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On the completeness of multifractal spectra

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Abstract. We study the behavior of multifractal spectra on the boundary of their domains of definition. In particular, we show that the dimension of the set of points having minimal (or maximal) pointwise dimension is not necessarily zero. However, this situation may be neglected in the sense of Baire category for Gibbs measures on expanding conformal repellers.

1. *Introduction*

Multifractal analysis, i.e. the analysis of invariant sets and measures with multifractal structure, has recently been developed as a powerful tool for the numerical study of dynamical systems. These spectra capture information about various dimensions associated with the dynamics. Among them are the well-known Hausdorff dimension, correlation dimension and information dimension of invariant measures.

Another example of multifractal spectra is entropy spectra introduced in [**2**]. They provide an integrated information on the distribution of topological entropy associated with local entropies.

In [**2, 3**] it is demonstrated that multifractal spectra can be used in a sense to 'restore' the dynamics—the phenomenon that we call *multifractal rigidity*.

Multifractal analysis is essentially measuring the 'size' (in the sense of Hausdorff dimension or topological entropy) of the level sets of special discontinuous measurable functions (such as the local entropy, the pointwise dimension, etc.) on geometrically complicated objects (supports of invariant measures). This is expressed in a function f which is called the *spectrum*. In this sense multifractal analysis can be considered as a non-smooth Morse theory. Our main concern is the further investigation of the dimension spectrum (i.e. the Hausdorff dimension of the level sets of the pointwise dimension) and the entropy spectrum (i.e. the topological entropy of the level sets of local entropies).

This analysis was investigated in several different situations and there is a huge amount of literature on this subject (see [**7**] for references and more details). One of the main

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results in this theory is that there is an interval (α_1, α_2) on which the spectrum $f(\alpha)$ (for definition see below) is analytic, convex and can be continuously extended to the boundary provided it is not a point spectrum (see §2.3).

A natural question which arises is what happens outside this interval? Are there level sets which are not covered by the analytic function f on (α_1, α_2)? We will see that for expanding conformal repellers the closed interval $[\alpha_1, \alpha_2]$ coincides with the domain of definition of the spectrum (*completeness of the spectrum*), i.e. the function f contains the information about all level sets (Theorem 3.1).

Another question we are interested in is which intervals occur as the domain of definition of spectra, and which values does f take on the boundary of $[\alpha_1, \alpha_2]$. This means that we are interested in the description of the extremal level sets. As we will see later these level sets (actually, certain Gibbs measures supported on them) can be interpreted as limit states of the system as the temperature increases or decreases, respectively. We will give a complete answer to this question. We will prove that for a given expanding conformal repeller $\mathcal{J}$ any interval in $\mathbb{R}^+$ containing the point $\dim_H \mathcal{J}$ is the domain of definition of the dimension spectrum of some Gibbs measure, i.e. it serves as the interval $[\alpha_1, \alpha_2]$ for the dimension spectrum of some Gibbs measure (Theorem 4.1). We call those intervals *admissible*.

This statement is supported by experimental and numerical observations of chaotic systems. Although the majority of these studies seem to indicate that the dimension spectrum vanishes at its boundary (i.e. $f(\alpha_1) = f(\alpha_2) = 0$) there is no rigorous result in this direction known to the author besides some results on geometric constructions with Bernoulli measures supported on their limit sets (see, for example, [**5, 11**]). In fact, we will show that this observation does not hold in general. On the contrary, for a given admissible interval (α_1, α_2) and a given pair of *admissible* boundary values on f_1, f_2 we construct a Hölder continuous potential for which the spectrum takes these values at the endpoints of its domain of definition $[\alpha_1, \alpha_2]$ (Theorem 4.1). Hereby the admissibility of a pair of boundary values is determined by canonical restrictions given by the general shape of the graph of the spectrum (see §4). We will call spectra with at least one strictly positive boundary value *degenerate*. In particular, this means that those systems have positive dimension (entropy) on their extremal level sets. Note that point spectra are degenerate in this sense.

On the other hand, in §5 (Theorem 5.1) we prove that degenerate spectra are not very likely. Namely, for a typical (in the sense of Baire) Hölder continuous potential the dimension or entropy spectrum of its corresponding Gibbs measure is *non-degenerate*. This justifies the experimental and numerical observations. It also shows that despite the spectrum itself (since it is defined in terms of the analytic pressure functional) the graph of the spectrum does not depend continuously on the potential.

The methods used in the proofs of the results are essentially combinations of methods from the thermodynamic formalism and symbolic dynamics. Since the multifractal analysis of expanding conformal repellers can be formulated in terms of the thermodynamic formalism (see §§2.2 and 2.3) our results may be considered as statements within the thermodynamic formalism. In particular, the results in this paper are connected to the behavior of the pressure functional as the temperature tends to zero.

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2. *Multifractal spectra of Gibbs measures for conformal repellers*

In this section we briefly illustrate the general concept of multifractal spectra. See [**1, 2, 3, 7, 9, 10, 13**] for more details.

We consider Gibbs measures invariant under conformal expanding maps, and describe the associated multifractal spectra for dimensions and entropies.

2.1. *Coding of expanding repellers.*

Let M be a smooth Riemannian manifold and let $F : M \to M$ be a C^1 map. Consider a compact subset $\mathcal{J}$ of M. We say that F is *expanding* and $\mathcal{J}$ is a *repeller* of F if:

(1) there are constants $C > 0$ and $\beta > 1$ such that $\|D_x F^n u\| \geq C\beta^n \|u\|$ for all $x \in \mathcal{J}$, $u \in T_x M$, and $n \geq 1$;

(2) $\mathcal{J} = \bigcap_{n\geq 0} F^{-n} V$ for some open neighborhood V of $\mathcal{J}$.

One can easily show that $F\mathcal{J} = \mathcal{J}$.

It is well known that repellers admit Markov partitions of arbitrarily small diameter. Markov partitions are used to build symbolic models of repellers by subshifts of finite type (see §2.2).

Let us assume that F is topological mixing. Then there is a subshift of finite type Σ_A^+ with a transitive transfer matrix A and a continuous surjective finite-to-one map $\chi : \Sigma_A^+ \to \mathcal{J}$ such that the following diagram is commutative

$$\begin{array}{ccc} \Sigma_A^+ & \xrightarrow{\sigma} & \Sigma_A^+ \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\chi} \\ \mathcal{J} & \xrightarrow{F} & \mathcal{J} \end{array}$$

A smooth map $F : M \to M$ is called *conformal* if $D_x F$ is a multiple of an isometry at every point $x \in M$. Well-known examples of conformal expanding maps include one-dimensional Markov maps and holomorphic maps. We write $\hat{a}(x) = \|D_x F\|$ for each $x \in M$ and denote by a some lift of $\hat{a}$ to Σ_A^+ via χ, i.e. $\hat{a}(x) = a(\chi(\underline{x}))$ for $x \in \mathcal{J} \setminus \bigcup_{i,j}(R_i \cap R_j)$.

The numbers

$$\lambda(x) = \underline{\lim}\, \frac{1}{n+1} \sum_{j=0}^{n} \log \hat{a}(F^j x) = \underline{\lim}\, \frac{1}{n+1} \sum_{j=0}^{n} \log a(\sigma^j \underline{x})$$

are called the *Lyapunov exponents*. It is well known that the lower limit is actually the limit on a set of total measure one. If μ is ergodic then $\lambda(x)$ is constant almost everywhere. This common value is denoted by λ_μ.

The multifractal spectra for $\mathcal{J}$ do not depend on the ambiguity at the boundaries of the rectangles and we have the set $\{x \in \mathcal{J} : F^n x \notin \bigcup_{i,j}(R_i \cap R_j) \text{ for all } n\}$ in mind when we write $\mathcal{J}$. On this set the function χ is invertible and the point $\underline{x} = \chi^{-1}(x)$ is uniquely defined.

A *Moran cover of depth n* associated to a positive number B and a set $\mathcal{X} \subset \mathcal{J}$ is a finite set of points $\underline{x}_i \in \Sigma_A^+$ such that $\mathcal{X} \subset \bigcup_i \chi(C_{n_i}(\underline{x}_i))$, where $C_{n_i}(\underline{x}_i)$ are the cylinder sets

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of length $n_i \geq n$ containing $\underline{x}_i$ and

$$B^{-1} \prod_{j=0}^{n_i-1} a(\sigma^j \underline{x}_i) \leq \operatorname{diam} \chi(C_{n_i}(\underline{x}_i)) \leq B \prod_{j=0}^{n_i-1} a(\sigma^j \underline{x}_i).$$

Sometimes we will also view the Moran cover as a set of cylinder sets $\{C_{n_i}(\underline{x}_i)\}$. An expanding conformal repeller $\mathcal{J}$ admits Moran covers of arbitrary depths and multiplicity Q for any subset $\mathcal{X} \subset \mathcal{J}$ and some positive real numbers B and Q not depending on the depth or the subset (see [**9**] for details).

2.2. *Subshifts of finite type.* Let A be a $p \times p$ matrix whose entries are either 0 or 1. The topological Markov chain Σ_A^+ consists of the sequences $\underline{x} = (i_1 i_2 \cdots) \in \{1, \ldots, p\}^{\mathbb{N}}$ such that $a_{i_k i_{k+1}} = 1$ for every $k \geq 1$. Let $\sigma(i_1 i_2 \cdots) = (i_2 i_3 \cdots)$ be the shift map on Σ_A^+. We assume that A is *transitive*, i.e. there exists a positive integer M such that all entries of A^M are positive (this holds if and only if $\sigma|_{\Sigma_A^+}$ is topologically mixing).

Let $\underline{x} \in \Sigma_A^+$. We consider the set $V_\sigma(\underline{x})$ of accumulation points of the sequence of measures $\{\frac{1}{n}\sum_{j=0}^{n-1}\}\delta_{\sigma^j \underline{x}}$. Clearly, any measure $\mu \in V_\sigma(\underline{x})$ is invariant. For a given set X we denote the set of invariant, respectively ergodic, measures supported on X by $M_{\text{inv}}(X)$ and $M_{\text{e}}(X)$. If $\mu \in M_{\text{e}}(\Sigma_A^+)$ then $m = \mu \circ \chi^{-1} \in M_{\text{e}}(\mathcal{J})$.

Fix $a > 1$ and define a metric on Σ_A^+ by

$$d(\underline{x}, \underline{x}') = \sum_{k=1}^{\infty} a^{-k} |i_k - i_k'|.$$

Notice that $d(\sigma \underline{x}, \sigma \underline{x}') = a \cdot d(\underline{x}, \underline{x}')$ for all $\underline{x}, \underline{x}' \in \Sigma_A^+$ with $d(\underline{x}, \underline{x}') < a^{-1}$.

Given a continuous function φ on Σ_A^+, a measure μ on Σ_A^+ is said to be a *Gibbs measure* for φ if there exist constants $C_1, C_2 > 0$, such that for every $\underline{x} = (i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbb{N}$,

$$C_1 \leq \frac{\mu(C_{i_1 \cdots i_n})}{\exp(-nP(\varphi) + \sum_{k=0}^{n-1} \varphi(\sigma^k \underline{x}))} \leq C_2,$$

where $C_{i_1 \cdots i_n} = C_n(\underline{x}) = \{\underline{x}' \in \Sigma_A^+ : i_k = i_k' \text{ for } 1 \leq k \leq n\}$ is the cylinder set of length n containing $\underline{x}$, and P is the topological pressure with respect to σ.

Let $\mathcal{F}_\theta$ be the space of Hölder continuous functions on Σ_A^+ with Hölder exponent θ. We can decompose $\mathcal{F}_\theta$ into

$$\mathcal{F}_\theta = \bigcup_{K>0} \mathcal{F}_\theta^K,$$

where

$$\mathcal{F}_\theta^K = \{\varphi \in C^0(\Sigma_A^+) : |\varphi(\underline{x}) - \varphi(\underline{x}')| \leq K d(\underline{x}, \underline{x}')^\theta \text{ for all } \underline{x}, \underline{x}' \in \Sigma_A^+\}.$$

For a Hölder continuous function $\varphi \in \mathcal{F}_\theta$ on Σ_A^+ we define its norm $\|\varphi\|_\theta$ by

$$\|\varphi\|_\theta = \sup |\varphi| + \inf\{K : \varphi \in \mathcal{F}_\theta^K\}.$$

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If $\varphi \in \mathcal{F}_\theta$ and $\underline{x}$, $\underline{x}'$ are contained in the same cylinder C_n of length n then

$$\left|\sum_{j=0}^{n-1}\varphi(\sigma^j\underline{x}) - \sum_{j=0}^{n-1}\varphi(\sigma^j\underline{x}')\right| \le \sum_{j=0}^{n-1}\|\varphi\|_\theta d(\sigma^j\underline{x}, \sigma^j\underline{x}')^\theta$$
$$\le \|\varphi\|_\theta \sum_{j=0}^{\infty} a^{-\theta j} = \|\varphi\|_\theta \frac{1}{1-a^{-\theta}}.$$

Let φ be a Hölder continuous function on Σ_A^+ and let μ be the corresponding Gibbs measure; it exists and is unique (because $\sigma|_{\Sigma_A^+}$ is topologically mixing). It is more convenient to work with the 'normalized' function $\log\psi$ on Σ_A^+ defined by $\log\psi = \varphi - P(\varphi)$. Note that μ is also the Gibbs measure for $\log\psi$.

For a given subset $S \in \Sigma_A^+$ we write $C_m(S)$ for the set $\bigcup_{\underline{x}\in S} C_m(\underline{x})$ and $U_m(S) = \{\underline{x} \in \Sigma_A^+ \mid \sigma^k\underline{x} \in C_m(S) \text{ for all } k \in \mathbb{N}\}$.

The number of times that the trajectory $\{\sigma^k\underline{x}\}_{k=0}^N$ of the point $\underline{x}$ hits a given set U is denoted by $A(U, N, \underline{x}) = \text{Card}\{0 \le k \le N : \sigma^k\underline{x} \in U\}$.

In the next section we are going to establish the main results for the dimension spectrum $f(\alpha) = f^D(\alpha)$ on $\mathcal{J}$. The corresponding statements for the entropy spectrum can be obtained from the dimension spectrum results by setting $\log a \equiv 1$.

2.3. *Multifractal spectra.* The considerations in this section are taken from [**1**]. Let $u : \Sigma_A^+ \to \mathbb{R}$ be a strictly positive continuous function. For each cylinder set C_n of length n, we write

$$u(C_n) = \begin{cases} \sup\left\{\sum_{k=0}^{n} u(f^k x) : x \in C_n\right\} & \text{if } C_n \neq \varnothing \\ -\infty & \text{if } C_n = \varnothing. \end{cases}$$

For each set $Z \subset \Sigma_A^+$ and each real number α we define

$$M(Z, \alpha, u) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{C\in\Gamma} \exp(-\alpha u(C)), \tag{1}$$

where the infimum is taken over all finite or countable collections Γ of cylinder sets of length at least n that cover Z.

When α goes from $-\infty$ to $+\infty$, the quantity in (1) jumps from $+\infty$ to 0 at a unique critical value. Hence, we can define the number

$$\dim_u Z = \inf\{\alpha : M(Z, \alpha, u) = 0\} = \sup\{\alpha : M(Z, \alpha, u) = +\infty\}.$$

We will present a complete multifractal analysis for the u-dimension, specified by a one-sided subshift of finite type $\sigma|\Sigma_A^+$. Let μ be a Borel probability measure on Σ_A^+. For every $x \in \Sigma_A^+$ we write

$$\underline{d}_{\mu,u}(x) = \underline{\lim}_{n\to\infty} -\frac{\log\mu(C_n(x))}{S_n u(x)} \quad \text{and} \quad \overline{d}_{\mu,u}(x) = \overline{\lim}_{n\to\infty} -\frac{\log\mu(C_n(x))}{S_n u(x)}.$$

For every real number α, set $K_\alpha = \{x \in \Sigma : \underline{d}_{\mu,u}(x) = \overline{d}_{\mu,u}(x) = \alpha\} \neq \varnothing$. We set

$$\mathcal{D}_u(\alpha) = \dim_u K_\alpha.$$

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The function $\alpha \mapsto \mathcal{D}_u(\alpha)$ is called the *u-dimension spectrum for u-pointwise dimensions* (with respect to the measure μ). Let φ be a continuous function on Σ. For every real number q, we define the function

$$\varphi_q = -T_u(q)u + q\varphi,$$

where the number $T_u(q)$ is chosen such that $P(\varphi_q) = 0$. We denote by ν_q and m_u, respectively, the equilibrium measures of φ_q and $-\dim_u \Sigma \cdot u$ with respect to σ.

The following is a complete multifractal analysis of the spectrum $\mathcal{D}_u$ for subshifts of finite type. It is taken from [**1**] as a consequence of results in [**2, 9**].

THEOREM 2.1. *Let $\sigma|\Sigma$ be a one-sided topologically mixing subshift of finite type, let u and φ be Hölder continuous functions on Σ, such that u is positive and $P(\varphi) = 0$, and let μ be the equilibrium measure of φ with respect to σ. Then, the following properties hold:*

(1) *For μ-almost every $x \in \Sigma$, the u-pointwise dimension of μ at x exists and*

$$d_{\mu,u}(x) = -\frac{\int_\Sigma \varphi \, d\mu}{\int_\Sigma u \, d\mu} = \frac{h_\mu(\sigma)}{\int_\Sigma u \, d\mu}.$$

(2) *The function $q \mapsto T_u(q)$ is real analytic on $\mathbb{R}$ and satisfies $T_u'(q) \le 0$ and $T_u''(q) \ge 0$ for every $q \in \mathbb{R}$. Moreover, $T_u(0) = \dim_u \Sigma$ and $T_u(1) = 0$.*

(3) *The domain of the function $\alpha \mapsto \mathcal{D}_u(\alpha)$ is a closed interval in $[0, +\infty)$ and coincides with the range of the function $\alpha_u(q) = -T_u'(q)$. For every $q \in \mathbb{R}$, we have*

$$\mathcal{D}_u(\alpha_u(q)) = T_u(q) + q\alpha_u(q),$$

and

$$\alpha_u(q) = -\frac{\int_\Sigma \varphi \, d\nu_q}{\int_\Sigma u \, d\nu_q}.$$

(4) *If $\mu \ne m_u$, then $\mathcal{D}_u$ and T_u are analytic strictly convex functions and, hence, $(\mathcal{D}_u, T_u)$ is a Legendre pair with respect to the variables α, q.*

(5) *If $\mu = m_u$, then $\mathcal{D}_u$ is the delta function*

$$\mathcal{D}_u(\alpha) = \begin{cases} \dim_u \Sigma & \text{if } \alpha = \dim_u \Sigma \\ 0 & \text{if } \alpha \ne \dim_u \Sigma. \end{cases}$$

(6) *For every $q \in \mathbb{R}$, $K_{\alpha_u(q)}$ is everywhere dense, $\nu_q(K_{\alpha_u(q)}) = 1$, and*

$$d_{\nu_q,u}(x) = T_u(q) + q\alpha_u(q)$$

for ν_q-almost all $x \in K_{\alpha_u(q)}$. Moreover, $\mathcal{D}_u(\alpha_u(q)) = \dim_u \nu_q$ for every $q \in \mathbb{R}$.

We call m_u the *measure of maximal u-dimension* (in fact, m_u is the unique σ-invariant measure such that $\dim_u m_u = \dim_u \Sigma$) and ν_q the *full measure* for the spectrum $\mathcal{D}_u$ at the point $\alpha_u(q)$, for each q.

Examples. The next theorem is a combination of results in [**1, 2, 9**].

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THEOREM 2.2.

(1) *If $u \equiv 1$, the spectrum $\mathcal{D}_u$ coincides with the usual entropy spectrum for local entropies, and m_u is the measure of maximal entropy.*

(2) *If $u = \log a$ for some Hölder continuous function a, the spectrum $\mathcal{D}_u$ coincides with the usual dimension spectrum for pointwise dimensions on a repeller of a $C^{1+\varepsilon}$ conformal expanding map f such that $a(x) = \|d_x f\|$ (expressed in terms of its underlying symbolic representation by a subshift of finite type) and m_u is the measure of maximal dimension.*

In the first case we denote $d_{\mu,u}(x) = d_\mu(x)$ and the decomposition into level sets

$$D_\alpha := K_\alpha = \{x : d_\mu(x) = \alpha\}$$

is called the multifractal decomposition of the pointwise dimension.

We also consider the sets

$$D_\alpha^- = \{x : \underline{d}_\mu(x) = \alpha\} \quad \text{and} \quad D_\alpha^+ = \{x : \overline{d}_\mu(x) = \alpha\}.$$

We obtain the multifractal spectrum by defining $f^D(\alpha) = f_\mu^D(\alpha) = G_D(D_\alpha) = \dim_H D_\alpha$. The spectrum f^D is known in the literature as the *dimension spectrum* or *$f(\alpha)$-spectrum for dimensions*. We will omit the subscript μ if there is no confusion. The concept of a multifractal analysis was suggested by a group of physicists in [**6**] (see [**7**] for more references and details).

In the second case $d_{\mu,u}(x)$ equals the local entropy $h_\mu(x)$ with respect to the partition into cylinder sets. The decomposition into level sets consists of the sets

$$E_\alpha := K_\alpha = \{x : h_\mu(F, \xi, x) = \alpha\}.$$

This decomposition is called the multifractal decomposition of the local entropies. We obtain the multifractal spectrum $f^E = f_\mu^E = G_E(E_\alpha) = h(F|_Z)$.

3. *Multifractal spectra are complete*

In this section we are going to prove that $D_\alpha = \varnothing$ if $\alpha \notin [\alpha_1, \alpha_2]$, where the numbers α_1 and α_2 are defined as $\alpha_1 = \lim_{q\to\infty} \alpha(q)$ and $\alpha_2 = \lim_{q\to-\infty} \alpha(q)$ (see §2.3). In fact we show that there is no limit point of $\log \mu(B(x, r))/\log r$ as $r \to 0$ outside the interval $[\alpha_1, \alpha_2]$.

THEOREM 3.1. *We have $\alpha_1 = \inf_{x\in\mathcal{J}} \underline{d}_m(x)$ and $\alpha_2 = \sup_{x\in\mathcal{J}} \overline{d}_m(x)$. Hence,*

$$\mathcal{D}_\alpha = \varnothing \quad \textit{iff } \alpha \notin [\alpha_1, \alpha_2].$$

Remark. Theorem 3.1 says that the theory discussed in §§2.2 and 2.3 reflects the full multifractal decomposition for local entropy and pointwise dimension, i.e. the interval $[\alpha_1, \alpha_2]$ contains all non-empty levels of the measurable functions g_D and g_E, respectively. In particular, the spectrum is an analytic convex function on all but the extremal levels.

Let φ be a Hölder continuous function on Σ_A^+, μ its Gibbs measure on Σ_A^+ and $m = \mu \circ \chi^{-1}$ the corresponding Gibbs state on $\mathcal{J}$. For the proof of the above theorem we need the following lemma.

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LEMMA 3.2.

$$\inf_{\rho\in\mathcal{M}_e(\mathcal{J})} -\frac{\int \log\psi\circ\chi\, d\rho}{\int \log\hat{a}\, d\rho} = \alpha_1 \quad \textit{and} \quad \sup_{\rho\in\mathcal{M}_e(\mathcal{J})} -\frac{\int \log\psi\circ\chi\, d\rho}{\int \log\hat{a}\, d\rho} = \alpha_2.$$

Proof. We present the proof of the first equality. In view of Theorem 2.1 we only have to prove that

$$\inf_{\rho\in\mathcal{M}_e(\mathcal{J})} -\frac{\int \log\psi\circ\chi\, d\rho}{\int \log\hat{a}\, d\rho} \le \alpha_1.$$

Simpelaere proved in [**12**] that

$$T(q) = \inf_{\rho\in\mathcal{M}_e(\mathcal{J})} \frac{h_\rho + q\int \log\psi\circ\chi\, d\rho}{\int \log\hat{a}\, d\rho} = \frac{h_{\nu_q^D} + q\int_{\Sigma_A^+}\log\psi\, d\nu_q^D}{\int_{\Sigma_A^+}\log a\, d\nu_q^D}.$$

For ε fixed and $q<0$ such that $\alpha(q)-\alpha_1<\varepsilon$, this gives

$$\begin{aligned}(1-q)\alpha_1 &\ge (q-1)\frac{\int\log\psi\, d\nu_q}{\int\log a\, d\nu_q} + q\varepsilon = \dim_H \nu_q + q\frac{\int\log\psi\, d\nu_q}{\int\log a\, d\nu_q} + q\varepsilon = T(q)+q\varepsilon\\ &= \inf_{\rho\in\mathcal{M}_e(\mathcal{J})}\left(\dim_H\rho + q\frac{\int\log\psi\circ\chi\, d\rho}{\int\log\hat{a}\, d\rho}\right) + q\varepsilon\\ &\ge -q\inf_{\rho\in\mathcal{M}_e(\mathcal{J})} -\frac{\int\log\psi\circ\chi\, d\rho}{\int\log\hat{a}\, d\rho} + q\varepsilon.\end{aligned}$$

Dividing by $-q$ and letting $q\to-\infty$ gives the first assertion of the lemma. The other equality can be proved in a similar fashion. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. In view of Theorem 2.1 we only have to prove that $\alpha_1 \le \inf_{x\in\mathcal{J}} \underline{d}_m(x)$. Let us assume that this is not the case. By Theorem 2.2 there is a $\Delta>0$, a point $\underline{x}\in\Sigma_A^+$ with $\chi(\underline{x})=x\in\mathcal{J}$ and a subsequence n_k of the natural numbers such that

$$-\frac{\sum_{j=0}^{n_k}\log\psi(\sigma^j\underline{x})}{\sum_{j=0}^{n_k}\log a(\sigma^j\underline{x})} < \alpha_1-\Delta.$$

Let ρ be an accumulation point of the sequence of measures $\rho_k = \frac{1}{n_k}\sum_{j=0}^{n_k-1}\delta_{\sigma^j\underline{x}}$. Obviously,

$$-\frac{\int\log\psi\, d\rho_k}{\int\log a\, d\rho_k} \le \alpha_1-\Delta.$$

Hence,

$$-\frac{\int\log\psi\, d\rho}{\int\log a\, d\rho} \le \alpha_1-\Delta$$

which contradicts Lemma 3.2. This proves the first statement. The proof of the second assertion follows the same lines. □

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4. *There are degenerate multifractal spectra*

We will show that for arbitrary 'allowed' boundary values (i.e. quadruples $(\alpha_1, f(\alpha_1), \alpha_2, f(\alpha_2))$) we can find a Hölder continuous potential whose multifractal dimension spectrum attains these values at its boundary. This means that such a system can be degenerated. Moreover, we always can find a potential such that the system at 'high' or 'low' temperatures has positive entropy or dimension. In particular, the set of points with extremal pointwise dimension (local entropy) may be large in terms of Hausdorff dimension (topological entropy).

Let us fix an expanding repeller $\mathcal{J}$ together with its metric. This gives rise to the symbolic coding space Σ_A^+ with transition matrix A and to the Hölder continuous potential $\log a = \log \|DF\|$ as explained in §2.3. Since the spectrum $f(\alpha) = f^D(\alpha)$ is a convex function and is not above the diagonal (i.e. $f(\alpha) \leq \alpha$ for all $\alpha \in (\alpha_1, \alpha_2)$), its maximum value is equal to $\dim_H \mathcal{J}$ and its graph touches the diagonal to the left of its maximum at $\alpha = \alpha(1) = f(\alpha(1)) = \dim_H m$ we get the obvious restrictions

$$(\alpha_1, f(\alpha_1), \alpha_2, f(\alpha_2)) \in \mathbb{B} \subset \mathbb{R}^4$$

with

$$\mathbb{B} = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : y_1 \leq x_1 \leq \dim_H \mathcal{J}, y_2 \leq \dim_H \mathcal{J} \leq x_2\}.$$

Let

$$\mathbb{B}^0 = \mathbb{B} \cap \{(x_1, y_1, x_2, y_2) : y_1 < x_1 < \dim_H \mathcal{J}, y_2 < \dim_H \mathcal{J} < x_2\}$$

be the set of *admissible boundary values*. The main result of this section is the following.

THEOREM 4.1. *Let $\mathcal{J}$ be an expanding conformal repeller. Then for any quadruple $(x_1, y_1, x_2, y_2) \in \mathbb{B}^0$ there is a Hölder continuous potential $\log \psi$ with dimension spectrum $f(\alpha)$ satisfying*

$$\alpha_i = x_i, \quad f(\alpha_i) = y_i, \quad i = 1, 2.$$

Remark. If $(x_1, y_1, x_2, y_2) \in \mathbb{B}$ and $x_1 = y_1$ or $x_2 = y_2$ or $y_1 = \dim_H \mathcal{J}$ or $y_2 = \dim_H \mathcal{J}$ then the measure m is the measure of maximal dimension m_D and has a degenerate spectrum consisting of the point $(\dim_H \mathcal{J}, \dim_H \mathcal{J})$. We are not going to prove this explicitly but one can derive that no boundary values outside $\mathbb{B}^0$ can be attained from the proof of the above Theorem 4.1 and the well-known fact that any proper subshift (i.e. closed shift-invariant proper subset of Σ_A^+) has topological entropy strictly less than $h_{\text{top}}(\Sigma_A^+)$.

The proof of the theorem relies on a series of lemmas. Some of them are generalizations of well-known facts in symbolic dynamics. However, we choose to prove them with the help of the multifractal analysis in order to show that this analysis can be used to derive results in symbolic dynamics.

The next lemma is due to Simpelaere for repellers equipped with Gibbs measures. It has been proven by several other authors for 'geometric constructions'(see, for example, [**5, 11**]).

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LEMMA 4.2. ([**12**]) *Let* $\log\psi$ *be a Hölder continuous function on* Σ_A^+, m *the corresponding Gibbs measure on* $\mathcal{J}$ *and* f *its dimension spectrum. There exist probability measures* ρ_1 *and* ρ_2 *concentrated on* D_{α_1} *and* D_{α_2}, *respectively, such that*

$$\dim_H(D_{\alpha_1}) = \lim_{q\to+\infty} f(\alpha(q)) = \dim_H \rho_1$$

and

$$\dim_H(D_{\alpha_2}) = \lim_{q\to-\infty} f(\alpha(q)) = \dim_H \rho_2.$$

We also need the following fact.

LEMMA 4.3. *Let* $\log\psi$ *be a Hölder continuous function on* Σ_A^+, m *the corresponding Gibbs measure on* $\mathcal{J}$ *and* f *its dimension spectrum. We have:*

(i) $\dim_H D^-_{\alpha(q)} \le f(\alpha(q))$ *for* $q \ge 0$;

(ii) $\dim_H D^+_{\alpha(q)} \le f(\alpha(q))$ *for* $q \le 0$.

In particular, $\dim_H D^-_{\alpha_i} = \dim_H \rho_i := f_i^D = f_i$, $i = 1, 2$.

Proof. Fix $q \ge 0$. We are going to prove that the s-dimensional Hausdorff measure of $D^-_{\alpha(q)}$ is finite provided that $s > f(\alpha(q)) = T(q) + \alpha(q)q$.

We set $s = f(\alpha(q)) + \varepsilon = T(q) + \alpha(q)q + \varepsilon$, where $\varepsilon > 0$. If $x \in D^-_{\alpha(q)}$ then by Theorem 2.2

$$-\frac{\sum_{j=0}^{n_k}\log\psi(\sigma^j\underline{x})}{\sum_{j=0}^{n_k}\log a(\sigma^j\underline{x})} \le \alpha(q) + \frac{1}{k}$$

for some subsequence $n_k = n_k(x)$ of the naturals. Therefore, we can find a Moran cover (see §2.3) $\mathfrak{C} = \{\underline{x}_1, \dots, \underline{x}_l\}$ of $D^-_{\alpha(q)}$ with arbitrary large depth and multiplicity Q such that

$$\begin{aligned}
\sum_{\underline{x}_i\in\mathfrak{C}} (\operatorname{diam} C_{n_k(\underline{x}_i)})^s &\le \sum_i QB \prod_{j=0}^{n_k(\underline{x}_i)-1} a(\sigma^j\underline{x}_i)^{-s} \\
&\le \sum_i QB \prod_{j=0}^{n_k(\underline{x}_i)-1} a(\sigma^j\underline{x}_i)^{-(T(q)+\alpha(q)q+\varepsilon)} \\
&\le QB \sum_i \prod_{j=0}^{n_k(\underline{x}_i)-1} \psi(\sigma^j\underline{x}_i)^q a(\sigma^j\underline{x}_i)^{-T(q)} \min(\psi)^{-q/k} \max a^{\varepsilon} \\
&\le QB \sum_{\underline{x}_i\in\mathfrak{C}} \mu_q(C_i) \le QB < \infty
\end{aligned}$$

if k is sufficiently large. A similar proof works for the set $D^+_{\alpha(q)}$ when $q \ge 0$. □

In the following sections we derive estimates on the entropy (which equals the Hausdorff dimension in the symbolic space Σ_A^+ equipped with the metric corresponding to

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the constant $a \equiv e$) of larger sets than $D_{\alpha_i}^{\pm}$. These estimations are based on the variational principle for non-compact sets developed by Pesin and Pitskel' (see [**8**] and the Appendix).

Define the *maximum* and *minimum* sets for φ by

$$\overline{\mathcal{M}}(\varphi) = \{\underline{x} \in \Sigma_A^+ : \varphi(\underline{x}) = \max \varphi\} \quad \text{and} \quad \underline{\mathcal{M}}(\varphi) = \{\underline{x} \in \Sigma_A^+ : \varphi(\underline{x}) = \min \varphi\}.$$

LEMMA 4.4. *Let φ be a Hölder continuous function on Σ_A^+ and let f^E be the entropy spectrum for its Gibbs measure. Let us assume that $\overline{\mathcal{M}}(\varphi)$ and $\underline{\mathcal{M}}(\varphi)$ are compact invariant sets. Then*

$$f_1^E = h_{\text{top}}(\overline{\mathcal{M}}(\varphi)) \quad \textit{and} \quad f_2^E = h_{\text{top}}(\underline{\mathcal{M}}(\varphi)).$$

Proof. Since

$$\sup_{\underline{x} \in \Sigma_A^+} \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(\sigma^j \underline{x}) \le \max(\varphi) = \frac{1}{n} \sum_{j=0}^{n-1} \varphi(\sigma^j \underline{y})$$

for all $\underline{y} \in \overline{\mathcal{M}}(\varphi)$, the set $\overline{\mathcal{M}}(\varphi)$ is contained in the set D_{α_1}. Similarly, $\underline{\mathcal{M}}(\varphi) \in D_{\alpha_2}$. Hence,

$$f_1 = f_1^E \le h_{\text{top}}(\overline{\mathcal{M}}(\varphi)) \quad \text{and} \quad f_2 = f_2^E \le h_{\text{top}}(\underline{\mathcal{M}}(\varphi)).$$

For the reverse inequality we observe that

$$\underline{x} \in D_{\alpha_1}^- \quad \text{if and only if} \quad \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(\sigma^j \underline{x}) = \max \varphi.$$

But this is only possible if $V_\sigma(\underline{x}) \cap M_{\text{inv}}(\overline{\mathcal{M}}(\varphi)) \neq \varnothing$, where $V_\sigma(\underline{x})$ is the set of accumulation points of the sequence of measures $\frac{1}{n}\sum_{j=0}^{n-1} \delta_{\sigma^j \underline{x}}$. By Theorem A.2 from the Appendix this yields $f_1 = h_{\text{top}}(\overline{\mathcal{M}}(\varphi))$. The arguments for f_2 are similar. □

Examples. Let $\Sigma_A^+ = \Sigma_N$ be the full shift on N symbols, let $\log a \equiv$ constant $= 1$ and let μ be the Bernoulli measure generated by the probability vector $(p_1, \ldots, p_N)$ (i.e. $\log \psi(\underline{x}) = \log \psi(x_1) = \log p_{x_1}$ where $\underline{x} = x_1, x_2, x_3, \ldots$). Then $f_i = 0$, $i = 1, 2$, if and only if there is an $i_{\max}$ and an $i_{\min}$ such that $p_{i_{\min}} < p_j < p_{i_{\max}}$ for all $j \notin \{i_{\min}, i_{\max}\}$. In this case $\overline{\mathcal{M}}(\log \psi)$ and $\underline{\mathcal{M}}(\log \psi)$ consist of the periodic point $\underline{x} = i_{\min}, i_{\min}, i_{\min}, \ldots$ or $\underline{x} = i_{\max}, i_{\max}, i_{\max}, \ldots$, respectively. Otherwise the sets $\overline{\mathcal{M}}(\log \psi)$ and $\underline{\mathcal{M}}(\log \psi)$ are subshifts of finite type generated by the sets of symbols $\{i : p_i = \min p_j\}$ or $\{i : p_i = \max p_j\}$, respectively. It is not hard to see that the extremal sets $\overline{\mathcal{M}}(\log \psi)$ and $\underline{\mathcal{M}}(\log \psi)$ for measures defined by potentials depending only on a finite number of coordinates (i.e. $\log \psi(\underline{x}) = \log \psi(x_0, \ldots, x_n)$) are subshifts of finite type. This need not to be true for arbitrary Hölder potentials.

PROPOSITION 4.5. *Let $\mathcal{J}$ be an expanding conformal repeller. For any pair of numbers $d_i < \dim_H \mathcal{J}$, $i = 1, 2$, there are disjoint closed invariant subsets $\mathcal{S}_i \subset \mathcal{J}$ such that $\dim_H \mathcal{S}_i = d_i$ and $\lim_{n\to\infty} \frac{1}{n} \log \|DF^n(x)\| = \lambda_i$ exists if $x \in \mathcal{S}_i$.*

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Proof. We fix d_1 and d_2 according to the assumption of the proposition. The mechanism of the proof is to construct inductively closed invariant subsets $\mathcal{S}_i^{(n)}$ which are images of subshifts of finite type in Σ_A^+ under the projection χ and approximate the final sets $\mathcal{S}_i$. Let us denote by μ_D the pull back of the measure of maximal dimension on $\mathcal{J}$ to Σ_A^+. Let $\lambda_D = \lambda_{\mu_D}$, $h_D = h_{\mu_D} = \lambda_D \dim_H \mathcal{J}$ and $\overline{\lambda} = \max\{\log a, 1\}$; $\underline{\lambda} = \min \log a$. We also fix $\varepsilon < \min(\dim_H \mathcal{J} - d_i)\lambda_D$. For $m \in \mathbb{N}$ let $L = 1/\min 4(d_i + 1)$,

$$\Lambda_\varepsilon^m = \left\{ \underline{x} \in \Sigma_A^+ : \left| \frac{1}{n} \sum_{j=0}^{n-1} \log a(\sigma^j \underline{x}) - \lambda_D \right| \le L\varepsilon \text{ for all } n \ge m \right\}$$

and

$$H_\varepsilon^m = \left\{ \underline{x} \in \Sigma_A^+ : \left| h_D - \frac{1}{n} \log \mu_D(C_n \underline{x}) \right| \le \varepsilon \text{ for all } n \ge m \right\}.$$

Finally we put

$$\Gamma_\varepsilon^m = \Lambda_\varepsilon^m \cap H_\varepsilon^m$$

and set

$$\hat{m}^{(1)} = \min\{m \in \mathbb{N} : \mu_D(\Gamma_\varepsilon^m) > 1/2\}.$$

In view of the Birkhoff ergodic theorem and the Shannon–McMillan–Breiman theorem the number $\hat{m}^{(1)}$ is finite and we can define

$$m^{(1)} = \max\left\{ \left\lfloor \frac{4(K+M)\overline{\lambda}(d_i+1)}{\varepsilon} \right\rfloor + 1, \left\lfloor \frac{1 + \log 4}{(\dim_H \mathcal{J} - d_i)\lambda_D - \varepsilon} \right\rfloor, \hat{m}^{(1)} \right\},$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a real number, $K = \|\log a\|_\theta e^{\overline{\lambda}}/(1 - e^{\underline{\lambda}})$ and M is the smallest number such that $A^M > 0$.

For $m \ge m^{(1)}$ we consider

$$\mathfrak{C}_m = \{C_m(\underline{x}) : \underline{x} \in \Gamma_\varepsilon^{m^{(1)}}\}.$$

Then

$$\mu_D(C_m(\underline{x})) \le e^{mh_D + m\varepsilon}$$

and

$$\text{Card}\{\mathfrak{C}_m\} \ge \frac{\mu_D(\Gamma_\varepsilon^m)}{\max \mu_D(C_m(\underline{x}))} \ge \tfrac{1}{2} e^{mh_D - m\varepsilon}.$$

By the choice of ε and $m^{(1)}$ we can choose a subset $\mathfrak{C}_1^{(1)} \in \mathfrak{C}_{m^{(1)}}$ with cardinality $\lfloor \exp\{m^{(1)}(\lambda_D d_1 + d_1\varepsilon/4)\} \rfloor + 1$. We define the subshift $S_1^{(1)}$ as the set of points $\underline{x} \in \Sigma_A^+$ with the property that there is a $l < m^{(1)} + M$ such that for all $j \in \mathbb{N}$ the cylinders $C_{m^{(1)}}(\sigma^{l + j(m^{(1)}+M)} \underline{x})$ are from the set $\mathfrak{C}_1^{(1)}$. According to this construction the set $S_1^{(1)}$ is a subshift of finite type. We set $\mathcal{S}^{(i)} = \chi(S^{(i)})$. We note that all entries in the transition matrix A^M are positive. This means that $S_1^{(1)} \ne \varnothing$ and

$$\underline{\lim}_{k\to\infty} \frac{1}{k} \log \text{Card}\{C_k(\underline{x}) : \underline{x} \in S_1^{(1)}\} \ge \underline{\lim}_{k\to\infty} \frac{1}{k} \log \text{Card}(\mathfrak{C}_1^{(1)})^{\lfloor k/(m^{(1)}+M) \rfloor - 1} \ge \lambda_D d_1 + \frac{\varepsilon}{2}$$

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and

$$\begin{aligned}\overline{\lim_{k\to\infty}}\,\frac{1}{k}\log \mathrm{Card}\{C_k(\underline{x}) : \underline{x}\in S_1^{(1)}\}&\\ \le \overline{\lim_{k\to\infty}}\,\frac{1}{k}\log \mathrm{Card}(\mathfrak{C}_1^{(1)})^{\lfloor k/(m^{(1)}+M)\rfloor}\,\mathrm{rank}(A)^{\lfloor k/(m^{(1)}+M)\rfloor+l}&\\ \le \lambda_D d_1 + \tfrac{3}{2}\varepsilon.&\end{aligned}$$

Hence,

$$\lambda_D d_1 + \tfrac{1}{2}\varepsilon \le h_{\mathrm{top}}(\mathcal{S}_1^{(1)}) \le \lambda_D d_1 + \tfrac{3}{2}\varepsilon.$$

For all $\underline{x}\in S_1^{(1)}$ we have

$$\begin{aligned}\overline{\lim_{k\to\infty}}\,\frac{1}{k}\sum_{j=0}^{k-1}\log a(\sigma^j\underline{x})&\\ \le \overline{\lim_{k\to\infty}}\sum_{l=0}^{\lfloor (m^{(1)}+M)/k\rfloor+1}\frac{1}{m^{(1)}+M}\sum_{j=0}^{m^{(1)}-1}\log a(\sigma^{j+l(m^{(1)}+M)}\underline{x})+M\overline{\lambda})&\\ \le \lambda_D + L\varepsilon + \frac{K}{m^{(1)}+M} + \frac{M}{m^{(1)}+M} \le \lambda_D + (L+1)\varepsilon&\end{aligned}$$

and

$$\begin{aligned}\underline{\lim_{k\to\infty}}\,\frac{1}{k}\sum_{j=0}^{k-1}\log a(\sigma^j\underline{x})&\\ \ge \underline{\lim_{k\to\infty}}\sum_{l=0}^{\lfloor (m^{(1)}+M)/k\rfloor+1}\frac{1}{m^{(1)}+M}\sum_{j=0}^{m^{(1)}-1}\log a(\sigma^{j+l(m^{(1)}+M)}\underline{x})+M\overline{\lambda})&\\ \ge \lambda_D - L\varepsilon - \frac{K}{m^{(1)}+M} - \frac{M}{m^{(1)}+M} \ge \lambda_D - (L+1)\varepsilon.&\end{aligned}$$

Combining the above estimates on the entropy and the Lyapunov exponents and setting $\mathcal{S}_1^{(1)} = \chi(S_1^{(1)})$ we can conclude that

$$d_1 + \frac{\varepsilon}{4} < \dim_H(\mathcal{S}_1^{(1)}) < d_1 + 2\varepsilon.$$

Now we will construct the set $S_2^{(1)} \in \Sigma_A^+ \setminus S_1^{(1)}$. Since $\varepsilon < \min(\dim_H \mathcal{J} - d_i)\lambda_D$ we can choose a subset $\mathfrak{C}_2^{(1)} \in \mathfrak{C}_{m^{(1)}}$ with cardinality $\lfloor \exp\{m^{(1)}(\lambda_D d_2 + d_2\varepsilon/4)\}\rfloor + 1$. We note that there is a cylinder $\hat{C} \in \Sigma_A^+$ of length $m^{(1)}$ which has empty intersection with the subshift $S_1^{(1)}$. Let $m = 8m^{(1)}/\varepsilon$. Then we define the subshift $S_2^{(1)}$ as consisting of all points $\underline{x}\in\Sigma_A^+$ such that there is a $l < m + M$ such that for all $j\in\mathbb{N}$ the cylinders $C_{m+m^{(1)}}(\sigma^{l+j(m+m^{(1)}+M)}\underline{x})$ are of the form

$$C_{m^{(1)},1} * \cdots * C_{m^{(1)},m} * \hat{C},$$

where $C_{m^{(1)},i}\in\mathfrak{C}_2^{(1)}$, $i = 1,\ldots,m$, and $C_1 * C_2$ is the cylinder $C_1 \cap \sigma^{-|C_1|}C_2$. Obviously, $\mathcal{S}_1^{(1)} \cap \mathcal{S}_2^{(1)} = \varnothing$ and $S_2^{(1)}$ is of finite type.

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Proceeding with the same estimations for the entropy and the Lyapunov exponents for the subshift $S_2^{(1)}$ as for the subshift $S_1^{(1)}$ yields

$$\lambda_D d_2 + \tfrac{3}{8}\varepsilon \le h_{\mathrm{top}}(\mathcal{S}_2^{(1)}) \le \lambda_D d_2 + \tfrac{13}{8}\varepsilon.$$

For all $\underline{x} \in S_2^{(1)}$ we have

$$\overline{\lim_{k\to\infty}} \frac{1}{k}\sum_{j=0}^{k-1} \log a(\sigma^j \underline{x}) \le \lambda_D + (L + \tfrac{9}{8})\varepsilon$$

and

$$\underline{\lim_{k\to\infty}} \frac{1}{k}\sum_{j=0}^{k-1} \log a(\sigma^j \underline{x}) \ge \lambda_D - (L + \tfrac{9}{8})\varepsilon.$$

Combining the above estimates we see that for $\mathcal{S}_2^{(1)} = \chi(S_2^{(1)})$,

$$d_2 + \tfrac{\varepsilon}{8} < \dim_H(\mathcal{S}_2^{(1)}) < d_2 + \tfrac{17}{8}\varepsilon.$$

Now let us assume that for given sufficiently small δ and each $l \le n$ we have constructed two subshifts of finite type $S_1^{(l)}$ and $S_2^{(l)}$ with the properties:

(i) $\mathcal{S}_i^{(l)} \subset \mathcal{S}_i^{(l-1)}$;

(ii)

$$\lambda_D d_i + \frac{\delta}{2^{l+1}} \le h_{\mathrm{top}}(\mathcal{S}_i^{(l)}) \le \lambda_D d_i + \frac{\delta}{2^{l-2}};$$

(iii) for all $\underline{x} \in S_i^{(l)}$ we have

$$\overline{\lim_{k\to\infty}} \frac{1}{k}\sum_{j=0}^{k-1} \log a(\sigma^j \underline{x}) \le \lambda_i^{(l)} + \frac{\delta}{2^l} \quad \text{and} \quad \underline{\lim_{k\to\infty}} \frac{1}{k}\sum_{j=0}^{k-1} \log a(\sigma^j \underline{x}) \ge \lambda_i^{(l)} - \frac{\delta}{2^l};$$

(iv) for $\mathcal{S}_i^{(l)} = \chi(S_i^{(l)})$,

$$d_i + \frac{\delta}{2^{l+1}} < \dim_H \mathcal{S}_i^{(l)} < d_i + \frac{\delta}{2^{l-2}},$$

where $i = 1, 2$, $l = 1, \ldots, n$ and $\lambda_i^{(l)}$ are some real numbers. In particular, if we set $\delta = \varepsilon$ we can assume that $\lambda_i^{(1)} = \lambda_D$.

Now we repeat the construction of $S_1^{(1)}$ in Σ_A^+ by substituting Σ_A^+ by $\mathcal{S}_1^{(n)}$ and setting $\varepsilon = \delta/2^n$. The role of the measure of maximal dimension μ_D is played by the lift $\mu_n^{(1)}$ to Σ_A^+ of the measure of maximal dimension of $\mathcal{S}_1^{(n)}$. This means that we construct subshifts of finite type $S_1^{(n+1)}$ inside the subshifts of finite type $S_1^{(n)}$ in the same manner as we constructed $S_1^{(1)}$ inside Σ_A^+. We then get a subshift $S_1^{(n+1)}$ which satisfies inequalities (i)–(iv) for $l = n + 1$.

The subshift $S_2^{(n+1)}$ is constructed in a similar manner by substituting Σ_A^+ by $S_2^{(n)}$ and proceeding as above. We define $\mathcal{S}_i^{(n+1)} = \chi(S^{(n+1)})$.

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In this way for each natural number we have two subshifts of finite type with properties (i)–(iv) and which form two nested sequences. We observe that the numbers $\lambda_i^{(n)}$ converge to a number $\lambda_i = \lambda_i^{(\infty)}$ as n tends to infinity. Let

$$\mathcal{S}_i = \bigcap_{n\geq 1} \mathcal{S}_i^{(n)} \quad \text{and} \quad S_i = \bigcap_{n\geq 1} S_i^{(n)}, \quad i = 1, 2.$$

Then for all $\underline{x} \in S_i$,

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \log a(\sigma^j \underline{x}) = \lambda_i^{(\infty)}$$

and for $\mathcal{S}_i = \chi(S_i)$,

$$\dim_H \mathcal{S}_i \leq \inf_n \dim_H \mathcal{S}_i^{(n)} = d_i.$$

Moreover, $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$ since $\mathcal{S}_1^{(1)} \cap \mathcal{S}_2^{(1)} = \varnothing$. But on the other hand, any accumulation point $\mu_\infty^{(i)}$ of the sequence of measures $\{\mu_n^{(i)}\}$ sits on S_i. By the upper semicontinuity of the metric entropies we derive

$$\dim_H \mathcal{S}_i \geq \dim_H \mu_\infty^{(i)} \circ \chi^{-1} \geq \underline{\lim} \dim_H \mu_n^{(i)} \circ \chi^{-1} = \underline{\lim} \dim_H \mathcal{S}_i^{(n)} = d_i.$$

This completes the proof of the proposition. □

COROLLARY 4.6. $h_{\text{top}}(\mathcal{S}_i) = \lambda_i d_i$.

We are going to define a potential φ on Σ_A^+ with maximum on S_1 and minimum on S_2. We will show that the corresponding Gibbs measure has an entropy spectrum with $f_i = d_i$, $i = 1, 2$.

Let d_1, d_2, S_1 and S_2 be as in the proof of the proposition, i.e. $\mathcal{S}_i = \chi(S_i)$. For $z_1 > z_2$ we consider the following set of functions:

$$\mathcal{F}(z_1, z_2) = \left\{ \varphi \in \mathcal{F}_\theta : \varphi(\underline{x}) = \begin{cases} z_1 & \text{if } \underline{x} \in S_1 \\ z_2 & \text{if } \underline{x} \in S_2 \\ \varphi(\underline{x}) \in (z_2, z_1) & \text{else} \end{cases} \right\}.$$

LEMMA 4.7. *If $d_1 = \dim_H \mathcal{S}_1 < -z_1 < \dim_H \mathcal{J}$ and $-z_2 > \dim_H \mathcal{J}$ then there is a function $\varphi_0 \in \mathcal{F}(z_1, z_2)$ with $P(\varphi_0 \log a) = 0$.*

Proof. Let $U_n^{(1)}$ and $U_n^{(2)}$ be two nested sequences of open sets converging to S_1 and S_2, respectively. We specify for $n \in \mathbb{N}$ two functions

$$\varphi_n^{(1)}(\underline{x}) = \begin{cases} z_1 & \text{if } \underline{x} \notin U_n^{(2)} \\ z_2 & \text{if } \underline{x} \in S_2 \\ \varphi(\underline{x}) \in (z_2, z_1) & \text{else} \end{cases}$$

and

$$\varphi_n^{(2)}(\underline{x}) = \begin{cases} z_1 & \text{if } \underline{x} \in S_1 \\ z_2 & \text{if } \underline{x} \notin U_n^{(1)} \\ \varphi(\underline{x}) \in (z_2, z_1) & \text{else.} \end{cases}$$

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We note that we can find such functions because the sets S_i are closed. The functions $\varphi_n^{(i)}$ are in the closure of the set $\mathcal{F}(z_1, z_2)$. If we denote by $P_n^{(i)}$, $i = 1, 2$, the topological pressure of the Gibbs measures corresponding to $\varphi_n^{(i)} \log a$, respectively we have

$$P_n^{(1)} = \max_{\mu \in M_e(\Sigma_A^+)} \left\{ h_\mu + \int \varphi_n^{(1)} \log a \, d\mu \right\} \geq h_{\mu_D} + \int \varphi_n^{(1)} \log a \, d\mu_D$$
$$\geq \dim_H \mathcal{J} + z_2 \mu_D(U_n^{(2)}) + z_1 \lambda_D (1 - \mu_D(U_n^{(2)})).$$

Because $\dim_H \mathcal{S}_2 < \dim_H \mathcal{J}$ and, consequently, $m_D(\mathcal{S}_2) = 0$ and the fact that μ_D is a Borel measure we see that $\lim_{n\to\infty} \mu_D(U_n^{(2)}) = 0$. Hence, by the assumptions of the lemma

$$\lim_{n\to\infty} P_n^{(1)} \geq \lambda_D \dim_H \mathcal{J} + z_1 > 0.$$

On the other hand

$$P_n^{(2)} = \max_{\mu \in M_e(\Sigma_A^+)} \left\{ h_\mu + \int \varphi_n^{(2)} \log a \, d\mu \right\}$$
$$= \max \left\{ \max_{\mu \in M_1} \left\{ h_\mu + \int \varphi_n^{(2)} \log a \, d\mu \right\}; \max_{\mu \in M_2} \left\{ h_\mu + \int \varphi_n^{(2)} \log a \, d\mu \right\} \right\}$$
$$\leq \max \left\{ h_{\mu_D} + \int \varphi_n^{(2)} \log a \, d\mu_D; h_{\text{top}}(S_1) + \max_{\mu \in M_2} \left\{ \int \varphi_n^{(2)} \log a \, d\mu \right\} \right\}$$
$$\leq \max \left\{ \dim_H \mathcal{J} + z_2 + z_1 \mu_D(U_n^{(1)}); d_1 \lambda_1 + z_1 \lambda_1 + \max_{\mu \in M_2} \{ z_2 \mu(U_n^{(1)}) \} \right\},$$

where $M_1 = \{\mu \in M_e(\Sigma_A^+) : \mu(S_1) = 0\}$ and $M_2 = \{\mu \in M_e(\Sigma_A^+) : \mu(S_1) = 1\}$. Hence, by the assumptions of the lemma and the definition of the sets $U_n^{(1)}$ and $U_n^{(2)}$,

$$\lim_{n\to\infty} P_n^{(1)} \leq \max\{\dim_H \mathcal{J} + z_2; (d_1 + z_1)\lambda_1\} < 0.$$

Since the topological pressure is a continuous functional on the connected set $\mathcal{F}(z_1, z_2)$ the assertion of the lemma follows. □

The following lemma gives us information on the boundary values of the entropy spectrum for the above considered potentials. It will also be used in the following section.

LEMMA 4.8. *Let d_i, S_i, z_i, $i = 1, 2$, be as above and let $\varphi_0 \in \mathcal{F}(z_1, z_2)$ be a potential with $P_\sigma(\varphi_0 \log a) = 0$ according to the previous lemma. If $f^D(\alpha)$ is the dimension spectrum of the Gibbs measure corresponding to $\varphi_0 \log a$ then $f_i^D = d_i$ and $\alpha_i = -z_i$.*

Proof. We are going to show that if $\underline{x} \in D_{\alpha_i}$ then $V_\sigma(\underline{x}) \cap M_{\text{inv}}(S_i) \neq \varnothing$. Since each $\underline{x} \in S_i$ has the property that $V_\sigma(\underline{x}) \cap M_{\text{inv}} S_i \neq \varnothing$, Corollary A.3 of the variational principle yields the assertion, because $\dim_H(\mathcal{S}_i) = d_i$ and

$$\dim_{\mu_{\varphi_0 \log a}}(x) = \frac{\int_{S_i} \varphi_0 \log a \, d\rho}{\int_{S_i} \log a \, d\rho} = z_i$$

for $x \in \mathcal{S}_i$, where $\rho \in V_\sigma(\underline{x})$ and $\mu_{\varphi_0 \log a}$ is the Gibbs measure for the potential $\varphi_0 \log a$. For the latter equality we used the fact that the pressure of $\varphi_0 \log a$ is zero, $\varphi_0 = z_1$ is constant on S_i and Theorems 2.1(1) and 2.2.

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Let us fix $\underline{x} \in \Sigma_A^+$ and assume that $V_\sigma(\underline{x}) \cap M_{\text{inv}}(S_1) = \varnothing$. This means that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{j=0}^{n-1} \varphi_0(\sigma^j \underline{x}) = w < z_1 \quad \text{and} \quad \overline{\lim_{n\to\infty}} A(n, \underline{x}) = \gamma < 1,$$

where $A(n, \underline{x}) = \text{Card}\{G(n)\}$ and $G(n) = \{0 \le j \le n-1 : \sigma^j \ge w\}$. We can conclude that

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \frac{\sum_{j=0}^{n-1} \varphi_0(\sigma^j \underline{x}) \log a(\sigma^j \underline{x})}{\sum_{j=0}^{n-1} \log a(\sigma^j \underline{x})} \\
&\quad \le \overline{\lim_{n\to\infty}} \frac{\sum_{j\in G(n)} \varphi_0(\sigma^j \underline{x}) \log a(\sigma^j \underline{x}) + \sum_{j\notin G(n)} \varphi_0(\sigma^j \underline{x}) \log a(\sigma^j \underline{x})}{\sum_{j=0}^{n-1} \log a(\sigma^j \underline{x})} \\
&\quad \le z_1 - (z_1 - w) \overline{\lim_{n\to\infty}} \frac{(1 - A(n, \underline{x})) \min(\log a)}{\max(\log a)} < z_1.
\end{aligned}$$

Therefore, $\underline{x} \notin D_{\alpha_1}$. The proof for α_2 is similar. □

Proof of Theorem 4.1. Let φ_0 be as in Lemma 4.7 with $z_1 = x_1$, $z_2 = x_2$, $f_1 = y_1$ and $f_2 = y_2$. The proof of Theorem 4.1 is now a concatenation of Proposition 4.5 and Lemma 4.8. □

5. *Typical multifractal spectra are non-degenerate*

In this section we are going to show that a typical (in the sense of Baire category) Hölder continuous potential gives rise to a Gibbs measure with non-degenerate spectrum, i.e. $f_1 = f_2 = 0$. This goes along with the majority of numerical and experimental observations. From this point of view this section contains the most interesting result for applications. We note that the space of Hölder continuous functions on $\mathcal{J}$ as well as the space $\mathcal{F}_\theta$ are Baire spaces with the topology of uniform convergence. We first prove that the above situation holds for entropy spectra of Hölder continuous potentials on Σ_A^+ and then use the same approach as in the proof of Theorem 4.1 to conclude the result for dimension spectra on $\mathcal{J}$. The result can be stated as follows.

THEOREM 5.1. *Let $\mathcal{J}$ be an expanding repeller for the map F. Then there is a residual subset $\mathcal{R} \subset \mathcal{F}_\theta$ such that every Gibbs state on $\mathcal{J}$ which corresponds to a potential in $\mathcal{R}$ has a non-degenerate spectrum.*

We will use the next lemma.

LEMMA 5.2. *For any invariant set S and any natural number m,*

$$\lim_{a\to 1} h_{\text{top}} \left\{ \underline{x} \in \Sigma_A^+ : \underline{\lim_{N\to\infty}} A(C_m(S), N, \underline{x}) \ge aN \right\} = h_{\text{top}}(U_m(S)).$$

Proof. Let $m \in \mathbb{N}$ and $S \in \Sigma_A^+$, $\sigma(S) = S$, be fixed. Since S is invariant the set $U_m(S) \ne \varnothing$. Moreover, $C_m(S)$ is a finite union of cylinder sets and, hence, closed. This implies that the characteristic function $\chi = \chi_{C_m(S)}$ of the set $C_m(S)$ is a Hölder continuous

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potential on Σ_A^+. It is easy to see that the multifractal decomposition D_α of the entropy spectrum of the Gibbs measure corresponding to χ has the following expression:

$$D_\alpha = \left\{ \underline{x} \in \Sigma_A^+ : \lim_{N\to\infty} \frac{A(U_m, N, \underline{x})}{N} = -\alpha + P_\sigma(\chi) \right\}$$

and

$$D_\alpha^- = \left\{ \underline{x} \in \Sigma_A^+ : \overline{\lim_{N\to\infty}} \frac{A(U_m, N, \underline{x})}{N} \geq -\alpha + P_\sigma(\chi) \right\}.$$

Because relative frequencies are bounded in between 0 and 1 we get $\alpha_1 = P_\sigma(\chi) - 1$, $\alpha_2 = P_\sigma(\chi)$ and

$$U_m(S) \subset D_{\alpha_1} \subset \bigcup_{\alpha\in(\alpha_1,\alpha_1+(1-a))} D_\alpha^- = D^-_{\alpha_1+(1-a)}$$
$$= \left\{ \underline{x} \in \Sigma_A^+ : \underline{\lim_{N\to\infty}} A(C_m(S), N, \underline{x}) \geq aN \right\}.$$

Together with Lemma 4.3 this implies the assertion of the lemma. □

The following proposition deals with the statements of Theorem 5.1 in the case of the entropy spectrum of Gibbs measures on Σ_A^+.

PROPOSITION 5.3. *There is a residual subset $\Theta \subset \mathcal{F}_\theta$ such that the entropy spectra for Gibbs measures corresponding to potentials in Θ are non-degenerated.*

Proof. The idea of the proof is to find a residual set $\Theta \subset \mathcal{F}_\theta$ such that any function $\varphi \in \Theta$ has $h_{\text{top}}(\overline{\mathcal{M}}(\varphi)) = h_{\text{top}}(\underline{\mathcal{M}}(\varphi)) = 0$. The assertion then follows from Lemma 4.4. First we find for any $\varepsilon > 0$ an open dense subset $\Theta_1^\varepsilon \subset \mathcal{F}_\theta$.

Let us fix $\varepsilon > 0$, $r \in \mathbb{N}$ and $\varphi \in \mathcal{F}_\theta$. Then there is a periodic point $\underline{x}_0$ with some period n_0 such that

$$\text{Max}_\varphi - \frac{\varepsilon}{r} := \sup_{\underline{x}\in\Sigma_A^+} \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(\sigma^j \underline{x}) - \frac{\varepsilon}{r} < \frac{1}{n_0} \sum_{j=0}^{n_0-1} \varphi(\sigma^j \underline{x}_0) < \text{Max}_\varphi$$

We consider the set $C = C_{n_0}(S_{\underline{x}_0}) = \bigcup_{j=0}^{n_0-1} C_{n_0}(\sigma^j \underline{x}_0)$, where $S_{\underline{x}_0}$ is the trajectory of the periodic point $\underline{x}_0$. We observe that $U = U_{n_0}(S_{\underline{x}_0}) = S_{\underline{x}_0}$ and hence has zero entropy. We define a new function close to φ by

$$\hat{\varphi}(\underline{x}) = \begin{cases} \varphi(\underline{x}) & \text{if } \underline{x} \in C \\ \varphi(\underline{x}) - \varepsilon & \text{else.} \end{cases}$$

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Clearly, $\hat{\varphi} \in \mathcal{F}_\theta$ and $\|\varphi - \hat{\varphi}\|_\theta < \varepsilon$. We will see that this function $\hat{\varphi}$ has the property that $h_{\text{top}}(\overline{\mathcal{M}}(\hat{\varphi})) < \varepsilon$. For $\underline{x} \in \Sigma_A^+$ and $n \in \mathbb{N}$ we estimate

$$\begin{aligned}
\frac{1}{n}\sum_{j=0}^{n-1}\hat{\varphi}(\sigma^j\underline{x}) &= \frac{1}{n}\Bigg(\sum_{\sigma^j\underline{x}\in C}\hat{\varphi}(\sigma^j\underline{x}) + \sum_{\sigma^j\notin C}\hat{\varphi}(\sigma^j\underline{x})\Bigg)\\
&\le \frac{1}{n}\Bigg(\sum_{\sigma^j\underline{x}\in C}\varphi(\sigma^j\underline{x}) + \sum_{\sigma^j\notin C}\varphi(\sigma^j\underline{x}) - (n - A(C,n,\underline{x}))\varepsilon\Bigg)\\
&\le \frac{1}{n}\sum_{j=1}^{n-1}\varphi(\sigma^j\underline{x}) - \left(1 - \frac{A(C,n,\underline{x})}{n}\right)\varepsilon \le \text{Max}_\varphi - \left(1 - \frac{A(C,n,\underline{x})}{n}\right)\varepsilon\\
&\le \text{Max}_{\hat{\varphi}} + \frac{\varepsilon}{r} - \left(1 - \frac{A(C,n,\underline{x})}{n}\right)\varepsilon
\end{aligned}$$

because

$$\text{Max}_{\hat{\varphi}} \ge \frac{1}{n_0}\sum_{j=0}^{n_0-1}\varphi(\sigma^j\underline{x}_0) \ge \text{Max}_\varphi - \frac{\varepsilon}{r}.$$

Hence,

$$\begin{aligned}
\overline{\mathcal{M}}(\hat{\varphi}) &\subset \left\{\underline{x}\in\Sigma_A^+ : \overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(\sigma^j\underline{x}) = \text{Max}_{\hat{\varphi}}\right\}\\
&\subset \left\{\underline{x}\in\Sigma_A^+ : \overline{\lim_{n\to\infty}}\frac{A(C,n,\underline{x})}{n} \ge 1 - \frac{1}{r}\right\}.
\end{aligned}$$

In view of Lemma 5.2 we can find a $\varepsilon > 0$ such that

$$\varepsilon > h_{\text{top}}\left\{\underline{x}\in\Sigma_A^+ : \overline{\lim_{n\to\infty}}\frac{A(C,n,\underline{x})}{n} \ge 1 - \frac{1}{r}\right\} \ge h_{\text{top}}(\overline{\mathcal{M}}(\hat{\varphi})).$$

We set

$$\Omega_1^\varepsilon(\varphi) := \{\psi \in \mathcal{F}_\theta : \|\hat{\varphi} - \psi\|_\theta < \varepsilon\}.$$

The above estimations still hold with $\varepsilon' = 2\varepsilon$ for all potentials $\psi \in \Omega_1^\varepsilon(\varphi)$. Therefore, for all $\varepsilon > 0$ the set

$$\Theta_1^\varepsilon := \bigcup_{\varphi\in\mathcal{F}_\theta}\Omega_1^\varepsilon(\varphi)$$

is an open and dense set in $\mathcal{F}_\theta$ with the property that the entropy of the maximum sets $\overline{\mathcal{M}}(\varphi)$ of its elements φ is less then 2ε. We repeat the same procedure for the set $\underline{\mathcal{M}}$ and produce a set Θ_2^ε with the property that $h_{\text{top}}(\underline{\mathcal{M}}(\varphi)) < 2\varepsilon$ for all $\varphi \in \Theta_2^\varepsilon$. Let

$$\Theta := \bigcap_{N=0}^{\infty}\bigcup_{n\ge N}(\Theta_1^{1/n} \cap \Theta_2^{1/n}).$$

Now we observe that functions $\varphi \in \Theta$ have the property that

$$h_{\text{top}}(\overline{\mathcal{M}}(\varphi)) = h_{\text{top}}(\underline{\mathcal{M}}(\varphi)) = 0.$$

Moreover, by the way of constructing the set Θ we immediately have that Θ is a residual subset of $\mathcal{F}_\theta$. □

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Proof of Theorem 5.1. As we have seen in the proof of Theorem 4.1 and of Lemma 4.8, the boundary values of the dimension spectrum of the Gibbs measure $m = \mu \circ \chi^{-1}$ on $\mathcal{J}$ adjoint to the potential $\psi(\underline{x}) = \varphi(\underline{x}) \log a(\underline{x})$ coincide with those of the entropy spectrum of the Gibbs measure μ on Σ_A^+ corresponding to the potential φ. The proof of Theorem 5.1 concludes by observing that the operator $\mathcal{L}_a$ on $\mathcal{F}_\theta$ defined by $\mathcal{L}_a(\varphi(\underline{x})) = \varphi(\underline{x}) \log a(\underline{x})$ is in fact a homeomorphism of the space $\mathcal{F}_\theta$ (in particular, it transforms residual sets into residual sets) because $\log a$ is bounded away from zero. □

6. *Concluding remarks*

In the previous sections we proved results about the behavior of the dimension spectrum of a Gibbs state on an expanding conformal repeller at its boundary. We used an approach which is related to the entropy spectrum of the underlying symbolic space, but stated the results only in terms of the dimension spectrum. However, the corresponding results for the entropy spectrum can easily be derived from the dimension results by considering the symbolic space Σ_A^+ itself as an expanding conformal repeller equipped with the metric $d(\underline{x}, \underline{x}') = \sum_{k=1}^{\infty} e^{-k}|i_k - i_k'|$. This metric corresponds to the stretching rate by the constant factor e. Then the dimension spectrum for a Gibbs measure μ on σ_A^+ coincides with the entropy spectrum for the Gibbs measure $m = \mu \circ \chi$ on $\mathcal{J}$. Moreover, the same considerations are successful in a more general situation. Namely, we can derive the same results for the entropy spectrum for *any* map that admits a finite Markov partition, i.e. admits a coding by some symbolic space Σ_A^+. Axiom A basic sets in any finite dimension belong to this class. The situation with the dimension spectrum is more complicated. The main problem in generalizing the results for expanding conformal repellers to non-conformal maps in higher dimensions is that there is no general theory of computing the dimension of an invariant set or measure. In particular, the Bowen–Ruelle dimension formula $T_D(0) = \dim_H \mathcal{J}$ need not hold. However, the results on multifractal dimension spectra can be verified for two-dimensional hyperbolic horseshoes. This investigation was done in [**3, 10, 12**]. Using the same techniques for two-sided subshifts instead of one-sided, one can derive in exactly the same way the corresponding results for two-dimensional hyperbolic horseshoes.

One could also consider the dimension spectrum for Lyapunov exponents. The corresponding multifractal analysis was investigated in [**13**]. The difference to our considerations is that the Lyapunov spectrum does not depend on the measure, so the spectrum is an invariant of the metric.

Appendix

For a subset $U \in \Sigma_A^+$ we set $Z = Z(U) = \{\underline{x} \in U : V_\sigma(\underline{x}) \cap M_{\text{inv}}(U) \neq \varnothing\}$, where $M_{\text{inv}}(U)$ is the set of invariant measures concentrated on U. We also use the notation $M_{\text{e}}(U)$ for the set of ergodic measures on U. Pesin and Pitskel' have proved the following variational principle.

THEOREM A.1. (Pesin and Pitskel' [**8**])

$$h_{\text{top}}(Z) = \sup_{\mu \in M_{\text{inv}}(U)} h_\mu.$$

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We will need the following result of Bowen [**4**].

THEOREM A.2. *Let $f : X \to X$ be a continuous map of a compact metric space. Then for*

$$Y_t := \{x \in X : \text{there is a } \mu \in V_f(x) \text{ with } h_\mu \leq t\}$$

the following inequality holds:

$$h_{\text{top}}(f, Y_t) \leq t.$$

We will state a simple consequence of this theorem.

COROLLARY A.3. *Let S be a compact invariant set. Then*

$$h_{\text{top}}\{\underline{x} \in \Sigma_A^+ : V_\sigma(\underline{x}) \cap M_{\text{inv}}(S) \neq \varnothing\} = h_{\text{top}}(S).$$

Proof. Since S is compact $S \subset \{\underline{x} \in \Sigma_A^+ : V_\sigma(\underline{x}) \cap M_{\text{inv}}(S) \neq \varnothing\}$, and we have $h_{\text{top}}\{\underline{x} \in \Sigma_A^+ : V_\sigma(\underline{x}) \cap M_{\text{inv}}(S) \neq \varnothing\} \geq h_{\text{top}}(S)$. On the other hand, $h_\mu \leq h_{\text{top}}(S)$ for all measures $\mu \in M_{\text{inv}}(S)$. Hence, $\{\underline{x} \in \Sigma_A^+ : V_\sigma(\underline{x}) \cap M_{\text{inv}}(S) \neq \varnothing\} \subset Y_{h_{\text{top}}(S)}$ and Bowen's Theorem A.2 implies the assertion. □

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