is clearly an equivalence relation on $\mathcal{I}$; it follows from our results that so is geometrical resonance. For more general compact sets there is no clear way to define resonance algebraically, but we can say that two compact sets K, K' of Hausdorff dimension at most $1/2$ are geometrically resonant if there is $s > 0$ such that

$$\dim(K + sK') < \dim(K) + \dim(K').$$

It turns out, however, that this is not an equivalence relation. To see this, fix two very close numbers $a < b$ such that C_a and C_b each have dimension less than $1/2$; from the proof of Theorem 4 it further follows that we can ensure that the dimension drop in $C_a + C_a$ and $C_b + C_b$ is at least 4η, where $\eta = \dim(C_b) - \dim(C_a)$. Construct a compact set K in the following way. Pick a rapidly increasing sequence n_j. Follow the construction of C_a for n_1 steps, then the construction of C_b for n_2 steps, and so on. If this is done carefully, then one can ensure that K has dimension $\dim(C_a)$ and

$$\max(\dim(K + C_a), \dim(K + C_b)) < 2\dim(C_b) - 2\eta,$$

whence K is resonant to both C_a and C_b, yet C_a and C_b need not be resonant.

(3) *Uncountably many resonances*. We can rephrase (2) in Theorem 3 as saying that a compact set cannot be resonant to a positive-measure set of central Cantor sets. In light of the results of this paper, a natural question is whether an arbitrary compact set is resonant to at most countably many of the C_a. The answer, however, turns out to be negative. We sketch the construction of a compact set resonant to uncountably many central Cantor sets.

First of all, let us say that two compact sets K, K' are γ-resonant at scale ρ if

$$\mathcal{L}(K_\rho + K'_\rho) \leq \rho^{-\gamma},$$

where K_ρ, K'_ρ denote the ρ-neighborhoods of K, K'. Pick $a_1 > a_2$ very close to each other so that, for some $\eta > 0$, we have:

(i) $\dim(C_{a_i}) < 1/2$;

(ii) $\dim(C_{a_i}) \in B_{\varepsilon/10}(\eta)$;

(iii) $2\dim(C_{a_i}) - \dim(C_{a_i} + C_{a_i}) > 10\varepsilon$ for $i = 1, 2$.

We construct the desired compact set K as follows (this is different from the set constructed in the previous remark). Starting from the unit interval, follow the construction of C_{a_1} for n_1 steps, so that at scale $\rho_1 = a_1^{n_1}$ the sets K and C_{a_1} are $(2\eta - 2\varepsilon)$-resonant. Observe that there exists a small interval I_1 around a_1 such that K and C_a are $(2\eta - \varepsilon)$-resonant at scale ρ_1 for all $a \in I_1$; we can assume that $a_2 \notin I_1$. Pick any $a_{11} < a_{12} < a_1$ in I_1.

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Next, we follow the construction of C_{a_2} for n_2 steps, with $n_2 \gg n_1$. In this way we can find a $\rho_2 \ll \rho_1$ such that C_{a_2} and K are $(2\eta - \varepsilon)$-resonant at scale ρ_2. We find an interval I_2 around a_2, disjoint from I_1, and points $a_{21} > a_{22} > a_2$ as above.
We repeat this process for a_{ij} to find intervals I_{ij} and points $a_{ijk} \in I_{ij}$ such that at some very small scales ρ_{ij}, the sets K and $C_{a_{ij}}$ are $(2\eta - \varepsilon)$-resonant. We continue this construction inductively. Then if we let

$$\mathcal{A} = \bigcap_{k=1}^{\infty} \bigcup_{i_1,\ldots,i_k \in \{1,2\}} I_{i_1 \ldots i_k},$$

it follows from the construction that if $a \in \mathcal{A}$, then K and C_a are $(2\eta - \varepsilon)$-resonant at arbitrarily small scales. This implies that $C_a + K$ has dimension at most $2\eta - \varepsilon$. Also, $\dim(K) \geq \eta - \varepsilon/10$. We conclude that

$$\begin{aligned} \dim(C_a + K) &< (\eta - \varepsilon/10) + (\eta - \varepsilon/10) - 4\varepsilon/5 \\ &< \dim(C_a) + \dim(K) - \varepsilon/2. \end{aligned}$$

The uncountable set of resonances in this example is of dimension zero. We do not know if there exists a compact set K which is geometrically resonant to the central Cantor set C_a for a set of parameters a of positive dimension.

(4) *The case* $\dim(C_a) + \dim(C_b) > 1$. In this paper we focused on the case where the sum of the dimensions is at most one. We do prove that if $\dim(C_a) + \dim(C_b) > 1$ and $\log b / \log a$ is irrational, then $\dim(C_a + C_b) = 1$. However, in light of Marstrand's projection theorem and Theorem 3(ii), it is natural to conjecture that $C_a + C_b$ actually has positive measure.
By analogy with the general results on intersections with lines (see [**12**, Ch. 10]) and [**19**, Theorem 1.2], we also conjecture that under the irrationality assumption, for all $\theta \in (0, \pi)\backslash\{\pi/2\}$ there is a set of positive measure of lines with direction θ which intersect $C_a \times C_b$ in a set of dimension $\dim(C_a) + \dim(C_b) - 1$. Note that this would imply that $C_a + C_b$ has positive measure.

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