Ergod. Th. & Dynam. Sys. (2016), **36**, 1839–1850 © *Cambridge University Press, 2015*
doi:10.1017/etds.2014.143

Positive area and inaccessible fixed points for hedgehogs

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(*Received* 13 *March* 2014 *and accepted in revised form* 8 *November* 2014)

Abstract. Let f be a germ of holomorphic diffeomorphism with an irrationally indifferent fixed point at the origin in $\mathbb{C}$ (i.e. $f(0) = 0$, $f'(0) = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} - \mathbb{Q}$). Perez-Marco showed the existence of a unique family of non-trivial invariant full continua containing the fixed point called Siegel compacta. When f is non-linearizable (i.e. not holomorphically conjugate to the rigid rotation $R_\alpha(z) = e^{2\pi i\alpha}z$) the invariant compacts obtained are called hedgehogs. Perez-Marco developed techniques for the construction of examples of non-linearizable germs [Fixed points and circle maps. *Acta Math.* **179**(2) (1997), 243–294]; these were used by the author to construct hedgehogs of Hausdorff dimension one, and adapted by Cheritat [Relatively compact Siegel disks with non-locally connected boundaries, *Preprint*, 2009] to construct Siegel disks with pseudo-circle boundaries. We use these techniques to construct hedgehogs of positive area and hedgehogs with inaccessible fixed points.

1. *Introduction*

A germ $f(z) = e^{2\pi i\alpha}z + O(z^2)$, $\alpha \in \mathbb{R} - \mathbb{Q}/\mathbb{Z}$ of holomorphic diffeomorphism fixing the origin in $\mathbb{C}$ is said to be linearizable if it is analytically conjugate to the rigid rotation $R_\alpha(z) = e^{2\pi i\alpha}z$. The number α is called the rotation number of f, and the maximal domain of linearization is called the Siegel disk of f. The linearizability of f is dependent on the arithmetic of α, and the optimal arithmetic condition for linearizability is known in this setting (see [**Sie42**, **Brj71**, **Yoc95**]).

In this article we are primarily concerned with non-linearizable germs. Perez-Marco studied the dynamics of such germs, proving the existence of non-trivial invariant full continua containing the fixed point called *hedgehogs* [**PM97**]. These have empty interior and are not locally connected at any point except perhaps the fixed point [**PM94**, **PM96**]. In [**PM93**, **PM95**, **PM00**], Perez-Marco developed techniques using 'tube-log Riemann surfaces' for the construction of interesting examples of indifferent dynamics. These were used by the author to construct examples of hedgehogs containing smooth combs and

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hedgehogs of minimal Hausdorff dimension one [**Bis05**, **Bis08**], and further developed by Cheritat to construct Siegel disks with pseudo-circle boundaries [**Che11**].

We use these techniques, incorporating Cheritat's modifications, to construct hedgehogs of positive area.

THEOREM 1.1. *There exists a non-linearizable germ with a hedgehog of positive area.*

As in the previous constructions of hedgehogs, the non-linearizable germ is obtained as a limit of a sequence of finite order germs which have decreasing linearization domains nesting down to a hedgehog of the limiting germ. Thanks to the modified construction introduced by Cheritat, these linearization domains can be constructed such that the loss of area at each step is as small as desired, ensuring that the intersection has positive area. While the above construction gives univalent maps with hedgehogs of positive area, an interesting question is whether this can be achieved in the setting of entire functions, in particular polynomials. For quadratic polynomials with a non-linearizable irrationally indifferent fixed point (or *Cremer point*), the hedgehogs belong to the Julia set, and Buff and Cheritat have constructed such Cremer quadratic polynomials with positive area Julia sets [**BC12**]; it is unclear, however, whether the corresponding hedgehogs have positive area.

In [**PM94**], Perez-Marco posed the question of whether the fixed point of a hedgehog is always accessible from its complement. The answer is negative.

THEOREM 1.2. *There exists a non-linearizable germ f with a hedgehog K such that the fixed point is inaccessible from $\hat{\mathbb{C}} - K$.*

The above example is constructed by controlling the geometry of the complements of the decreasing linearization domains in such a way that the fixed point becomes inaccessible from the union of these complementary domains.

The general scheme of the construction is described in §2, and then applied to construct examples in §3.

2. *Outline of the construction*

Let $E(z) = e^{2\pi i z}$ be the universal covering $E : \mathbb{C} \to \mathbb{C}^*$ with deck transformation $T(z) = z + 1$. We will denote by $\mathbb{D}_R$ the disk $\{|z| < R\}$, and by $\mathbb{H}_M$ the upper half-plane $\{\mathrm{Im}\, z > M\}$. For $\alpha > 0$ we let $M_\alpha(z) = \alpha z$. Let $\mathcal{S}_\alpha$ denote the set of univalent maps f on the unit disk $\mathbb{D}$ such that $f(0) = 0$, $f'(0) = e^{2\pi i \alpha}$, $\alpha \in \mathbb{R}$, and $\hat{\mathcal{S}}_\alpha$ be the set of lifts F of maps in $\mathcal{S}_\alpha$ to $\mathbb{H}$, satisfying $F(z) = z + \alpha + \phi(z)$, where ϕ is 1-periodic and tends to zero as $\mathrm{Im}\, z \to +\infty$. We will refer to the T-invariant pre-images by E of Jordan domains in $\mathbb{C}$ containing the origin as 1-periodic domains. By *linearization domain (respectively, hedgehog)* we will refer both to a linearization domain (respectively, hedgehog) $K \subset \mathbb{D}$ for f and to the T-invariant, F-invariant set $E^{-1}(K) \subset \mathbb{H}$. By *linearizable maps* we will refer both to maps $f \in \mathcal{S}_\alpha$ conjugate to rotations and to their lifts $F \in \hat{\mathcal{S}}_\alpha$. A *semi-flow* $\mathcal{F}$ of maps will refer to a commuting collection of maps $(f_t)_{t\in A}$ such that $f_t \in \mathcal{S}_t$ (or maps $(F_t)_{t\in A}$, $F_t \in \hat{\mathcal{S}}_t$) for some $A \subset \mathbb{R}$; $\mathcal{F}$ will be called linearizable if all maps in $\mathcal{F}$ are linearizable.

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2.1. *Approximation by linearization domains.* The key to the construction is the following approximation lemma, due to Cheritat [**Che11**], allowing one to approximate any Jordan domain by an invariant domain for a vector field such that the elements of the flow for small times extend univalently to a given large disk. We let Y be the holomorphic vector field $+1(\partial/\partial z)$ on $\mathbb{C}$. Any holomorphic map f with non-vanishing derivative on a domain U defines a non-vanishing holomorphic vector field f^*Y on U pulling back Y by f.

PROPOSITION 2.1. *Let $D \subset \mathbb{C}$ be a Jordan domain containing the origin, $\Lambda = E^{-1}(D)$ a 1-periodic domain, and ϵ, $M > 0$. There exists $\delta = \delta(\Lambda, \epsilon, M) > 0$, an entire function $\Psi = \Psi(\Lambda, \epsilon, M)$ with non-vanishing derivative of the form $\Psi(z) = z - iC + \psi(z)$ where $C > 0$, ψ is 1-periodic, $\psi(z) \to 0$ as $\operatorname{Im} z \to +\infty$, and a 1-periodic domain $\Omega = \Omega(\Lambda, \epsilon, M) \subset \Lambda$ such that the following hold for the vector field $X = \Psi^*Y$:*

Ψ maps Ω univalently to the upper half-plane $\mathbb{H}$, $\partial\Omega$ is contained in the ϵ-neighbourhood of $\partial\Lambda$, and for $|t| < \delta$, the time-t flow maps $F_t = \exp(tX)$ are univalent on $\mathbb{H}_{-M}$ and leave Ω invariant.

Proof. Let $\phi : \overline{D} \to \overline{\mathbb{D}}$ be the normalized Riemann mapping such that $\phi(0) = 0$, $\phi'(0) > 0$. Let $\Phi : \overline{\Lambda} \to \overline{\mathbb{H}}$ be a lift of ϕ to Λ so that Φ commutes with T and $\Phi(z) = z + \chi(z)$ where χ is 1-periodic. Then Φ' is non-zero and 1-periodic, and hence can be written in the form $\Phi' = e^G$ where G is 1-periodic, holomorphic on Λ, and tends to zero as $\operatorname{Im} z \to +\infty$, and is therefore of the form $G = g \circ E$ for some g holomorphic on D with $g(0) = 0$.

Fix $\kappa > 0$. By Runge's theorem we can approximate g uniformly by a polynomial P on any given compact subdomain of D such that $P(0) = 0$; it follows easily that we can approximate Φ uniformly on $\Lambda_\kappa = \{\operatorname{Im} \Phi(z) \geq \kappa\}$ by an entire function Ψ_1 commuting with T with nowhere-vanishing derivative $\Psi_1'(z) = e^{P \circ E(z)}$ on $\mathbb{C}$, such that $\Psi_1(z) = z + \psi(z)$, where ψ is 1-periodic. Choosing the approximation close enough, we may assume that Ψ_1 is univalent on Λ_κ and $\Psi_1(\Lambda_{3\kappa}) \subset \Phi(\Lambda_{2\kappa}) \subset \Psi_1(\Lambda_\kappa)$ so that, letting $\Omega = \{z \in \Lambda : \operatorname{Im} \Psi_1(z) > 2\kappa\}$, we have $\Lambda_{3\kappa} \subset \Omega \subset \Lambda_\kappa$. Let $\Psi = \Psi_1 - 2i\kappa$; then Ψ maps Ω univalently onto $\mathbb{H}$.

The pull-back $X = \Psi^*Y$ is an entire non-zero 1-periodic vector field. Hence there exists $\delta > 0$ such that the maps $F_t = \exp(tX)$, $|t| < \delta$, are defined and univalent on the upper half-plane $\mathbb{H}_{-M}$. Since the flow of Y leaves upper half-planes invariant, the maps $(F_t)_{|t|<\delta}$ leave the domain Ω invariant. Taking κ small enough initially, we may ensure that $\partial\Omega$ is within the ϵ-neighbourhood of $\partial\Lambda$. □

2.2. *Iterated inverse renormalization.* Fix a strictly decreasing sequence of constants $M_n > 1$, $n \geq 0$. Let $\Lambda_n \subset \mathbb{H}$, $n \geq 0$ be a given sequence of 1-periodic domains, and $\epsilon_n > 0$ a given sequence of positive reals. We then construct inductively a sequence of corresponding semi-flows $\mathcal{F}_n$; the flexibility of the construction derives from the fact that, in practice, the domains Λ_n can be chosen appropriately to obtain examples with specified properties.

We inductively construct the following:

- a sequence $(\Psi_n)_{n\geq 0}$ of entire functions with non-vanishing derivatives of the form $\Psi_n(z) = z - iC_n + \psi_n(z)$, where $C_n > 0$ and ψ_n is 1-periodic and tends to zero as $\operatorname{Im} z \to +\infty$;

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- a sequence of 1-periodic domains $(\Omega_n)_{n\geq 0}$ such that Ψ_n maps Ω_n univalently onto $\mathbb{H}$;
- a sequence $\delta_n > 0$, $n \geq 0$;
- a sequence of integers $a_n \geq 3$, $n \geq 0$;
- finite subsets $A_n \subset \mathbb{Q}$ and linearizable semi-flows $\mathcal{F}_n = (F_{n,t})_{t\in A_n}$ for $n \geq 0$ such that the sets A_n are increasing and the semi-flows $\mathcal{F}_n$ are univalent on $\mathbb{H}_{-M_n}$; and
- a decreasing sequence of 1-periodic domains $D_n \subset \mathbb{H}$, $n \geq 0$ such that D_n is invariant under $\mathcal{F}_n$.

At stage $n = 0$, given Λ_0, ϵ_0, M_0, we let $\Psi_0 = \Psi(\Lambda_0, \epsilon_0, M_0)$, $\Omega_0 = \Omega(\Lambda_0, \epsilon_0, M_0)$, and $\delta_0 = \delta(\Lambda_0, \epsilon_0, M_0)$ be as given by Proposition 2.1, and $X_0 = (\Psi_0)^* Y$. Let $a_0 \geq 3$ be an integer such that $1/a_0 < \delta_0$. We let $A_0 = \{\varepsilon/a_0 : \varepsilon \in \{-1, 0, 1\}\}$ and $\mathcal{F}_0 = (\exp(tX_0))_{t\in A_0}$. We let $D_0 = \Omega_0$. We assume a_0 is chosen large enough so that the maps $\exp(tX_0)$ map $\mathbb{H}_{-M_1}$ univalently into $\mathbb{H}_{-M_0}$ for $|t| \leq 1/a_0$.

At stage $n \geq 1$, we assume by induction that we have the following induction hypotheses:

(1) the maps $F_{n-1,t}$ in $\mathcal{F}_{n-1}$ map $\mathbb{H}_{-M_n}$ univalently into $\mathbb{H}_{-M_0}$;
(2) the map $\Phi_{n-1} := M_{a_{n-1}} \circ \Psi_{n-1} \circ \cdots \circ M_{a_0} \circ \Psi_0$ maps D_{n-1} univalently onto $\mathbb{H}$; and
(3) Φ_{n-1} semi-conjugates the maps in $\mathcal{F}_{n-1}$ to powers of T.

Each Ψ_k is of the form $\Psi_k = \psi_k \circ E$ for some entire function ψ_k, and hence the image of $\mathbb{H}_{-M_0}$ under Φ_{n-1} is contained in a half-plane $\mathbb{H}_{-\hat{M}_n}$ for some $\hat{M}_n > 0$. We let $\Psi_n = \Psi(\Lambda_n, \epsilon_n, \hat{M}_n)$, $\Omega_n = \Omega(\Lambda_n, \epsilon_n, \hat{M}_n)$ and $\delta_n = \delta(\Lambda_n, \epsilon_n, \hat{M}_n)$ be as given by Proposition 2.1. Let $\hat{X}_n = (\Psi_n)^* Y$ and $X_n = (\Phi_{n-1})^* \hat{X}_n$. For $|t| < \delta_n$ the maps $\exp(t\hat{X}_n)$ are univalent on $\mathbb{H}_{-\hat{M}_n}$; hence for $|t| < \delta_n$ the maps $\exp(tX_n)$ are univalent on $\mathbb{H}_{-M_0}$. Moreover, by the induction hypothesis (3) above, Φ_{n-1} semi-conjugates the maps $F_{n-1,t}$, $t \in A_{n-1}$ in $\mathcal{F}_{n-1}$ to powers of T, which commute with the maps $\exp(t\hat{X}_n)$, $|t| < \delta_n$; hence the maps $\exp(tX_n)$, $|t| < \delta_n$, commute with the maps $F_{n-1,t}$, $t \in A_{n-1}$.

We let $a_n \geq 3$ be an integer such that $1/a_n < \delta_n$. We let

$$A_n = \left\{ t = t_1 + \frac{t_2}{a_0 \cdots a_{n-1}} : t_1 \in A_{n-1},\, t_2 \in \{-1/a_n,\, 0,\, 1/a_n\} \right\},$$

$$\mathcal{F}_n = \left(F_{n,t} = F_{n-1,t_1} \circ \exp(t_2 X_n) : t = t_1 + \frac{t_2}{a_0 \cdots a_{n-1}} \in A_n \right)_{t\in A_n}.$$

We assume that a_n is chosen large enough so that the maps $\exp(tX_n)$ map $\mathbb{H}_{-M_{n+1}}$ univalently into $\mathbb{H}_{-M_n}$ for $|t| \leq 1/a_n$. Then the induction hypotheses (1), (3) above are satisfied. We let $\Phi_n := M_{a_n} \circ \Psi_n \circ \Phi_{n-1}$. Since Φ_{n-1} maps D_{n-1} univalently onto $\mathbb{H}$, we can define the domain $D_n := \Phi_{n-1}^{-1}(\Omega_n)$, where by Φ_{n-1}^{-1} we mean the inverse of the diffeomorphism $\Phi_{n-1} : D_{n-1} \to \mathbb{H}$. Since $M_{a_n} \circ \Psi_n$ maps $\Omega_n \subset \mathbb{H}$ univalently onto $\mathbb{H}$, it follows that induction hypothesis (2) is satisfied and the induction can proceed, giving a sequence of semi-flows $\mathcal{F}_n$.

Let $A = \bigcup_{n\geq 0} A_n$. We observe that $F_{n,t} = F_{n+1,t}$ for $t \in A_n$ so the semi-flows $\mathcal{F}_n$ are increasing, giving a well-defined semi-flow $\mathcal{F} = (F_t = F_{n,t} : t \in A)_{t\in A}$ univalent on $\mathbb{H}_{-1}$ (since $M_n > 1$ for all n). The maps in $\mathcal{F}$ are lifts of maps in a semi-flow $\overline{\mathcal{F}} = (f_t)_{t\in A}$

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univalent in the disk $\mathbb{D}_{e^{2\pi}}$ such that $f_t(0) = 0$, $f_t'(0) = e^{2\pi i t}$. Let $K_n = \overline{E(D_n)}$ and

$$A_\infty = \overline{A} = \left\{ \sum_{n=0}^{\infty} \frac{\varepsilon}{a_0 \cdots a_n} : \varepsilon \in \{-1, 0, 1\} \right\},$$
$$D_\infty = \overline{\bigcap_{n \geq 0} D_n},$$
$$K_\infty = E(D_\infty).$$

LEMMA 2.2. *There are unique semi-flows $\mathcal{F}_\infty = (F_t)_{t \in A_\infty}$ (respectively, $\overline{\mathcal{F}}_\infty = (f_t)_{t \in A_\infty}$) containing $\mathcal{F}$ (respectively, containing $\overline{\mathcal{F}}$). The semi-flows $\mathcal{F}_\infty$ (respectively, $\overline{\mathcal{F}}_\infty$) leave D_∞ (respectively, K_∞) invariant.*

Proof. Let $t_n \in A$, $t_n \to t \in A_\infty$. The maps F_{t_n} are lifts of maps f_{t_n} univalent in $\mathbb{D}$ such that $f_{t_n}(0) = 0$, $f_{t_n}'(0) = e^{2\pi i t_n}$ and hence form a normal family. Any subsequential limit F of F_{t_n} is a lift of a map f univalent in $\mathbb{D}$ with $f(0) = 0$, $f'(0) = e^{2\pi i t}$; moreover, f commutes with all the maps in f_{t_n}. It is easy to see that these two properties determine the Taylor expansion of f at the origin (and hence f) uniquely (because A accumulates at zero). Consequently, F is the unique lift of f tangent to the translation $z \mapsto z + t$ as $\operatorname{Im} z \to +\infty$. Hence, for any $t_n \to t$, the sequence F_{t_n} has a unique limit depending only on t, say F_t. We thus obtain semi-flows $\mathcal{F}_\infty := (F_t)_{t \in A_\infty}$, $\overline{\mathcal{F}}_\infty := (f_t)_{t \in A_\infty}$ as required. Since D_m is invariant under $\mathcal{F}_n$ for $m \geq n$, it follows that D_∞ is invariant under $\mathcal{F}_\infty$. □

We note that, in the above construction, at each stage n the choices of Λ_n, ϵ_n are independent of the objects constructed up to stage $n - 1$, and hence the approximating domains Ω_n can be constructed almost arbitrarily. This allows us to control the geometry of the limiting invariant compact K_∞. If $K_\infty \neq \{0\}$, then $\overline{\mathcal{F}}_\infty$ will be linearizable if and only if the origin is an interior point of K_∞; otherwise, all irrational elements of $\overline{\mathcal{F}}_\infty$ will be non-linearizable and will have K_∞ as a common hedgehog.

3. *Construction of examples*

Varying the choices of the domains Λ_n in the above construction will allow us to construct the required examples. We fix a sequence $M_n > 1$ as in the previous section.

3.1. *Positive area hedgehogs.*

Fix $0 < r < 1$ and sequences $0 < r_n < r$, $\alpha_n < 1$ such that $r_n \to 0$ and $\prod_{n \geq 0} \alpha_n > 0$.

LEMMA 3.1. *We can choose the domains Λ_n and constants $\epsilon_n > 0$ inductively so that the Jordan domains $K_n = \overline{E(D_n)}$ given by the construction satisfy the following:*

(1) *∂K_n intersects $\{|z| > r\}$ for all $n \geq 0$;*
(2) *∂K_n intersects $\{|z| < r_n\}$ for all $n \geq 0$; and*
(3) *$\lambda(K_n)/\lambda(K_{n-1}) > \alpha_n$ for all $n \geq 1$ where λ is Lebesgue measure on $\mathbb{C}$.*

Proof. In the construction of the previous section, the domains K_n are given by $K_n = \overline{E(D_n)}$ where $D_n = \Phi_{n-1}^{-1}(\Omega_n)$. The domains Ω_n are given by Proposition 2.1 as

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approximations to the domains Λ_n. At each stage n we will choose Λ_n such that the compact $K_n' = \overline{E(D_n')}$, $D_n' = \Phi_{n-1}^{-1}(\Lambda_n)$, satisfies conditions (1), (2) above (with K_n replaced by K_n'), and condition (3) with K_n replaced by K_n', namely, $\lambda(K_n')/\lambda(K_{n-1}) > \alpha_n$. This will suffice as the approximating domain Ω_n can then be chosen close enough so that the compact K_n satisfies (1)–(3).

Let $h_n > h > 0$ be such that $e^{-2\pi h} = r$, $e^{-2\pi h_n} = r_n$. For $H > 0$ let $S(H)$ be the open strip $\{0 < \operatorname{Im} z < H\}$.

At stage $n = 0$, we choose $\Lambda_0 \subset \mathbb{H}$ a 1-periodic domain such that $\partial\Lambda_0$ intersects $S(h)$ and $\mathbb{H}_{h_0}$. Then conditions (1), (2) are satisfied by $K_0' = E(\Lambda_0)$ (condition (3) is empty for $n = 0$).

At stage $n \geq 1$, if we choose Λ_n such that $\partial\Lambda_n$ intersects the images of the non-empty open sets $S(h) \cap D_{n-1}$ (non-empty by induction) and $\mathbb{H}_{h_n} \cap D_{n-1}$ under Φ_{n-1}, then conditions (1), (2) will be satisfied by K_n'.

Since $K_{n-1} = \overline{E(\Phi_{n-1}^{-1}(\mathbb{H}))}$ and the diffeomorphism Φ_{n-1} is fixed, we can choose Λ_n so that conditions (1), (2) will be satisfied and so that the Jordan domain $K_n' = E(\Phi_{n-1}^{-1}(\Lambda_n))$ satisfies condition (3), $\lambda(K_n')/\lambda(K_{n-1}) > \alpha_n$; for example, Λ_n can be taken to be of the form $\mathbb{H}_\delta - \{d(\operatorname{Re} z, \mathbb{Z}) \leq \epsilon, \operatorname{Im} z \leq H\}$, where choosing $H > 0$ large enough ensures condition (2) is satisfied, and then choosing $\delta > 0$ and $\epsilon > 0$ small enough ensures conditions (1), (3) are satisfied. This completes the induction. □

Proof of Theorem 1.1. Since ∂K_n intersects $\{z = r\}$ and $\{|z| = r_n\}$ for all n, it follows that $K_\infty \neq \{0\}$ and the origin is not an interior point of K_∞. Therefore, all elements f_t of the semi-flow $\overline{\mathcal{F}}_\infty$ are non-linearizable for t irrational, and K_∞ is a common hedgehog for these elements.

Since the compacts K_n decrease to K_∞, $\lambda(K_n) \to \lambda(K_\infty)$ as $n \to \infty$, and

$$\frac{\lambda(K_N)}{\lambda(K_0)} = \prod_{n=1}^{N} \frac{\lambda(K_n)}{\lambda(K_{n-1})} \geq \prod_{n=1}^{N} \alpha_n \to \prod_{n=1}^{\infty} \alpha_n > 0,$$

so $\lambda(K_\infty) > 0$. □

3.2. *Hedgehogs with inaccessible fixed points.* We give first a brief outline of the construction. We wish to choose the domains Λ_n inductively, controlling the geometry of the complementary 1-periodic domains $W_n := \mathbb{C} - \overline{D_n}$ so that there is no continuous curve $\gamma : [0, +\infty) \to W_\infty$ with $\lim_{t \to +\infty} \operatorname{Im} \gamma(t) = +\infty$, but $\sup_{z \in W_\infty} \operatorname{Im} z = +\infty$, where $W_\infty = \bigcup_{n \geq 1} W_n$ is the pre-image under E of the complement $\mathbb{C} - K_\infty$ of the invariant compact obtained from the construction. Then K_∞ will be a hedgehog such that the origin is inaccessible from $\mathbb{C} - K_\infty$.

Given W_n, we construct W_{n+1} inductively by choosing an equipotential curve γ_n' of D_n very close to $\gamma_n := \partial D_n = \partial W_n$, then constructing a Jordan arc α_n lying inside the 'upper' subdomain of $\mathbb{C}$ bounded by γ_n', such that α_n starts from a point of γ_n', follows γ_n' closely for a full period, then goes up to a height much larger than the largest height of any point of γ_n, and such that the images of α_n under the semi-flow $\mathcal{F}_n$ are pairwise disjoint. We then choose D_{n+1} such that W_{n+1} is very close to the domain given by the union of the

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'lower' subdomain of $\mathbb{C}$ bounded by γ_n' together with the images under $\mathcal{F}_n$ of a small neighbourhood of α_n.

We refer to these small neighbourhoods of the Jordan arcs $F(\alpha_n)$, $F \in \mathcal{F}_n$, as 'tubes'. Roughly speaking, each domain W_{n+1} is obtained from the preceding domain W_n by adding a collection of tubes. Thus, each W_n may be written as a union of tubes, where the tubes are divided into n groups, the jth group containing the tubes added at level j of the construction. The tubes are constructed such that the tubes of level $n+1$ go up to a height h_{n+1} much larger than the height h_n reached by the level n tubes, but the level $n+1$ tubes initially follow the tubes of the previous levels so closely that they must pass through a fixed height $0 < C < 5$ before reaching a height h_{n+1}. This property of the tubes forces any curve γ in W_∞ with $\operatorname{Im}\gamma$ accumulating $+\infty$ to also have $\operatorname{Im}\gamma$ accumulating $(-\infty, 2]$, and hence there is no curve with $\lim_{t\to+\infty} \operatorname{Im}\gamma(t) = +\infty$.

This ends the outline. We now give the details of the actual construction.

The building block for the construction will be a Jordan arc $\alpha = \alpha(q, \tau, H)$ depending on three parameters q, τ, H, where $q \geq 1$ is an integer and $\tau, H > 0$. The curve α is given by the straight line segment from 0 to $q + i\tau$, followed by the vertical segment from $q + i\tau$ to $q + iH$. We note that the images of α under the translation by one $T : z \mapsto z + 1$ are pairwise disjoint. In practice, q will be given, and τ and H will have to be chosen small enough and large enough, respectively.

We now define inductively the domains Λ_n, the constants $\epsilon_n, h_n, \tau_n, H_n > 0$, and the tubes referred to above, such that at each stage n we have a decomposition of W_n into a union of tubes of levels 1 to n, as follows.

At stage $n = 0$, we let $H_0 = 10$, $\tau_0 = 1/2$, $\alpha_0 = \alpha(2, \tau_0, H_0)$. We let U_0 be a 1/8-neighbourhood of α_0 and let Λ_0 be the union of the lower half-plane with the open sets $T^k(U_0)$, $k \in \mathbb{Z}$. Recall that ϵ_0 is one of the parameters of Proposition 2.1 (controlling the proximity of D_0 to Λ_0), and that $D_0 \subset \Lambda_0$. We choose $\epsilon_0 > 0$ small so that we may express the complement W_0 of D_0 as

$$W_0 = G_0 \cup \left(\bigsqcup_k T^k(V_0) \right),$$

where G_0 is the 'lower' subdomain of $\mathbb{C}$ bounded by a 1-periodic curve very close to the real axis, and V_0 is a Jordan domain, with boundary given by a Jordan arc following closely the boundary of U_0 together with a cross-cut C_0 of W_0 of diameter less than two joining two points z and $z+1$ with z close to $-1/2$, as shown in Figure 1 (we recall that a cross-cut of a domain W is an embedding $\gamma : [a, b] \subset \mathbb{R} \to \overline{W}$ such that $\gamma((a, b)) \subset W$, $\gamma(\{a, b\}) \subset \partial W$).

Since the choice of Ψ_0 is fixed by the above and $\Phi_0 = M_{a_0} \circ \Psi_0$, we can ensure that $a_0 \geq 3$ is chosen large enough so that $|\Phi_0^{-1}(x) - \Phi_0^{-1}(x')| \leq 1/2^5$ for all $x, x' \in \mathbb{R}$ with $|x - x'| \leq 2$.

We define the height h_0 by $h_0 := \sup_{z\in W_0} \operatorname{Im} z$.

We refer to the sets $V_{k,0,0} := T^k(V_0)$ as the level zero tubes of stage zero of the construction, and denote the bounding cross-cuts $T^k(C_0)$ by $c_{k,0,0}$.

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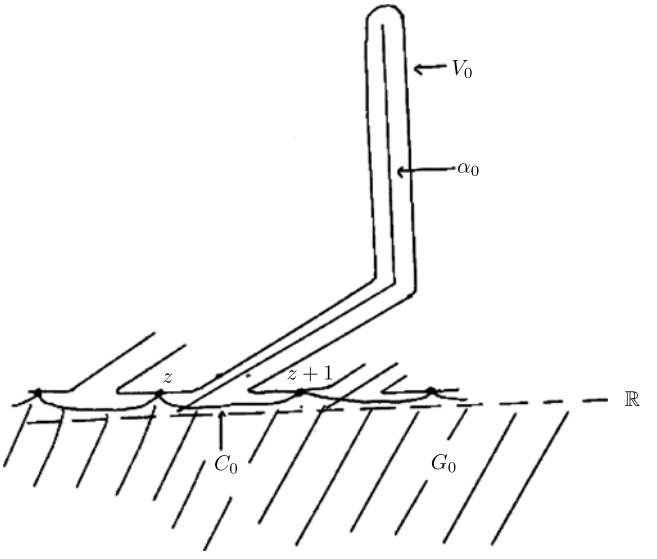


FIGURE 1. Stage $n = 0$ of the construction.

At stage $n \geq 1$, we assume by induction that the 1-periodic domain D_{n-1} (and hence also its complement W_{n-1}) has been constructed such that we have a decomposition

$$W_{n-1} = G_{n-1} \cup \left(\bigsqcup_{j=0}^{n-1} \bigsqcup_{k \in \mathbb{Z}} V_{k,j,n-1} \right),$$

where G_n is the 'lower' subdomain of $\mathbb{C}$ bounded by a 1-periodic curve close to the real axis, and $V_{k,j,n-1}$ are the level j tubes of the stage $n-1$ of the construction, with bounding cross-cuts $c_{k,j,n-1}$.

We assume by induction that the tubes and cross-cuts are arranged as follows.

Given a domain $W \subset \mathbb{C}$ bounded by a Jordan curve ∂W in $\hat{\mathbb{C}}$ with $\infty \in \partial W$, and a cross-cut $c \subset \mathbb{C}$ of W, we denote by $B(W, c)$ the bounded component of $W - c$. We assume by induction the nesting property that for $0 \leq j \leq j' \leq n-1$, $k, k' \in \mathbb{Z}$, either $B(W_{n-1}, c_{k',j',n-1}) \cap B(W_{n-1}, c_{k,j,n-1}) = \emptyset$ or $B(W_{n-1}, c_{k',j',n-1}) \subset B(W_{n-1}, c_{k,j,n-1})$. Moreover, each level j tube may be written as

$$V_{k,j,n-1} = B(W_{n-1}, c_{k,j,n-1}) - \left(\bigcup_{j'>j, k' \in \mathbb{Z}} B(W_{n-1}, c_{k',j',n-1}) \right).$$

Each level zero tube $V_{k,0,n-1}$ intersects G_{n-1} precisely along the cross-cut $c_{k,0,n-1}$. Also, each tube $V_{k,j,n-1}$ of level $j \geq 1$ intersects exactly one tube of level $j-1$ along the cross-cut $c_{k,j,n-1}$, and only intersects tubes of levels $j' > j$ for $j' = j+1$ along cross-cuts of level $j+1$. We assume by induction that the heights of the tubes satisfy

$$\sup_{z \in V_{k,j,n-1}} \operatorname{Im} z \leq h_j + 1/2^j + \cdots + 1/2^{n-1}.$$

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We assume by induction that $|\Phi_{n-1}^{-1}(x)-\Phi_{n-1}^{-1}(x')|\leq 1/2^{n+4}$ for $x, x' \in \mathbb{R}$, $|x-x'|\leq 2$, and that the interiors of the cross-cuts $c_{j,k,n-1}$ are pairwise disjoint, and all of them are of diameter less than $1/2^{n-2}$.

By induction we also have a univalent map Φ_{n-1} mapping D_{n-1} to $\mathbb{H}$ and conjugating $\mathcal{F}_{n-1}$ to powers of T.

We choose $\tau_n > 0$ small enough so that

$$|\Phi_{n-1}^{-1}(x+iy) - \Phi_{n-1}^{-1}(x'+iy')| < 1/2^{n+4} \quad \text{where } |x-x'| \leq 2 \text{ and } y, y' \in [0, 10\tau_n].$$

We choose $H_n > 0$ large enough so that

$$\inf_{x\in\mathbb{R}} \operatorname{Im} \Phi_{n-1}^{-1}(x+iH_n) > 2h_{n-1} + 10.$$

We define $\alpha_{n,n}$ to be the curve $\alpha(3a_0 \cdots a_{n-1}+2, \tau_n, H_n)$ translated by $i\tau_n$, and choose $U_{n,n}$ to be a small neighbourhood of $\alpha_{n,n}$ such that $U_{n,n} \cap \{0 \leq \operatorname{Re} z \leq a_0 \cdots a_{n-1}+1\} \subset \{0 < \operatorname{Im} z < 3\tau_n\}$, and such that the translates $U_{k,n,n} := T^k(U_{n,n})$ are pairwise disjoint.

We define Λ_n to be the domain

$$\Lambda_n := \mathbb{H} - \left(\{\operatorname{Im} z < \tau_n\} \cup \left(\bigsqcup_{k\in\mathbb{Z}} U_{k,n,n}\right)\right).$$

The end-points $e_{k,j,n-1}^-$, $e_{k,j,n-1}^+$ of the cross-cuts $c_{k,j,n-1}$, $j = 0, \ldots, n-1$ of W_{n-1} lie on $\partial W_{n-1} = \partial D_{n-1}$, and are mapped by Φ_{n-1} to points $x_{k,j,n-1}^- < x_{k,j,n-1}^+$ lying on $\mathbb{R}$. The points $x_{k,j,n-1}^-$, $x_{k,j,n-1}^+$ satisfy $x_{k,j,n-1}^+ - x_{k,j,n-1}^- \geq a_{n-1} - 1 \geq 2$ for all k, j, and can be joined to points on $\{\operatorname{Im} z = \tau_n\}$ by small disjoint vertical segments. Moreover for $j' \geq j$, $k, k' \in \mathbb{Z}$, we have the nesting property that either $(x_{k',j',n-1}^-, x_{k',j',n-1}^+) \cap (x_{k,j,n-1}^-, x_{k,j,n-1}^+) = \emptyset$ or $(x_{k',j',n-1}^-, x_{k',j',n-1}^+) \subset (x_{k,j,n-1}^-, x_{k,j,n-1}^+)$. We choose ϵ_n small enough so that the boundary of the domain Ω_n is close enough to $\partial\Lambda_n$ so that the points $x_{k,j,n-1}^\pm$ can be joined to points of $\partial\Omega_n$ by small almost vertical segments $l_{k,j,n-1}^\pm$ lying in $\{0 \leq \operatorname{Im} z \leq 3\tau_n\}$. We choose the segments such that $l_{k,j,n-1}^\pm$ and $l_{k',j',n-1}^\pm$ are disjoint if $x_{k,j,n-1}^\pm \neq x_{k',j',n-1}^\pm$, while $l_{k,j,n-1}^\pm = l_{k',j',n-1}^\pm$ if $x_{k,j,n-1}^\pm = x_{k',j',n-1}^\pm$.

We also assume ϵ_n is small enough so that we may make almost horizontal cross-cuts $h_{k,n,n}$ of $\mathbb{H} - \Omega_n$ with disjoint interiors lying in $\{0 \leq \operatorname{Im} z \leq 2\tau_n\}$, each joining points of $\partial\Omega_n$ close to the points $(k-1/2)+i\tau_n$ and $(k+1/2)+i\tau_n$ such that the Jordan domains $U'_{k,n,n} := B(\mathbb{H} - \Omega_n, h_{k,n,n})$ are close to the neighbourhoods $U_{k,n,n}$ of the curves $\alpha_{k,n,n}$. We may assume ϵ_n is small enough so that $U'_{k,n,n} \cap \{0 \leq \operatorname{Re} z \leq a_0 \ldots a_{n-1}\} \subset \{0 \leq \operatorname{Im} z \leq 4\tau_n\}$. The cross-cuts $h_{k,n,n}$ are chosen such that the starting and ending points of $h_{k,n,n}$ are the ending and starting points, respectively, of $h_{k-1,n,n}$ and $h_{k+1,n,n}$, so that the complement of the cross-cuts $h_{k,n,n}$ in $\mathbb{H} - \Omega_n$ is the disjoint union of the domains $B(\mathbb{H} - \Omega_n, h_{k,n,n})$ and a simply connected subdomain of $\mathbb{H} - \Omega_n$ whose closure intersects $\partial\Omega_n$ only at the endpoints of the cross-cuts $h_{k,n,n}$.

Define $D_n := \Phi_{n-1}^{-1}(\Omega_n)$, $W_n := \mathbb{C} - D_n$. We define the tubes $V_{k,j,n}$ of level j and stage n and the cross-cuts $c_{k,j,n}$, for $0 \leq j \leq n$, as follows.

Define the cross-cuts $c_{k,n,n} := \Phi_{n-1}^{-1}(h_{k,n,n})$, $k \in \mathbb{Z}$. The induction hypothesis $|\Phi_{n-1}^{-1}(x) - \Phi_{n-1}^{-1}(x')| \leq 1/2^{n+4}$ for $x, x' \in \mathbb{R}$, $|x-x'| \leq 2$ implies that the cross-cuts $c_{k,n,n}$ all have diameter less than $1/2^{n-1}$.

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For $0 \le j \le n-1$, $k \in \mathbb{Z}$, we first define cross-cuts $c'_{k,j,n}$ to be the union of $c_{k,j,n-1}$ with $\Phi_{n-1}^{-1}(l^-_{k,j,n-1})$, $\Phi_{n-1}^{-1}(l^-_{k,j,n-1})$. We note that cross-cuts $c'_{k,j,n}$, $c'_{k',j',n}$ are disjoint if $c_{k,j,n-1}$, $c_{k',j',n-1}$ are disjoint, while if $c_{k,j,n-1}$ and $c_{k',j',n-1}$ intersect at an endpoint, then $c'_{k,j,n}$, $c'_{k',j',n}$ intersect along a small segment $\Phi_{n-1}^{-1}(l^{\pm}_{k,j,n})$. Clearly the cross-cuts $c'_{k,j,n}$ may thus be slightly deformed near their endpoints to obtain cross-cuts with disjoint interiors but with the same endpoints. We define the cross-cuts $c_{k,j,n}$ to be the cross-cuts thus obtained from the cross-cuts $c'_{k,j,n}$. The induction hypothesis on the diameters of the cross-cuts $c_{k,j,n-1}$ being less than $1/2^{n-2}$ implies that the diameters of the cross-cuts $c_{k,j,n}$ are less than $1/2^{n-1}$.

For $0 \le j \le n$, $k \in \mathbb{Z}$, define the tube $V_{k,j,n}$ by

$$V_{k,j,n} := B(W_n, c_{k,j,n}) - \Bigg(\bigcup_{j'>j, k' \in \mathbb{Z}} B(W_n, c_{k',j',n}) \Bigg),$$

and define $G_n := W_n - (\bigcup_{k,j} V_{k,j,n})$. We note that, for $0 \le j \le n-1$, the tubes $V_{k,j,n}$ are given by the union of $V_{k,j,n-1}$ together with a domain contained in $\Phi_{n-1}^{-1}(\{0 \le \operatorname{Im} z \le 4\tau_n\})$. Hence, by the hypothesis on τ_n and the induction hypothesis on the heights of the tubes $V_{k,j,n-1}$, the heights of the tubes $V_{k,j,n}$ satisfy

$$\sup_{z \in V_{k,j,n}} \operatorname{Im} z \le h_j + 1/2^j + \cdots + 1/2^n, \quad k \in \mathbb{Z},\ 0 \le j \le n-1.$$

We note that any tube $V_{k,j,n}$ for $0 \le j \le n-1$ is contained in the one neighbourhood of $V_{k,j,j}$, and that the tubes $V_{k,n,n}$ are given by the pre-images of the domains $U'_{k,n,n}$ under Φ_{n-1}^{-1}.

We define the height $h_n := \sup_{z \in W_n} \operatorname{Im} z$ and note that the hypothesis on H_n implies that $h_n \ge 2h_{n-1} + 2$.

Finally, we assume that a_n is chosen large enough so that $|\Phi_n^{-1}(x) - \Phi_n^{-1}(x')| \le 1/2^{n+5}$ for $x, x' \in \mathbb{R}$, $|x - x'| \le 2$.

This ends the inductive construction.

We let $W_\infty := \bigcup_{n \ge 0} W_n$, observing that $W_\infty = \mathbb{C} - E^{-1}(K_\infty)$. We remark that it is easy to show by induction, using the hypothesis on the heights τ_n, that every tube of W_n intersects $\{\operatorname{Im} z \le 2\}$.

We note that two tubes $V_{k,j,n}$, $V_{k',j',n}$ can intersect along a cross-cut $c_{k'',j'',n}$ only if $|j - j'| = 1$. For each $n \ge 0$, let Γ_n be the graph with vertices the tubes $V_{k,j,n}$, $k \in \mathbb{Z}$, $0 \le j \le n$ and the domain G_n, where two vertices are joined by an edge if and only if the corresponding domains intersect along a cross-cut. We define the level of a vertex to be j if it corresponds to a tube $V_{k,j,n}$ and to be -1 if it corresponds to G_n. It is clear from the above construction that Γ_n is a rooted tree with root the vertex corresponding to G_n, and for $j \ge 0$ each vertex of level j is connected to exactly one vertex of level $j-1$, and is only connected otherwise to vertices of level $j+1$.

We note that any curve $\gamma : [a, b] \to W_n$ with only finitely many transverse intersections with the union of the cross-cuts $c_{k,j,n}$, $k \in \mathbb{Z}$, $0 \le j \le n$, gives rise to a finite path $\tau(\gamma)$ in the graph Γ_n, obtained from γ by considering the tubes of W_n through which γ passes. We call $\tau(\gamma)$ the trace of γ in Γ_n.

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LEMMA 3.2. *Let* $\gamma : [0, T] \to W_n$ *be a curve such that* $\operatorname{Im} \gamma(t) > 10$ *for all* $0 \le t \le T$. *Then there exists* $M > 10$ *depending only on* $\gamma(0)$ *such that* $\operatorname{Im} \gamma(T) \le M$.

Proof. Let $1 \le n_0 \le n$ be the level of the tube of W_n containing $\gamma(0)$. We will show the statement of the lemma holds for $M = h_{n_0} + 10$.

We first perturb γ slightly to a curve γ_1 such that γ_1 has only finitely many transverse intersections with the union of the cross-cuts $c_{k,j,n}$, $k \in \mathbb{Z}$, $0 \le j \le n$, $\gamma_1(T)$ belongs to the interior of a tube of W_n, $|\gamma_1(T) - \gamma(T)| < 1/2$, and $\operatorname{Im} \gamma_1(t) > 10$ for all $0 \le t \le T$. Consider the trace $\tau = \tau(\gamma_1)$ of γ_1 in Γ_n.

Then, since Γ_n is a tree, τ contains the unique simple path α between its endpoints. Moreover, γ_1 consists of arcs lying in tubes corresponding to vertices of α, together with arcs lying in tubes corresponding to vertices of $\tau - \alpha$. Each such latter arc β starts from and returns to a bounding cross-cut c of a tube V corresponding to a vertex of α. We replace each arc β of γ_1 by a small arc β' with the same endpoints which is contained in the cross-cut c, to obtain a curve $\gamma_2 : [0, T] \to W_n$ contained entirely in the tubes corresponding to the vertices of α. We then perturb γ_2 slightly near each β', replacing a subarc of γ_2 containing β' with endpoints in the interior of V by an arc lying entirely in the interior of V with the same endpoints to obtain a curve $\gamma_3 : [0, T] \to W_n$ such that the trace of γ_3 in W_n is α, and such that $\gamma_3(T) = \gamma_1(T)$. Since the diameters of all cross-cuts are less than two, the hypothesis on γ implies that we may assume the choices of γ_1, γ_2, γ_3 above have been made so that $\operatorname{Im} \gamma_3(t) > 8$ for $0 \le t \le T$.

Let the vertices of α correspond to tubes $V_1, \ldots, V_m$, and let $0 = t_0 < \cdots < t_m = T$ be a partition of $[0, T]$ such that $\gamma_3([t_{i-1}, t_i]) \subset \overline{V_i}$. We note that the level of V_1 is n_0. Since α is a simple path, either the levels of the tubes V_i are strictly increasing, or they are strictly decreasing, or $m = 1$. If either of the latter two cases hold, then the level of V_m is less than or equal to n_0; hence by the hypothesis on the height of the tubes $\operatorname{Im} \gamma_3(T) \le h_{n_0} + 4$ and we are done.

We may therefore assume that $m \ge 2$ and the levels of the tubes V_i are strictly increasing. Let V_0 be the unique tube of level $n_0 - 1$ connected to V_1. Let c_i be the cross-cut of W_n which is the intersection of V_{i-1} with V_i, for $i = 1, \ldots, m-1$.

There is an initial subarc of the boundary of V_1 which follows closely the boundary of D_{n_0-1} for a full period. Since ∂D_{n_0-1} intersects $\{\operatorname{Im} z \le 2\}$, it follows that we may make a cross-cut d_1 of V_1 such that $\sup_{z \in d_1} \operatorname{Im} z \le 2 + 1/2^{n_0}$. Of the two Jordan subdomains of V_1 bounded by d_1, we let Z_1 be the subdomain whose boundary contains c_1, and note that Z_1 is contained in $\{\operatorname{Im} z \le h_{n_0}\}$. Since $\operatorname{Im} \gamma_3(t) > 8$ for all t, γ_3 cannot intersect d_1 and it follows that $\gamma_3(t_1) \in \overline{Z_1}$, and hence $c_2 \subset \partial Z_1$.

There is an initial subarc of the boundary of V_2 which follows closely the boundary of Z_1; hence we may make a cross-cut d_2 of V_2 close to d_1 such that $\sup_{z \in d_2} \operatorname{Im} z \le 2 + 1/2^{n_0} + 1/2^{n_0+1}$, and such that of the two Jordan subdomains of V_2 bounded by d_2, the subdomain Z_2 whose boundary contains c_2 satisfies $Z_2 \subset \{\operatorname{Im} z \le h_{n_0} + 1/2^{n_0+1}\}$. As before, γ_3 cannot intersect d_2, and hence $\gamma_3(t_2) \in \overline{Z_2}$, so $c_3 \subset \partial Z_2$.

It follows, using the hypothesis $\operatorname{Im} \gamma_3(t) > 8$ for all t and arguing by induction, that we may make cross-cuts $d_1, d_2, \ldots, d_m$ of the domains $V_1, V_2, \ldots, V_m$, respectively,

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bounding subdomains $Z_1, Z_2, \ldots, Z_m$, with $c_i \subset \partial Z_i$, such that $\sup_{z \in d_m} \operatorname{Im} z \leq 2 + 1/2^{n_0} + 1/2^{n_0+1} + \cdots + 1/2^{n_0+m-1}$, and such that $Z_m \subset \{\operatorname{Im} z \leq h_{n_0} + 1/2^{n_0+1} + \cdots + 1/2^{n_0+m-1}\}$. Since $\gamma_3(t_{m-1}) \in c_m \subset \partial Z_m$, and γ_3 cannot intersect d_m, we must have $\gamma_3(T) = \gamma_3(t_m) \in \overline{Z_m}$.

It follows that $\operatorname{Im} \gamma(T) \leq \operatorname{Im} \gamma_3(T) + 1 \leq h_{n_0} + 1 + 1 < M$. □

Proof of Theorem 1.2. Since $D_n \cap \{\operatorname{Im} z \leq 2\} \neq \emptyset$ for all n, the compact K_∞ is non-trivial, and since $h_n \to +\infty$ it is a hedgehog. Suppose the fixed point is accessible from the complement of K_∞. Then the lift of a curve landing at the fixed point gives a curve $\gamma : [0, +\infty) \to W_\infty$ such that $\operatorname{Im} \gamma(t) \to +\infty$ as $t \to +\infty$. Let $t_0 > 0$ be such that $\gamma(t) > 10$ for all $t \geq t_0$. Given $T > t_0$, choose $n \geq 1$ such that $\gamma([t_0, T]) \subset W_n$; then by the previous lemma there exists M depending only on $\gamma(t_0)$ such that $\operatorname{Im} \gamma(T) \leq M$. Hence, $\operatorname{Im} \gamma(t) \leq M$ for all $t \geq t_0$, a contradiction. □

Acknowledgements. This research was partly supported by Department of Science and Technology research project grant DyNo. 100/IFD/8347/2008-2009. The author thanks R. Perez-Marco and A. Cheritat for helpful discussions.

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