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Discriminant functions for isotropic configurations in robot manipulators

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SUMMARY
In this paper, a family of discriminant functions is defined for searching isotropic configurations in robot manipulators. A subfamily of the functions extensively exploits properties of symmetric polynomials derived from a manipulability matrix. The complexity analysis of computing the discriminant functions is provided. Possible applications of the functions are mentioned and illustrated.

KEYWORDS: Manipulator; Isotropy; Discriminant function; Complexity; Robotics; Applications

1. Introduction

In order to design a structure and/or physical parameters of a robot and/or to select its desirable behavior in a factory floor or even to analyze biological systems from a technical perspective, quality criteria are required to verify a design, a trajectory, or a biological system. Usually, the criteria are based on physical principles and reflect desired properties of a system (for example, avoiding boundary configurations, avoiding obstacles, minimizing an energy consumption, preserving motion ability in each direction, and avoiding singularities). Sometimes, based on observations of natural systems, one is eager to know what a criterion is minimized when series of data gather from a nature are collected.[1] Quality criteria concern with various aspects of a task analyzed. They can deal with kinematics,[2] dynamics, force transmission,[3] and others.[4] Most of them are based on a manipulability matrix (introduced in robotics by Yoshikawa[5,6]) derived from the Jacobian matrix of a robot.

In robotics, isotropic configurations are desirable because at those configurations, a short-distance motion in each direction in the taskspace is equicostly. In practice, some coordinates of the taskspace differ in nature (position, orientation) and their ranges. In this paper, it will be assumed that all coordinates are comparable (i.e., they are scaled, by redefining appropriately robot kinematics) and isotropic configurations and their neighborhood will be investigated for a stationary manipulator. Some research has been done in this domain. Quality criteria were proposed and analyzed[2] at first for manipulators then expanded to other classes of robotic systems (mobile platform, mobile manipulators,[7] and parallel manipulators[8,9]). Kircanski[10] has calculated isotropic configurations for simple (planar) structures where analytic (closed-form) formulas can be derived.

In this paper, it will be shown that isotropic configurations are relatively rare. Consequently, approaching rather reaching them is advisable. To achieve this goal, reliable measures are required describing how close is a given configuration to a set of isotropic configurations.

The outline of the paper follows. In Section 2, necessary robotic preliminaries are presented. Kinematics, Jacobian, and manipulability matrix are defined. Some qualitative questions on existence, in generic case, of isotropic configurations are posed and answered. In Section 3, a variety of discriminant functions is defined. In Section 4, computational complexity issues are addressed. Moreover, an equivalence of a particular discriminant function based on symmetric polynomials with those discriminant functions based on zeroing off-diagonal elements of the manipulability matrix is proved. Section 5 presents two possible applications of discriminant functions. First one concerns with finding a configuration that is as close to isotropic one as possible. The goal of the second application is to plan a motion along a given curve in the taskspace with minimization of a discriminant function as a secondary criterion. In Section 6, the two applications were illustrated on the model of 3D planar pendulum manipulator. Section 7 concludes the paper.

2. Preliminaries

A manipulator, when considered at a kinematic level, is described by its kinematics

$$\mathbb{Q} \ni \boldsymbol{q} \to \boldsymbol{x} = \boldsymbol{k}(\boldsymbol{q}) \in \mathbb{X}, \quad \dim(\mathbb{Q}) = n \geq m = \dim(\mathbb{X}), \tag{1}$$

which assigns to each configuration $\boldsymbol{q} = (q_1, \ldots, q_n)$ from the configuration space $\mathbb{Q}$, a point $\boldsymbol{x}$ in the taskspace $\mathbb{X}$, which is a subset of the Euclidean group $SE(3)$. Elements of the group are composed of matrices $\left[\begin{smallmatrix} \boldsymbol{R} & \boldsymbol{d} \\ \boldsymbol{0} & 1 \end{smallmatrix}\right]$, $\boldsymbol{R} \in SO(3), \boldsymbol{d} = (x, y, z)^T \in \mathbb{R}^3$, where $SO(3)$ denotes a special orthogonal group. Because $\dim SO(3) = 3$, $\boldsymbol{R}$ is usually coded with only three items, for example, Euler or RPY angles.[11] Kinematics (1) is called nonredundant one when $n = m$ and redundant one if only $n > m$. This classical concept of redundancy, respected in this paper, assumes that each joint is driven by a single motor. Recently, nonholonomic manipulators have been constructed, [12] which preserve a full

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motion ability in the taskspace with smaller than m number of controls.

Many kinematic quality measures exploit a manipulability matrix $\boldsymbol{M}(\boldsymbol{q})$ based on a Jacobian matrix of the manipulator $\boldsymbol{J}(\boldsymbol{q}) = \partial \boldsymbol{k}/\partial \boldsymbol{q}$. The manipulability matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q}) \tag{2}$$

is square, $(m \times m)$, and nonnegatively defined. It is well known[13] that any (also nonsquare) matrix $\boldsymbol{A}_{m\times n}$ can be decomposed with the Singular Value Decomposition (SVD) algorithm into a product of three matrices

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{V}^T, \tag{3}$$

where $\boldsymbol{U}_{m\times m} \in SO(m)$, $\boldsymbol{V}_{n\times n} \in SO(n)$ are rotation matrices in m and n dimensional spaces, respectively, while $\boldsymbol{D}_{m\times n}$ is a diagonal matrix with nonnegative numbers, on the main diagonal that are called singular values, σ_i, $i = 1, \ldots, m$. When a manipulability matrix (2) is decomposed with the SVD algorithm (3), on the main diagonal of the matrix $\boldsymbol{D}(\boldsymbol{q})\boldsymbol{D}^T(\boldsymbol{q})$ stand squares of singular values of $\boldsymbol{J}(\boldsymbol{q})$ as

$$\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{U}(\boldsymbol{q})\boldsymbol{D}(\boldsymbol{q})\boldsymbol{D}^T(\boldsymbol{q})\boldsymbol{U}^T(\boldsymbol{q}). \tag{4}$$

Based on the manipulability matrix $\boldsymbol{M}(\boldsymbol{q})$, well-known quality indices were introduced:[10]

– a conditional number

$$cond(\boldsymbol{M}(\boldsymbol{q})) = \frac{\max_i \sigma_i(\boldsymbol{q})}{\min_i \sigma_i(\boldsymbol{q})}, \tag{5}$$

– a volume of the manipulability ellipsoid

$$\begin{aligned} vol(\boldsymbol{M}(\boldsymbol{q})) &= \sqrt{\det(\boldsymbol{M}(\boldsymbol{q}))} \\ &= \sqrt{\det(\boldsymbol{D}(\boldsymbol{q})\boldsymbol{D}^T(\boldsymbol{q}))} = \prod_{i=1}^{m} \sigma_i(\boldsymbol{q}). \end{aligned} \tag{6}$$

At isotropic configurations, all singular values attain the same value

$$\sigma = \sigma_i, \quad i = 1, \ldots, m, \tag{7}$$

while at singular configurations at least one of them takes the value of 0. Nonisotropic (anisotropic) are those configurations that are not isotropic. Isotropic configurations are important because at those configurations, a motion in each direction in the taskspace is equicostly and the manipulability ellipsoid becomes a perfect m-dimensional sphere. It is easy to verify that at isotropic configurations, the condition number attains its minimal value of 1. Both of quality indices (5) and (6) can serve as detectors of singular configurations. At first, an easy to prove fact can be stated as follows:

Fact 1: Singular values do not depend on the first coordinate of the configuration vector.

Proof: Two cases will be considered depending on the type of the first joint (revolute or prismatic one). Let $O_0x_0y_0z_0$ denotes fixed, right-oriented base coordinate frame with its origin placed at O_0 with axes x_0, y_0, z_0. Without loosing generality, it is assumed that kinematics of a manipulator considered is described by the Denavit–Hartenberg convention.[11] In this case, transformations between consecutive $(i-1) \to i$, $i = 1, \ldots, n$ frames are expressed as follows:

$$\begin{aligned} SE(3) \ni T_{i-1}^{i} &= Rot(z, \Theta_i)Trans(z, d_i) \\ &\quad \times Trans(z, a_i)Rot(x, \alpha_i), \end{aligned} \tag{8}$$

where $Rot(a, \beta)$ represents the rotation matrix around axis a by the angle β while $Trans(a, v)$ denotes the matrix representation of the shift by the vector v along the axis a.

It is well known from the matrix analysis[14] that singular values do not change when the base coordinate frame is rotated by any angle around any axis passing through the origin O_0 of that frame. Consequently, if the first joint is a revolute one, the singular values do not depend on the value of the first rotational component of $\boldsymbol{q}$. If the first joint is prismatic one ($T_0^1 = Rot(z, \Theta_1)Trans(z, q_1)Trans(x, a_1)Rot(x, \alpha_1)$), we can rotate the base coordinate frame by $Rot(z, \Theta_1)$ without influencing the singular values. In the new base coordinate frame, the coordinate q_1 impacts only the z-coordinate of the generalized position vector in a taskspace. The z-coordinate is equal to $z = q_1 + f(q_2, \ldots, q_n)$. The translational coordinate q_1 does impact neither rotational coordinates of the taskspace nor positional coordinates x, y. It can influence the manipulability matrix $\boldsymbol{M}(\boldsymbol{q})$ only by $\partial \boldsymbol{z}/\partial q_1$. Due to the form of the z-coordinate of the taskspace, the differentiation excludes the impact of q_1 on the manipulability matrix and consequently on singular values as well. □

Now, the meaning of the generic case will be explained. As a rule, to determine isotropic configurations, a set of algebraic equations with an independent vector $\boldsymbol{q}$ is to be formulated and solved. The generic case appears when the equations are independent of each other, i.e., none of them can be expressed as a (linear or nonlinear) combination of the others. However, at some configurations, they may become dependent. Practically, in the generic case, a set of equations with n independent variables has not got any solution when the number of equations is larger than n.

In order to formulate conditions defining isotropic configurations, it is assumed that an isotropic configuration $\boldsymbol{q}$ was found and $\sigma_1 = \ldots = \sigma_m = \sigma$. In this case, cf. Eq. (4)

$$\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{U}(\boldsymbol{q})\sigma^2(\boldsymbol{q})\boldsymbol{U}^T(\boldsymbol{q}) = \sigma^2(\boldsymbol{q})\boldsymbol{I}_m, \tag{9}$$

where $\boldsymbol{I}_m$ is the $(m \times m)$ identity matrix.

Now, for a fixed taskspace dimensionality m, it will be established how large must be a configuration space to have a chance to contain isotropic configurations.

Fact 2: In generic case, $C(m)$ conditions,

$$C(m) = (m-1)m/2 + m - 1 = (m-1)(m+2)/2, \tag{10}$$

have to be satisfied to get isotropic configurations.

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Table I. Dimensionality of the minimal isotropic configuration space $n(m)$ for selected values of the taskspace dimensionality m. $n(m) - m$ denotes the minimal redundancy required to get isotropic configurations in a generic case.

m	2	3	4	5	6	7	8
C(m)	2	5	9	14	20	27	35
n(m)	3	6	10	15	21	28	36
n(m)-m	1	3	6	10	15	21	28

Proof: There is $(m-1)m/2$ conditions to cancel elements out of the main diagonal in a symmetric $(m \times m)$ manipulability matrix $\boldsymbol{M}$, cf. Eq. (9), while $m-1$ conditions are due to identical elements on the main diagonal of the matrix. □

Because each condition imposed on an isotropic configuration requires one dimension in the configuration space and the first coordinate of $\boldsymbol{q}$ does not impact singular values of the manipulability matrix, cf. Fact 1, thus the configuration space must be at least

$$n(m) = C(m) + 1 \tag{11}$$

dimensional to have a chance to contain isotropic configurations. The dimensionality $n(m)$ will be called the generic dimensionality of a minimal isotropic configuration space. In Table I, dimensionality of the minimal isotropic configuration space was displayed for some values of $m > 1$ together with a required redundancy $n(m) - m$ of a manipulator. The $m = 1$ case was excluded from considerations because a single singular value σ_1 cannot be compared with any other and the concept of isotropy is meaningless. Based on Eq. (10) exemplified in Table I, the following fact can be stated:

Fact 3: In generic case, there is no isotropic configurations in nonredundant manipulators.

Proof: From Eqs. (10) and (11), it results that $n(m) > C(m) \geq m$, in particular, $n(m) > m$. Consequently, in generic case, there is no isotropic configurations for nonredunant manipulators as for the manipulators $n = m$. □

Fact 4: In generic case, isotropic configurations are also rare for redundant manipulators.

Proof: Respecting Eqs. (10) and (11), in a generic case, the minimal redundancy required to obtain isotropic configurations equals $n(m) - m = m(m-1)/2$. It appears that for $m > 1$ this value is positive and increases rapidly with m. □

Fact 4 is illustrated in the last raw of Table I. For a manipulator with a full (position and orientation) motion ability $m = 6$, the redundancy is enormous.

Till now, only qualitative questions were stated on a possible existence of isotropic configurations without paying attention how to compute them. A straightforward attempt to derive analytically singular values of a Jacobian matrix for a given manipulator is difficult if ever possible for nontrivial manipulators. In robotic literature,[10] only for low-dimensional and planar structures closed-form expressions for singular values have been obtained. A numerical computation of singular values of $J(\boldsymbol{q})$ may be utilized to check whether a given configuration (in this case, the Jacobian matrix is filled out with constant rather than configuration-dependent values) is isotropic or not. As isotropic configurations are really rare, there is a practical need to determine configurations close to isotropic ones. Consequently, there is a necessity to find some nonnegative functions, which vanish only at isotropic configurations and take values close to zero when the configurations are approached to. Such functions will be called discriminant functions for isotropic configuration (or discriminant functions for short). The next section will be devoted to a construction of the functions.

3. Discriminant Functions for Isotropic Configurations

In order to find discriminant isotropic functions, some preliminary definitions and properties of symmetric polynomials are required. Symmetric polynomials will be constructed based on singular values of the manipulability matrix (equivalently squares of the singular values of the original Jacobian matrix, cf. Eq. (4)) taken as independent variables $\boldsymbol{y}$. The kth elementary symmetric polynomial in m variables $y_1, \ldots, y_m$ is defined as

$$e_k(y_1, \ldots, y_m) = \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq m} y_{j_1} \cdot y_{j_2} \cdot \ldots \cdot y_{j_k}, \quad k = 1, \ldots, m, \tag{12}$$

while the kth elementary symmetric mean equals

$$S_k(y_1, \ldots, y_m) = \frac{e_k(y_1, \ldots, y_m)}{\binom{m}{k}}, \quad k = 1, \ldots, m, \quad S_0 = 1, \tag{13}$$

where $\binom{m}{k}$ denotes the binomial symbol. Two chains of inequalities are known in literature.[15] The Newton inequalities, valid for all real numbers, state that

$$S_i^2 \geq S_{i-1} S_{i+1}, \quad i = 1, \ldots, m-1, \tag{14}$$

while the Maclaurin chain of inequalities

$$S_1 \geq \sqrt{S_2} \geq \sqrt[3]{S_3} \geq \ldots \geq \sqrt[m]{S_m} \tag{15}$$

is satisfied for nonnegative real numbers.

Fact 5:[15] *In Eqs.* (14) *and* (15)*, equality holds, if and only if, all variables in Eq.* (12) *are equal to each other.*

In robotic terms, Fact 5 says that at isotropic configurations all $S_i, i = 1, \ldots, m$, take the same value, and the same value of S_i's is attained at isotropic configurations. Consequently, construction of discriminant functions can be based on S_i's rather on computing singular values themselves. Later on a physical meaning will be given to symmetric polynomials S_i.

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Rather informally (denominators must take nonzero values) Eq. (13) will be rewritten in the form

$$\frac{S_1}{S_0} \geq \frac{S_2}{S_1} \geq \frac{S_3}{S_2} \geq \cdots \geq \frac{S_m}{S_{m-1}} \tag{16}$$

to see that many equivalent conditions for isotropic configurations can be derived. Selecting the ith and the jth quotient from the chain of inequalities (16), the family of discriminant functions can be defined

$$\frac{S_i}{S_{i-1}} \geq \frac{S_j}{S_{j-1}} \Rightarrow B_{ij} = S_i S_{j-1} - S_{i-1} S_j \geq 0, \quad 1 \leq i < j \leq m. \tag{17}$$

B_{ij} is a discriminant function because it is nonnegative and takes the value of zero only at isotropic configurations ($S_0 = \cdots = S_m$), cf. Fact 4. The family (17) is composed of $m(m-1)/2$ items. Obviously, many more discriminant functions can be defined taking more numerous set of conditions. It is naturally to define the order of a discriminant function expressed in polynomial form by the degree of the polynomial. According to this definition, discriminant functions (17) are of the minimal order equal to 2. An example of discriminant functions of order 4 follows:

$$\frac{S_2}{S_1} \cdot \frac{S_4}{S_3} \geq \frac{S_3}{S_2} \cdot \frac{S_5}{S_4} \Rightarrow S_2^2 S_4^2 - S_1 S_3^2 S_5 \geq 0. \tag{18}$$

Discriminant functions of order higher than two are redundant ones and will not be used later on as they are computationally more expansive and carry out no more information that the functions of order two.

Discriminant functions (17) are based on singular values of the Jacobian matrix and singular values are difficult to get in a closed form. Fortunately, there is no need to compute singular values explicitly.

Let a $(k \times k)$ main submatrix $\boldsymbol{M_a}$ of $\boldsymbol{M}$ arises from the matrix by selecting k indices out of $m \geq k$ and taking into the submatrix elements from crossings of rows and columns corresponding to the selection set. There is $\binom{m}{k}$ main submatrices. Let us define

$$E_k(\boldsymbol{M}) = \frac{1}{\binom{m}{k}} \sum_{\boldsymbol{a} \in C_m^k} \det(\boldsymbol{M_a}), \tag{19}$$

where C_m^k are k element combinations taken from the set $\{1, \ldots, m\}$. In particular, $E_1 = \mathrm{tr}(\boldsymbol{M})/m$ and $E_m = \det(\boldsymbol{M})$, where $\mathrm{tr}(\boldsymbol{M})$ is a trace of the manipulability matrix while $\det(\boldsymbol{M})$ denotes its determinant. The theorem presented in[14] and mentioned by Niculescu[15] states that for any square matrix $\boldsymbol{M}$ (in our case, the manipulability matrix)

$$S_k(\boldsymbol{M}) = E_k(\boldsymbol{M}). \tag{20}$$

Consequently, substituting E_k for S_k into Eq. (17), a family of discriminant functions can be generated that depends on configuration-dependent items of the manipulability matrix. Now, a physical explanation of $S_k = E_k(\boldsymbol{M})$ can be given. According to Eq. (19), $E_k(\boldsymbol{M})$ can be interpreted as an averaged volume of all possible k-dimensional parallelepipeds, $1 \leq k \leq m$, spanned by singular values of the manipulability matrix. In this interpretation, $E_1 = \mathrm{tr}(\boldsymbol{M})$ is an averaged value from all singular values while E_m is a volume of the manipulability ellipsoid having singular values as the length of its semiaxes.

Another useful and analytic method to determine conditions for isotropic configurations relies on zeroing upper-triangular elements of the manipulability matrix $\boldsymbol{M} = [a_{ij}]$, $i, j = 1, \ldots, m$ and forcing the same values on the main diagonal of the matrix, cf. Eq. (9). This approach generates the following (exemplary) discriminant function:

$$B = \left(\max_i a_{ii} - \min_i a_{ii}\right) + \sum_{1 \leq i < j \leq m} |a_{ij}| \geq 0, \tag{21}$$

or another discriminant function, this time in a differentiable form

$$B = \sum_{i=1}^{m-1} (a_{ii} - a_{i+1,i+1})^2 + \sum_{1 \leq i < j \leq m} a_{ij}^2 \geq 0. \tag{22}$$

In Eqs. (21) and (22), most of the terms can be weighted with positive coefficients preserving properties constituting discriminant functions.

One can write down more discriminant functions of type (22). For example,

$$B_j = \sum_{i=1, i \neq j}^{m} (a_{jj} - a_{ii})^2 + \sum_{1 \leq i < j \leq m} a_{ij}^2 \geq 0. \tag{23}$$

4. Computational Complexity and Equivalence of Discriminant Functions

Having defined a lot of discriminant functions, it is instructive to evaluate their computational complexity. For a particular function A, its computational complexity will be denoted as $cmplx(A)$ and measured by a number of elementary operations: additions ($\oplus$) and multiplications ($\odot$). For functions based on symmetric polynomials, cf. Eqs. (17), (19), and (20), the most time consuming operation is to compute determinants in Eq. (22). Therefore,

$$cmplx(E_k) = \binom{m}{k} \cdot cmplx(\det M(\boldsymbol{\alpha})). \tag{24}$$

Using the Lagrange's expansion formula with respect to a fixed row (column) to calculate determinants of a $(k \times k)$ matrix M_k (with k varied), the following recursive expression for the computational complexity of M_i, $i = 1, \ldots, k$

$$\begin{aligned} &cmplx(\det M_1) = 0, \quad cmplx(\det M_2) = 2 \odot +1\oplus, \\ &cmplx(\det M_k) = k \cdot cmplx(\det M_{k-1}) + (k-1)\oplus, \end{aligned} \tag{25}$$

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or in the closed form

$$cmplx(\det M_k) = k! \odot +(k! - 1)\oplus, \quad k = 2, \ldots \tag{26}$$

Finally, taking into account (24), one can estimate the number of additions and multiplications required to compute E_k

$$cmplx(E_k) \simeq \frac{m!}{(m-k)!}(\odot + \oplus), \quad k = 1, \ldots, m. \tag{27}$$

For fixed value of m, one can deduce from Eq. (27) that the smallest computational complexity for discriminant functions based on symmetric polynomials is for small k and increases rapidly as k tends to m.

For other presented discriminant functions based on zeroing off-diagonal terms of the manipulability matrix, cf. Eqs. (21)–(23), the complexity is of the type $m(m-1)/2$, which, in turn, equals to complexity for the functions based on symmetric polynomials with terms E_1, E_2, E_{m-1}, E_m. The similarity between the two classes of discriminant functions is even deeper. To show this, let us compute an expression for discriminant function $B_{12} = E_1^2 - E_2$ for the manipulability matrix $\boldsymbol{M} = [a_{ij}]$, $i, j = 1, \ldots, m$.

$$\begin{aligned} E_1^2 - E_2 &= \frac{1}{m^2}\left(\sum_{i=1}^{m} a_{ii}\right)^2 - \frac{2}{m(m-1)} \\ &\quad \times \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\left(a_{ii}a_{jj} - a_{ij}^2\right) = \ldots \\ &= \frac{1}{m^2(m-1)}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(a_{ii} - a_{jj})^2 \\ &\quad + \frac{2}{m(m-1)}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} a_{ij}^2. \end{aligned} \tag{28}$$

Comparing conditions (23) and (28), one can note their similar shapes and slight differences in details (for example, weighting factors). In fact, condition (28) is a symmetric version of Eq. (23), i.e., it does not prefer any of the items involved in the formula.

5. Applications of Discriminant Functions

Discriminant functions, B, can be applied at least at two prominent applications in robotics. First one is to find either an isotropic configuration or a configuration as close to isotropic one as possible. An iterative process, initialized at a given configuration $\boldsymbol{q}_0$, can be described as follows:

$$\boldsymbol{q}_{i+1} = \boldsymbol{q}_i - \xi \left.\frac{\partial B}{\partial \boldsymbol{q}}\right|_{\boldsymbol{q}_i}, \tag{29}$$

where ξ is a positive real number. The resulting, final configuration is obtained when a stop condition is satisfied. Two stop conditions can be formulated. Obviously, $B(\boldsymbol{q}) = 0$ completes the computations in any case as an isotropic configuration was reached. However, in most applications, less restrictive conditions should be applied. The computations will be finished when either a point in the configuration space moves from one iteration to another very slowly ($||\boldsymbol{q}_{i+1} - \boldsymbol{q}_i|| < \epsilon_1$) or the criterion function decreases too slowly in the quality function space ($B(\boldsymbol{q}_i) - B(\boldsymbol{q}_{i+1}) < \epsilon_2$) (positive real constants ϵ_1, ϵ_2 are given). As the antigradient technique exploited in Eq. (29) guarantees only a local optimality, a multistart (in the configuration space) approach is advised to increase a chance of reaching the global optimum.

Another application is to plan a motion of a manipulator along a prescribed path in the taskspace (a special instance of the task is to reach a given point in the taskspace). The manipulator described by its kinematics $\mathbb{X} \ni \boldsymbol{x} = k(\boldsymbol{q} \in \mathbb{Q})$ should pass through a sequence of current goal points $\boldsymbol{x}_f^j$ $j = 1, \ldots$ in the taskspace $\mathbb{X}$ while minimizing a secondary quality (discriminant) function B in the null space of the Jacobian matrix $J(\boldsymbol{q}) = \partial k/\partial \boldsymbol{q}$. This iterative scheme to reach a current goal $\boldsymbol{x}_f$ while starting at a given configuration $\boldsymbol{q}_0$ is described by

$$\begin{aligned} \boldsymbol{q}_{i+1} &= \boldsymbol{q}_i + \xi \boldsymbol{J}^{\#}(\boldsymbol{q}_i)(\boldsymbol{x}_f - k(\boldsymbol{q}_i)) \\ &\quad - \eta(\boldsymbol{I} - \boldsymbol{J}^{\#}(\boldsymbol{q}_i)\boldsymbol{J}(\boldsymbol{q}_i))\left.\frac{\partial B}{\partial \boldsymbol{q}}\right|_{\boldsymbol{q}_i}, \end{aligned} \tag{30}$$

where $\boldsymbol{J}^{\#} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}$ is the Moore–Penrose pseudo-inverse of the Jacobian matrix $\boldsymbol{J}$ and ξ, η are positive constants.[16] When many goal points $\boldsymbol{x}_f^j$ should be visited, the final configuration to reach the jth goal becomes the initial configuration for the next $(j+1)$st subtask.

The algorithm described by Eq. (30) admits a few versions as different scenarios to select ξ, η can be utilized. The following variant of the algorithm was applied later on.

In the ith iteration of the algorithm (29) at the current configuration $\boldsymbol{q}_i$, two subtasks were solved. The first one was to determine $\xi^\star = \xi > 0$ ($\eta = 0$), in a one-dimensional optimization process, to move the $k(\boldsymbol{q}_{i+1})$ as close to the goal $\boldsymbol{x}_f$ as possible. The second task was to establish, also in one-dimensional optimization process, $\eta^\star > 0$ ($\xi = 0$) to maximally reduce the quality function B (at this stage, a slight increase of the distance to the goal is allowed $||k(\boldsymbol{q}_i) - \boldsymbol{x}_f|| + \rho < ||k(\boldsymbol{q}_{i+1}(\eta^\star)) - \boldsymbol{x}_f||$, where ρ is given). Finally, the subspace of the parameter space $[0, \xi^\star] \times [0, \eta^\star]$ is covered by a mesh of points. At each point, the quality function B is evaluated. The optimal pair (ξ, η) must minimize B still decreasing the distance to the goal to preserve convergence of the algorithm. The pair determines the next configuration $\boldsymbol{q}_{i+1}$.

6. Simulations

Simulations of algorithms (29) and (30) were carried out on the 3D planar pendulum presented in Fig. 1. The manipulator is redundant one and resulting characteristics are easy to visualize on 2D plots. Kinematics of the 3D planar pendulum is given by

$$x = a_1c_1 + a_2c_{12} + a_2c_{123}, \quad y = a_1s_1 + a_2s_{12} + a_2s_{123}. \tag{31}$$

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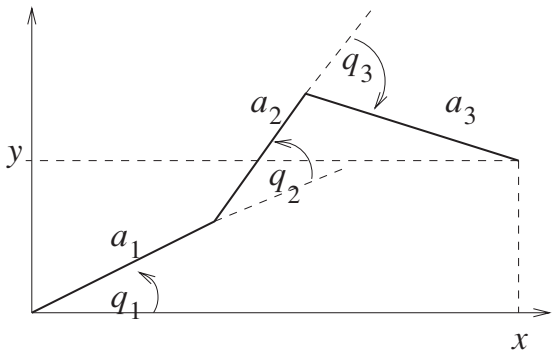


Fig. 1. The 3D planar pendulum.

The taskspace is a subset of the plane (x, y) $(m = 2)$, while the configuration is given by $\boldsymbol{q} = (q_1, q_2, q_3)$ $(n = 3)$. In Eq. (31), a standard robotic convention was used to denote trigonometric functions, i.e., c_{12} stands for $\cos(q_1 + q_2)$, while s_{123} for $\sin(q_1 + q_2 + q_3)$. In all simulations, lengths of links were set to $a_1 = a_2 = a_3 = 1$.

The discriminant function B was based on the formula $B = E_1^2 - E_2 = \frac{2}{m(m-1)}B$ (the constant, for a given m, multiplying B was missed as it does not influence optimal configurations). After simplification, with the use of Mathematica package, it is described by the following expression:

$$\begin{aligned} B = {} & 38 + 28\cos(q_2) + 4\cos(2q_2) + 8\cos(q_2 - q_3) \\ & + 48\cos(q_3) + 12\cos(2q_3) + 32\cos(q_2 + q_3) \\ & + 6\cos(2(q_2 + q_3)) + 8\cos(2q_2 + q_3) \\ & + 12\cos(q_2 + 2q_3). \end{aligned} \tag{32}$$

The first, optimization task of the discriminant function (32) was performed with the use of algorithm (29). ξ was selected dynamically (and varied from one iteration to another) to change configuration at most by $0.6^\circ > ||\boldsymbol{q}_{i+1} - \boldsymbol{q}_i||$ in the ith iteration. Ten tasks were run with initial configurations selected randomly. The optimal configuration obtained $(\star, 60^\circ, 120^\circ)$ (with any value of q_1 as the coordinate does not impact B, see Eq. (32)) realizes the isotropic configuration $(B = 0)$.

The second task to perform was to trace a semicircle path in the taskspace centered at the point $(1, 0)$ with the radius $R = 1$. Consecutive subgoals were selected by varying the parameter $\alpha \in [0^\circ, 180^\circ]$ with a step of 5°. $\alpha = 0^\circ$ corresponds to the point $(1, -1)$ and the path is traced counterclockwise. The accuracy of reaching subgoals was set to 10^{-5} while the initial configuration for the first subgoal planning was $\boldsymbol{q}_0 = (45^\circ, 0^\circ, 0^\circ)$. The mean value of

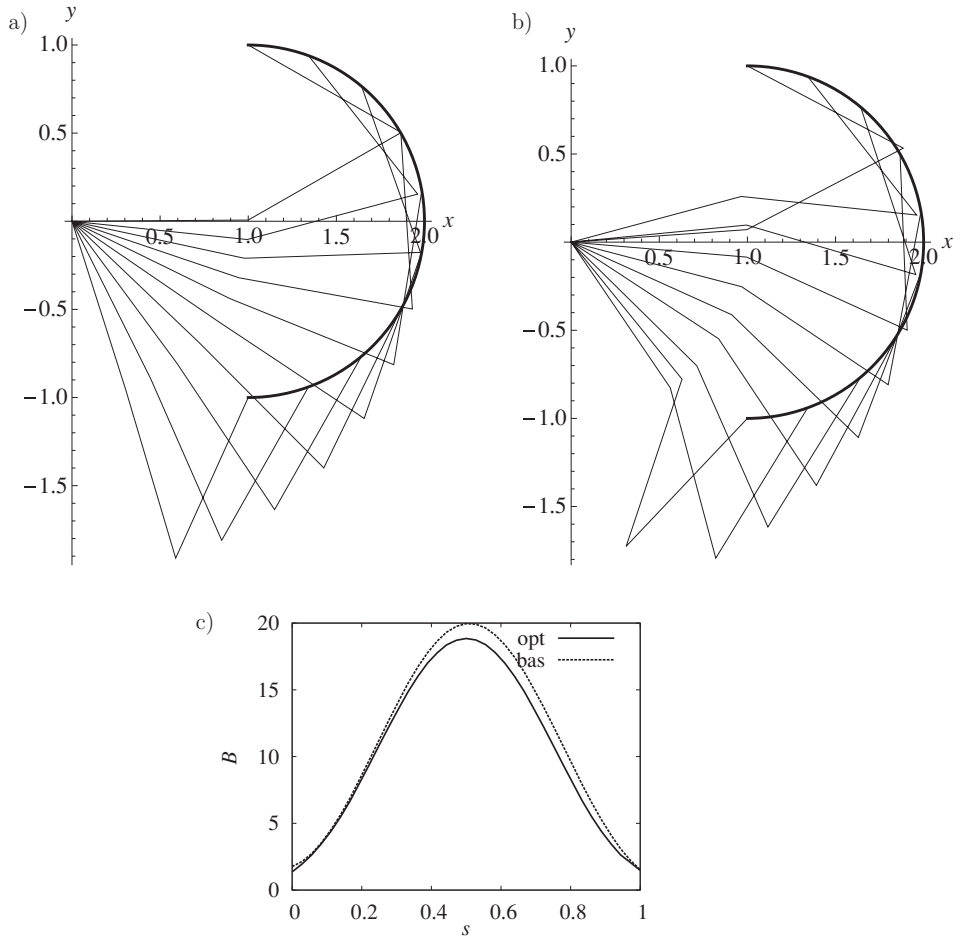


Fig. 2. (a) Postures generated with the basic algorithm, (b) and the algorithm with optimization in the null space, (c) the discriminant function B along the path for the two versions of the algorithm. The independent variable $s = \alpha/180^\circ$.

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the discriminant function B collected from all configurations corresponding to consecutive subgoals along the path was equal to 11.18 for the basic (without optimization) algorithm and 10.40 for its optimized version. Resulting characteristics are presented in Fig. 2 for two versions of the algorithm (30) with ($\eta \neq 0$) and without $\eta = 0$ optimization of B. Comparing Fig. 2(a) and (b), one can deduce that minimization of the discriminant function prefers configurations with bended links. Also, for all subgoals along the path, the optimized version of the Newton algorithm (30) provides better (smaller) values of B.

In order to validate the claim that isotropic configurations are really rare, let us consider the 2D planar pendulum ($a_3 = 0$ in Eq. (31)). From the Kircanski's paper, [10] the Jacobian matrix $J^{(2)}$ corresponding to task coordinates (x, y) expressed in the end-effector frame of the pendulum has been taken

$$J^{(2)} = \begin{bmatrix} a_1 s_2 & 0 \\ a_2 + a_1 c_2 & a_2 \end{bmatrix}. \tag{33}$$

In the quoted paper,[10] it was proved that at isotropic configurations, the following equation:

$$(k^2 + 2 + 2kc_2)^2 = 4k^2 s_2^2, \quad \text{where } k = \frac{a_1}{a_2} \tag{34}$$

is satisfied. Taking advantage of the trigonometric identity $s_2^2 = 1 - c_2^2$, Eq. (34) becomes a quadratic equation with independent variable $c_2 = \cos(q_2)$. It appears that its discriminant equals $\Delta = -16k^2(k^2 - 2)^2 \leq 0$ and real solutions exist only for a single value of $k = \sqrt{2}$ (in this case $c_2 = -1/\sqrt{2}$ and $s_2 = \pm 1/\sqrt{2}$). Consequently, isotropic configurations are not generic in this case, because they exist only for a single value of k among infinite many possible.

7. Conclusions

In this paper, discriminant functions for isotropic configurations were formulated. It was shown that, in generic case, for nonredundant manipulators, there is no isotropic configurations at all. Two families of discriminant functions were designed. First family was derived based on the theory of symmetric polynomials and Newton and Mclaurin inequalities. The other family was based on minimizing the distance of a manipulability matrix from a diagonal matrix with the same elements on the diagonal. It was shown that the simplest discriminant function from the first family corresponds to a specific member of the second family (with slightly different coefficients). Two possible applications of discriminant functions were presented. First, one was aimed at detection of optimal configurations while the second one at motion planning of a manipulator with a discriminant function optimization in the null space of the Jacobian matrix of the manipulator.

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