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Adding propositional connectives to countable infinitary logic

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Abstract. For countable admissible α, one can add a new infinitary propositional connective to $\mathscr{L}_\alpha$ so that the extended language obeys the Barwise compactness theorem, and the set of valid sentences is complete α-r.e.

Aside from obeying the compactness theorem and a completeness theorem, ordinary finitary predicate calculus is also truth-functionally complete.

In (1), Barwise shows that for countable admissible A, $\mathscr{L}_A$ provides a fragment of $\mathscr{L}_{\omega_1\omega}$ which obeys a compactness theorem and a completeness theorem. However, we of course lose truth-functional completeness, with respect to infinitary propositional connectives that operate on infinite sequences of propositional variables. This raises the question of studying extensions of the language $\mathscr{L}_A$ obtained by adding infinitary propositional connectives, in connexion with the Barwise compactness and completeness theorems, and other metatheorems, proved for $\mathscr{L}_A$. Some aspects of this project are proposed in (3).‡ It is the purpose of this paper to answer a few of the more basic questions which arise in this connexion.

We have not attempted to study the preservation of interpolation or implicit definability. This could be quite interesting if done systematically.

The formulae of $\mathscr{L}^+_{\omega_1\omega}$ are given by (1) all atomic formulae of $\mathscr{L}_{\omega_1\omega}$ are formulae of $\mathscr{L}^+_{\omega_1\omega}$, (2) if ϕ is a formula of $\mathscr{L}^+_{\omega_1\omega}$ then so are $\sim\phi$, $(\forall x)\,\phi$, (3) if X is a countable set of formulae of $\mathscr{L}^+_{\omega_1\omega}$ (with finitely many free variables altogether) then $\&(X)$ is a formula

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‡ In (3), p. 147, it is erroneously claimed that there is no infinitary propositional connective which, when added to $\mathscr{L}_\alpha$ for the first admissible α, results in a new language, whose set of valid sentences is α-r.e. Theorem 1 gives a strong negative answer to the conjecture on (3), p. 147.

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of $\mathscr{L}^+_{\omega_1\omega}$ (4) if $(\phi_0, \phi_1, \ldots)$ is a sequence of formulae (with finitely many free variables altogether) then $S(\phi_0, \phi_1, \ldots)$ is a formula of $\mathscr{L}^+_{\omega_1\omega}$.

For admissible A, take the formulae of $\mathscr{L}^+_A$ to be those formulae of $\mathscr{L}^+_{\omega_1\omega}$ which are encoded by elements of A. By convention, we assume that every admissible set contains the element ω.

An infinitary propositional connective, ρ, will be taken to be a subset of $\mathscr{P}(\omega)$. An infinitary propositional connective gives rise to a semantics for the language $\mathscr{L}^+_{\omega_1\omega}$. The semantics for $\mathscr{L}^+_{\omega_1\omega}$ will be the same as for $\mathscr{L}_{\omega_1\omega}$, except for the additional clause: $M \models S(\phi_0, \phi_1, \ldots)[f]$ if and only if $\{n: M \models \phi_n[f]\} \in \rho$. For $x \subset \omega$, let $\bar{x}$ be the set of all subsets of ω that are finitely different from x.

LEMMA 1. *Let (A, R) be a countable admissible pair. Then there are countable admissible $(B, R), x \in (B - A) \cap \mathscr{P}(\omega)$ such that the intersection with A of every $B - \Sigma_1$ predicate with parameter $\bar{x}$, is A-r.e. Furthermore, $\mathscr{L}^+_A$ (using the $\bar{x}$ semantics) and $\mathscr{L}_A$ are not semantically equivalent.*

Proof. Take $x \subset \omega$ to be Cohen generic with respect to all statements, limited and unlimited, in the forcing language appropriate for adding a subset of ω to A, where a binary relation symbol for R is used. Take B to be the set of all values of the forcing terms, using x. Verify that (B, R) is admissible, using the admissibility of (A, R). Then apply the homogeneity of Cohen forcing, together with the fact that forcing for Σ_1 statements is Σ_1. To see that $\mathscr{L}^+_A$ is stronger than $\mathscr{L}_A$, it is enough to show that $S(p_0, p_1, \ldots)$ is not equivalent to any sentence ϕ in $\mathscr{L}_A$, where the p_i are 0-ary predicate symbols. This is clear, since the class of all $y \subset \omega$ such that ϕ in $\mathscr{L}_A$ comes out true under the assignment $p_i = T$ if $i \in y$; F if $i \notin y$, is the set of solutions to some limited statement in the forcing language for A.

Following terminology of Barwise and Moschovakis, a resolved admissible set is an admissible pair (A, F), where $F: On \to A$, $\cup \operatorname{Rng}(f) = A$; F is identified with its characteristic relation. An admissible set A is resolvable just in case (A, F) is a resolved admissible set, for some A.

THEOREM 1. *Let A be a countable admissible set. Then there is an infinitary propositional connective ρ such that* (1) *$\mathscr{L}^+_A$ (using the ρ semantics) is not sematically equivalent to $\mathscr{L}_A$* (2) *the set of valid sentences of $\mathscr{L}^+_A$ (using the ρ semantics) is A-r.e. If in addition A is resolvable, then we may add* (3) *every inconsistent A-r.e. set of sentences of $\mathscr{L}^+_A$ (using the ρ semantics) has an A-finite inconsistent subset.*

Proof. Apply Lemma 1 to (A, R), where R is the empty relation. Set $\rho = \bar{x}$. Clause (1) is by Lemma 1. Since the set of valid sentences of $\mathscr{L}_B$ is $B - \Sigma_1$, clearly the set of valid sentences of $\mathscr{L}^+_B$ (with the ρ semantics) is $B - \Sigma_1$ with parameter $\bar{x}$. Hence the valid sentences of $\mathscr{L}^+_A$ (using the ρ semantics) is A-r.e.

Now let (A, F) be a resolved admissible set, and apply Lemma 1 to (A, F). Let K be an A-r.e. inconsistent set of sentences of $\mathscr{L}^+_A$, using the $\bar{x}$ semantics. Obviously K may be viewed as a (B, F)-r.e. set of sentences of $\mathscr{L}_B$. By applying the compactness theorem for $\mathscr{L}_B$, and transferring back to K, we have an inconsistent B-finite $K_0 \subset K$.

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The admissibility of (B, F) produces an ordinal $\alpha \in A$ such that $K_0 \subset \bigcup_{\beta < \alpha} F(\beta)$. Clearly $\bigcup_{\beta < \alpha} F(\beta)$ is the desired A-finite subset of K.

For each set x, let x^+ be the least admissible set containing x as an element.

THEOREM 2. *Let A be a countable admissible set. The following are equivalent*: (*a*) *validity for sentences in $\mathscr{L}_A$ is A-recursive*, (*b*) $x \in A \to x^+ \in A$, (*c*) *there is an infinitary propositional connective ρ such that validity for sentences of $\mathscr{L}_A^+$ (using the ρ semantics) is A-recursive, and $\mathscr{L}_A^+$ (using the ρ semantics) is not semantically equivalent to $\mathscr{L}_A$. Furthermore, if A is in addition resolvable, and validity for sentences of $\mathscr{L}_A$ is A-recursive, then there is an infinitary propositional connective ρ such that* (*c*) *above and* (3) *of Theorem* 1 *hold.*

Proof. (*a*) $\to$ (*b*). Assume $x \in A$, $x^+ \notin A$. We may assume x is transitive. By known techniques, every element of x^+ is, in x^+, the unique solution to some existential predicate involving finitely many parameters from $TC(\{x\})$. For each $a \in TC(\{x\})$, we will use the constant c_a. Let ϕ be the sentence of $\mathscr{L}_A$ consisting of the conjunction of the KP axioms together with $(\forall x)(x \in c_a \leftrightarrow \bigvee_{b \in a} x = c_b)$, for $a \in TC(\{x\})$. Every model of ϕ must be an end extension of x^+, and so the set K of existential consequences (involving parameters from $TC(\{x\})$) must be the same as the set of true existential statements (involving parameters from $x \cup \{x\}$) in x^+. Now clearly K must not be A-recursive, for if it were, then it would be A-finite. This in turn is impossible, since using K, we could code up x^+, and obtain x^+ in A. However, K will be A-recursive if the set of consequences in $\mathscr{L}_A$ of ϕ is A-recursive. Hence validity for $\mathscr{L}_A$ is not A-recursive.

(*b*) $\to$ (*c*). The proof follows the proof of Theorem 1. In Lemma 1, if A obeys (*b*), then the B defined there will also obey (*b*), and so in the proof of Theorem 1, validity for sentences of $\mathscr{L}_B$ will be not only $B - \Sigma_1$, but also $B - \Pi_1$.

(*c*) $\to$ (*a*) is trivial. The rest of the Theorem comes from the proof of the second part of Theorem 1. We are done.

Now fix A to be a countable admissible set, $A \cap On = \alpha$. Let $\alpha \cdot (1 + Q)$ be the linear ordering consisting of all pairs (β, q), where $\beta < \alpha$, and q is a rational $\geqslant 0$, ordered first by q and second by β. We now describe '$\alpha \cdot (1 + Q)$-forcing over A'.

The generic symbol will be b, which stands for a subset of ω^2. The forcing conditions are finite one-one maps $p: \omega \to \alpha \cdot (1 + Q)$. We say $p \Vdash \langle \overline{n}, \overline{m} \rangle \in b$ just in case $q(m) \leqslant q(n)$ in $\alpha \cdot (1 + Q)$, for no condition $q \supset p$. The forcing language is the usual one for adjoining a subset of ω^2 to A, and the inductive clauses for this forcing are standard. For $\alpha \cdot (1 + Q)$-forcing, we will assume all forcing statements are limited.

Our first aim is to show that $\{\psi : \phi \Vdash \psi\}$ is A-r.e., for $\alpha \cdot (1 + Q)$-forcing over A. To this end, let (B, E) be any model of ZFC, possibly nonstandard. We now describe 'modified Levy forcing over (B, E)'.

This forcing motion is defined within (B, E). The generic symbol is again b, for a subset of ω^2, and the conditions are finite one-one maps $p: \omega \to \omega_1$. We say $p \Vdash \langle \overline{n}, \overline{m} \rangle \in b$ just in case $q(m) \leqslant q(n)$ for no condition $q \supset p$. The forcing language is the usual one for

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adjoining a subset of ω^2 to (B, E), and the inductive clauses are standard. For modified Levy forcing, we will allow both limited and unlimited statements.

LEMMA 2. *Suppose* $(B, E) \models ZFC$, (B, E) *is an end extension of* A, *the ordinal of the standard part of* (B, E) *is* α, *and the* ω_1 *of* (B, E) *is nonstandard. Then* $\phi \Vdash \psi$ *with* $\alpha \cdot (1+Q)$*-forcing over* A, *if and only if* $(B, E) \models$ "$\phi \Vdash \psi$", *with modified Levy forcing over* (B, E), *for any forcing statement* ψ *in* $\alpha \cdot (1+Q)$*-forcing over* A.

Proof. By Theorem 3·2 of (2), if B is countable, then the order type of the ordinals of (B, E) less than the ω_1 of (B, E), must be that of $\alpha \cdot (1+Q)$. Therefore the two forcing notions are isomorphic, as far as limited statements are concerned. The uncountable case is derived from the countable case by a Skolem hull argument.

LEMMA 3. *Lemma 2 holds without the hypothesis that the ordinal of the standard part of* (B, E) *is* α.

Proof. Suppose this is false. It is readily seen that this would give a set of models of ZFC, the ordinal of the standard part of each being $> \alpha$, which is the set of models of some A-recursive theory in $\mathscr{L}_A$. This contradicts Theorem 2·2 of (2).

LEMMA 4. *The relation* $\phi \Vdash \psi$, *for* $\alpha \cdot (1+Q)$*-forcing over* A, *is* A-r.e.

Proof. This is clear from Lemma 3, using the fact that the set of consequences of an A-recursive theory in $\mathscr{L}_A$, is A-r.e.

Take an $\alpha \cdot (1+Q)$-generic function to be a map $f: \omega \to \alpha \cdot (1+Q)$ such that every forcing statement is decided by some condition included in f. An $\alpha \cdot (1+Q)$-generic object is a $b \subset \omega^2$ such that $b = \{\langle n, m\rangle : f(n) < f(m) \text{ in } \alpha \cdot (1+Q)\}$, for some $\alpha \cdot (1+Q)$-generic function f. $A[b]$ is the collection of values of all forcing terms, using the generic object $b \subset \omega^2$.

It is clear that every $\alpha \cdot (1+Q)$-generic b is a linear ordering with field ω, of type $\alpha \cdot (1+Q)$. For linear orderings b on ω, let $\hat{b}$ be the set of all linear orderings which are isomorphic to b via an isomorphism which moves at most finitely many numbers.

LEMMA 5. *Let* b *be an* $\alpha \cdot (1+Q)$*-generic object over* A. *Then the intersection with* A *of every* $A[b]$-Σ_1 *predicate with parameter* $\hat{b}$, *is* A-r.e.

Proof. By the homogeneity of $\alpha \cdot (1+Q)$-forcing, and that forcing with the ϕ condition is A-r.e.

LEMMA 6. *Let* (A, F) *be a resolved admissible set, with* A *countable. Then there is a countable* $(B, E) \models ZFC$, (B, E) *an end extension of* A, *the ordinal of the standard part of* (B, E) *is* α, *and for some object* $g \in B$ *we have* $(B, E) \models$ 'g *is a function and* $g(\bar{\beta}) = \overline{F(\beta)}$', *for each* $\beta > \alpha$.

Proof. Let (A, F) be a resolved admissible set. Consider the constants c and c_a, for each $a \in A$. Let T be the axioms (1) ZFC, (2) $(\forall x)\,(x \in c_a \leftrightarrow \bigvee_{y \in a} x = c_y)$, (3) c is a function, (4) $c(c_\beta) = c_{F(\beta)}$, for $\beta < \alpha$. Then apply (the obvious generalization of) Theorem 2·2 in (2), (together with a Skolem hull argument to insure countability of B).

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LEMMA 7. *Let (A, F) be a resolved admissible set, with A countable. Then $(A[b], F)$ is admissible, for some $\alpha \cdot (1+Q)$-generic object b over A.*

Proof. Let (B, E) and g be as in Lemma 6. Let f be a Levy generic function over (B, E). Then $(B, E)[f]$ is a model of ZFC, and $b = \{\langle n, m\rangle : f(n)\, Ef(m)\}$ is an $\alpha \cdot (1+Q)$-generic object (over A). We claim that $(A[b], F)$ is admissible. The hard part to verify is collection. Let $a \in A[b]$, and suppose $(\forall x \in a)(\exists y \in A[b])(\phi(x, y))$, where ϕ is Δ_0, and can mention parameters from A, and F. Now we can rewrite this as

$$(\forall x \in a)(\exists \langle \beta, z\rangle \in A)(\exists y \in L_\beta(b, z))(\phi(x, y)).$$

Replace all occurrences of A in ϕ with $\mathrm{Rng}(g)$, and F in ϕ with g, and view this as a statement in (B, E): $(\forall x \in a)(\exists \langle \beta, z\rangle)(\exists y \in L_\beta(b, z))(\phi^*(x, y))$. Within (B, E), we can define the function $h : y \to On$ given by $h(x) = (\mu\gamma)(\exists \langle \beta, z\rangle \in g(\gamma))(\exists y \in L_\beta(b, z))(\phi^*(x, y))$. The range of h in (B, E) is clearly a subset of α. Since the ordinal of the standard part of (B, E) is α, clearly h is bounded by some $\delta < \alpha$. Let $w = \cup\{L_\beta(b, z) : \langle \beta, z\rangle \in \bigcup_{\beta<\delta} F(\beta)\}$. It is easily seen that $w \in A[b]$, and that $(\forall x \in a)(\exists y \in z)(\phi(x, y))$.

LEMMA 8. *Let (A, F) be a resolved admissible set, with A countable. Then there is an admissible (B, F) and a linear ordering $b \in B$ on ω of type $\alpha \cdot (1+Q)$, such that the intersection with A of every $B - \Sigma_1$ predicate with parameter $\hat{b}$, is A-r.e. Furthermore, $\mathscr{L}_A^+$ (with the $\hat{b}$ semantics) is not equivalent to $\mathscr{L}_A$.*

Proof. By Lemmas 5 and 7. The last sentence is proved in a similar way as the last sentence of Lemma 1.

Call an admissible set C projectible just in case there is a one-one C-recursive function from C into an element of C.

THEOREM 3. *Let A be a countable resolvable admissible set which has a projectible admissible subset $L_\alpha(x)$, for some $x \in A$, $\alpha = A \cap On$. Or, alternatively let A be of the form $A = L_\alpha(x)$. Then there is an infinitary propositional connective ρ such that (1) $\mathscr{L}_A^+$ (using the ρ semantics) is not semantically equivalent to $\mathscr{L}_A$, (2) the set of valid sentences of $\mathscr{L}_A^+$ (using the ρ semantics) is A-r.e., but not A-recursive, (3) every inconsistent A-r.e. set of sentences of $\mathscr{L}_A^+$ (using the ρ semantics) has an A-finite inconsistent subset.*

Proof. Let A, F, B, b be as in Lemma 8. Set $\rho = \hat{b}$. (1) is by Lemma 8. (3), and the first part of (2), are as in the proof of Theorem 1. We have only to prove the second part of (2).

First assume $L_\alpha(x) \subset A$, $A \cap On = \alpha$, $x \in A$, $L_\alpha(x)$ projectible. Let y be the range of some one-one $L_\alpha(x)$-recursive function on $L_\alpha(x)$, $y \subset d$, $d \in L_\alpha(x)$. Let

$$y = \{a : P(a, z_1, \ldots, z_k) \quad \text{holds in} \quad L_\alpha(x)\},$$

where $z_1, \ldots, z_k \in L_\alpha(x)$, P is Σ_1. We use constants c_a, for each $a \in TC(\{x, z_1, \ldots, z_k, d, \omega\})$. Let ϕ be the conjunction of KP with $(\exists z)(S(c_0 \in z, c_1 \in z, c_2 \in z, \ldots))$, and the axioms $(\forall z)(z \in c_a \leftrightarrow \bigvee_{e \in a} z = c_e)$, for $a \in TC(\{x, z_1, \ldots, z_k, d, \omega\})$. Under the $\hat{b}$ semantics, each model of ϕ must be an end extension of $L_\alpha(x)$, and $A[\hat{b}]$ is a model of ϕ. Hence $y = \{a \in d : \phi$ semantically implies "$P(c_a, c_{z_1}, \ldots, c_{z_k})$ holds in $L(c_x)$"$\}$. If validity for $\mathscr{L}_A^+$ is A-recursive, then y would be an A-recursive subset of d, and hence A-finite. This would

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violate the admissibility of A, since y is the range of some one-one partial A-recursive function with domain $L_\alpha(x)$. Hence validity for $\mathscr{L}_A^+$ is not A-recursive.

Secondly, assume that $A = L_\alpha(x)$. Let $K = \{a \in L_\alpha(x)\colon P(a, z_1, \ldots, z_n)\}$ be an A-r.e. but not A-recursive, subset of A, where $z_1, \ldots, z_n \in A$, and P is existential. Add a constant c_a for each $a \in TC(\{x, z_1, \ldots, z_n, \omega\})$, and let ϕ be KP together with

$$(\exists z)\,(S(c_0 \in z, c_1 \in z, c_2 \in z, \ldots)),$$

and the axioms $(\forall z)\,(z \in C_a \leftrightarrow \bigvee_{e \in a} z = c_e)$. For each $u \in A$, let $\psi(u)$ be the conjunction of all sentences $(\forall z)\ (z \in c_y \leftrightarrow \bigvee_{w \in y} z = c_w)$, for $y \in TC(\{u\})$. Then $u \in K$ if and only if $\phi \,\&\, \psi(u)$ semantically implies '$P(c_u, c_{z_1}, \ldots, c_{z_n})$ holds in $L(c_x)$'. If validity in $\mathscr{L}_A^+$ were A-recursive, then K would be an A-recursive subset of A.

We remark that, by the proof of Theorem 3, we can furthermore insist that validity for $\mathscr{L}_A^+$ (using the ρ semantics) is complete A-r.e., in case $A = L_\alpha(x)$.†

REFERENCES

(1) BARWISE, J. Infinitary logic and admissible sets. *J. Symbolic Logic*, **34** (1969), 226–252.

(2) FRIEDMAN, H. *Countable models of set theories*. Lecture Notes in Mathematics, v. 337 (1973), 539–573.

(3) KREISEL, G. *Choice of infinitary languages by means of definability criteria; generalized recursion theory*, Lecture Notes in Mathematics, v. 72 (1968), 139–151.

† This is related to the problem posed in (3), p. 147.

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