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Dynamics of hyperbolic correspondences

CARLOS SIQUEIRA

Department of Mathematics, Federal University of Bahia, Salvador CEP 40170115, Brazil
(e-mail: carlos.siqueira@ufba.br)

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Abstract. This paper establishes the geometric rigidity of certain holomorphic correspondences in the family $(w-c)^q = z^p$, whose post-critical set is finite in any bounded domain of $\mathbb{C}$. In spite of being rigid on the sphere, such correspondences are J-stable by means of holomorphic motions when viewed as maps of $\mathbb{C}^2$. The key idea is the association of a conformal iterated function system to the return branches near the critical point, giving a global description of the post-critical set and proving the hyperbolicity of these correspondences.

Key words: complex dynamics, holomorphic correspondences, rigidity.
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1. *Introduction*

Holomorphic correspondences are relations $z \mapsto w$ given by $p(z, w) = 0$, where p is a polynomial in two complex variables. Finitely generated Kleinian groups and rational maps are two types of correspondences. *Matings* between Kleinian groups and rational maps provide a third type. An important topic of the theory involves deciding when an abstract mating between a polynomial map and a Kleinian group Γ can be realized by a holomorphic correspondence. For example, if Γ is Hecke group, matings between pinched polynomial-like maps f and Γ are generated by holomorphic correspondences if and only if f is Chebyshev-like [3]. Bullett and Penrose [5] discovered a family of matings whose connectedness locus is (conjecturally) homeomorphic to the Mandelbrot set. This phenomenon suggests an interesting *dictionary* between the dynamics of certain holomorphic correspondences (e.g. matings), rational maps and Kleinian groups [6]. The theory of holomorphic motions [14] has already been extended to the family $\mathbf{f}_c$ of all multifunctions $z \mapsto w$ determined by

$$(w-c)^q = z^p, \tag{1}$$

where $p > q$. This is a *formal* generalization of the quadratic family. If $\beta = p/q$ in reduced form, then $\mathbf{f}_c(z) = z^\beta + c$, where $z^\beta = \exp \beta \log(z)$. We shall see in this paper that each

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$\mathbf{f}_c$ presents features of both polynomial maps and *conformal iterated function systems* (CIFSs).

1.1. *Rigidity.* A dynamical system is described as rigid if every deformation either keeps the system unchanged (up to isomorphism), or else radically alters its dynamics, for example by destroying topological conjugacy.

Post-critically finite rational maps are rigid. Critically finite correspondences (those for which the full orbit of every critical point is finite) are rigid; indeed, in the quadratic case there are only 11 critically finite correspondences up to conformal conjugacy (see Bullett [2]).

Julia sets of hyperbolic rational maps f are *geometrically rigid* in the sense that if $J(f)$ is a Jordan curve, then either $J(f)$ is a circle in $\hat{\mathbb{C}}$ or

$$\dim_{\mathrm{H}} J(f) > 1,$$

where $\dim_{\mathrm{H}}$ stands for Hausdorff dimension. (More recently, Urbański [16] has extended this result to a class of meromorphic functions with finitely many singularities). In this example, we cannot deform the circle without transforming it into a fractal. In Theorem 1.1 we have another example of geometric rigidity where a finite set cannot be deformed without transforming it into a Cantor set.

1.2. *Geometric rigidity for correspondences.* The Julia set J_c can also be defined for $\mathbf{f}_c$; it is the closure of all repelling cycles (see page 2691 for a detailed reference to all terminology). The analysis of the post-critical set plays a central role in the dynamics of rational maps, mainly because of the following two properties: (a) the set of attracting cycles is always finite for rational maps f; and (b) every attracting cycle attracts the orbit of a critical point of f. In general, the closure of the set of attracting cycles of $\mathbf{f}_c$ is uncountably infinite. Property (b) still holds for the family of holomorphic correspondences $\mathbf{f}_c$. Therefore, the set of attracting cycles is still very important to study the dynamics of $\mathbf{f}_c$, but its structure can be very complicated, as the following theorem indicates. For this reason, we define the *dual Julia set* J_c^* as the closure of all attracting cycles of $\mathbf{f}_c$, except ∞.

The following result will be restated as Corollary 3.2.1.

THEOREM 1.1. (Geometric rigidity) *If the critical point of $\mathbf{f}_a$ has only one bounded forward orbit, and this orbit is a cycle containing zero, then the dual Julia set J_a^* is finite, and J_c^* is a Cantor set for every $c \neq a$ in a neighbourhood of a.*

Under the conditions of Theorem 1.1, the post-critical set P_a is finite in any bounded domain of $\mathbb{C}$ and P_c is accumulated in a Cantor set when $c \neq a$ is sufficiently close to a. Therefore, Theorem 1.1 is somewhat related to the post-critically finite rigidity of rational maps. As remarked above, critically finite correspondences are rigid. If a correspondence is not critically finite, there is still the chance that the post-critical set is finite in every bounded domain, that is, that the only possible accumulation point is ∞. This is precisely the structure of the post-critical set P_a under the hypothesis of Theorem 1.1. However, P_c

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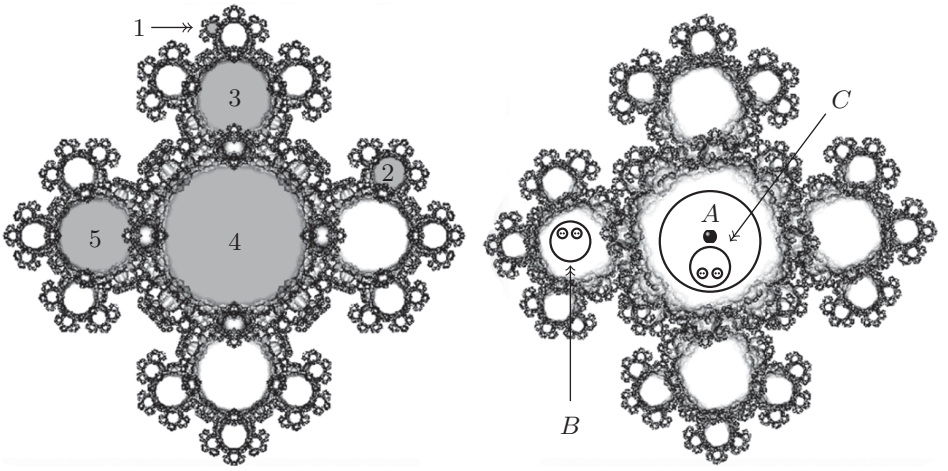


FIGURE 1. Julia sets of $(w+c)^2 = z^4$ for $c = -1$ and $c = -1 + i/10$, respectively.

presents uncountably many accumulation points, for any infinitesimal perturbation c of a. This fact could be used to show that $\mathbf{f}_a : \mathbb{C} \to \mathbb{C}$ is not structurally stable (a precise meaning of *structural stability of multifunctions* involves $\mathbb{C}^2$ extensions;see [14] for a detailed reference).

1.3. *Conformal iterated function systems.* The geometric rigidity established in Theorem 1.1 comes from the association of an iterated function system whose limit set is precisely the closure of attracting periodic orbits. We will illustrate this idea in a particular example (see Figure 1). The interplay between iterated function systems and multifunctions is a well-known topic in conformal dynamics and the situation described in Figure 1 is just one instance. See, for example, Bousch [1], Münzner and Rasch [11], and Przytycki and Urbański [12].

In Figure 1, the Julia set of $c_1 = -1$ is drawn in black, and some of the bounded components of the Fatou set $\hat{\mathbb{C}} - J_{c_1}$ (which are conformally isomorphic to the unit disk) are coloured in grey, with labels $1, 2, \ldots, 5$.

If we denote the ith component in the figure by C_i, then the sequence of maps of components given by

$$C_1 \xrightarrow{\varphi_1} C_2 \xrightarrow{\varphi_2} C_3 \xrightarrow{\varphi_3} C_4 \xrightarrow{\mathbf{f}_c} C_5 \xrightarrow{\varphi_5} C_4$$

is eventually periodic, where each φ_i is a conformal isomorphism, for $i = 1, 2, 3, 5$.

Since the critical point 0 belongs to C_4, there is no univalent branch defined on C_4. Under $\mathbf{f}_{c_1}$, every point z in C_4 has two images in C_5, except when $z = 0$. The fifth component is mapped back onto the fourth by a univalent branch φ_5, making a cycle of components of length 2. We define the *filled Julia set* K_c as the union of J_c with all bounded components of the Fatou set. The structure of K_{c_1} can be deduced from Figure 1.

It turns out that every point in K_c has one bounded forward orbit; on the other hand, every forward orbit of a point in the complement of K_c converges to ∞ exponentially fast. Except for the fifth component (the central one in the figure), every bounded component

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U of the Fatou set of $\mathbf{f}_{c_1}$ which does not contain the critical point determines two univalent branches of the correspondence; one of them sends U onto another bounded component contained in K_c, and the other sends U to the complement of K_c. This gives a global description of the dynamics on the Fatou set.

For this particular example, every component is pre-periodic and is eventually mapped into the cycle of components (fourth and fifth disks). What makes this example especial is that

$$0 \mapsto -1 \mapsto 0 \tag{2}$$

is the only bounded orbit of the critical point. Under this combinatorial condition, it is possible to prove that every forward orbit in the Fatou set either escapes to infinity or converges to the cycle (2). Since every non-zero point has uncountably many forward orbits, such well-organized behaviour is the first indication of the hyperbolicity of $\mathbf{f}_{c_1}$.

The right-hand side of this figure displays the Julia set of $\mathbf{f}_{c_2}$, where c_2 is a small perturbation of the parameter c_1. A Cantor set is produced near the cycle (2), and every forward orbit of a point in the Fatou set of $\mathbf{f}_{c_2}$ either escapes to infinity or accumulates in this Cantor set. In the following paragraph we shall explain the construction that yields this Cantor set.

In Figure 1, the correspondence $\mathbf{f}_{c_2}$ maps the disk A around the critical point onto the disk B, which is centred at c_2. Clearly, the mapping $\mathbf{f}_{c_2}$ is not single-valued and every point in A has two images in B, except for the critical point, which is mapped to c_2. In a second step, $\mathbf{f}_{c_2}$ determines two univalent branches on B; one of them sends B into the complement of K_{c_2}, and the other branch, say φ, maps B onto the disk C. The following fact is very important: C is contained in A and also avoids the critical point. Furthermore, the image of C under $\mathbf{f}_{c_2}$ splits into two disks *inside of* B. Now φ sends the two disks contained in B onto two disks contained in C. The process can be repeated indefinitely, producing two conformal iterated function systems $\mathcal{G}_i$ whose limit sets $\Lambda(\mathcal{G}_i)$ are Cantor sets contained in B and C, respectively.

The general version of the following result is Theorem 3.1.

THEOREM 1.2. *The limit set Λ obtained from the two conformal iterated function systems $\mathcal{G}_i$ above is precisely the closure of all attracting cycles; that is,*

$$J_{c_2}^* = \Lambda(\mathcal{G}_1) \cup \Lambda(\mathcal{G}_2).$$

Furthermore, the post-critical set P_{c_2} is accumulated in this Cantor set, and $\mathbf{f}_{c_2}$ expands the hyperbolic metric of the Riemann surface $\mathbb{C} - P_{c_2}$. As a consequence, the Julia set is hyperbolic.

Every correspondence $(w-a)^2 = z^{2d}$ with $a^{d-1} = -1$ and $d > 1$ displays the same features described in Figure 1 and Theorem 1.2. In all such examples, a is a simple centre; that is, the critical point has only one bounded orbit and this orbit is a cycle. In the next section we explain why simple centres are important.

1.4. *Combinatorial description of the parameter space.* In the quadratic family, the post-critically finite rigidity is related to a special coding of the hyperbolic components

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of the interior of the Mandelbrot set M. Every hyperbolic component U of the interior of M has a *centre*, corresponding to a post-critically finite f_c determined by the unique $c \in U$ such that $f_c(z) = z^2 + c$ has an attracting cycle with multiplier $\lambda = 0$. See Douady and Hubbard [7, 8]. Every centre is thus determined by a solution of $f_c^n(0) = 0$. The period n is not fixed, and by collecting all possible solutions of $f_c^n(0) = 0$ we obtain a coding of infinitely many components of the interior of M.

Although $f_c : \mathbb{C} \to \mathbb{C}$ is rigid at the centre $c \in U$, all quadratic maps in U are J-stable by means of holomorphic motions; see Mañé *et al.* [10]. (Somewhat similar results for $\mathbf{f}_c$ are Theorems 1.1 and 1.4).

Following the same principle, we say that c is a *centre* for the family $\mathbf{f}_c$ if only finitely many forward orbits of 0 return to 0 and the others are eventually contained in the basin of infinity. Equivalently, we say that c is a centre if $\mathbf{g}_c^n(0) = 0$, for some $n > 0$, where $\mathbf{g}_c$ is the multifunction defined on the filled Julia set K_c, given by $\mathbf{g}_c(z) = \mathbf{f}_c(z) \cap K_c$. (Since K_c consists of every z with at least one bounded forward orbit under $\mathbf{f}_c$, we conclude that $\mathbf{g}_c$ is well defined.)

The study of centres is the easiest approach to the hard problem of topologically classifying the dynamics of all correspondences in the family. This problem is related to the topological properties of the *connectedness locus* M_β of the family $(w - c)^q = z^p$, where $\beta = p/q$ and $\beta > 1$. By definition, M_β is the set of all parameters c such that K_c is connected.

The set $M_{\beta,0}$ consists of all $c \in \mathbb{C}$ such that $0 \in K_c$. For the quadratic family, this set is equal to M_β. Therefore, the two sets corresponding to $\beta = 2$ are equal to the Mandelbrot set M. Topology and dynamics are related by the well-known fact that if the boundary of M is locally connected then hyperbolic maps are dense in the quadratic family.

In this paper we show that $M_{\beta,0} \subset M_\beta$. Moreover, every centre belongs to $M_{\beta,0}$ and our results concern the dynamical properties of correspondences near centres (such as Theorem 1.3 below), providing an initial picture of M_β as a space of dynamical systems. (For more information on M_β and a new class of Julia sets named *carpets*, see [4].)

The following result is also presented as Theorem 5.11.

THEOREM 1.3. *There is an open set H_β containing $\mathbb{C} - M_{\beta,0}$ and every simple centre, such that $c \mapsto J_c$ and $c \mapsto J_c^*$ are continuous on H_β with respect to the Hausdorff distance.*

According to Siqueira and Smania [14, Theorem 3.4], this theorem implies structural stability and the existence of branched holomorphic motions, as stated in Theorem 5.10, which is the general version of the following result.

THEOREM 1.4. (Holomorphic motions) *If c_0 is in H_β, there is a normal branched holomorphic motion $\mathbf{h}_c$ based at c_0 and parameterized on U, such that $\mathbf{h}_c(J_{c_0}) = J_c$, for every $c \in U$.*

Branched holomorphic motions and structural stability are related by $\mathbb{C}^2$ extensions, as explained in [14, Theorem 2.1]. Indeed, to every c in the parameterization domain U of the branched motion $\mathbf{h}$ there corresponds a holomorphic map $f_c : V \to \mathbb{C}^2$, where V is an open subset of $\mathbb{C}^2$. The closure of periodic points of f_c is denoted by $J(f_c)$. The dynamics

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of $\mathbf{f}_c$ on J_c is a factor of

$$f_c : J(f_c) \to J(f_c).$$

The sets $J(f_c)$ are related by a (non-branched) holomorphic motion $h : U \times J(f_{c_0}) \to J(f_c)$ whose projection onto $\mathbb{C}$ is precisely $\mathbf{h}$.

Since every $h_c : J(f_{c_0}) \to J(f_c)$ is a topological conjugacy, this construction in dimension 2 reveals that $\mathbf{f}_c$ is structurally stable on J_c when viewed as a map of $\mathbb{C}^2$, at every parameter in H_β.

1.5. *Hyperbolic correspondences.* In this paper we give a precise meaning of hyperbolicity for correspondences. Recall that (i) a quadratic map f is hyperbolic if the critical point zero is attracted to an attracting cycle. This definition presents little information on the dynamics, but it is well known that if f is hyperbolic then (ii) the union of all the attracting cycles (including the one at infinity) attracts the whole Fatou set. The two properties are equivalent, and the second can be extended to correspondences, yielding some results which are well known for the quadratic family. There is also a generalization of hyperbolicity for finitely generated holomorphic families of rational semigroups

$$G_c = \langle f_{1,c}, \ldots, f_{n,c} \rangle$$

satisfying certain conditions. Sumi has proved that if G_a is hyperbolic, then the Julia set $J(G_c)$ moves holomorphically at $c = a$ and G_c is hyperbolic, for every c in a neighbourhood of a (see [**15**]).

In our context, we say that the correspondence $\mathbf{f}_c$ is hyperbolic if

$$\hat{\mathbb{C}} - J_c = \{z \in \hat{\mathbb{C}} : \omega(z, \mathbf{f}_c) \subset J_c^* \cup \{\infty\}\},$$

where $\omega(z, \mathbf{f}_c)$ denotes the ω-limit set of a point z.

The following result is also stated as Corollary 5.7.1.

THEOREM 1.5. (Hyperbolicity) *There is an open set H_β containing $\mathbb{C} - M_{\beta,0}$ and every simple centre such that $\mathbf{f}_c$ is hyperbolic, for every c in H_β. Moreover, backward orbits of any point in $\mathbb{C} - P_c$ accumulate on J_c when $c \in H_\beta$.*

This result could be used to show that $\mathbf{f}_c$ has no wandering domains when $c \in H_\beta$, as Figure 1 indicates for the particular example $(w+1)^q = z^p$. We do not prove this result in this paper.

The second assertion of Theorem 1.5 is the basis of the algorithm used to generate Figure 1 and reveals that the notion of Julia set introduced in this paper (closure of repelling periodic orbits) displays an expected feature already known for limit sets of Kleinian groups.

Recall that for a non-elementary Kleinian group Γ, there are several equivalent definitions of its limit set $\Lambda(\Gamma)$: it is the complement of its domain of discontinuity $\Omega(\Gamma)$, and also the closure of repelling fixed points; moreover, $\Lambda(\Gamma)$ is the set of accumulation points of any full orbit under Γ.

The foundations for a systematic study of regular and limit sets of general holomorphic correspondences are given by Bullett and Penrose in [**6**]. The generalizations of the

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various equivalent definitions of $\Lambda(\Gamma)$ are no longer equivalent in the larger category of holomorphic correspondences. It is possible to give a general definition of *regular domain* [5], and a notion of *equicontinuity set* for a correspondence [6]. The Julia set of a correspondence as defined in this paper satisfies at least two of the equivalent definitions of the Julia set of a rational function: it is the closure of repelling periodic cycles, and for every c in H_β, backward orbits of any point in $\mathbb{C} - P_c$ accumulate on J_c.

2. *First steps: recurrence near simple centres*

We start with some preliminary results on the speed of convergence of iterates in the basin of infinity.

Recall that a *multifunction* from A to B is any relation $\mathbf{h} \subset A \times B$ such that, for every z in A, there is w in B such that (z, w) belongs to $\mathbf{h}$. The notation $\mathbf{h} : A \to B$,

$$\mathbf{h}(z) = \{w \in B : (z, w) \in \mathbf{h}\} \quad \text{and} \quad \mathbf{h}(A) = \bigcup_{z \in A} \mathbf{h}(z)$$

is commonly used. The composition $\mathbf{h}_1 \circ \mathbf{h}_2$ of multifunctions is defined by the relation consisting of all pairs (z, w) for which there is x such that $(z, x) \in \mathbf{h}_2$ and $(x, w) \in \mathbf{h}_1$. If a multifunction is surjective, that is, $\mathbf{h}(A) = B$, then its inverse $\mathbf{h}^{-1} : B \to A$ always exists. In this case, $\mathbf{h}^{-1}(w)$ is the set of all $z \in A$ such that $(z, w) \in \mathbf{h}$.

Each member of the one-parameter family of holomorphic correspondences $(w - c)^q = z^p$ (where the integers $p > q \geq 1$ are fixed) is a multifunction $\mathbf{f}_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Therefore, $\mathbf{f}_c(z)$ is the set of all w such that $(w - c)^q = z^p$.

A sequence (z_i) is a *forward orbit* (or simply, orbit) if $z_{i+1} \in \mathbf{f}_c(z_i)$, for every i. If $z_{i+1} \in \mathbf{f}_c^{-1}(z_i)$ for every i, then (z_i) is a *backward orbit*. A set B is called *forward invariant* if $\mathbf{f}_c(B) \subset B$. If $\mathbf{f}_c^{-1}(B) \subset B$, then B is *backward invariant.* If $k > 0$, then

$$\mathbf{f}_c^k = \underbrace{\mathbf{f}_c \circ \cdots \circ \mathbf{f}_c}_{k} \quad \text{and} \quad \mathbf{f}_c^{-k} = (\mathbf{f}_c^{-1})^k.$$

Fix any $\lambda > 1$. Given $c \in \mathbb{C}$, the equation $x^{p/q} - \lambda x - |c| = 0$ has two solutions in x. Let x_0 be the greatest solution. Any $R > x_0$ is, by definition, *an escaping radius of* $\mathbf{f}_c$. The following result shows that the region determined by $|z| > R$ is forward invariant under $\mathbf{f}_c$, and that every orbit in this region converges exponentially fast to ∞.

LEMMA 2.1. (Escaping radius) *An escaping radius R can be chosen locally constant at every $c \in \mathbb{C}$. If $|z| > R$, then any forward orbit (z_i) of z under $\mathbf{f}_c$ satisfies*

$$\cdots |z_n| > \lambda |z_{n-1}| > \lambda^2 |z_{n-2}| > \cdots > \lambda^n |z|.$$

Consequently, (z_i) converges exponentially fast to ∞, and

$$\mathbf{f}_c^{-m-1}(B_R) \Subset \mathbf{f}_c^{-m}(B_R),$$

where $B_R = \{z \in \mathbb{C} : |z| < R\}$.

Proof. Let R be an escaping radius of $\mathbf{f}_c$. If w is an image of z under $\mathbf{f}_c$ and $|z| > R$, then $|w| \geq |w - c| - |c| = |z|^{p/q} - |c| > \lambda |z|$. □

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As a corollary, the *filled Julia set*

$$K_c = \bigcap_{k=0}^{\infty} \mathbf{f}_c^{-k}(B_R) = \bigcap_{k=0}^{\infty} \overline{\mathbf{f}_c^{-k}(B_R)} \tag{3}$$

is non-empty. This definition is independent of R. Due to the following lemma, a point belongs to K_c if and only if it has at least one bounded forward orbit.

LEMMA 2.2. (Choice) *Suppose $z \in A$ and, for every $n > 0$, there is a forward orbit $z \mapsto z_1 \mapsto \cdots \mapsto z_n$ contained in A. Then there is an infinite forward orbit of z in A (and similarly for backward orbits).*

Proof. Use the fact that every point has at most q images to find repeated terms and extract an infinite orbit out of infinitely many finite orbits. □

2.1. *Connectedness locus.* It follows from (3) that $\mathbf{f}_c^{-1}(K_c) = K_c$. Hence every point in K_c has one infinite forward orbit in K_c and $\mathbf{g}_c : K_c \to K_c$ is the well-defined multifunction given by

$$\mathbf{g}_c(z) = \mathbf{f}_c(z) \cap K_c,$$

for every $z \in K_c$. We let M_β be the set of all c such that K_c is connected, and

$$M_{\beta,0} = \{c \in \mathbb{C} : 0 \in K_c\}.$$

We intend to show that M_β contains $M_{\beta,0}$. For the quadratic family, both sets are equal to the Mandelbrot set and $M_2 - M_{2,0}$ is empty. (If β is sufficiently close to 1, then the complement of $M_{\beta,0}$ in M_β represents a large class of filled Julia sets called carpets (see [4] for more details).

THEOREM 2.3. *For any rational $\beta > 1$, the set $M_{\beta,0}$ is contained in the connectedness locus M_β.*

We shall need the following lemma. (Recall that a *region* is a non-empty open connected set.)

LEMMA 2.4. *If $V = \mathbf{f}_c(U)$ and $0 \in U$, then U is a region if and only if V is a region.*

Proof. Connectedness in the main concern. Notice that U is a union of connected sets L containing 0 such that $\mathbf{f}_c(L)$ is connected (e.g., if L is a line segment, then $\mathbf{f}_c(L)$ consists of q line segments). The sets $\mathbf{f}_c(L)$ have a point in common, namely, c. The union of connected sets with a point in common is also connected. Hence V is connected if U is connected. □

Proof of Theorem 2.3. If $0 \in K_c$, then c has a bounded forward orbit and, in particular, c is in $\mathbf{f}_c^{-m}(B_R)$, for every m, where R is an escaping radius. It follows from Lemma 2.4 that every $\mathbf{f}_c^{-m}(B_R)$ is connected. Hence the nested intersection $K_c = \bigcap_m \overline{\mathbf{f}_c^{-m}(B_R)}$ is connected. (The nested intersection of compact and connected subsets of a metric space is compact and connected.) This proves that $M_{\beta,0} \subset M_\beta$. □

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2.2. *Recurrence.* A forward orbit (z_i) which is also a periodic sequence is called a *cycle*. *Centres* are those parameters c for which every forward orbit of 0 under $\mathbf{f}_c$ either escapes to ∞ or comes back to 0, forming a cycle. More precisely, a parameter c is a *centre* if $\mathbf{g}_c^n(0) = \{0\}$, for some $n > 0$. In particular, if there is only one bounded forward orbit of 0 under $\mathbf{f}_c$ and this orbit is necessarily a cycle, then c is by definition a *simple centre*. (See §1 for examples of simple centres.)

A holomorphic map φ is a *branch* of $\mathbf{f}_c$ if $(\varphi(z) - c)^q = z^p$, for every z in the domain of φ.

A cycle $(z_i)_0^n$ of period n is said to be *attracting* in two situations: (i) when zero belongs to the cycle (in this case, it is called *super-attracting)*; or (ii) when the *multiplier*

$$\lambda = \prod_{i=0}^{n-1} \varphi_i'(z_i)$$

satisfies $|\lambda| < 1$, where each φ_i is a branch of $\mathbf{f}_c$ taking z_i to z_{i+1} (φ_i exists only if z_i is non-zero). If $|\lambda| > 1$, the cycle is said to be *repelling*.

Let $B_d = \{|z| < d\}$. Every region which is conformally isomorphic to B_d is a *conformal disk*, or simply a disk. Suppose $c = a$ is a simple centre. Then there is only one bounded forward orbit (z_i) of zero under $\mathbf{f}_a$, and this orbit is necessarily a cycle of length n. We say that n is a minimal period if no other element of the cycle is zero, except for the first one.

THEOREM 2.5. *Suppose a is a simple centre, with a cycle (z_i) of minimal period n. For every d in some interval $(0, \epsilon)$ and c in a neighbourhood V of a, there is only one sequence of maps*

$$B_d \xrightarrow{\mathbf{f}_c} D_{1,c} \xrightarrow{\varphi_{1,c}} D_{2,c} \to \cdots \xrightarrow{\varphi_{n-1,c}} D_{n,c}, \quad c \in V, \tag{4}$$

satisfying: (a) every $D_{i,c}$ is a disk and $\varphi_{i,c} : D_{i,c} \to D_{i+1,c}$ is a conformal isomorphism; (b) for every c fixed, the disks $D_{i,c}$ are pairwise disjoint and $D_{n,c} \Subset B_d$; (c) every $\varphi_{i,c}$ is a branch of $\mathbf{f}_c$, and there are branches ϕ_i of $\mathbf{f}_0$ such that $\varphi_{i,c} = \phi_i + c$ on $D_{i,c}$, for every $c \in V$ (in this sense, we say that $\varphi_{i,c}$ is a perturbation *of $\varphi_{i,0}$); and (d) the orbit of zero under successive compositions of the maps $\varphi_{i,a}$ is precisely the cycle (z_i).*

Proof. A sequence of disks satisfying $D_{n,a} \Subset B_d$ exists because the cycle (z_i) contains zero and is super-attracting. The maps $\varphi_{i,a}$ are uniquely determined. By continuity, a small perturbation of each $\varphi_{i,a}$ yields a family of sequences (4) with $D_{n,c} \Subset B_d$, for every c in a neighbourhood of a. □

The *dual Julia set* J_c^* is the closure of all attracting cycles except ∞. For rational maps, this set is always finite, but for correspondences its geometry can be as puzzling as the Julia set J_c, which is by definition, the closure of all *repelling cycles* of $\mathbf{f}_c$.

We shall prove that J_c^* is a Cantor set associated with a CIFS when c is sufficiently close to a simple centre.

The first step towards this assertion is Theorem 2.5. The second step is provided by Theorem 2.7 below. It can be regarded as an improvement of the previous theorem, in the sense that $D_{n,c}$ is now proved to avoid the critical point.

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Since $\mathbf{f}_c(z) = c + \exp((\log z^p)/q)$, for every disk D avoiding 0 there are q univalent branches $\varphi_j : D \to \mathbb{C}$ determining $\mathbf{f}_c(D) = \bigcup_j \varphi_j(D)$. If this union is pairwise disjoint, D is said to be a *univalent disk*. The polar coordinates form of $\mathbf{f}_c$ is $(r, \theta) \mapsto (r^{p/q}, (p/q)\theta) + c$. Thus $\mathbf{f}_c$ bijectively transforms sectors of angle $2\pi/p$ at 0 to sectors of angle $2\pi/q$ at c. This fact can be used to prove the following lemma.

LEMMA 2.6. *Any disk D contained in an open sector of angle $2\pi/p$ and vertex 0 is a univalent disk.*

The family of sequences of Theorem 2.5 depends on two parameters d and c. For this reason we shall denote it by $\mathcal{S}_{c,d}$ in the following theorem.

THEOREM 2.7. (Recurrence) *For every c in a punctured neighbourhood $V - \{a\}$ of a simple centre a, there is $d(c)$ such that the sequence $\mathcal{S}_{c,d}$ satisfies: (a) the radius $d(c)$ of the first disk B_d approaches zero as $c \to a$; (b) $D_{n,c} \Subset B_d - \{0\}$ is a univalent disk; (c) some iterate $\mathbf{f}_c^{-k}(B_R)$ is a connected set containing the union of all disks of the sequence $\mathcal{S}_{c,d}$; and (d) every $D_{i,c}$ is mapped into q different disks under $\mathbf{f}_c$, and only one of them, $D_{i+1,c}$, is contained in $\mathbf{f}_c^{-k}(B_R)$.*

We include here an analysis of Figure 1 to illustrate this theorem. In that particular example, $n = 2$ and B_d is the disk labelled A in Figure 1. By Theorem 2.7(a), the size of the disk A is very small when c is close to the simple centre -1. As a consequence, the Cantor set J_c^* described in Figure 1 is very close to the cycle $0 \mapsto -1 \mapsto 0$ when c is sufficiently close to -1.

Statement (b) of Theorem 2.7 simply means that the disk $C = D_{2,c}$ avoids 0 and its image splits into two small disks contained in $B = D_{1,c}$. The region $\mathbf{f}_c^{-k}(B_R)$ contains the filled Julia set, and can be visualized in Figure 1 as a good approximation of K_c for high values of k.

Proof of Theorem 2.7. For $c \in V$, let $g_c : D_{1,c} \to D_{n,c}$ be given by $g_c(z) = \varphi_{n-1,c} \circ \cdots \circ \varphi_{1,c}(z)$. If $(z_{i,c})_{i=0}^n$ denotes the orbit of $z_{0,c} = 0$ under the sequence of maps in (4), then $z_{c,n} = g_c(c)$. Let $f : V \to \mathbb{C}$ be given by $f(c) = g_c(c)$. Since $\varphi_{i,c}(z) = \phi_i(z) + c$ for a certain branch ϕ_i of $z \mapsto z^\beta$ with $\beta = p/q$, it is possible to show that $f(z)$ is indeed holomorphic and non-constant. Hence, $f : V \to \mathbb{C}$ is an open map. (For example, if $n = 4$ then $f(z)$ is a branch of the multifunction $((z^\beta + z)^\beta + z)^\beta + z$, and therefore $f(z)$ cannot be constant.)

There is a constant C_0 such that $|g_c'(z)| \le C_0$, for every $c \in V$ and $z \in D_{1,c}$. Since $D_{1,c} = \mathbf{f}_c(B_d)$, it follows that $D_{1,c}$ is the open ball $B(c, d^\beta)$. Notice that g_c maps the centre of $D_{1,c}$ to $f(c)$. Using the mean value inequality applied to $g_c : D_{1,c} \to D_{n,c}$, we conclude that $D_{n,c}$ is contained in the open ball of radius $C_0 d^\beta$ and centre $f(c)$.

The main argument is as follows. in order to show (a) and (b), we shall prove that there is a neighbourhood V_1 of a such that, for every c in $V_1 - \{a\}$, there is d satisfying (i) $|f(c)| + C_0 d^\beta < d$ and (ii) $C_0 d^\beta < |f(c)| \sin(\pi/p)$. Since $D_{n,c}$ is contained in the open ball of radius $C_0 d^\beta$ and centre $f(c)$, the first inequality implies $D_{n,c} \Subset B_d$. Using basic trigonometry, we conclude that (ii) implies $D_{n,c} \Subset S_{2\pi/p}$, for some open sector $S_{2\pi/p}$

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of angle $2\pi/p$ and vertex 0. According to Lemma 2.6, the disk $D_{n,c}$ is univalent and compactly contained in $B_d - \{0\}$.

Recall that $\beta = p/q$ and let $\gamma = \beta - 1$. There is $d_0 > 0$ with $3C_0 d_0^{\gamma} < \sin(\pi/p)$. Let $r = C_0 d_0^{\beta}/\sin(\pi/p)$. Since a is an isolated zero of f, the set $f^{-1}(B_r)$ contains an open set $V_1 \subset V$ around a such that $0 < |f(c)| < r$, for every $c \in V_1 - \{a\}$. If we choose d with $y := C_0 d^{\beta}/\sin(\pi/p)$ sufficiently close to $|f(c)|$, then $y < |f(c)| < 2y$. Since $y < |f(c)| < r$, it follows that $d < d_0$. By the assumption, $3C_0 d_0^{\gamma} < \sin(\pi/p)$. Since $d < d_0$, we have $3C_0 d^{\beta} < d \sin(\pi/p)$. Therefore,

$$\frac{C_0 d^{\beta}}{\sin(\pi/p)} < |f(c)| < \frac{2C_0 d^{\beta}}{\sin(\pi/p)} < d - \frac{C_0 d^{\beta}}{\sin(\pi/p)} < d - C_0 d^{\beta},$$

which completes the main argument.

We now show that we may reduce V_1 even more so that, for any c in $V_1 - \{a\}$ and $d = d(c)$, the set $\mathbf{f}_c^{-k}(B_R)$ satisfies assertions (c) and (d).

Since the cycle $(z_{i,a})_{i=0}^{n}$ is the only orbit of zero under $\mathbf{g}_a$, $z_{i+1,a}$ is the only image of $z_{i,a}$ under $\mathbf{f}_a$ which is contained in K_a.

Hence $\mathbf{f}_a(z_{i,a}) - \{z_{i+1,a}\}$ is contained in the complement of K_a. By (3), there is $m > 0$ such that $\mathbf{f}_a^m$ maps $\mathbf{f}_a(z_{i,a}) - \{z_{i+1,a}\}$ into the complement of B_R, for every $0 < i < n$. By continuity, if c is sufficiently close to a, then $\mathbf{f}_c^m$ maps

$$\mathbf{f}_c(D_{i,c}) - D_{i+1,c}$$

into the complement of B_R (notice that the diameter of $D_{i,c}$ tends to zero as $c \to a$). Therefore, $\mathbf{f}_c$ maps $D_{i,c}$ into q different disks, and only one of them, $D_{i+1,c}$, is contained in $\mathbf{f}_c^{-m}(B_R)$. The set $\mathbf{f}_c^{-m}(B_R)$ is connected (by Lemma 2.4) and satisfies assertions (c) and (d). □

Property (b) of Theorem 2.7 states that $D_{n,c}$ avoids the critical point when $d = d(c)$. The following result reveals that $D_{n,c}$ contains the critical point in many situations where $d \neq d(c)$.

THEOREM 2.8. *Let a be a simple centre. For every d in a small interval $(0, \epsilon)$, there is a neighbourhood V_d of a such that, if $c \in V_d$, then the sequence $\mathcal{S}_{c,d}$ of Theorem 2.5 satisfies*

$$0 \in D_{n,c} \Subset B_d,$$

and properties (c) and (d) of Theorem 2.7 still hold in this case.

Proof. Fix $c = a$ and find an interval $(0, \epsilon)$ such that $0 \in D_{n,a} \Subset B_d$, for every d in $(0, \epsilon)$. Since $D_{n,c}$ is open and varies continuously with c, for every d in $(0, \epsilon)$ there is a neighbourhood V_d such that $0 \in D_{n,c} \Subset B_d$, whenever $c \in V_d$. □

Definition 2.9. (Critical and non-critical systems) Any sequence of maps satisfying the conditions of Theorem 2.7 is a *non-critical system* and is denoted by $\mathcal{F}_c$. Any sequence satisfying the conditions of Theorem 2.8 is a *critical system* and is denoted by $\mathcal{A}_c$.

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As described in Theorem 2.5, in both cases the sequence is given by disks $D_{i,c}$. We define

$$\mathrm{dom}(\mathcal{F}_c) = \bigcup_{i=1}^{n} D_{i,c} \quad \text{and} \quad \mathrm{dom}\,(\mathcal{A}_c) = \bigcup_{i=0}^{n-1} D_{i,c}.$$

The main difference between $\mathrm{dom}(\mathcal{F}_c)$ and $\mathrm{dom}(\mathcal{A}_c)$ is that the critical point belongs only to the latter. However, both sets are contained in K_c, and iterations *inside* them will always converge to the Cantor set J_c^*, as we shall see in Theorems 3.1, 3.4 and equation (8).

Iterating inside $\mathrm{dom}(\mathcal{F}_c)$ naturally produces a CIFS.

3. *Conformal iterated function systems*

In view of Theorem 2.7, to every $c \neq a$ sufficiently close to a simple centre a there corresponds a non-critical system $\mathcal{F}_c$ given by (4). Since $D_{n,c}$ is simply connected and $0 \notin D_{n,c}$, there are q holomorphic maps $\psi_k : D_{n,c} \to \mathbb{C}$ such that $\mathbf{f}_c(D_{n,c}) = \bigcup_k \psi_k(D_{n,c})$. Let

$$f_k(z) = \varphi_{n-1,c} \circ \cdots \circ \varphi_{1,c} \circ \psi_k(z), \tag{5}$$

where the sequence of maps $\varphi_{i,c}$ is determined by $\mathcal{F}_c$.

Since $f_k(D_{n,c}) \Subset D_{n,c}$, for every $0 < k \leq q$, by the Schwarz–Pick lemma each f_k is a contraction of the hyperbolic metric, and $\{f_k\}_1^q$ defines a CIFS without overlaps on $D_{n,c}$. Since $D_{n,c}$ is univalent (see Theorem 2.7(b)), the limit set

$$\Lambda_0 = \bigcap_{j>0} H^j(D_{n,c}) \tag{6}$$

is a Cantor set, where H is the Hutchinson operator given by $H(A) = \bigcup_{k=1}^{q} f_k(A)$, for $A \subset D_{n,c}$.

By Theorem 2.7(c)–(d), the region $\mathbf{f}_c^{-k}(B_R)$ contains $\mathrm{dom}(\mathcal{F}_c)$, and

$$\mathbf{g}_c(\mathrm{dom}(\mathcal{F}_c)) \subset \mathrm{dom}(\mathcal{F}_c).$$

Therefore, it makes sense to define

$$\Lambda(\mathcal{F}_c) = \bigcup_{j=0}^{n-1} \mathbf{g}_c^j(\Lambda_0) \tag{7}$$

as the *limit set* of $\mathcal{F}_c$. In (8) and (9) we present a dynamical interpretation of $\Lambda(\mathcal{F}_c)$.

By definition, a point z_* belongs to $\omega(z, \mathbf{g}_c)$ if and only if there is a forward orbit under $\mathbf{g}_c$ starting at z which has a subsequence converging to z_*. If we allow only orbits under compositions of maps of $\mathcal{F}_c$, then we obtain $\omega_{\mathcal{F}_c}(z)$. Notice that every orbit of a point outside $\mathbf{f}_c^{-k}(B_R)$ is attracted to infinity. Combining this fact with Theorem 2.7(d), we conclude that, if $z \in \mathrm{dom}(\mathcal{F}_c)$ or $z \in B_d$, then any forward orbit of z under $\mathbf{g}_c$ is also an orbit under the maps of $\mathcal{F}_c$, and

$$\omega(z, \mathbf{g}_c) = \omega_{\mathcal{F}_c}(z) = \Lambda(\mathcal{F}_c). \tag{8}$$

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The system $\mathcal{F}_c$ induces a CIFS not only on $D_{n,c}$, but also on every $D_{i,c}$. If Λ_i denotes the limit set of the CIFS induced on $D_{i,c}$, then

$$\Lambda(\mathcal{F}_c) = \bigcup_{i=0}^{n-1} \Lambda_i. \tag{9}$$

3.1. *Hyperbolic sets.* We say that Λ is a *hyperbolic attractor* of $\mathbf{g}_c$ if $\mathbf{g}_c(\Lambda) = \Lambda$ and $\mathbf{g}_c$ attracts a conformal metric $\|\cdot\|$ defined on a neighbourhood of Λ; that is, $\sup \|\varphi'(z)\| < 1$, where sup is taken over all $z \in \Lambda$ and every branch φ of $\mathbf{g}_c$ such that $\varphi(z) \in \Lambda$. (If we replace $\mathbf{g}_c$ by $\mathbf{f}_c$ in this definition, we obtain a *hyperbolic attractor* of $\mathbf{f}_c$.)

Recall that J_c^* was defined as the closure of all attracting cycles. Under the presence of a non-critical system $\mathcal{F}_c$, we shall prove that J_c^* is a hyperbolic attractor obtained from the limit set $\Lambda(\mathcal{F}_c)$. In the following theorem, we say that a cycle $(z_i)_0^n$ attracts a forward orbit $(w_i)_0^\infty$ of the critical point $w_0 = 0$ if $\{z_i\}_0^n$ is the set of limits of convergent subsequences of $(w_i)_0^\infty$.

THEOREM 3.1. *Any attracting cycle of $\mathbf{f}_c$ attracts a forward $\mathbf{f}_c$-orbit of the critical point. If $\mathcal{F}_c$ is a non-critical system, then the Cantor set $\Lambda(\mathcal{F}_c)$ is a hyperbolic attractor of $\mathbf{g}_c$ and*

$$J_c^* = \Lambda(\mathcal{F}_c). \tag{10}$$

We shall use the following lemma.

LEMMA 3.2. *If U is simply connected and $c \notin U$, then for any $w_0 \in U$ and z_0 in $\mathbf{f}_c^{-1}(w_0)$ there is a unique univalent branch $\phi : U \to \mathbb{C}$ of $\mathbf{f}_c^{-1}$ such that $\phi(w_0) = z_0$.*

Proof. A consequence of analytic continuation applied to the algebraic curve determined by $\mathbf{f}_c^{-1}$. □

Proof of Theorem 3.1. Let Λ_0 be the limit set defined by (6). Then $\Lambda_0 = h(\Sigma_q)$, where Σ_q denotes $\{1, \ldots, q\}^{\mathbb{N}_0}$ and

$$h(\underline{k}) = \bigcap_j f_{k_0} \circ f_{k_1} \circ \cdots \circ f_{k_j}(D_{n,c}),$$

for any sequence $\underline{k} = (k_j)$ in Σ_q. Give Σ_q the product topology.

Periodic points of the left shift σ on Σ_q are dense in Σ_q, and h maps a periodic $\underline{k}$ into a periodic $z \in \Lambda_0$, since $h(\sigma\underline{k}) = \bigcup_j f_{k_1} \circ \cdots \circ f_{k_j}(D_{n,c})$.

If $\sigma^j(\underline{k}) = \underline{k}$, then $g(z) = z$, where $g = f_{k_0} \circ \cdots \circ f_{k_{j-1}}$. Therefore, z is an attracting periodic point of $\mathbf{f}_c$. Since $h : \Sigma_q \to \Lambda_0$ is continuous and surjective, Λ_0 is contained in J_c^*. Using the same reasoning, the limit set Λ_i in (9) is also contained in J_c^*. Hence, $\Lambda(\mathcal{F}_c) \subset J_c^*$.

We will now show that attracting cycles attract a forward orbit of the critical point. Let $z_k \to \cdots \to z_1 \to z_0$ be an attracting cycle of $\mathbf{f}_c$ of period k, that is, $z_k = z_0$. Suppose this cycle does not attract any orbit of the critical point, and that no element of this cycle is 0.

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Any point z in a sufficiently small neighbourhood B_0 of z_0 is attracted to the attracting cycle. By a repeated application of Lemma 3.2, we construct a sequence of conformal isomorphisms

$$\cdots \xrightarrow{\phi_{j+1}} B_j \xrightarrow{\phi_j} B_{j-1} \xrightarrow{\phi_{j-1}} \cdots B_1 \xrightarrow{\phi_1} B_0$$

where each B_j is a topological disk containing z_j and the maps ϕ_j satisfy the following extension property: $B_{(k+1)n+j} \supset B_{kn+j}$, and $\phi_{(k+1)n+j}$ restricted to B_{kn+j} is ϕ_{kn+j}. This construction is possible because at each step j, we have $c \notin B_j$ (otherwise the cycle would attract an orbit of the critical point). Hence the family of disks B_j avoid $\{0, c, \infty\}$ (we may suppose $c \neq 0$, for if $c = 0$, then every point outside the unit circle is attracted to either 0 or ∞. In this case there is nothing to prove, since $\mathbf{f}_0$ has only one attracting cycle, namely, the fixed point $z = 0$.)

It follows that $B_* = \bigcup_{j=0}^{\infty} B_{jn}$ is a nested union of disks; hence B_* is also a simply connected set avoiding a three-point set $\{0, c, \infty\}$. Therefore, B_* is isomorphic to the unit disk. There is a conformal isomorphism $\phi : B_* \to B_*$ such that

$$\phi|_{B_{jk}} = \phi_{(j-1)k+1} \circ \cdots \circ \phi_{jk},$$

for every j. In particular, ϕ has an attracting fixed point at z_0.

By the Schwarz lemma, the map $\phi : B_* \to B_*$ is conformally conjugate to a rotation of the unit disk, and the existence of an attracting fixed point yields a contradiction. Hence the cycle must attract at least one orbit of the critical point. In other words, every attracting cycle is contained in $\omega(0, \mathbf{g}_c)$. Since $\omega(0, \mathbf{g}_c)$ is closed, $J_c^* \subset \omega(0, \mathbf{g}_c)$. A final application of (8) yields

$$J_c^* \subset \omega(0, \mathbf{g}_c) = \Lambda(\mathcal{F}_c) \subset J_c^*,$$

thereby proving (10).

It is clear from $\mathbf{g}_c(\mathrm{dom}(\mathcal{F}_c)) \subset \mathrm{dom}(\mathcal{F}_c)$ that $\mathbf{g}_c$ attracts a conformal metric defined on a neighbourhood of $J_c^* = \Lambda(\mathcal{F}_c)$. Therefore, J_c^* is a hyperbolic attractor of $\mathbf{g}_c$. □

The combination of Theorems 3.1 and 2.7 establishes the geometric rigidity of dual Julia sets at simple centres by proving that simple centres are isolated in the parameter space.

COROLLARY 3.2.1. (Geometric rigidity) *The dual Julia set J_c^* is finite at every simple centre $c = a$, and a Cantor set for any $c \neq a$ sufficiently close to a.*

There is a theorem analogous to Theorem 3.1 for critical systems $\mathcal{A}_c$.

THEOREM 3.3. *Let $\mathcal{A}_c$ be a critical system given by (4), that is, a sequence $B_d \to \cdots \to D_{n,c}$. Then $\mathrm{dom}(\mathcal{A}_c)$ is forward invariant under $\mathbf{g}_c$ and $\mathbf{g}_c^n(B_d) \Subset B_d$. If some iterate $\mathbf{g}_c^{kn}(B_d)$ is a hyperbolic Riemann surface contained in a disk $D \Subset B_d - \{0\}$, then the restriction of $\mathbf{g}_c^n$ to $\mathbf{g}_c^{kn}(B_d)$ is given by finitely many conformal maps $g_j : \mathbf{g}_c^{kn}(B_d) \to \mathbf{g}_c^{kn}(B_d)$ which are contractions with respect to the hyperbolic metric of*

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$\mathbf{g}_c^{kn}(B_d)$. *Therefore,* $\{g_j\}$ *is a CIFS. If* Λ_0 *denotes its limit set, then*

$$\omega(z, \mathbf{g}_c) = \bigcup_{i=0}^{n-1} \mathbf{g}_c^i(\Lambda_0) = J_c^*, \tag{11}$$

for every z *in* $\mathrm{dom}(\mathcal{A}_c)$.

Proof. A repetition of the arguments used to prove (5), (7), (8) and Theorem 3.1. □

The following is a version of Theorem 2.7 for critical systems.

THEOREM 3.4. *Let* a *be a simple centre. For every* d *in some interval* $(0, \epsilon)$ *there is a neighbourhood* V_d *of* a *such that, for any* c *in* V_d, *the sequence* $\mathcal{S}_{c,d}$ *of Theorem 2.5 is a critical system* $\mathcal{A}_c$ *and*

$$\omega(z, \mathbf{g}_c) = J_c^*, \tag{12}$$

for every $z \in \mathrm{dom}(\mathcal{A}_c)$.

Proof. The proof is divided into two steps.

Step 1: preliminaries. By Theorem 2.8, for every d in some interval $(0, \epsilon_1)$, there is a neighbourhood W_d of the simple centre a such that the sequence $\mathcal{S}_{c,d}$ in (4) is a critical system, for every $c \in W_d$.

For $c \in W_d$ and $z \in D_{1,c}$, let $g_c(z) = \varphi_{n-1,c} \circ \cdots \circ \varphi_{1,c}(z)$, where $(\varphi_{i,c})_i$ is the sequence of maps of $\mathcal{S}_{c,d}$. Therefore, the domain of g_c is $D_{1,c}$ and g_c maps $D_{1,c}$ onto $D_{n,c}$. Recall that $D_{1,c} = \mathbf{f}_c(B_d)$. The following properties hold for the family g_c:

(a) there exists a constant $C_0 > 1$ such that

$$|g_c'(z)| \le C_0,$$

for every $d \in (0, \epsilon_1)$, $c \in W_d$ and $z \in \mathbf{f}_c(B_d)$;

(b) $f(c) = g_c(c)$ defines an open holomorphic map $f : W_d \to \mathbb{C}$ with an isolated zero at $c = a$.

Notice that $f(c)$ is the final point of the orbit of zero under the sequence of maps in $\mathcal{S}_{c,d}$. Now fix d_1 in $(0, \epsilon_1)$, $\lambda = 1/3$, and $0 < \epsilon < d_1$ such that $C_0\epsilon^{\beta-1} < \lambda$ where $\beta = p/q > 1$. Let

$$V_d = W_d \cap W_{d_1},$$

for every $d \in (0, \epsilon_1)$.

Step 2: the V_d *just defined satisfies the properties of the theorem.* Indeed, let $d \in (0, \epsilon)$. Then $a \in V_d$ and $\omega(z, g_c) = J_c^*$ if $c = a$. (As a matter of fact, every orbit under $\mathbf{g}_a$ inside $\mathrm{dom}(\mathcal{S}_{a,d})$ is attracted to the only cycle $(z_i)_0^n$ of $\mathbf{g}_a$ containing zero. In this case, J_a^* is precisely $\{z_1, \ldots, z_{n-1}\}$.)

We will show that $\omega(z, \mathbf{g}_c) = J_c^*$ if $c \in V_d$ and $c \neq a$. Let $\delta = |f(c)|$. Since a is an isolated zero, $\delta > 0$. If $z \in B_d$, then $|z| < \epsilon$ and $|z|^{\beta-1} < \lambda$. Since $\mathbf{g}_c^n = g_c \circ \mathbf{f}_c$ on B_d, from (a) of step 1 we conclude that

$$|\mathbf{g}_c^n(z) - f(c)| = |\mathbf{g}_c^n(z) - \mathbf{g}_c^n(0)| \le C_0|z - 0|^\beta < \lambda|z|, \tag{13}$$

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where $|\cdot|$ is also used to denote the diameter of a set. Since $\mathbf{g}_c^n$ is a multifunction, $\mathbf{g}_c^n(z)$ is a set and (13) means that if we replace $\mathbf{g}_c(z)$ by any w in $\mathbf{g}_c(z)$, then (13) holds.

It follows that $\mathbf{g}_c^n$ maps B_d into a subset of the ball $B(f(c), r)$ of radius $r = \lambda d$. Hence $\mathbf{g}_c^n(B_d)$ is contained in B_{r_1}, where $r_1 = \delta + \lambda d$. Using induction, we conclude that $\mathbf{g}_c^{kn}(B_d)$ is contained in the ball B_{r_k} where

$$r_k = \delta + \lambda\delta + \cdots + \lambda^{k-1}\delta + \lambda^k d.$$

Choose k such that $\lambda^{k+1} d < \delta/2$. It follows that $\mathbf{g}_c^{(k+1)n}(B_d)$ is contained in the ball $B(f(c), s)$, where

$$s = \lambda r_k = \lambda\delta + \lambda^2\delta + \cdots + \lambda^k\delta + \lambda^{k+1} d < \frac{\lambda}{1-\lambda}\delta + \lambda^{k+1} d < \delta.$$

Combining this with Lemma 2.4, we conclude that $\mathbf{g}_c^{(k+1)n}(B_d)$ is a connected open set contained in a disk $D \Subset B_d - \{0\}$. A final application of Theorem 3.3 yields the desired result. □

The following corollary gives an approximation of J_c^* in terms of $\mathbf{g}_c^n(B_d)$.

COROLLARY 3.4.1. *If $\mathcal{A}_c$ is a critical system in the conditions of Theorem 3.4 and B_d denotes the disk of $\mathcal{A}_c$ containing zero, then for every open set U containing J_c^* there is $n_0 > 0$ such that*

$$J_c^* \subset \bigcup_{n>n_0} \mathbf{g}_c^n(B_d) \subset U.$$

Proof. Use Theorems 3.4, 3.3 and the following general fact: if a CIFS is represented by finitely many maps $f_k : X \to X$, then any high iterate $H^n(X)$ is an approximation of the limit set, where $H(A) = \bigcup_k f_k(A)$ is the Hutchinson operator. □

4. *Post-critical set*

The *post-critical set* of $\mathbf{f}_c$ is $P_c = \overline{\bigcup_{n>0} \mathbf{f}_c^n(0)}$. Since $\mathbf{f}_c(P_c) \subset P_c$, the complement $\mathbb{C} - P_c$ is backward invariant under $\mathbf{f}_c$.

The set of *accumulation points* of P_c is denoted by P_c'. Therefore $z \in P_c'$ if and only if there is a sequence $z_i \neq z$ in P_c converging to z.

THEOREM 4.1. (Post-critical sets) *Let a be a simple centre of the family $\mathbf{f}_c$.*

(a) *Suppose $c = a$. If $P_a \subset K_a$, then $P_a' = \emptyset$; otherwise, $P_a' = \{\infty\}$.*
(b) *Suppose $c \neq a$ is sufficiently close to a. Then P_c' is a Cantor set. If $P_c \subset K_c$, then $P_c' = J_c^*$; otherwise, $P_c' \supsetneq J_c^* \cup \{\infty\}$.*
(c) *If c is in the complement of $M_{\beta,0}$, then $P_c' = \{\infty\}$.*

Proof. (a) If $P_a \subset K_a$, then P_a is bounded. Hence every forward orbit of zero is trapped into the critical system and, consequently, P_a consists of the finitely many points of an attracting cycle containing zero. Thus, $P_a' = \emptyset$.

If P_a is not contained in K_a, then some orbit of zero escapes to infinity. Therefore, ∞ belongs to P_a'. Since a is a simple centre, there is a region $\mathbf{f}_a^{-k}(B_R)$ satisfying properties (c) and (d) of Theorem 2.7. As a consequence of Theorem 2.8, the intersection of P_a with a

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certain iterate $\mathbf{f}_a^{-k}(B_R)$ is a cycle containing zero; and since every point of P_a which is not in $\mathbf{f}_a^{-k}(B_R)$ is attracted to infinity exponentially fast, there can be no other accumulation point except ∞. Thus, $P_a' = \{\infty\}$.

(b) Suppose $c \neq a$ is sufficiently close to a. If $\mathbf{f}_c^{-k}(B_R)$ is a region satisfying properties (c) and (d) of Theorem 2.7, then every forward orbit of zero is either trapped into the non-critical system $\mathcal{F}_c$ or eventually leaves $\mathbf{f}_c^{-k}(B_R)$. Hence,

$$P_c' \cap \mathbf{f}_c^{-k}(B_R) = \omega(0, \mathbf{g}_c) = J_c^*,$$

where the second inequality comes from Theorem 3.1 and equation (8). By Corollary 3.2.1, J_c^* is a Cantor set. Hence, $P_c' \cap \mathbf{f}_c^{-k}(B_R)$ is a Cantor set. If $P_c \subset K_c$, then in particular P_c is contained in $\mathbf{f}_c^{-k}(B_R)$, and P_c' is therefore a Cantor set.

Now suppose P_c is not contained in K_c. Then

$$W = P_c' \cap (\mathbb{C} - \mathbf{f}_c^{-k}(B_R))$$

is non-empty. Indeed, $\overline{\varphi(D \cap J_c^*)}$ is a Cantor set contained in W, for every univalent branch $\varphi : D \to \mathbb{C}$ defined in an open set D intersecting J_c^*. Iterations of $\overline{\varphi(D \cap J_c^*)}$ are still contained in W and converge to ∞ in the Hausdorff distance. It can be shown that the whole set W can be described in this way. Hence both W and P_c' are Cantor sets containing ∞.

(c) If c is in the complement of $M_{\beta,0}$, then 0 is not in K_c. Hence every orbit of zero escapes to ∞. It follows that $P_c' = \{\infty\}$. □

Definition 4.2. We shall use H_β to denote the union of the complement of $M_{\beta,0}$ with a sufficiently small neighbourhood of the set of all simple centres $a \in M_{\beta,0}$. We shall see in Lemma 5.3 that H_β is open.

Remark 4.3. If H_β appears in two different statements, then we have two different sets H_β; their intersection is again another H_β which works for both statements.

In Corollary 5.7.1 we show that every parameter in H_β is hyperbolic, for some H_β.

THEOREM 4.4. (Hyperbolic metric) *For every non-zero parameter c in H_β, the set $\mathbb{C} - P_c$ is a hyperbolic Riemann surface and $\mathbf{f}_c$ expands its hyperbolic metric. Equivalently,*

$$d_c(\varphi(z), \varphi(w)) > d_c(z, w), \tag{14}$$

for any $z, w \in U$ and any forward holomorphic branch $\varphi : U \to \mathbb{C} - P_c$, where d_c is the distance function associated with the hyperbolic metric of $\mathbb{C} - P_c$.

For every family $\mathcal{B}_K$ of branches φ as above such that $\varphi(U)$ is contained in the same compact set $K \subset \mathbb{C} - P_c$, there is a constant $\rho > 1$ such that every $\varphi \in \mathcal{B}_K$ expands d_c by the uniform factor ρ.

This theorem does not hold for $c = 0$ because $\mathbb{C} - P_0 = \mathbb{C}^*$ is not hyperbolic.

Proof. A key role is played by the Schwarz–Pick lemma. We have to show that if $c \notin M_{\beta,0}$ or c is sufficiently close to a simple centre, then (a) $\mathbb{C} - P_c$ is connected and P_c contains

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at least two points; (b) $\mathbf{f}_c^{-1}(P_c) \supsetneq P_c$; and (c) every holomorphic branch φ whose image is contained in $\mathbb{C} - P_c$ is indeed an expansion of the hyperbolic metric.

(a) If $c = a$ is a simple centre, then P_a contains $\{0, a\}$, and $a \neq 0$. By Theorem 4.1, the set of accumulation points P_a' is either empty or $\{\infty\}$. Hence $\mathbb{C} - P_a$ is connected and its complement in $\hat{\mathbb{C}}$ contains at least three points. We conclude that $\mathbb{C} - P_a$ is a hyperbolic Riemann surface.

If $c \neq a$ is close to a simple centre, then P_c' is a Cantor set, and therefore $\mathbb{C} - P_c$ is connected. (All Cantor sets are homeomorphic. Furthermore, a Cantor set never disconnects the plane, as a consequence of the Schoenflies theorem, which states that any homeomorphism between plane Cantor sets can be extended to a homeomorphism of $\mathbb{C}$.)

By Theorem 4.1, if $c \notin M_{\beta,0}$, then P_c' is a single point set and P_c contains an infinite forward orbit of zero converging to ∞. It follows that $\mathbb{C} - P_c$ is a hyperbolic Riemann surface.

(b) Our second claim is $\mathbf{f}_c^{-1}(P_c) \supsetneq P_c$.

Case 1. Suppose $c \neq a$ is sufficiently close to a simple centre a. Then $0 = \mathbf{f}_c^{-1}(c)$ is a point of $\mathbf{f}_c^{-1}(P_c)$, which has a bounded orbit contained in $\mathrm{dom}(\mathcal{F}_c)$. In particular, $0 \in K_c$. Since $P_c \cap K_c \subset \mathrm{dom}(\mathcal{F}_c)$ and $\mathrm{dom}(\mathcal{F}_c)$ avoids 0, we conclude that 0 is not in P_c. Hence $\mathbf{f}_c^{-1}(P_c) \supsetneq P_c$ in this case.

Case 2. Now $c = a$ is a simple centre. Suppose for a moment that $\mathbf{f}_a^{-1}(P_a) = P_a$. Recall that $p \geq 2$ and $q \geq 1$. Every $z \neq a$ has at least two pre-images and $C = P_a \cap K_a$ is a cycle containing 0. Since both P_a and K_a are backward invariant, $\mathbf{f}_a^{-1}(C) = C$. Since $a \neq 0$, the cycle C must contain at least two points; thus $C = \{a\} \cup C_1$, where $\mathrm{card}(C_1) = N \geq 1$ and every element of C_1 has $p \geq 2$ pre-images in C. Different points x, y in C determine disjoint sets $\mathbf{f}_a^{-1}(x)$ and $\mathbf{f}_a^{-1}(y)$ contained in C. Proving that $\mathbf{f}_a^{-1}(x)$ and $\mathbf{f}_a^{-1}(y)$ are disjoint is not immediate because $\mathbf{f}_a$ is a multifunction. In this specific case, since $\mathbf{f}_a^{-1}(x)$ and $\mathbf{f}_a^{-1}(y)$ are contained in C, if the two sets are not disjoint, then there is a point z of the cycle C with two different images in C. This implies that 0 has two different bounded orbits under $\mathbf{f}_a$, but this is impossible because a is a simple centre.

Therefore

$$1 + N = \mathrm{card}(C) = \mathrm{card}\, \mathbf{f}_a^{-1}(C) \geq 1 + 2N,$$

which is a contradiction. Therefore, $\mathbf{f}_a^{-1}(P_a) \supsetneq P_a$.

Case 3, Suppose $c \notin M_{\beta,0}$. Equivalently, 0 is not in K_c. Since 0 belongs to $\mathbf{f}_c^{-1}(P_c)$, it suffices to show that 0 is not in P_c. Since $P_c' = \{\infty\}$, if $0 \in P_c$ then $0 \in \mathbf{f}_c^i(0)$, for some $i > 0$. Hence $0 \in P_c$ implies the existence of a cycle containing 0, which contradicts the fact $0 \notin K_c$.

(c) Suppose $c \in H_\beta$. Consider the Riemann surface

$$X = \{(z, w) \in \mathbb{C}^2 : w \notin P_c\}.$$

The two projections $\pi_1(z, w) = z$ and $\pi_2(z, w) = w$ define holomorphic maps on X. Since $\pi_2|_X$ omits at least two points of $\mathbb{C}$, it follows that X is hyperbolic. Since $\pi_2 : X \to \mathbb{C} - P_c$ is a covering map, it is an isometry (by the Schwarz lemma). On the other hand, $\pi_1 : X \to \mathbb{C} - P_c$ is not onto because $\mathbf{f}_c^{-1}(P_c) \supsetneq P_c$. Hence π_1 is a strict contraction with respect to the hyperbolic metrics of X and $\mathbb{C} - P_c$. If $\varphi : U \to \mathbb{C} - P_c$ is a holomorphic branch,

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then $U \subset \mathbb{C} - P_c$ and φ is also a branch of $\pi_2 \circ \pi_1^{-1}$; therefore φ expands the hyperbolic metric of $\mathbb{C} - P_c$. □

5. *Stability and hyperbolicity*

A subset Λ of the plane is a *hyperbolic repeller* for $\mathbf{f}_c$ if $\mathbf{f}_c^{-1}(\Lambda) = \Lambda$ and $\mathbf{f}_c$ expands a conformal metric $\|\cdot\|$ defined on a neighbourhood of Λ; that is, $\sup \|\varphi'(z)\| > 1$, where sup is taken over all $z \in \Lambda$ and every branch φ of $\mathbf{f}_c$ such that $\varphi(z) \in \Lambda$.

The main goal of this section to prove that, if $c \in H_\beta$, then J_c is a hyperbolic repeller which is stable by means of holomorphic motions. In spite of the several complications of iterations of multivalued functions (e.g., uncountably many orbits of a point) the dynamics of $\mathbf{f}_c$ is quite well understood when $c \in H_\beta$. The hyperbolic geometry of $\mathbb{C} - P_c$ is responsible for such nice behaviour.

We define $\alpha(z, \mathbf{f}_c)$ as the set of all $\zeta \in \mathbb{C}$ such that ζ is a limit point of a backward orbit of z under $\mathbf{f}_c$. The following result provides approximations of J_c, and reveals that $\alpha(z, \mathbf{f}_c)$ is independent of z.

THEOREM 5.1. *For every c in H_β, the Julia set J_c is a hyperbolic repeller for $\mathbf{f}_c$, and J_c^* is a hyperbolic attractor for $\mathbf{g}_c$. If z is in $\mathbb{C} - P_c$, then*

$$\alpha(z, \mathbf{f}_c) = J_c. \tag{15}$$

In particular, backward orbits are dense in J_c, and for any ϵ-neighbourhood U of J_c there is $n_0 > 0$ such that $\mathbf{f}_c^{-n}(z) \subset U$, for every $n > n_0$.

The proof of Theorem 5.1 will be given in a sequence of lemmas.

LEMMA 5.2. *If c is sufficiently close to a simple centre, then $J_c^* \subset \mathrm{dom}(\mathcal{A}_c)$ and J_c is contained in the complement of* $\mathrm{dom}(\mathcal{A}_c)$, *for any critical system $\mathcal{A}_c$. In particular, J_c is disjoint from J_c^*.*

Proof. The first assertion follows from Corollary 3.4.1, since $\mathbf{g}_c^n(B_d)$ is always contained in $\mathrm{dom}(\mathcal{A}_c)$. For the second, notice that if $\mathrm{dom}(\mathcal{A}_c)$ contains one point of a repelling cycle, then the whole cycle must be contained in $\mathrm{dom}(\mathcal{A}_c)$. But any cycle contained in $\mathrm{dom}(\mathcal{A}_c)$ is attracting. □

LEMMA 5.3. *If c is in the complement of $M_{\beta,0}$, then J_c^* is empty. Furthermore, for every neighbourhood U of ∞ there is $n_0 > 0$ such that $\mathbf{f}_c^n(0) \subset U$, for every $n > n_0$. Consequently, $M_{\beta,0}$ is closed and H_β is open.*

Proof. The dual Julia set is empty because any attracting cycle attracts a bounded orbit of the critical point (Theorem 3.1), and in this case no forward orbit of zero is bounded.

Let $U_1 \subset U$ be a forward invariant neighbourhood of ∞. We will show that $\mathbf{f}_c^{n_0}(0) \subset U_1$, for some n_0. (Consequently, $\mathbf{f}_c^n(0) \subset U$, for every $n \geq n_0$.). If no iterate $\mathbf{f}_c^{n_0}(0)$ is contained in U_1, than by Lemma 2.2 there is a forward orbit of zero contained in $\mathbb{C} - U_1$, contradicting the fact that every orbit of zero converges to ∞ when c is not in $M_{\beta,0}$.

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If c_0 is not in $M_{\beta,0}$, then there exist n and a neighbourhood of infinity U such that $\mathbf{f}_{c_0}^n(0) \subset U$; by continuity, $\mathbf{f}_c^n(0) \subset U$ for every c in a neighbourhood V of c_0. Hence, $0 \notin K_c$ for every $c \in V$. Therefore, c_0 is an interior point of the complement of $M_{\beta,0}$. Since c_0 is arbitrary, $M_{\beta,0}$ is closed and H_β is consequently open. □

LEMMA 5.4. *For any parameter c in H_β there is a region $\mathcal{D}_c$ such that*

$$\mathbf{f}_c^{-1}(\mathcal{D}_c) \subset \mathcal{D}_c \quad \textit{and} \quad J_c \subset \mathcal{D}_c \Subset \mathbb{C} - P_c.$$

Moreover, for any bounded set $A \Subset \mathbb{C} - P_c$ there is $n > 0$ such that $\mathbf{f}_c^{-n}(A) \subset \mathcal{D}_c$.

Proof. Since J_c is the closure of repelling cycles (which are contained in K_c) and K_c is closed, it follows that $J_c \subset K_c$. Therefore, $J_c \subset \mathbf{f}_c^{-m}(B_R)$, for every $m > 0$.

First suppose c is sufficiently close to a simple centre a. By Theorem 2.8 there exist a critical system $\mathcal{A}_c$ and a region $\mathbf{f}_c^{-k}(B_R)$ containing $\mathrm{dom}(\mathcal{A}_c)$ satisfying properties (c) and (d) of Theorem 2.7. Since $\mathrm{dom}(\mathcal{A}_c)$ is a disjoint union of disks, $\mathcal{D}_c = \mathbf{f}_c^{-n}(B_R) - \mathrm{dom}(\mathcal{A}_c)$ is connected. From Lemma 5.2 we conclude that $\mathcal{D}_c$ contains J_c. Since c is sufficiently close to a simple centre, $P_c \cap \mathbf{f}_c^{-n}(B_R) \Subset \mathrm{dom}(\mathcal{A}_c)$. Taking complements, we obtain $\mathcal{D}_c \Subset \mathbb{C} - P_c$.

Let $A \Subset \mathbb{C} - P_c$ be a bounded set. It remains to show that some backward iterate of A is contained in $\mathcal{D}_c$. Since A is bounded, there is $r > 0$ such that $A \subset B_r$. From Lemma 2.1, there is k_0 such that $\mathbf{f}_c^{-k_0}(B_r) \Subset B_R$. If we let $k_1 = k_0 + n$, and define A_1 to be $\mathbf{f}_c^{-k_1}(A)$, then $A_1 \Subset \mathbf{f}_c^{-n}(B_R)$. Denote the Euclidean distance $d(A_1, J_c^*)$ by δ. Since $\mathbb{C} - P_c$ is backward invariant, A_1 is compactly contained in $\mathbb{C} - P_c$. By Theorem 3.1, we have $J_c^* \subset P_c$. Thus, $\delta > 0$. Let U denote $\{z : d(z, J_c^*) < \delta\}$.

Let $E_0 = A_1 \cap \mathrm{dom}(\mathcal{A}_c)$. If $E_0 = \emptyset$, then $A_1 \subset \mathcal{D}_c$ and there is nothing to prove. Otherwise, let

$$E_1 = \mathbf{f}_c^{-1}(A_1) \cap \mathrm{dom}(\mathcal{A}_c) = \mathbf{f}_c^{-1}(E_0) \cap \mathrm{dom}(\mathcal{A}_c).$$

(In the second equality, we have used the definition of $\mathcal{A}_c$ and the following facts: $A_1 \subset \mathbf{f}_c^{-n}(B_R)$, and $\mathrm{dom}(\mathcal{A}_c)$ is forward invariant under $\mathbf{g}_c$). Inductively, we obtain a sequence $E_k \subset \mathrm{dom}(\mathcal{A}_c)$ such that

$$E_k = \mathbf{f}_c^{-1}(E_{k-1}) \cap \mathrm{dom}(\mathcal{A}_c) = \mathbf{f}_c^{-k}(A_1) \cap \mathrm{dom}(\mathcal{A}_c).$$

By Corollary 3.4.1, there is j_0 such that, if z is any point having a backward orbit of length greater than j_0 contained in $\mathrm{dom}(\mathcal{A}_c)$, then z is in U. As it turns out, if E_k is non-empty, then there is an orbit $z_k \to \cdots \to z_1 \to z$ in $\mathrm{dom}(\mathcal{A}_c)$ with $z \in A_1$. If $k > j_0$, then $z \in U$; and since U is disjoint from A_1, it follows that $E_k = \emptyset$ if $k > j_0$. We conclude that $\mathbf{f}_c^{-k}(A_1) \subset \mathcal{D}_c$, and a backward iterate of A is contained in $\mathcal{D}_c$ when c is sufficiently close to a simple centre.

Now suppose that $c \notin M_{\beta,0}$. By Lemma 5.3, some backward iterate $\mathbf{f}_c^{-n}(B_R)$ is compactly contained in $\mathbb{C} - P_c$. We may take $\mathcal{D}_c = \mathbf{f}_c^{-n}(B_R)$ in this case. If $A \subset \mathbb{C} - P_c$ is bounded, then A is contained in some ball B_r, for which there is m such that $\mathbf{f}_c^{-m}(B_r) \Subset B_R$. Therefore, $\mathbf{f}_c^{-(n+m)}(A) \Subset \mathcal{D}_c$. □

Proof of Theorem 5.1. Suppose $c \in H_\beta$ and let $z \in \mathbb{C} - P_c$.

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First claim: $\alpha(z, \mathbf{f}_c)$ is contained in J_c. If z_* is in $\alpha(z, \mathbf{f}_c)$ and (z_n) is a backward orbit of z with a subsequence converging to z_*, then in view of Lemma 5.4, there is n_1 such that $z_n \in \mathcal{D}_c$ for $n \geq n_1$. Consider the distance function d_c given by the hyperbolic metric of $\mathbb{C} - P_c$ (Theorem 4.4). By Lemma 5.4, an ϵ-neighbourhood of $\mathcal{D}_c$ is compactly contained in $\mathbb{C} - P_c$. By Theorem 4.4, there is $\mu < 1$ such that every branch $\varphi : B(w, \epsilon) \to \mathbb{C}$ of $\mathbf{f}_c^{-1}$ with $w \in \mathcal{D}_c$ contracts distances by μ. Since $z_{n_j} \to z_*$, we may suppose z_{n_1} and z_{n_2} are contained in $B(z_*, \epsilon/4)$ and $\mu^{n_2 - n_1} < 1/4$. Since $B(z_{n_1}, \epsilon) \subset \mathbb{C} - P_c$, there is a backward orbit

$$\cdots U_j \xrightarrow{\varphi_j} \cdots U_{n_1+1} \xrightarrow{\varphi_{n_1+1}} U_{n_1} = B(z_{n_1}, \epsilon)$$

where every φ_j is a conformal isomorphism and $z_j \in U_j$. It follows that U_j is contained in $B(z_j, \mu^{j-n_1}\epsilon)$, and so U_{n_2} is contained in $B(z_{n_2}, \epsilon/4) \subset B(z_{n_1}, 3\epsilon/4)$. Hence $U_{n_2} \Subset U_{n_1}$, and by the Banach fixed point theorem, there is a fixed point of $\varphi_{n_1+1} \circ \cdots \circ \varphi_{n_2}$ in U_{n_2}, which necessarily comes from a repelling cycle of $\mathbf{f}_c$. Since $U_{n_2} \subset B(z_{n_2}, \epsilon/4) \subset B(z_*, \epsilon/2)$ and ϵ is arbitrary, we conclude that $z_* \in J_c$.

Second claim: $\alpha(z, \mathbf{f}_c) \subset \mathcal{D}_c$ is closed and independent of z in $\mathbb{C} - P_c$. Since P_c' is either a single-point set or a Cantor set (see Theorem 4.1), for every two points z and w in $\mathbb{C} - P_c$ there is a simply connected set $U \subset \mathbb{C} - P_c$ such that $z, w \in U$. Given an infinite backward orbit (z_n) of z, a repeated application of Lemma 3.2 yields a sequence of univalent branches of $\mathbf{f}_c$,

$$\cdots \to U_n \xrightarrow{\varphi_n} U_{n-1} \cdots \xrightarrow{\varphi_1} U_0 = U,$$

where each $\varphi_i : U_i \to U_{i-1}$ is a conformal isomorphism taking z_i to z_{i-1}. This sequence of maps also produces a backward orbit (w_n) of w with $\varphi_i(w_i) = w_{i-1}$. By Lemma 5.4 and Theorem 4.4, there are n_0 and $\mu < 1$ such that $U_n \subset \mathcal{D}_c$ and

$$\operatorname{diam}(U_n) \leq \mu^{n-n_0} \operatorname{diam}(U_{n_0}),$$

for every $n > n_0$. Since z_n and w_n belong to U_n, and the diameter of U_n goes to zero as $n \to \infty$, every limit point of (z_n) is a limit point of (w_n). We conclude that $\alpha(z, \mathbf{f}_c) \subset \alpha(w, \mathbf{f}_c)$. The same argument may be used to prove the other inclusion. Hence $\alpha(w, \mathbf{f}_c) = \alpha(z, \mathbf{f}_c)$.

In order to prove that $\alpha(z, \mathbf{f}_c)$ is closed, let x be the limit of a convergent sequence (x_n) in $\alpha(z, \mathbf{f}_c)$. Let ϵ_n be a sequence of positive real numbers converging to zero. Since x_1 is in $\alpha(z, \mathbf{f}_c)$, there is a backward orbit

$$z_{n_1} \to z_{n_1 - 1} \to \cdots \to z_1 \to z$$

such that z_{n_1} is in $\mathbb{C} - P_c$ and $d(z_{n_1}, x_1) < \epsilon_1$. Since x_2 is in $\alpha(z_{n_1}, \mathbf{f}_c) = \alpha(z, \mathbf{f}_c)$, there is a backward orbit $z_{n_2} \to \cdots \to z_{n_1}$ such that $d(z_{n_2}, x_2) < \epsilon_2$. Inductively, we construct a backward orbit (z_n) of z such that $d(z_{n_j}, x_j) < \epsilon_j$, from which we conclude that x is in $\alpha(z, \mathbf{f}_c)$. In other words, $\alpha(z, \mathbf{f}_c)$ is closed.

Third claim: the set $\alpha(z, \mathbf{f}_c)$ is backward invariant, that is, $\mathbf{f}_c^{-1}(\alpha(z, \mathbf{f}_c)) = \alpha(z, \mathbf{f}_c)$. Let x be in $\alpha(z, \mathbf{f}_c)$ and $\epsilon_n \to 0$. By Lemma 5.4, x is in $\mathcal{D}_c \subset \mathbb{C} - P_c$. Any pre-image of x is determined by a univalent branch φ of $\mathbf{f}_c^{-1}$ at x. We will show that $\varphi(x)$ is in $\alpha(z, \mathbf{f}_c)$. There is a backward orbit $z_{n_1} \to \cdots \to z_1 \to z$ such that $|x - z_{n_1}| < \epsilon_1$. Define z_{n_1+1} as

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$\varphi(z_{n_1})$. Since

$$\alpha(\varphi(z_{n_1}), \mathbf{f}_c) = \alpha(z, \mathbf{f}_c),$$

there is another backward orbit $z_{n_2} \to \cdots \to \varphi(z_{n_1}) \to z_{n_1}$ with $|x - z_{n_2}| < \epsilon_2$. In this way, we construct a backward orbit (z_k) of z such that $|z_{n_j} - x| < \epsilon_j$ and $z_{n_j+1} = \varphi(z_{n_j})$. By taking limits, we conclude that $\varphi(x)$ belongs to $\alpha(z, \mathbf{f}_c)$. This proves that $\mathbf{f}_c^{-1}(\alpha(z, \mathbf{f}_c)) \subset \alpha(z, \mathbf{f}_c)$. For the other inclusion it suffices to show that any ζ in $\alpha(z, \mathbf{f}_c)$ has at least one image in $\alpha(z, \mathbf{f}_c)$. Indeed, if ζ is in $\alpha(z, \mathbf{f}_c)$, then there is a backward orbit $(z_i)_0^\infty$ of z such that $\zeta = \lim_{n\in N_1} z_n$, where N_1 is an infinite subset of $\mathbb{N}$ ($\lim_{n\in N_1}$ means the limit of the subsequence $(z_n)_{n\in N_1}$). Since $\zeta \in \mathcal{D}_c$, we have $\zeta \neq 0$ and there are q univalent branches $\phi_1, \ldots, \phi_q$ of $\mathbf{f}_c$ at ζ. By deleting finitely many $n \in N_1$, if necessary, we may suppose that z_n is in the domain of ϕ_i, for every i and $n \in N_1$. If $n \in N_1$, then there is $k_n \in \{1, \ldots, q\}$ such that $\phi_{k_n}(z_n) = z_{n-1}$. The sequence $(k_n)_{n\in N_1}$ has an element that repeats infinitely many times, say $k_1 = k_n$, for every n in an infinite set $N_2 \subset N_1$. Hence $\phi_{k_1}(z_n) = z_{n-1}$ for every $n \in N_2$. We conclude that

$$\phi_{k_1}(\zeta) = \lim_{n\in N_2} \phi_{k_1}(z_n) = \lim_{n\in N_2} z_{n-1}$$

belongs to $\alpha(z, \mathbf{f}_c)$. Hence ζ has at least one image in $\alpha(z, \mathbf{f}_c)$, as desired.

Fourth claim: $J_c \subset \alpha(z, \mathbf{f}_c)$. Indeed, let $(z_i)_0^n$ be a repelling cycle. Recall that J_c is contained in $\mathbb{C} - P_c$ when c is in H_β. By the second claim, every z_i is in $\alpha(z_0, \mathbf{f}_c) = \alpha(z, \mathbf{f}_c)$. Hence $\alpha(z, \mathbf{f}_c)$ contains all repelling cycles. Since $\alpha(z, \mathbf{f}_c)$ is closed, J_c is a subset of $\alpha(z, \mathbf{f}_c)$.

Using the first, third and fourth claims, we conclude that $J_c \subset \mathbb{C} - P_c$ is a hyperbolic repeller. By Theorem 3.1, J_c^* is a hyperbolic attractor for $\mathbf{g}_c$.

Let U be an open set containing J_c and $z \in \mathbb{C} - P_c$. We want to show that there is n such that $\mathbf{f}_c^{-n}(z)$ is contained in U. Since J_c is a hyperbolic repeller, U contains a smaller backward invariant neighbourhood of J_c. It suffices to prove that $\mathbf{f}_c^{-n}(z)$ is contained in the smaller neighbourhood, which we denote by U as well. Suppose the assertion $\mathbf{f}_c^{-n}(z) \subset U$ is not true for every n. ,Since U is backward invariant, for every n there is a backward orbit of z with length n which is contained in the complement of U. Using Lemma 2.2 we extract an infinite backward orbit of z in the complement of U, with a limit point in $\mathbb{C} - J_c$, contradicting $J_c = \alpha(z, \mathbf{f}_c)$. □

COROLLARY 5.4.1. *If c is in H_β, then any $\mathcal{D}_c$ under the conditions of Lemma 5.4 satisfies*

$$J_c = \bigcap_{k>0} \mathbf{f}_c^{-k}(\mathcal{D}_c). \tag{16}$$

In particular, if w has an infinite forward orbit contained in a compact set $E \subset \mathbb{C} - J_c^*$, then $w \in J_c$.

Proof. Since J_c is backward invariant and contained in $\mathcal{D}_c$, we only need to prove that the nested intersection (16)—temporarily denoted by L—is contained in J_c. By Lemma 2.2, a point z belongs to L if and only if z has an infinite forward orbit contained in $\mathcal{D}_c$. If z has an infinite forward orbit in $\mathcal{D}_c$, we will show that $d(z, J_c) < \epsilon$, for every $\epsilon > 0$, where d denotes the hyperbolic distance function of $\mathbb{C} - P_c$. Indeed, the orbit (z_n)

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is bounded and has a limit point z_* in $\mathbb{C} - P_c$. By Theorem 4.4, every univalent branch of $\mathbf{f}_c^{-1}$ contract distances by a factor $\mu < 1$ on $\mathcal{D}_c \Subset \mathbb{C} - P_c$. Let w be a point of J_c. Since P_c' is either a single point set or a Cantor set, there is a bounded simply connected $U \subset \mathbb{C} - P_c$ containing $\{z_*, w\}$. By Lemma 5.4, there is n_0 such that $\mathbf{f}_c^{-n_0}(U) \subset \mathcal{D}_c$. Since z_* is a limit point, there is $n > 2n_0$ such that $z_n \in U$ and $\mu^{n-n_0} \operatorname{diam}(\mathcal{D}_c) < \epsilon$, where diam denotes diameter with respect to the hyperbolic metric of $\mathbb{C} - P_c$.

By Lemma 3.2, there is a backward orbit of U,

$$U_0 \xrightarrow{\varphi_0} U_1 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} U_n = U, \tag{17}$$

given by conformal isomorphisms $\varphi_i : U_i \to U_{i+1}$, with $z_i \in U_i \subset \mathbb{C} - P_c$. Every φ_i is a univalent branch of $\mathbf{f}_c$. Let $(w_i)_0^n$ be the orbit determined by $w_i \in U_i$, $\varphi_i(w_i) = w_{i+1}$ and $w_n = w$. Since $n > 2n_0$, we have

$$U_{n-n_0} \subset \mathbf{f}_c^{-n_0}(U) \subset \mathcal{D}_c.$$

Therefore,

$$\operatorname{diam}(U_0) \le \mu^{n-n_0} \operatorname{diam}(U_{n-n_0}) \le \mu^{n-n_0} \operatorname{diam}(\mathcal{D}_c) < \epsilon.$$

Since J_c is backward invariant (Theorem 5.1), every point of (w_i) is in J_c. Both $z = z_0$ and w_0 belong to U_0. Hence $d(z_0, w_0) < \epsilon$ and $d(z, J_c) < \epsilon$, as desired.

Now suppose (z_i) is an infinite forward orbit contained in a compact set $E \subset \mathbb{C} - J_c^*$. We want to show that every z_i belongs to J_c. Since every z_i has a bounded orbit, it follows that $z_i \in K_c$. In particular, the whole sequence (z_i) is contained in $\mathbf{f}_c^{-n}(B_R)$, for every $n > 0$. For the case where c is sufficiently close to a simple centre, there exists a critical system $\mathcal{A}_c$ and no point of (z_i) can be in $\operatorname{dom}(\mathcal{A}_c)$; indeed, if some z_k is in $\operatorname{dom}(\mathcal{A}_c)$, then there is a subsequence converging to a point of J_c^* (see Theorem 3.4), contradicting the fact that z_i never escapes E.

Recall from the proof of Theorem 5.4 that, if c is sufficiently close to a simple centre, then $\mathcal{D}_c = \mathbf{f}_c^{-n}(B_R) - \operatorname{dom}(\mathcal{A}_c)$. If c is not in $M_{\beta,0}$ then $\mathcal{D}_c = \mathbf{f}_c^{-n}(B_R)$, for some $n > 0$. In any case, the sequence (z_i) is contained in $\mathcal{D}_c$. Since we have already proved that (16) holds, it follows that every point of the sequence (z_i) is in J_c. □

THEOREM 5.5. *If c is not in $M_{\beta,0}$, then $J_c = K_c$ and the dual Julia set is empty. In this case, for any point z in the complement of J_c and any ϵ-neighbourhood U of ∞, there exists n_0 such that $\mathbf{f}_c^n(z) \subset U$, for every $n > n_0$.*

Proof. By Lemma 5.3, if c is not in $M_{\beta,0}$, then some backward iterate $A = \mathbf{f}_c^{-n_0}(B_R)$ is contained in $\mathbb{C} - P_c$. Since A is bounded, by Lemma 5.4 there is n_1 such that $\mathbf{f}_c^{-n_1}(A)$ is contained in $\mathcal{D}_c$. Let $k_1 = n_0 + n_1$. Since $\mathbf{f}_c^{-k_1}(B_R)$ is contained in $\mathcal{D}_c$, from Corollary 5.4.1 we have

$$K_c = \bigcap_{n>k_1} \mathbf{f}_c^{-n}(B_R) = \bigcap_{n>0} \mathbf{f}_c^{-n}(\mathbf{f}_c^{-k_1}(B_R)) \subset J_c.$$

By Theorem 5.1 the Julia set is backward invariant and, therefore, $J_c \subset K_c$. Hence $J_c = K_c$.

Let U be a neighbourhood of ∞ and $z \in \hat{\mathbb{C}} - J_c$. We may suppose, without loss of generality, that U is forward invariant. If no iterate $\mathbf{f}_c^n(z)$ is contained in U, then for every

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n there corresponds a forward orbit of z with length n which is in the complement of U. Using Lemma 2.2 we extract an infinite forward orbit of z which is in the complement of U. In particular, z has a bounded forward orbit and we conclude that $z \in K_c = J_c$, which contradicts the assumption. □

5.1. *Hyperbolicity.* Hyperbolic rational maps play a central role in holomorphic dynamics and can be defined in several equivalent ways. We shall present a dynamical definition first and then prove that any parameter in H_β is hyperbolic.

The basin of attraction of $J_c^* \cup \{\infty\}$ consists of every point z of the Riemann sphere such that $\omega(z, \mathbf{f}_c) \subset J_c^* \cup \{\infty\}$.

From the dynamical point of view, hyperbolic correspondences are those which display simplest possible behaviour, in which J_c is *source* and J_c^* is a *sink*. More precisely, we have the following definition.

Definition 5.6. (Hyperbolicity) We say that $\mathbf{f}_c$ is hyperbolic, or that c is a hyperbolic parameter, if the basin of attraction of $J_c^* \cup \{\infty\}$ is $\hat{\mathbb{C}} - J_c$.

Both J_c and J_c^* are contained in the filled Julia set K_c. For the quadratic family, for example, J_c is the boundary of K_c. Therefore, understanding the dynamics of $\mathbf{g}_c : K_c \to K_c$ is the first step in showing that $\mathbf{f}_c$ is hyperbolic when $c \in H_\beta$. Indeed, under the action of $\mathbf{g}_c$, any orbit of a point in $K_c - J_c$ is attracted to J_c^*.

THEOREM 5.7. *Suppose c is sufficiently close to a simple centre. Then J_c^* is non-empty and is contained in the interior of K_c. The Julia set J_c is properly contained in K_c. For every $z \in K_c - J_c$,*

$$\omega(z, \mathbf{g}_c) = J_c^*, \tag{18}$$

and for any open set $U \supset J_c^*$, there is $n_0 > 0$ such that $\mathbf{g}_c^n(z) \subset U$, for every $n > n_0$. In particular, forward orbits are dense in J_c^*.

Proof. Since J_c^* is contained in $\mathrm{dom}(\mathcal{A}_c) \subset K_c$ (see Theorem 5.2), it follows that J_c^* is contained in the interior of K_c. Since J_c is contained in the complement of $\mathrm{dom}(\mathcal{A}_c)$, we have $J_c \subsetneq K_c$.

Let $U \supset J_c^*$ be an open set and $z \in K_c - J_c$. We will show that there is $n > 0$ such that $\mathbf{g}_c^n(z) \subset U$. Since $\mathrm{dom}(\mathcal{A}_c)$ is forward invariant under $\mathbf{g}_c$, there is another open set $U_1 \subset U$ containing J_c^* which is forward invariant under $\mathbf{g}_c$. It suffices to show that $\mathbf{g}_c^n(z) \subset U_1$, for some $n > 0$. Indeed, if no $\mathbf{g}_c^n(z)$ is contained in U_1, then for every n there is a finite forward orbit $z_0 \to \cdots \to z_n$ of $\mathbf{g}_c$ with $z_0 = z$ and $z_n \notin U_1$. Since U_1 is forward invariant under $\mathbf{g}_c$, the whole sequence is in the complement of U_1. Using Lemma 2.2, we extract an infinite forward orbit (z_i) under $\mathbf{g}_c$ which is in $\mathbb{C} - U_1$. Since (z_i) is an orbit of $\mathbf{g}_c$, we have $z_i \in K_c$, for every i. Thus (z_i) is an infinite forward orbit contained in the compact set $E = K_c - U_1$. By Corollary 5.4.1, the sequence (z_i) is contained in J_c, which is a contradiction, since z_0 is in $K_c - J_c$.

Equation (18) follows from Theorem 3.3. □

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In the following corollary, recall from Definition 4.2 that H_β is the union of the complement of $M_{\beta,0}$ with a sufficiently small neighbourhood of the set of all simple centres.

COROLLARY 5.7.1. *If c is in H_β, then $\mathbf{f}_c$ is hyperbolic.*

Proof. Since any forward orbit converges exponentially fast to ∞ on $\hat{\mathbb{C}} - K_c$, the corollary is a consequence of Theorems 5.7 and 5.5. □

5.2. *Sensitive dependence on initial conditions.* The Julia set $J(f)$ of a rational function f satisfies the following property: for any open set U intersecting J, there is $n > 0$ such that

$$f^n(U \cap J) = J.$$

In general, any mapping $g : X \to X$ from a topological space that eventually maps every open set U onto the whole space, $g^n(U) = X$, is called *locally eventually onto* (LEO). This property implies sensitive dependence on initial conditions, leading to a very precise characterization of *chaos*.

It is the purpose of this section to prove a similar result for the Julia set of $\mathbf{f}_c$ when $c \in H_\beta$. Roughly speaking, this means that the interesting dynamics of $\mathbf{f}_c : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is concentrated in J_c when $c \in H_\beta$. We shall see that J_c is stable by means of holomorphic motions.

If Λ is a hyperbolic repeller of $\mathbf{f}_c$, then the restriction $\mathbf{f}_c|_\Lambda(z) = \mathbf{f}_c(z) \cap \Lambda$ is a well-defined multifunction from Λ to Λ (since every point of Λ has at least one image in Λ). We say that Λ is LEO (with respect to $\mathbf{f}_c$) if for every open set U intersecting Λ, there is $n > 0$ such that

$$\mathbf{f}_c^n|_\Lambda(U \cap \Lambda) = \Lambda. \tag{19}$$

Since the complement of a hyperbolic repeller is forward invariant under $\mathbf{f}_c$, it follows that (19) is equivalent to $\mathbf{f}_c^n(U) \supset \Lambda$.

THEOREM 5.8. *If $c \in H_\beta$, then J_c is a LEO hyperbolic repeller of $\mathbf{f}_c$.*

Proof. Due to Theorem 5.1, we only need to show that J_c is LEO. Let U be an open set intersecting J_c. Let d_c denote the hyperbolic distance function of $\mathbb{C} - P_c$ (Theorem 4.4). If $z \in J_c \cap U$, then some ball $B(z, \delta)$ of radius $\delta > 0$ and centre z with respect to d_c is contained in U. Consider a forward orbit (z_i) of z contained in J_c. (Recall that every point of J_c has at least one image in J_c). Since the sequence (z_n) is bounded, it has a limit point $z_* \in J_c$. For every $x \in J_c$, the set $\{x, z_*\}$ is contained in $\mathbb{C} - P_c \supset J_c$. Recall from Theorem 4.1 that P_c' is either a single point set or a Cantor set. Hence, there is a bounded simply connected set $V_x \supset \{x, z_*\}$ such that $V_x \subset \mathbb{C} - P_c$. The open cover $\{V_x\}_{x \in J_c}$ of the compact set J_c has a finite sub-cover $\{V_{x_i}\}_{i=1}^k$ with k open sets. According to Lemma 5.4, there is a backward invariant open set $\mathcal{D}_c \Subset \mathbb{C} - P_c$ such that $J_c \subset \mathcal{D}_c$.

By Theorem 4.4 and Lemma 5.4, there are $\mu < 1$ and $n_0 > 1$ such that every univalent branch of $\mathbf{f}_c^{-1}$ defined on a subset of $\mathcal{D}_c$ contracts the hyperbolic metric by the factor μ,

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and

$$k\mu^{n_0}\,\mathrm{diam}_c(\mathcal{D}_c) < \delta/2 \quad \text{and} \quad \bigcup_{i=1}^{k} \mathbf{f}_c^{-n_0}(V_{x_i}) \subset \mathcal{D}_c,$$

where diam_c denotes the diameter with respect to d_c. Let $X = \bigcap_{i=1}^{k} V_{x_i}$.

Since z_* belongs to X, there is $n_1 > 2n_0$ such that $z_{n_1} \in X$. By Lemma 3.2, every $V_{x_i} \subset \mathbb{C} - P_c$ determines a sequence of maps

$$U_{0,i} \xrightarrow{\varphi_{1,i}} U_{1,i} \xrightarrow{\varphi_{2,i}} \cdots \xrightarrow{\varphi_{n_1,i}} U_{n_1,i} := V_{x_i},$$

where each $\varphi_{j,i} : U_{j-1,i} \to U_{j,i}$ is a conformal isomorphism (and also a branch of $\mathbf{f}_c$), $z_j \in U_{j,i}$, and $\varphi_{j,i}$ sends z_{j-1} to z_j. (This is possible because $z_{n_1} \in V_{x_i}$). In particular, $z = z_0$ belongs to $U_{0,i}$, for every i.

Let $W = \bigcup_i U_{0,i}$. Since $n_1 > 2n_0$ and $\mathbf{f}_c^{-n_0}(V_{x_i})$ is contained in $\mathcal{D}_c$, we have $U_{n_0,i} \subset \mathcal{D}_c$. Thus,

$$\mathrm{diam}_c(U_{0,i}) \le \mu^{n_0}\,\mathrm{diam}_c(U_{n_0,i}) \le \mu^{n_0}\,\mathrm{diam}_c(\mathcal{D}_c);$$

and since $\bigcap_i U_{0,i}$ is non-empty,

$$\mathrm{diam}_c(W) \le \sum_{i=1}^{k} \mathrm{diam}_c(U_{0,i}) \le k\mu^{n_0}\,\mathrm{diam}_c(\mathcal{D}_c) < \delta/2.$$

It turns out that $W \subset B(z, \delta) \subset U$, and

$$\mathbf{f}_c^{n_1}(U) \supset \mathbf{f}_c^{n_1}(W) \supset \bigcup_i V_{x_i} \supset J_c.$$

Therefore, J_c is a LEO hyperbolic repeller. □

5.3. *Holomorphic motions.* Holomorphic motions arise naturally in the study of structurally stable rational maps. For holomorphic correspondences on the plane and holomorphic maps in higher dimensions we need a new definition which allows *branches*. This leads to the definition of branched holomorphic motion originally introduced by Dujardin and Lyubich [9] for dissipative polynomial automorphisms of $\mathbb{C}^2$. The following definition is an adaptation for holomorphic correspondences given in [14].

Definition 5.9. (Branched holomorphic motion) Let Λ and U be non-empty subsets of $\mathbb{C}$ and suppose U is open. A multifunction $\mathbf{h} : U \times \Lambda \to \mathbb{C}$ is a branched holomorphic motion with base point a in U if:

(i) $\mathbf{h}_a(z) = \{z\}$, for every $z \in \Lambda$;

(ii)

$$\bigcup_{z\in\Lambda} \{(c, w) : c \in U, w \in \mathbf{h}_c(z)\} = \bigcup_{f\in\mathcal{F}} G_f,$$

where $\mathcal{F}$ is a family of holomorphic functions $f : U \to \mathbb{C}$ and G_f is the graph of f. (Notice that the left-hand side of the above equation is a union of graphs of multifunctions given by $c \mapsto \mathbf{h}_c(z)$.) The holomorphic motion is said to be normal if $\mathcal{F}$ is normal.

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THEOREM 5.10. (Structural stability) *For every parameter c_0 in H_β there is a normal branched holomorphic motion*

$$\mathbf{h} : U \times J_{c_0} \to \mathbb{C}$$

with base point c_0 such that $\mathbf{h}_c(J_{c_0}) = J_c$, for every $c \in U$.

The dual Julia set and the Julia set move continuously for parameters in H_β (see Theorem 5.11). Since $\mathbf{h}_c$ is multivalued and J_c is the image of J_{c_0} under $\mathbf{h}_c$, the geometry of J_c is likely to change drastically as we move c close to c_0, and therefore we cannot expect J_c to be homeomorphic to J_{c_0}. In particular, the dynamics restricted to the Julia sets cannot be topologically conjugate. However, $\mathbf{h}_c$ is the projection of a holomorphic motion in $\mathbb{C}^2$ given by a family of conjugacies in the usual sense; see [14].

Every hyperbolic repeller (such as J_c when $c \in H_\beta$) is a particular example of a *hyperbolic set*, as defined in [14]. According to [14, Theorem 3.4], hyperbolic Julia sets of $\mathbf{f}_c$ move holomorphically, in the sense that if J_{c_0} is a hyperbolic set, then there exist a normal branched holomorphic motion $\mathbf{h} : U \times J_{c_0} \to \mathbb{C}$ with base point c_0, and an open set Ω containing J_{c_0} such that

$$\mathbf{h}_c(J_{c_0}) = \Omega \cap J_c,$$

for every $c \in U$. In addition, if $c \mapsto J_c$ is continuous at c_0, then J_c remains within Ω for every c sufficiently close to c_0. In this way, Theorem 5.10 is a corollary of the following result.

THEOREM 5.11. (Stability) *The set functions $c \mapsto J_c^*$ and $c \mapsto J_c$ are continuous at every $c \in H_\beta$.*

The proof of Theorem 5.11 will be given in the remaining pages of this paper, after the following lemma (the proof of which can be skipped at a first reading).

LEMMA 5.12. *If c_0 is in H_β, then there exist a neighbourhood V of c_0, a compact set E, and $\epsilon_0 > 0$ such that*

$$J_c \subset \operatorname{int}(E) \subset (E)_{3\epsilon_0} \subset \mathbb{C} - P_c, \tag{20}$$

for every $c \in V$, where $(E)_{3\epsilon_0}$ is the set of all z such that $d(z, E) < 3\epsilon_0$ and d is the Euclidean distance.

Proof. By Lemma 2.1, we can fix an escaping radius R such that $R - 1$ is also an escaping radius of $\mathbf{f}_c$, for every c in a neighbourhood V_0 of c_0. By taking R sufficiently large, we may suppose that

$$|z| \geq R - 1 \implies |\mathbf{f}_c(z)| > R, \tag{21}$$

for every $c \in V_0$. As usual, by $|\mathbf{f}_c(z)| > R$ we mean that $|w| > R$, whenever $w \in \mathbf{f}_c(z)$.

Step 1: sandwich. We will show that, for every $m > 1$, there is a neighbourhood $V_m \subset V_0$ of c_0 such that

$$\mathbf{f}_c^{-m-2}(B_R) \subset \mathbf{f}_{c_0}^{-m-1}(B_R) \subset \mathbf{f}_c^{-m}(B_R), \tag{22}$$

for every $c \in V_m$.

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Since $\mathbf{f}_{c_0}^{-m-2}(B_R) \Subset \mathbf{f}_{c_0}^{-m-1}(B_R)$ there exists $\epsilon > 0$ such that

$$(\mathbf{f}_{c_0}^{-m-2}(B_R))_\epsilon \subset \mathbf{f}_{c_0}^{-m-1}(B_R).$$

The set $E = \hat{\mathbb{C}} - (\mathbf{f}_{c_0}^{-m-2}(B_R))_\epsilon$ is compact. Since E is compact and contained in the complement of $(\mathbf{f}_{c_0}^{-m-2}(B_R))_\epsilon$, we have $|\mathbf{f}_{c_0}^{m+2}(z)| > R$, for every $z \in E$. The following is a standard compactness argument. By continuity, for every point z of E there are a neighbourhood U_z of z and a neighbourhood W_z of c_0 such that $|\mathbf{f}_c^{n+2}(w)| > R$, for every (c, w) in $W_z \times U_z$. Since E is compact, there is a finite cover $\{U_{z_i}\}$; and if V_m is the intersection of V_0 with $\cap_i W_{z_i}$, then $|\mathbf{f}_c^{n+2}(z)| > R$, for every (c, z) in $V_m \times E$. In other words, $E \subset \hat{\mathbb{C}} - \mathbf{f}_c^{-m-2}(B_R)$, for every $c \in V_m$. By taking complements,

$$\mathbf{f}_c^{-m-2}(B_R) \subset \hat{\mathbb{C}} - E = \mathbf{f}_{c_0}^{-m-2}(B_R) \subset \mathbf{f}_{c_0}^{-m-1}(B_R), \quad c \in V_m.$$

This proves the first inclusion of (22).

We shall need the following property: since R is an escaping radius of every $\mathbf{f}_c$ with $c \in V_0$,

(P) the set $A = \{|z| > R\}$ is contained in

$$\{z : |\mathbf{f}_c^{n+1}(z)| \geq R\} = \{z : |w| \geq R \text{ whenever } w \in \mathbf{f}_c^{n+1}(z)\},$$

for every $c \in V_m$.

It follows that $\{z : |\mathbf{f}_c^n(z)| \geq R\} - A$ is contained in the compact set $E_1 = \{|z| \leq R\}$, for every $c \in V_m$. By continuity, given $\epsilon > 0$ we may reduce V_m so that

$$d_{\mathrm{H}}(\mathbf{f}_c^n(z), \mathbf{f}_{c_0}^n(z)) < \epsilon, \tag{23}$$

whenever $c \in V_m$ and $z \in E_1$, where d_{H} denotes Hausdorff distance. (Here we have used another compactness argument to prove uniform continuity.) Take $\epsilon = 1$ and let $c \in V_m$. We want to show that $\{|\mathbf{f}_c^n(z)| \geq R\}$ is contained in $\{|\mathbf{f}_{c_0}^{n+1}(z)| \geq R\}$; equivalently, $\mathbf{f}_{c_0}^{-m-1}(B_R) \subset \mathbf{f}_c^{-m}(B_R)$, which completes step 1. Indeed, if z is in $\{|\mathbf{f}_c^n(z)| \geq R\} - A$, then $z \in E_1$, and by (23) we have $|\mathbf{f}_{c_0}^n(z)| \geq R - 1$. From (21) we conclude that $|\mathbf{f}_{c_0}^{n+1}(z)| > R$. On the other hand, if z is in $\{|\mathbf{f}_c^n(z)| \geq R\} \cap A$, then by (P) the point z is also in $\{|\mathbf{f}_{c_0}^{n+1}(z)| \geq R\}$. Step 1 is complete.

Step 2: prove (20) *for* c_0 *close to a simple centre.* Let a be a simple centre. Using Theorem 2.8, we construct four critical systems $\mathcal{A}_c^i$, $i = 1, \ldots, 4$, parameterized in a neighbourhood U of a, such that

$$\mathrm{dom}(\mathcal{A}_{c_4}^4) \subset \overline{\mathrm{dom}(\mathcal{A}_{c_3}^3)} \subset \mathrm{dom}(\mathcal{A}_{c_2}^2) \subset \mathrm{dom}(\mathcal{A}_{c_1}^1), \tag{24}$$

for every c_1, c_2, c_3 and c_4 in U.

According to Theorem 2.8, there is $m > 1$ such that $\mathbf{f}_c^{-m}(B_R)$ is a region satisfying the conditions (c) and (d) of Theorem 2.7 with respect to all critical systems $\mathcal{A}_c^i$, for any i and $c \in U$. Hence, the iterates of the critical point either remain in $\mathrm{dom}(\mathcal{A}_c^i)$ or are sent to the complement of $\mathbf{f}_c^{-m}(B_R)$. In particular, $\mathbf{f}_c^{-m}(B_R) - \mathrm{dom}(\mathcal{A}_c^4)$ is contained in $\mathbb{C} - P_c$.

According to step 1, there is a neighbourhood $V_m \subset U$ of c_0 such that $\mathbf{f}_{c_0}^{-m-1}(B_R) \subset \mathbf{f}_c^{-m}(B_R)$, for every $c \in V_m$. If $n > m$, then $\mathbf{f}_{c_0}^{-n-1}(B_R) \Subset \mathbf{f}_{c_0}^{-m-1}(B_R)$. In view of step 1,

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we may reduce V_m so that $\mathbf{f}_c^{-n-2}(B_R) \subset \mathbf{f}_{c_0}^{-n-1}(B_R)$, for every c in V_m. By Lemma 5.2,

$$\begin{aligned} J_c \subset \mathbf{f}_c^{-n-2}(B_R) - \operatorname{dom}(\mathcal{A}_c^1) &\subset \mathbf{f}_{c_0}^{-n-1}(B_R) - \operatorname{dom}(\mathcal{A}_{c_0}^2) \\ &\Subset \mathbf{f}_{c_0}^{-m-1}(B_R) - \overline{\operatorname{dom}(\mathcal{A}_{c_0}^3)} \\ &\subset \mathbf{f}_c^{-m}(B_R) - \operatorname{dom}(\mathcal{A}_c^4) \subset \mathbb{C} - P_c, \end{aligned} \tag{25}$$

for every $c \in V_m$. If E is the closure of $\mathbf{f}_{c_0}^{-n-1}(B_R) - \operatorname{dom}(\mathcal{A}_{c_0}^2)$, then E satisfies (20) for some ϵ_0.

Step 3: prove (20) *when* c_0 *is not in* $M_{\beta,0}$. According to Lemma 2.1, there is an escaping radius R that holds for every $\mathbf{f}_c$ with c sufficiently close to c_0. Since $|\mathbf{f}_{c_0}^m(0)| > R$, for some $m > 1$, by continuity we conclude that $|\mathbf{f}_c^m(0)| > R$, for every c in a neighbourhood V of c_0. This implies $\mathbf{f}_c^{-m}(B_R) \subset \mathbb{C} - P_c$, for if z belongs to $P_c \cap \mathbf{f}_c^{-m}(B_R)$, then in particular it belongs to $P_c \cap B_R$; and since R is an escaping radius, z must be in $\mathbf{f}_c^i(0)$, for some natural number $i < m$. Hence $|\mathbf{f}_c^m(z)| > R$, which contradicts $z \in \mathbf{f}_c^{-m}(B_R)$. By step 1, we can find $n > m + 1$ and replace V by a smaller neighbourhood of c_0 such that, for every $c \in V$,

$$J_c \subset \mathbf{f}_c^{-n-1}(B_R) \subset \mathbf{f}_{c_0}^{-n}(B_R) \Subset \mathbf{f}_{c_0}^{-m-1}(B_R) \subset \mathbf{f}_c^{-m}(B_R) \subset \mathbb{C} - P_c.$$

This proves (20) with $E = \overline{\mathbf{f}_{c_0}^{-n}(B_R)}$. The proof of Lemma 5.12 is complete. □

Proof of Theorem 5.11. Since a is a simple centre, there is only one bounded orbit of zero under $\mathbf{f}_a$, and this orbit is a cycle $(z_i)_0^n$. For every $c \neq a$ in a neighbourhood V of a,

$$J_c^* = \Lambda(\mathcal{F}_c) = \bigcup_{i=0}^{n-1} \Lambda_{i,c},$$

where each $\Lambda_{i,c}$ is a Cantor set with $\Lambda_{i,c} \to \{z_i\}$ as $c \to a$ (in the Hausdorff topology). This follows from Theorem 3.1, equation (9) and Theorem 2.7. Since J_a is just the cycle $(z_i)_0^n$ containing zero, we conclude that $c \mapsto J_c^*$ is continuous at $c = a$. The continuity at points $c_0 \in V - \{a\}$ follows from $J_c^* = \Lambda(\mathcal{F}_c)$ and Theorem 2.5(c).

By Lemma 5.3, if c_0 is in $\mathbb{C} - M_{\beta,0}$ then J_c^* is empty on a neighbourhood of c_0 and the map $c \mapsto J_c^*$ is clearly continuous at c_0. We conclude that $c \mapsto J_c^*$ is continuous at every point of H_β.

On the continuity of $c \mapsto J_c$. The continuity of $c \mapsto J_c$ at $c_0 = 0$ is a delicate subject, since $\mathbb{C} - P_0$ is not hyperbolic. It must be handled separately. Fortunately, this proof has already been presented in [**14**, Theorem 4.1]. From now on, assume $c_0 \neq 0$.

Step 1: preliminary assumptions. Suppose c_0 is in H_β. Let E be a compact set satisfying the properties stated in Lemma 5.12. For every c in a neighbourhood of c_0, let d_c denote the hyperbolic distance function of $\mathbb{C} - P_c$. Since any two conformal metrics are equivalent on compact sets, there exist a constant $C > 1$ and a neighbourhood V of c_0 such that

$$d_c(z, z + w) \leq C|w|, \tag{26}$$

for every z in $(E)_{\epsilon_0}$, $c \in V$, and $|w| \leq \epsilon_0$. (Recall that $(A)_\delta$ is the set of all points which are within distance δ of A, with respect to the Euclidean metric.) We may reduce V if

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necessary and consider a constant $C_1 > 1$ such that, for every $c \in V$,

$$\{z : d_c(z, E) < 2\epsilon_0/C_1\} \subset (E)_{2\epsilon_0} \tag{27}$$

and

$$C_1^{-1} d_e(z, w) \leq d_c(z, w) \leq C_1 d_e(z, w), \quad z, w \in (E)_{2\epsilon_0}, \tag{28}$$

where d_e denotes the Euclidean distance. Using Theorem 4.4 and a compactness argument, we find $\mu < 1$ such that, for any $c \in V$ and any holomorphic branch $\varphi : U \to \mathbb{C}$ of $\mathbf{f}_c^{-1}$ with $U \subset (E)_{2\epsilon_0}$, we have

$$d_c(\varphi(z), \varphi(w)) < \mu d_c(z, w), \tag{29}$$

for any z and w in U.

From Theorem 5.1, the Julia set J_{c_0} is a hyperbolic repeller and any sufficiently small neighbourhood U_{c_0} of J_{c_0} is backward invariant under $\mathbf{f}_{c_0}$. Since J_{c_0} is contained in the interior of E, we may fix one such neighbourhood $U_{c_0} \subset E$.

Fix $z \in U_{c_0}$. By Theorem 5.1, $\alpha(z, \mathbf{f}_{c_0}) = J_{c_0}$. Let $(z_i)_0^\infty$ be a backward orbit of z under $\mathbf{f}_{c_0}$. Hence $z_0 = z$ and $\mathbf{f}_{c_0}$ sends every z_{i+1} to z_i. Since U_{c_0} is backward invariant, every z_i belongs to E.

Let B_i denote the ball centred at z_i of radius $2\epsilon_0/C_1$ with respect to the metric d_{c_0}. Since $d_e \leq C_1 d_{c_0}$, the ball B_i is contained in $(E)_{2\epsilon_0}$. Hence B_i is contained in $\mathbb{C} - P_{c_0}$, and for every i there is a univalent branch φ_i of $\mathbf{f}_{c_0}^{-1}$ defined on B_i sending z_i to z_{i+1}. Since φ_i is a contraction, $\varphi_i(B_i)$ is contained in B_{i+1}.

The main idea of the proof is given in the following two steps. Given $0 < \epsilon < \epsilon_0$, we shall prove that if $c, c_0 \in V$ satisfy

$$|c - c_0| \sum_{i=0}^{\infty} \mu^i C < \epsilon/C_1, \tag{30}$$

then $d_{\mathrm{H}}(J_c, J_{c_0}) < \epsilon$, where d_{H} is the Hausdorff distance obtained from the Euclidean metric. Equivalently, we shall prove that $J_{c_0} \subset (J_c)_\epsilon$ and $J_c \subset (J_{c_0})_\epsilon$, establishing the continuity of the Julia set at every point of H_β.

Step 2: shadowing. We will define a backward orbit $(w_i)_0^\infty$ of z under $\mathbf{f}_c$ satisfying the following conditions: the sequence (w_i) must be contained in $(E)_{\epsilon_0}$ and

$$d_{c_0}(w_i, z_i) < \mu C|c - c_0| + \mu^2 C|c - c_0| + \cdots + \mu^i C|c - c_0|, \tag{31}$$

for every $i > 0$. The proof will be given by induction. Let $w_0 = z$. In order to define w_1, we shall use the following fact: for any branch φ of $\mathbf{f}_{c_0}^{-1}$, the map

$$z \mapsto \varphi(z - c + c_0)$$

is a branch of $\mathbf{f}_c^{-1}$. By (26) and (30), the distance $d_{c_0}(w_0 - c + c_0, w_0)$ is not greater than $C|c - c_0| < \epsilon/C_1$. Therefore, $w_0 - c + c_0$ is in the domain of φ_0 and we may define w_1 as $\varphi_0(w_0 - c + c_0)$. It follows that $\mathbf{f}_c$ sends w_1 to w_0. Since φ_0 contracts distances on $(E)_{2\epsilon_0}$, from (29) and (30) we have

$$d_{c_0}(w_1, z_1) < \mu d_{c_0}(w_0 - c + c_0, w_0) \leq \mu C|c - c_0| < \epsilon/C_1.$$

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Since $d_e \le C_1 d_{c_0}$ and $z_1 \in E$, the point w_1 belongs to $(E)_\epsilon$. Hence (31) is true for $i = 1$. If (31) is true for $i = n$, then

$$d_{c_0}(w_n - c + c_0, z_n) \le C|c - c_0| + d_{c_0}(w_n, z_n) \le \sum_{j=0}^{n} \mu^j C|c - c_0| < \frac{\epsilon}{C_1}, \tag{32}$$

and so $w_n - c + c_0$ is in B_n. Since the domain of φ_n contains B_n, we may define w_{n+1} by $\varphi_n(w_n - c + c_0)$. It follows from (32) that

$$d_{c_0}(w_{n+1}, z_{n+1}) < \mu d_{c_0}(w_n - c + c_0, z_n) \le \sum_{j=1}^{n+1} \mu^j C|c - c_0| < \epsilon/C_1.$$

Since $d_e \le C_1 d_{c_0}$, the point w_{n+1} is in $(E)_{\epsilon_0}$. This completes the induction process and step 2.

Step 3: continuity. It follows from step 2 that, for every limit point z_* of (z_j), there is a limit w_* of a subsequence of (w_j) such that $d_{c_0}(w_*, z_*) < \epsilon/C_1$. Since (z_j) is arbitrary, and taking into account that $\alpha(z, \mathbf{f}_{c_0}) = J_{c_0}$ and $\alpha(w_0, \mathbf{f}_c) = J_c$, we conclude that

$$J_{c_0} \subset \{z \in \mathbb{C} : d_{c_0}(z, J_c) < \epsilon/C_1\} \subset (J_c)_\epsilon.$$

Starting with c instead of c_0, the same argument of step 2 can be applied to construct a shadowing of any backward orbit under $\mathbf{f}_c$. In the same way, we obtain $J_c \subset (J_{c_0})_\epsilon$. The proof of Theorem 5.11 is complete. □

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A. *Appendix. List of symbols*

(a) The exponent β of the family $\mathbf{f}_c$ is a rational number $p/q > 1$ (see page 2661); $\mathbf{f}_c$ is a holomorphic correspondence and $R > 0$ is an escaping radius (page 2667); $z \mapsto z_1 \mapsto \cdots$ is a forward orbit (page 2667).

(b) $\mathbf{f}_c^{-1}(w)$ and $\mathbf{f}_c(z)$ are defined by $\{z : (w - c)^q = z^p\}$ and $\{w : (w - c)^q = z^p\}$ (page 2667); $\mathbf{f}_c(A)$ and $\mathbf{f}_c^{-1}(A)$ are defined by $\bigcup_{z \in A} \mathbf{f}_c(z)$ and $\bigcup_{z \in A} \mathbf{f}_c^{-1}(z)$ (page 2667); $A \Subset B$ means that $A \subset E \subset B$ for some compact set E (page 2667).

(c) M_β is the connectedness locus and $M_{\beta,0} = \{c \in \mathbb{C} : 0 \in K_c\}$ (page 2668); every open ball centred at zero is denoted by $B_r = \{z \in \mathbb{C} : |z| < r\}$ (page 2667); $\mathbf{g}_c$ is defined by $\mathbf{g}_c(z) = \mathbf{f}_c(z) \cap K_c$, where K_c is the filled Julia set of $\mathbf{f}_c$ (page 2668); φ is a branch of $\mathbf{f}_c$ or $\mathbf{f}_c^{-1}$ (page 2669).

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(d) $\mathcal{S}_{c,d}$ is the sequence of maps $B_d \xrightarrow{\mathbf{f}_c} D_{1,c} \xrightarrow{\varphi_{1,c}} \cdots \xrightarrow{\varphi_{n-1,c}} D_{n,c}$ (page 2670). J_c and J_c^* are the Julia set and dual Julia set of $\mathbf{f}_c$ (page 2669); $\mathrm{dom}(\mathcal{F}_c)$ is the domain of a non-critical system (page 2672).

(e) $\mathrm{dom}(\mathcal{A}_c)$ is the domain of a critical system (page 2671); $\mathcal{F}_c$ is a non-critical system and $\mathcal{A}_c$ is a critical system (page 2671); $\omega(z, \mathbf{g}_c)$ is the ω-limit set under $\mathbf{g}_c$ (page 2672).

(f) $\Lambda(\mathcal{F}_c)$ is the limit set of a non-critical system (page 2673); P_c is the post-critical set of $\mathbf{f}_c$ (page 2676); H_β is a subset consisting of hyperbolic parameters (page 2677).

(g) d_c is the hyperbolic distance function of $\hat{\mathbb{C}} - P_c$ (page 2677); $\alpha(z, \mathbf{f}_c)$ is the α-limit set of z (page 2679); $\mathbf{h}_c$ denotes a branched holomorphic motion (page 2686); $\{|\mathbf{f}_c(z)| > r\}$ is a short version of the set $\{z \in \mathbb{C} : |w| > r \text{ for any } w \in \mathbf{f}_c(z)\}$ (page 2687).

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