Math. Proc. Camb. Phil. Soc. (1976), **79**, 299
MPCPS 79–24
Printed in Great Britain

Three notes on proximity theory

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(*Received* 28 *January* 1975)

I. REMARKS ON SOME THEOREMS OF W. J. THRON

It seems worthwhile to point out that a few of the theorems in (2) can, as indeed one would expect in a paper of such length and scope, be somewhat generalized or simplified. This note uses the notation of (2) and all references are to that paper.

Remark 1. Theorem 5·9 requires only that $f \in A_0$ (Def. 4·2), and can also be slightly generalized so as to include the second part of Theorem 5·8, as follows:

If Π *is an f-proximity on* X, $f \in A_0$, *and* $\mathscr{F}$ *is any filter on* X, *then* $\mathscr{G} = \cap\{\Pi(A);\ A \in \mathscr{F}\}$ *is a grill and* $c(\mathscr{G})$ *is f-round.*

If $B_1, B_2 \notin \mathscr{G}$ then $\exists A_1, A_2 \in \mathscr{F}$, $B_i \notin \Pi(A_i)$, $i = 1, 2$; hence $B_1 \cup B_2 \notin \Pi(A_1 \cap A_2)$, $A_1 \cap A_2 \in \mathscr{F}$. Thus $\mathscr{G}$ is a grill.

Now, by A_0, $A \in \mathscr{F} \Rightarrow f(\Pi, \Pi(A)) \supset f(\Pi, \mathscr{G})$; so

$$f(\Pi, \mathscr{G}) \subset \cap\{f(\Pi, \Pi(A));\ A \in \mathscr{F}\} = \cap\{\Pi(A): A \in \mathscr{F}\} = \mathscr{G}.$$

Thus $\mathscr{G}$ is an f-grill and so $c(\mathscr{G})$ is f-round (Defs. 5·2, 5·3).

By applying the dualizing operator c it is easily seen that $\mathscr{F}$ is Π-round if and only if $c(\mathscr{G})$ (above) coincides with $\mathscr{F}$, so we have in particular Theorem 5·9 with A_0 in place of A_2.

Remark 2. The present writer found the enunciation of Theorem 6·5 ambiguous, in that it was not clear whether, in forming condition (*) as in Theorem 3·20, $\Delta(\mathscr{U})$ was to be taken in the natural sense $\{\mathscr{V}; \mathscr{V}\Delta\mathscr{U}\}$ or in the sense $\{A; \exists\mathscr{V}, \mathscr{V}\Delta\mathscr{U}, A \in \mathscr{V}\}$ of Theorem 3·21. It appeared on examination that the validity of the theorem requires the latter interpretation. The additional condition in the enunciation (that $\{\mathscr{V}: \mathscr{V}\Delta\mathscr{U}\}$ is closed in the standard topology of the ultrafilter space) is not necessary for the truth of the theorem stated, but is equivalent to the consistency of Δ and Δ_Π, where $\Pi = \Pi_\Delta$. For clarity we restate the second part of the theorem, amended as just indicated, with the conditions involved stated explicitly.

Theorem 6·5(*b*). *If an equivalence relation* Δ *on the space* $\Omega(X)$ (*of ultrafilters on* X) *satisfies the condition* (*):

$$\cap\{\cup\{\Delta(\mathscr{U}): A \in \mathscr{U}\}: A \in \mathscr{V}\} = \Delta(\mathscr{V}),$$

where $\Delta(\mathscr{U})$ *denotes* $\{B; \exists\mathscr{W}, \mathscr{W}\Delta\mathscr{U}, B \in \mathscr{W}\}$, *then the relation* $\Pi(= \Pi_\Delta)$, *defined by* $(A, B) \in \Pi$ *if and only if* $\exists\mathscr{U}, \mathscr{V}, A \in \mathscr{U}, B \in \mathscr{V}, \mathscr{U}\Delta\mathscr{V}$, *is an EF-proximity such that, for every ultrafilter* $\mathscr{U}$, $\Pi(\mathscr{U}) = \Delta(\mathscr{U})$.

If further Δ *is such that the condition* '$\forall A \in \mathscr{U}, \exists\mathscr{V}, A \in \mathscr{V}, \mathscr{V}\Delta\mathscr{W}$' *implies* $\mathscr{U}\Delta\mathscr{W}$, *then* $\mathscr{U} \subset \Pi(\mathscr{W})$ *if and only if* $\mathscr{U}\Delta\mathscr{W}$.

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Proof. A set A_0 belongs to the left-hand side of (*) if and only if, $\forall A \in \mathscr{V}$, $\exists \mathscr{U}$, $A \in \mathscr{U}$ and $\exists \mathscr{W}$, $A_0 \in \mathscr{W}$, such that $\mathscr{U}\Delta\mathscr{W}$, i.e. if and only if $(A, A_0) \in \Pi_\Delta$ for all $A \in \mathscr{V}$. This is precisely the definition of '$A_0 \in \Pi(\mathscr{V})$' when (as is easily verified to be the case with $\Pi = \Pi_\Delta$) Π is a basic proximity. Thus, on interchanging $\mathscr{U}$, $\mathscr{V}$ in the notation, (*) asserts precisely that $A \in \Pi(\mathscr{U})$ if and only if $A \in \Delta(\mathscr{U})$, i.e. if and only if $\exists \mathscr{V}$, $A \in \mathscr{V}$, $\mathscr{U}\Delta\mathscr{V}$. It follows that if A_1, A_2 are any two members of $\Pi(\mathscr{U})$, $\exists\, \mathscr{V}_1, \mathscr{V}_2$ such that $A_1 \in \mathscr{V}_1$, $A_2 \in \mathscr{V}_2$, and, since Δ is an equivalence relation, $\mathscr{V}_1\Delta\mathscr{V}_2$, so that by definition $(A_1, A_2) \in \Pi$. Thus $\Pi(\mathscr{U})$ is a Π-clan, and by Theorem 6·7 this means that Π is an EF-proximity. The last condition mentioned in the enunciation (which states that, in the ultrafilter-space, the set $\{\mathscr{V}: \mathscr{V}\Delta\mathscr{W}\}$ is closed) now asserts exactly that if, $\forall A \in \mathscr{U}$, $A \in \Pi(\mathscr{W})$ then $\mathscr{U}\Delta\mathscr{W}$; the converse is immediate from the definitions of Π and of $\Pi(\mathscr{W})$.

Remark 3. In Theorem 7·2, the given conditions ($\mathscr{G}_1$ and $\mathscr{G}_2$ are e-grills with respect to Π' and $\mathscr{G}_1 \times \mathscr{G}_2 \subset \Pi'$) imply that $\mathscr{G}_2 = \mathscr{G}_1$ and is a Π'-cluster, so that the theorem is rather less general than it appears.

Proof of remark. Suppose $A \notin \mathscr{G}_1$. Since $e(\Pi', \mathscr{G}_1) = \mathscr{G}_1$ we have, by definition of e, $\mathscr{N}(\Pi', A) \not\subset \mathscr{G}_1$, so $\exists B \notin \mathscr{G}_1$, $(X \backslash B) \notin \Pi'(A)$. Hence $(X \backslash B) \in \mathscr{G}_1$ while $(X \backslash B, A) \notin \Pi'$, so as $\mathscr{G}_1 \times \mathscr{G}_2 \subset \Pi'$ we have $A \notin \mathscr{G}_2$. The converse also holds since, Π' being symmetrical, $\mathscr{G}_2 \times \mathscr{G}_1$ is also included in Π'. Thus $\mathscr{G}_1 = \mathscr{G}_2$.

If $\mathscr{U}$ is any ultrafilter $\subset \mathscr{G}_1$ we have

$$e(\Pi', \mathscr{U}) \subset e(\Pi', \mathscr{G}_1) = \mathscr{G}_1.$$

Now $\mathscr{G}_1$ is a Π'-clan since we now know that $\mathscr{G}_1 \times \mathscr{G}_1 \subset \Pi'$. By Theorem 6·2 (*e*), $e(\Pi', \mathscr{U})$ is a Π'-cluster and, being contained in $\mathscr{G}_1$, must coincide with $\mathscr{G}_1$.

II. THE COMPACT HYPERSPACE OF ROUND FILTERS

While I am unable to give an exact reference, it has for some years been known to me, and I believe to many other workers in the field, that the space of all round filters in a given uniform space, considered as a subspace of $\mathscr{P}[\mathscr{P}(X)]$ with the uniformity obtained by twice applying the Hausdorff process, is separated and complete and, if the original space is separated, contains the classical completion by round Cauchy filters as a subspace. It follows that the space of all round filters in a given separated EF-proximity space, with the appropriate topology, is compact and contains the T2 compactification associated with the given proximity. The following direct proof of this fact may be of interest.

THEOREM. *Let X^* be the set of all round filters $\mathscr{R}$ in a separated proximity space (X, δ) and $\mathscr{T}^*$ the topology with sub-basic sets*

$$G(A) = \{\mathscr{R}: A \in \mathscr{R}\} \quad \text{and} \quad H(A) = \{\mathscr{R}: \exists B, A < B, B \notin \mathscr{R}\}$$

where A runs through all subsets of X and as usual $A < B$ means $A\delta(X \backslash B)$. Then $(X^*, \mathscr{T}^*)$ is compact T2 and hence has a unique associated EF-proximity. Defining the injection $i: X \to X^*$, by $xi = \mathscr{N}_x = \{N; \{x\} < N\}$ we have $\mathscr{R} \in (Ai)^-$ if and only if $\mathscr{R}$ is compressed

https://doi.org/10.1017/S0305004100052294 Published online by Cambridge University Press

and $\mathscr{R} \notin G(X\backslash A)$; *in particular* $(Xi)^-$ *is the set of all compressed round filters. Finally,* $A\delta B$ *if and only if*

$$(Ai)^- \cap (Bi)^- \neq \varnothing .$$

Proof. We recall that a filter $\mathscr{F}$ is compressed with respect to a proximity if and only if $A < B$ implies that either B or $X\backslash A$ is in $\mathscr{F}$; that is, a round filter $\mathscr{R}_0$ is compressed if and only if, for all A, $\mathscr{R}_0 \in H(A)$ implies $\mathscr{R}_0 \in G(X\backslash A)$. We note that in any case $G(X\backslash A) \subset H(A)$, for if $X\backslash A \in \mathscr{R}$ then $\exists C \in \mathscr{R}$, $C < X\backslash A$, so $(X\backslash C) > A$ and $(X\backslash C) \notin \mathscr{R}$.

Now let $\mathscr{U}^*$ be any ultrafilter on X^*. Then it is clear that

$$\mathscr{F}_0 = \{A;\, G(A) \in \mathscr{U}^*\}$$

is a filter in X, since $G(A \cap B) = G(A) \cap G(B)$. Then

$$\mathscr{R}_0 = \{B;\, \exists A < B, A \in \mathscr{F}_0\}$$

is an element of X^*: we shall show that every sub-basic neighbourhood of $\mathscr{R}_0$ is in $\mathscr{U}^*$ so that $\mathscr{U}^* \to \mathscr{R}_0$. This is immediate from the definitions if the neighbourhood is of form $G(A)$, since $\mathscr{R}_0 \subset \mathscr{F}_0$. Now consider a neighbourhood $H(A)$; $\mathscr{R}_0 \in H(A)$ so $\exists B$, $B > A$, $B \notin \mathscr{R}_0$. Then $\exists B_1$, $A < B_1 < B$, so $B_1 \notin \mathscr{F}_0$, whence $G(B_1) \notin \mathscr{U}^*$; since $\mathscr{U}^*$ is an ultrafilter this gives

$$\{\mathscr{R};\, B_1 \notin \mathscr{R}\} = [X^*\backslash G(B_1)] \in \mathscr{U}^*.$$

But this family of filters is clearly included in $H(A)$ whence $H(A) \in \mathscr{U}^*$ as required.

Now let $\mathscr{R}_1 \neq \mathscr{R}_2$, say $\mathscr{R}_1 \not\subset \mathscr{R}_2$ so that $\exists B_1 \in \mathscr{R}_1 \backslash \mathscr{R}_2$ and then $\exists A_1 < B_1$, $A_1 \in \mathscr{R}_1$. That is, $\mathscr{R}_1 \in G(A_1)$, $\mathscr{R}_2 \in H(A_1)$; these two open sets are obviously disjoint. We have now shown that we have a compact T2 topology.

If $\mathscr{R}_0$ is not compressed, we can find A, B and then B_1 such that $A < B_1 < B$ and neither $X\backslash A$ nor B is in $\mathscr{R}_0$. Since $(X\backslash B_1) < (X\backslash A)$, $H(B_1) \cap H(X\backslash B_1)$ is a neighbourhood of $\mathscr{R}_0$. Now, for any x, either $x \in B_1$, in which case all sets $> B_1$ are in $\mathscr{N}_x$ and $\mathscr{N}_x \notin H(B_1)$, or else $x \in X\backslash B_1$ and so similarly $\mathscr{N}_x \notin H(X\backslash B_1)$. That is, the above-mentioned neighbourhood of $\mathscr{R}_0$ does not meet Xi, so that $(Xi)^-$, and *a fortiori* $(Ai)^-$, $A \subset X$, consists only of compressed round filters. By the remark at the beginning of the proof, if $C^* \subset X^*$ is the set of compressed round filters then $C^* \cap H(A) = C^* \cap G(X\backslash A)$, so that the topology of this subspace (which includes $(Xi)^-$) is determined by sets of the form $G(A) \cap C^*$; these in fact form a base since $G(A_1 \cap A_2) = G(A_1) \cap G(A_2)$. That is, C^*, with the subspace topology, is just the space of round compressed (= maximal round) filters with the usual topology. The rest is well known from the standard compactification theory. (It is almost immediate that, for $\mathscr{R} \in C^*$, $\mathscr{R} \in (Ai)^-$ if and only if $\mathscr{R} \notin G(X\backslash A)$, i.e. if and only if $B \in \mathscr{R} \Rightarrow \exists x \in A, \{x\} < B$; and hence that

$$\mathscr{R} \in (Ai)^- \cap (Bi)^- \Rightarrow A\delta B$$

since otherwise $A < X\backslash B$ and neither $X\backslash A$ nor $X\backslash B$ could be in $\mathscr{R}$: the proof of the converse is rather longer but is omitted here.)

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III. SEMI-FILTERS AND SEMI-GRILLS IN RELATION TO PROXIMITIES AND UNIFORMITIES

We shall call a (non-empty) family $\mathscr{S}$ of subsets of a given set X a *semi-filter* (on X) if and only if it satisfies

(i) $A \in \mathscr{S}$ and $A \subset B \Rightarrow B \in \mathscr{S}$;

(ii) $A_1, A_2 \in \mathscr{S} \Rightarrow A_1 \cap A_2 \neq \varnothing$.

In the language of (2), $\mathscr{S}$ is a semi-filter if and only if it is a stack and

$$\mathscr{S} \subset c\mathscr{S} = \{A;\ X \backslash A \notin \mathscr{S}\}.$$

We shall denote the last-mentioned family by $\mathscr{S}^d$ (cf. (1), p. 34, Definition 6· (c)). Dually, $\mathscr{S}$ will be called a *semi-grill* if and only if it satisfies (i) and also

(ii$d$) $A_1 \cup A_2 = X \Rightarrow A_1 \in \mathscr{S}$ or $A_2 \in \mathscr{S}$.

In relation to a given EF-proximity (resp. a uniformity) on X, a surprisingly large number of the properties of filters and grills are shared by semi-filters and semi-grills respectively; in particular, they can be used to form from X a compact (resp. complete) hyperspace including as a closed subspace the space of round filters considered in the second of these notes. For Lodato proximities, as we shall briefly show, the position is less neat. Where no indication of proof is given, the proof of statements below is either obvious, or virtually identical with a standard proof for filters.

Since this paper was prepared it has come to the author's notice that set-families which in the above notation would be called 'semi-filter bases' were introduced, under the name of 'linked-systems', by J. de Groot and J. M. Aarts (4); and compact 'super-extensions' of a topological space, formed by a method of Wallman type using such linked systems, were studied by A. Verbeek (5). These superextensions however were formed by reference to closed sub-bases of the topology concerned, not by reference to any proximity, and do not appear to have any close relation with the compact spaces defined in this note.

1. *Set-theoretic properties.* Clearly $\mathscr{S}$ is a semi-filter if and only if its dual $\mathscr{S}^d$ is a semi-grill. The following lemma is immediate:

(1) *If $\mathscr{S}$ is a semi-filter and $X \backslash A \notin \mathscr{S}$ then $\mathscr{S} \cup \{E: E \supset A\}$ is also a semi-filter.*

COROLLARY. *A semi-filter is maximal if and only if $\mathscr{S} = \mathscr{S}^d$, i.e. if and only if it is also a semi-grill; it is then a minimal semi-grill.*

In contrast to the position for filters, we can easily give both trivial and non-trivial examples of actual definitions of maximal semi-filters.

Example A. a, b, c any three distinct points of X; $\mathscr{S} = \{A:$ at least two of a, b, c in $A\}$ is a maximal semi-filter.

Example B. Let ϕ be a non-decreasing real-valued function on $\mathscr{P}(X)$: e.g. for $X = \{\xi;\ 0 \leqslant \xi \leqslant 1\}$, $\phi(A)$ may be taken as diam (A) or (alternatively) as $m(\mathrm{Int}\, A)$. Then $\mathscr{S} = \{A;\ \phi(A) > \phi(X \backslash A)$ or $\phi(A) = \phi(X \backslash A)$ and $x_0 \in A\}$, where x_0 is an arbitrarily chosen point of X, is a maximal semi-filter.

(2) PROPOSITION. *If X has at least three points, and $\mathscr{S}_0$ is a semi-filter and not an ultrafilter, there there exists a semi-filter $\mathscr{S} \supset \mathscr{S}_0$ containing three sets A_1, A_2, A_3 such that*

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$A_1 \cap A_2 \cap A_3 = \varnothing$, *and hence a maximal semi-filter* $\mathscr{M} \supset \mathscr{S}_0$ *which is not an ultrafilter.*

Proof. By (1) we may suppose that $\mathscr{S}_0$ contains a set E_0 which is neither X nor a singleton. Unless $\mathscr{S}_0$ is maximal (when the proposition is trivial) there exists an $A \subset X$ such that neither A nor $X \backslash A$ is in $\mathscr{S}_0$, so that every set of $\mathscr{S}_0$ meets both $X \backslash A$ and A. We may choose the notation so that $E_0 \not\supset A$; then by (1) $\mathscr{S}_1 = \mathscr{S}_0 \cup \{E; E \supset A\}$ is a semi-filter, and $E_0 \cap A \notin \mathscr{S}_1$; again applying (1) we see that $\mathscr{S} = \mathscr{S}_0 \cup \{E: E \supset A\} \cup \{E: E \supset X \backslash (E_0 \cap A)\}$ is a semi-filter containing A, E_0, and $X \backslash (E_0 \cap A)$ whose intersection is empty.

2. *Properties connected with a topology or proximity on* X. If X has a topology $\mathscr{T}$, and $\mathscr{S}$ is a semi-filter, we define, just as for filters, $\mathscr{S} \to x_0$ if and only if $\mathscr{S} \supset \mathscr{N}_{x_0}$, the neighbourhood filter, and $x_0 \in Ad\,\mathscr{S}$ if and only if $A \in \mathscr{S} \Rightarrow x_0 \in \bar{A}$, that is, if and only if $\mathscr{S}^d \supset \mathscr{N}_{x_0}$. Proposition 2 shows that in general convergent semi-filters exist which are not filters.

If X has a symmetric (Čech) proximity δ, we define a *δ-family* as any family $\mathscr{A}$ of subsets of X such that $A_1, A_2 \in \mathscr{A} \Rightarrow A_1 \delta A_2$, a *$\delta$-semi-clan* as a semi-grill which is a δ-family, a *δ-compressed semi-filter* as one whose dual is a δ-semi-clan, and a *round semi-filter* as one, $\mathscr{S}$ say, such that $\mathscr{S} = r(\mathscr{S})$, where $r(\mathscr{S})$ denotes

$$\{B: \exists A \in \mathscr{S},\ A \delta (X \backslash B)\},$$

which is clearly a semi-filter.

(3) Proposition. *For any Čech proximity:*

(*a*) *the maximal semi-clans and maximal δ-families coincide;*

(*b*) *a round compressed semi-filter $\mathscr{R}$ such that A_1, A_2 and $A_3 \in \mathscr{R} \Rightarrow A_1 \cap A_2 \cap A_3 \neq \varnothing$ is necessarily a filter;*

(*c*) *a round compressed semi-filter is maximal round and minimal compressed;*

(*d*) *if $\mathscr{S}_1$, $\mathscr{S}$ are semi-filters with $\mathscr{S}_1 \supset \mathscr{S}$, $\mathscr{S}$ round compressed, then $\mathscr{S} = r(\mathscr{S}_1)$.*

Proof. If $\mathscr{A}$ is a maximal δ-family and $A \in \mathscr{A}$, then clearly $A_1 \supset A \Rightarrow \mathscr{A} \cup \{A_1\}$ a δ-family, so that $A_1 \in \mathscr{A}$. If $X \backslash A_1 \notin \mathscr{A}$ then $\exists A_0, A_0 \in \mathscr{A}$, $(X \backslash A_1) \delta A_0$: *a fortiori* $(X \backslash A_1) \cap A_0 = \varnothing$, $A_1 \supset A_0$, so $A_1 \in \mathscr{A}$. Thus every maximal δ-family is a δ-semi-clan, whence (*a*) follows. Under the conditions of (*b*), $\exists B_1, B_2 \in \mathscr{R}$, $B_1 < A_1$ (i.e. $B_1 \delta (X \backslash A_1)$), $B_2 < A_2$, so $B_1 \cap B_2 < A_1 \cap A_2$. By the data, $X \setminus (B_1 \cap B_2) \notin \mathscr{R}$, so as $\mathscr{R}$ is compressed $A_1 \cap A_2 \in \mathscr{R}$. (*c*) and (*d*) are proved just as for filters.

(4) Theorem. *If δ is a separating EF-proximity on X, and $X^\dagger$ is the space of round semi-filters $\mathscr{R}$ on X, with the topology $\mathscr{T}^\dagger$ with sub-base $\{G(A); A \subset X\}$, $\{H(A); A \subset X\}$, where $G(A) = \{\mathscr{R}; A \in \mathscr{R}\}$ and $H(A) = \{\mathscr{R}; \exists B > A, B \notin \mathscr{R}\}$ (cf.* (3)) *then $(X^\dagger, \mathscr{T}^\dagger)$ is a compact T2 space. The subsets $X_{\mathscr{F}}^\dagger$ of round filters and $X_{\mathscr{C}}^\dagger$ of round compressed semi-filters are each closed, hence compact (so that their intersection is the usual compactification of (X, δ) by round compressed filters).*

The proof of compactness and of the T2 property is almost exactly as in note II above, except that if $\mathscr{U}^\dagger$ is an ultrafilter on $X^\dagger$ then $\{A; G(A) \in \mathscr{U}^\dagger\}$ is a semi-filter instead of a filter. (Just as for filters, if $\mathscr{S}$ is a semi-filter and δ is an EF-proximity, then

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$r(\mathscr{S})$ is round and is compressed if $\mathscr{S}$ is compressed. Moreover, the maximal round semi-filters, minimal compressed semi-filters (which we know are the duals of maximal δ-families), round compressed semi-filters, and filters of the type $r(\mathscr{M})$, $\mathscr{M}$ a maximal semi-filter, all coincide.) The space $(X^*, \mathscr{T}^*)$ of round filters, constructed in note II, is a compact, hence closed, subspace of the T2 space $(X^\dagger, \mathscr{T}^\dagger)$. If $\mathscr{S}_0$ is a round, but not compressed, semi-filter, $\exists A, B, A < B$, such that neither B nor $X\backslash A$ is in $\mathscr{S}_0$. If $A < A_1 < B$ then $H(A_1) \cap H(X\backslash A_1)$ is a neighbourhood of $\mathscr{S}_0$ not meeting $X^\dagger_\mathscr{C}$, so that this last set is closed.

In the case of an LO-proximity, we have the following analogous but considerably weaker result (which can be stated either in terms of minimal compressed semi-filters or, as here, in terms of their duals).

(5) THEOREM. *Let δ be an LO-proximity on X, X^* the set of all maximal δ-families $\mathscr{M}$ on X, and $\mathscr{T}^*$ the topology on X^* with sub-base $\{\{\mathscr{M}; A \notin \mathscr{M}\}: A \subset X\}$. Then $(X^*, \mathscr{T}^*)$ is $T1$ and every compressed filter in X^*, with respect to δ_0^*, where*

$$A^* \delta_0^* B^* \text{ if and only if } \bar{A}^* \cap \bar{B}^* \neq \varnothing,$$

is convergent. Moreover, if $i: X \to X^*$, is defined by $xi = \mathscr{M}_x = \{A: \{x\} \delta A\}$, then $A \delta B$ if and only if $(Ai)^- \cap (Bi)^- \neq \varnothing$. However, $((Xi)^-, \mathscr{T}^*)$ is not in general equivalent to the maximal-bunch compactification of (X, δ), and $\{(Ai)^-; A \in X\}$ is not necessarily a base or sub-base of closed sets in $((Xi)^-, \mathscr{T}^*|(Xi)^-)$.

Proof. If $\mathscr{F}^*$ is a compressed filter in X^*, so that $(\mathscr{F}^*)^d$ is a δ_0^*-family, let

$$\mathscr{S}_0 = \{A; \{\mathscr{M}; A \in \mathscr{M}\} \in (\mathscr{F}^*)^d\};$$

then if $A_1, A_2 \in \mathscr{S}_0$ we see that $\{\mathscr{M}; A_r \in \mathscr{M}\}, r = 1, 2$, are intersecting closed sets so that $A_1 \delta A_2$; that is, $\mathscr{S}_0$ is a δ-family contained in some $\mathscr{M}_0$ say. Then $A \notin \mathscr{M}_0$ implies $A \notin \mathscr{S}_0, \{\mathscr{M}: A \notin \mathscr{M}\} \in \mathscr{F}^*$; hence $\mathscr{F}^* \to \mathscr{M}_0$ (*a fortiori*, $(X^*, \mathscr{T}^*)$ is compact). The T1 property is almost obvious, as is the fact that if $A \delta B$ then, for every $\mathscr{M}_0$, one of $\{\mathscr{M}: A \notin \mathscr{M}\}$ and $\{\mathscr{M}: B \notin \mathscr{M}\}$ is a neighbourhood of $\mathscr{M}_0$, so that $(Ai)^- \cap (Bi)^- = \varnothing$. Conversely, if $A \delta B$, by Thron's lemmas ((2) Theorems 3·14 and 3·15) there exists a maximal clan $\mathscr{C}$ containing A and B and hence a maximal δ-family $\mathscr{M}_0 \supset \mathscr{C}$. If $\bigcap_{r=1}^{n} \{\mathscr{M}; A_r \notin \mathscr{M}\}$ is a basic neighbourhood of $\mathscr{M}_0$, then $\forall r$ certainly $A_r \notin \mathscr{C}$, and so $A_1 \cup \ldots \cup A_n \notin \mathscr{C}$, $\bar{A}_1 \cup \ldots \cup \bar{A}_n \not\supset A$ (for every maximal clan with respect to a LO-proximity is a bunch). That is, $\exists x \in A$, $\forall r$, $A_r \notin \mathscr{M}_x$; whence $\mathscr{M}_0 \in (Ai)^-$ and similarly $\mathscr{M}_0 \in (Bi)^-$.

If we knew also that $\{(Ai)^-: A \subset X\}$ was a base of closed sets for $\mathscr{T}^*|(Xi)^-$ we could deduce (see (3)) that $(Xi)^-$ was equivalent to the maximal bunch compactification. Unfortunately neither of these need be true, as we see by the following example.

(6) *Example.* X consists of three sequences of points (a_n), (b_n) and (c_n), $(n = 1, 2, \ldots)$ and four special points x_1, x_2, y, z. All finite sets are closed: the closures of infinite sets are determined as follows: if $a_n \in E$ for an infinity of n, then y and $z \in \bar{E}$: similarly cyclically for z, x_1 and (b_n), x_1, y and (c_n). Finally, if either b_n or $c_n \in E$ for an infinity of n then $x_2 \in \bar{E}$. This defines a T1 topological space which is easily seen to be compact. Writing $A \delta_0 B$ if and only if $\bar{A} \cap \bar{B} \neq \varnothing$, we see that the only maximal clans (maximal bunches)

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are the point-clans $\{A: x \in \bar{A}\}$ and the family of all infinite sets (since if a finite set belongs to a clan, so does some singleton-set, and no x is in $\bar{A}$ for *all* infinite A). On the other hand, each of the families $\mathscr{M}_1 = \{A:$ either A is infinite or it contains at least two of $x_1, y, z\}$ and $\mathscr{M}_2$ (similar with x_2 for x_1) is a maximal δ-family. We easily check that $\mathscr{M}_1 \in (Ai)^-$ if and only if A is infinite; the same holds for $\mathscr{M}_2$. Thus the sets $(Ai)^-$ cannot even be a sub-base of the closed sets of the T1 topology $\mathscr{T}^*|(Xi)^-$. Again, it is obvious that there can be no homeomorphism of the maximal-bunch space on to $(Xi)^-$ which reduces to identity on the set of point-clans.

3. *Semi-filters on a uniform space.* If $(X, \mathscr{V})$ is a uniform space, with its associated EF-proximity, it is clear from the examples given earlier that, even if $(X, \mathscr{V})$ is totally bounded, a compressed semi-filter will not usually be Cauchy and so cannot be expected to converge in a complete uniform extension of $(X, \mathscr{V})$. We have in fact, as is easily verified:

(7) PROPOSITION. *A semi-filter $\mathscr{S}$ is Cauchy if and only if $r(\mathscr{S})$ (with respect to $\delta(\mathscr{V})$) is a Cauchy filter.*

Corresponding to (and indeed including, in view of the well-known relation between uniformities and proximities) Theorem (4) above, we have a result embedding the completion of a separated space $(X, \mathscr{V})$ in a complete space of semi-filters. To each $V \in \mathscr{V}$ we associate the set $V^\dagger = \{(\mathscr{S}_1, \mathscr{S}_2);\ A_1 \in \mathscr{S}_1 \Rightarrow A_1 \circ V \in \mathscr{S}_2$ and $A_2 \in \mathscr{S}_2 \Rightarrow A_2 \circ V \in \mathscr{S}_1$, (all $A_1, A_2 \subset X)\}$. Just as for filters, these sets form the base of a uniformity $\mathscr{V}^\dagger$ for the set of all semi-filters $\mathscr{S}$ on $(X, \mathscr{V})$, whose restriction to the set $X^\dagger$ of all round semi-filters $\mathscr{R}$ is separating, and we have:

(8) THEOREM. *The space $(X^\dagger, \mathscr{V}^\dagger)$ is separated and complete, and contains as closed sub-spaces the sets $X^\dagger_{\mathscr{F}}$ of all round filters and $X^\dagger_{\mathscr{C}}$ of all round compressed semi-filters.* (If $(X, \mathscr{V})$ is separated, $X^\dagger_{\mathscr{F}} \cap X^\dagger_{\mathscr{C}}$ of course contains the separated completion of $(X, \mathscr{V})$, being identical with it if and only if $(X, \mathscr{V})$ is totally bounded. In this case it is easy to prove that $(X^\dagger, \mathscr{V}^\dagger)$ also is totally bounded, hence compact, and we recover Theorem (4).)

The proof proceeds (as for filters) by showing that if $\mathscr{F}^\dagger$ is a Cauchy filter on $(X^\dagger, \mathscr{V}^\dagger)$ then $\mathscr{F}^\dagger \to \mathscr{R}_0 = r(\mathscr{S}_0)$, where $\mathscr{S}_0 = \{A: \{\mathscr{R} \in X^\dagger;\ A \in \mathscr{R}\} \in \mathscr{F}^\dagger\}$. The details are left to the reader, as is the proof that the stated sub-spaces are closed.

The work presented in these notes was carried out while the author was in receipt of a grant from funds of the National Research Council of Canada.

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https://doi.org/10.1017/S0305004100052294 Published online by Cambridge University Press