A NOTE ON THE AIRY-INTEGRAL FUNCTION

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In view of the importance of the Airy-integral function in theoretical physics, it was thought that the following elementary treatment of this function may be of some interest. The arguments used are similar to those often employed in the teaching of elementary physics.

The example chosen for illustration is from optics, since this was used by Airy (1) in his original paper. The intensity of light near a caustic will be found by using Huyghens's principle to construct 'amplitude-phase' diagrams for a series of neighbouring points. Each of these diagrams is a spiral, somewhat analogous to Cornu's spiral. Its vector resultant gives the amplitude and phase of the light at the point considered.

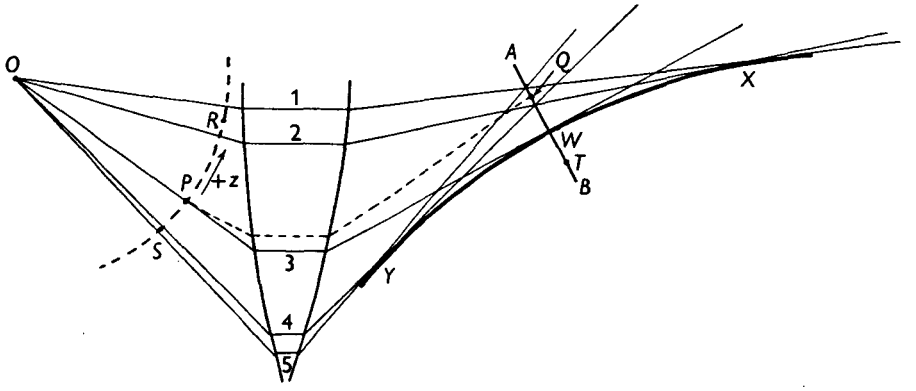


Fig. 1. Caustic formed by a beam of light incident on a lens.

Consider a beam of light emerging from a point source O, and incident from the left on a convex lens (Fig. 1). For simplicity the lens is assumed to be cylindrical, and the problem is treated as though it were in two dimensions only. It is well known that the rays of the beam are tangents to a caustic curve XY. It is required to find the intensity of the light at the points of a line AB perpendicular to the caustic. Let Q be a typical point of this line. Consider a cylindrical wave front of the beam (RS in Fig. 1), just before it reaches the lens. Imagine this wave front to be divided into infinitesimal strips of equal width dz, which all radiate cylindrical waves of the same intensity. On arrival at Q these waves all have the same amplitude, but their phases depend on the position, z, of the strip from which each originates. If Q is inside the caustic, as shown in Fig. 1, two geometrical rays pass through it. Assume that one of these, RQ, is included in the narrow pencil between rays 1 and 2, and the other, SQ, in the pencil between rays 4 and 5.

The pencil enclosed by the rays 1 and 2 (Fig. 1) comes to a focus at X. Q is within this pencil, and the light reaches it before it reaches the focus X. It is then easily shown

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that for the ray RQ within this pencil Fermat's principle is a principle of *least* time. The variation with z of the phase $\phi(z)$ therefore has a minimum for Huyghens's wavelets which originate near R. The pencil enclosed by the rays 4 and 5 comes to a focus at Y. Q is within the pencil and beyond the focus, and it can then similarly be shown that for the ray SQ Fermat's principle is a principle of *greatest* time, so that the phase $\phi(z)$ has a maximum for Huyghens wavelets which originate near S. $\phi(z)$ has no other turning points (Fig. 2, curve Q). If Q is outside the caustic, at T, say, no geometrical rays pass through it. Then $\phi(z)$ has no turning points, and varies as shown in Fig. 2, curve T. If Q is on the caustic, at W say, the two turning points coincide (Fig. 2, curve W).

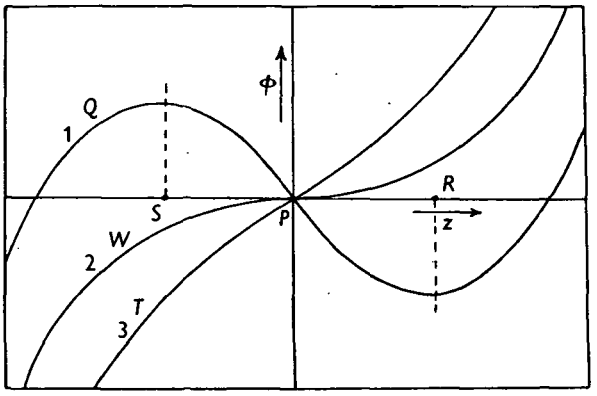


Fig. 2. Variation of phase, $\phi(z)$, of Huyghens's wavelet with position, z, of source on initial wavefront.

Let the zero of z be chosen midway between the two turning points, or at the point of least slope when there are no turning points. Then the form of the curves in Fig. 2 is given by the cubic law

$$\phi = K(az + \tfrac{1}{3}z^3), \tag{1}$$

where a depends on the distance of Q from the caustic, and K is a constant. a is negative when Q is on the illuminated side of the caustic, as in Fig. 1, zero when Q is on the caustic, and positive when Q is outside the caustic. Clearly a could be expressed by an ascending power series in $u = QW$, and if a is small enough, powers higher than the first can be neglected.

For each position of the point Q an amplitude-phase diagram may now be constructed. The curvature of this at a given point is proportional to the slope at the corresponding point in the (ϕ, z)-curve. Since this slope eventually increases indefinitely both for z increasing and for z decreasing, the amplitude-phase diagrams end in spirals. Typical diagrams are shown in Fig. 3, corresponding to the three curves of Fig. 2. There is an obvious analogy between these diagrams and Cornu's spiral, for which the (ϕ, z)-curve is a parabola. Cornu's spiral always has the same form. In diffraction problems whose solution involves a Cornu spiral, variations of intensity occur, because different fractions of the spiral are used to determine the intensity reaching different points. In the present problem, however, the shape of the spiral is different for different values of a. For all points of the line AB, the resultant of the

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whole of the spiral is found, but the intensity varies because the shape of the spiral varies from point to point.

In Fig. 3, let the x-axis be chosen parallel to the element corresponding to $z = 0$. Then the diagrams are symmetrical about the y-axis, and the resultant vector is always parallel to the x-axis. When a is positive (first diagram of Fig. 3), the curvature of the

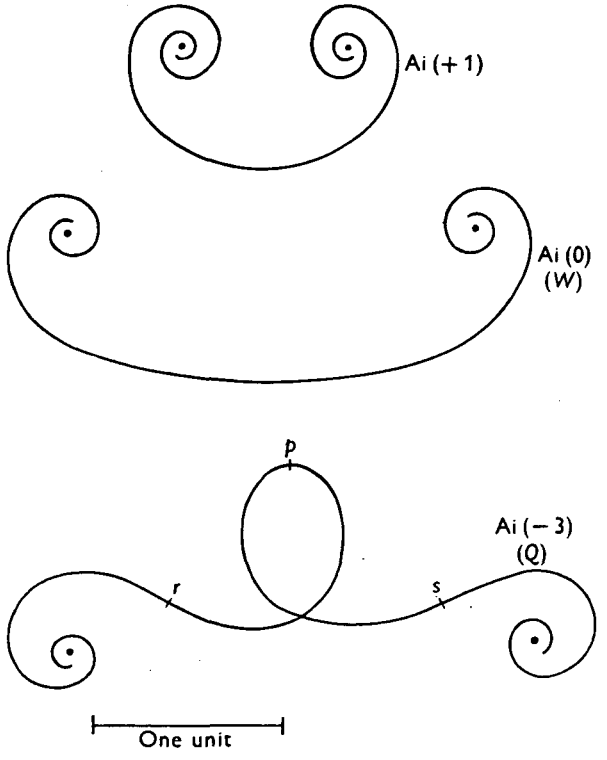


Fig. 3. Spirals associated with the Airy-integral function. In the third spiral the points p, r, s correspond roughly to the points P, R, S in Figs. 1 and 2.

spiral always has the same sign and the length of the resultant decreases monotonically as a increases. When a is negative (third diagram of Fig. 3), the middle section of the spiral has opposite curvature from the rest, and as a decreases the limiting points of the spiral repeatedly move completely round the origin. The resultant thus alternates in sign, and in the third diagram of Fig. 3 it is negative.

The contribution of an element dz of the wavefront to the resultant vector is proportional to $dz \cos\phi$, and hence the resultant vector is proportional to

$$\int_{-\infty}^{\infty} \cos\phi \, dz = \int_{-\infty}^{\infty} \cos(az + \tfrac{1}{3}z^3)\, dz.$$

Apart from a constant, this is the standard expression for the Airy integral function, Ai (a).

When Q is well beyond the caustic on the illuminated side (a large and negative), the turning points of the (ϕ, z)-curve are widely separated, and near each turning point the curve is approximately a parabola. Hence the portion of the amplitude-phase

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diagram corresponding to the neighbourhood of each turning point is approximately a Cornu spiral. The two Cornu spirals curve opposite ways, and together constitute the more complex spiral whose resultant is the Airy-integral function (see Fig. 4). When a is large and negative, the Airy-integral function $\mathrm{Ai}(-a)$ is proportional approximately to

$$(-a)^{-\frac{1}{4}}\exp\left[i\{\tfrac{2}{3}(-a)^{\frac{3}{2}}-\tfrac{1}{4}\pi\}\right]-(-a)^{-\frac{1}{4}}\exp\left[-i\{\tfrac{2}{3}(-a)^{\frac{3}{2}}-\tfrac{1}{4}\pi\}\right]$$

(Miller (3)). These two terms may be regarded as the resultants of the two Cornu spirals.

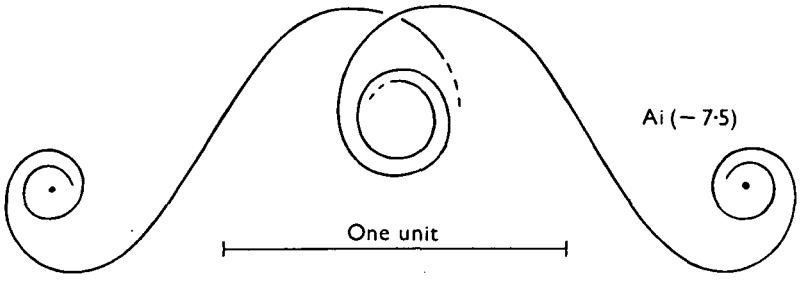


Fig. 4. Spiral associated with Airy-integral function of large negative argument, showing resemblance to two Cornu spirals. The part of the left half of the diagram where overlapping occurs is not shown.

An argument very similar to the above can be used to show that when a radio wave is reflected from the ionosphere, and is received at the ground near the 'skip' distance, the law of variation of the received signal over the ground is given by the Airy-integral function. The physical optics in the two cases is practically the same, but the geometry is different.

Another example occurs in the theory of propagation of waves through a homogeneous, dispersive, non-absorbing medium, in which the propagation constant, k, expressed as a function of the angular frequency p has a minimum value k_0 at frequency p_0. The impulse response of such a medium includes a disturbance consisting of a wave whose fundamental frequency is p_0, and whose modulation envelope is an Airy-integral function, $\mathrm{Ai}\{-Cx^{\frac{1}{3}}(t-x/U_0)\}$, where t is time, x is distance travelled through the medium, U_0 is the group velocity at frequency p_0, and C is a constant (see, for example, Baerwald (2)*). This result can be derived, for given values of x and t, by constructing a spiral whose elements are the infinitesimal Fourier components of the original impulse with their phases modified by propagation through the medium. The same series of spirals is obtained as in the first part of this note.

REFERENCES

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(2) Baerwald, H. G. *Ann. Phys., Lpz.*, (5), 6 (1930), 295.
(3) Miller, J. C. P. *British Association mathematical tables*, part-vol. B (Cambridge, 1946).

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* This author uses Bessel functions of order $\frac{1}{3}$, and it is well known that these can be expressed in terms of the functions Ai and Bi (Miller (3), p. B 9).

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